WEAK DISCERNIBILITY FOR QUANTA, THE RIGHT WAY

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Abstract: Muller and Saunders ([2008]) purport to demonstrate that, surprisingly, bosons and fermions are discernible; this paper disputes their arguments, then derives a similar conclusion in a more satisfactory fashion. After briefly explicating their proof, and indicating how it escapes earlier indiscernibility results, we note that the observables which Muller and Saunders argue discern particles are (i) non-symmetric in the case of bosons and (ii) trivial multiples of the identity in the case of fermions. Both problems undermine the claim that they have shown particles to be *physically* discernible. We then prove two results concerning observables that truly are physical: one showing when particles are discernible and one showing when they are not (categorically) discernible. Along the way we clarify some frequently misunderstood issues concerning the interpretation of quantum observables.

Organization:

1. Background

Muller and Saunders ([2008]) argue, against a prior consensus, for the discernibility of quantum particles. While we agree (largely) with this conclusion and (wholly) with the intuition behind it, we dispute the argument that they give for it. Thus the primary aims of this paper are to explain what goes wrong in the way that they formalize their intuitive idea and then to propose an alternative approach that avoids the difficulties, but establishes a very similar conclusion. Along the way, our secondary goal is to address some important misconceptions from the philosophical literature on the subject, and which cause problems for Muller and Saunders. In particular, we argue for more care concerning the correspondence between quantum operators and physical magnitudes.

First a note on methodology. Their stated aim (499) is to proceed with only minimal interpretational assumptions about quantum mechanics and without taking unnecessarily controversial positions on metaphysical issues, in order to give the strongest and most general case possible; we will follow suit. So rather than show the impossibility of their position, we will explain why it would require extra-ordinary assumptions about physics or metaphysics – and show how our alternative account requires only safe assumptions. No more is intended.

With that very general observation about the terms of the argument understood, let us quickly review some of the essential philosophical and physical background: for a more extended discussion the reader is referred back to Muller and Saunders' paper. First, Quine ([1976]) distinguishes three 'grades of discriminability'. The weakest is the least sufficient to define an identity relation satisfying the usual Fregean axioms (of reflexivity and substitutivity). Using a fairly standard terminology (which diverges from Quine's): a pair of objects a and b are ...

- … *strongly discernible* iff $\exists \mathcal{P}\ \mathcal{P}(a) \wedge \neg \mathcal{P}(b)$
- … *relatively discernible* iff $\exists \mathcal{R}, c\ \mathcal{R}(a, c) \wedge \neg \mathcal{R}(b, c)$
- … *weakly discernible* iff $\exists \mathcal{R}\ \mathcal{R}(a, b) \wedge \mathcal{R}(b, a) \wedge \neg \mathcal{R}(a, a) \wedge \neg \mathcal{R}(b, b)$.

Furthermore, Strongly Discernible $\Rightarrow$ Relatively Discernible $\Rightarrow$ Weakly Discernible, while neither of the converse implications hold.

As Saunders ([2003]) emphasized, in quantum mechanics (QM) discussions of discernibility and indiscernibility have focussed almost exclusively on strong and relative discernibility, with most commentators arguing that bosons and fermions (collectively 'quanta') are neither strongly nor relatively discernible. However, Saunders, singly and then jointly with Muller, has argued that fermions, and bosons sometimes, are weakly discernible.[1] The basic intuition is quite simple: consider a pair of electrons in a 'Bell-Bohm', state: $1/\sqrt{2}(|\uparrow\rangle|\downarrow\rangle - |\downarrow\rangle|\uparrow\rangle)$. Although neither has definite spin at all according to the standard interpretation, it appears true of each that it has opposite spin to the other, and equally false that it has opposite spin to itself. If so, the relation 'has opposite spin to' satisfies the conditions for weakly discerning the electrons.[2]

We accept that this intuition is correct, so that such quanta are weakly discerned; *however, we do not think that Muller and Saunders satisfactorily generalize and express it in the quantum formalism.* In the remainder of this section we will briefly review why quanta are neither strongly nor relatively discernible, and Muller and Saunders' argument for weak discernibility. Later in this paper we will explain what is wrong with their reasoning (§2), and then we will develop the intuition in a satisfactory way that avoids our objections (§3).

First, suppose that we have a collection of 'identical' particles, in the sense of having all their constant, species-determining, properties in common; in addition, indistinguishability means that any single particle of the species lives in the same single particle Hilbert space $\mathcal{H}_1$. Then let

$$\mathcal{H}_n \equiv \bigotimes^{n} \mathcal{H}_1, \tag{1}$$

the n-fold tensor product of the single particle space. Let Π_{ij} be the permutation operator that exchanges the states in the $i^{\mathbf{th}}$ and $j^{\mathbf{th}}$ slots of any vector in $\mathcal{H}_n$. Then a state is *symmetric* iff $\Pi_{ij}\psi = +\psi$, and *antisymmetric* iff $\Pi_{ij}\psi = -\psi$ (for all i, j, in both cases). Finally, let $\mathcal{H}_+$ and $\mathcal{H}_-$ be the sectors of $\mathcal{H}_n$ comprised of symmetric and antisymmetric states, respectively.

The *symmetrization postulate* (SP) says that all the pure states of any ensemble of identical particles either all lie in $\mathcal{H}_+$ (in which case the particles are bosons) or all lie in $\mathcal{H}_-$ (in which case the particles are fermions); such states are 'symmetrized'.[3] We refer to particles that satisfy the SP – i.e., bosons and fermions collectively – as 'quanta' (without any field theoretic connotations). Quanta are the only kinds of particle that we consider in this paper.

The *indistinguishability postulate* (IP) says that only operators that commute with all permutations are observables. Within a conventional understanding of quantum mechanics, the SP implies the IP, but not vice versa – we will scrutinize this conventional wisdom later. (See Huggett and Imbo ([2009]) for further explanation of these ideas and claims).

Let us review how typical indiscernibility proofs – for instance of strong indiscernibility – work (the prototype is French and Redhead ([1988])). First, suppose Q is an observable on $\mathcal{H}_1$; then a 'single particle observable' for the i^{th} particle out of n is typically (in the philosophical literature) taken to have the form

(2) $$Q_i \equiv I \otimes I \otimes \cdots \otimes Q \otimes \cdots \otimes I,$$

where Q is the i^{th} factor out of n, and I is the identity on $\mathcal{H}_1$. The reader may note that (2) violates the IP (unless $Q = I$); for now we will pass over this point without comment.

However, a weaker assumption suffices for the proof, namely the natural one that single particle observables form a family satisfying the *conjugacy condition* (CC) pairwise:

(3) $$\forall i, j \ Q_i = \Pi_{ij}^{\dagger} Q_j \Pi_{ij}.$$

CC simply says that the Q-observable for the j^{th} particle is obtained by a suitable permutation of the Q-observable for the i^{th} particle. That (2) satisfies the CC is manifest.

Then it is easy to show that in any symmetrized state the expectation values for an observable for the i^{th} quantum equals that for the same observable for the j^{th} quantum.

(4) $$\begin{aligned} \langle \Psi | Q_i | \Psi \rangle &= \langle \Pi_{ij} \Psi | Q_i | \Pi_{ij} \Psi \rangle && \text{by SP} \\ &= \langle \Psi | \Pi_{ij}^{\dagger} Q_i \Pi_{ij} | \Psi \rangle \\ &= \langle \Psi | Q_j | \Psi \rangle && \text{by CC.} \end{aligned}$$

But if we are to define physical properties, all hands agree, we must do it in terms of observables and their expectation values, so it's hard to see in what other way we could define physical properties except by schema such as:

(5) $$\mathcal{P}_t(a) \text{ iff } \langle \Psi | Q_a | \Psi \rangle = t.$$

But then (4) immediately implies that $\mathcal{P}_t(a)$ iff $\mathcal{P}_t(b)$ for any a, b, and hence particles are not strongly discerned by any such $\mathcal{P}$ – by any physical property.

The argument against relative discernibility compares observables such as $Q \otimes Q' \otimes I \otimes \cdots \otimes I$, which relates the Q-value of the first particle to the Q'-value of the second, to $Q' \otimes Q \otimes I \otimes \cdots \otimes I$, which relates the Q-value of the second particle to the Q'-value of the first. In other words, the 'relational-observables' involved are also related by permutations, and so satisfy the CC. Hence, given the SP, the reasoning of (4) still holds, and they will take the same expectation values in all states, and relative indiscernibility follows.[4] The proofs of strong and relative discernibility are typically presented in a more complicated way, but this very simple derivation shows that as long as the SP holds, and one asks whether observables related by permutations – hence satisfying the CC – will discern, the answer has to be 'no'. But the same reasoning does not imply weak indiscernibility: the question becomes whether particles bear a relation to each other that they don't bear to themselves. But $\mathcal{R}(a, b)$ and $\mathcal{R}(a, a)$ are not related by the permutation of a and b, and neither are the observables that represent the corresponding quantities in the formalism. So CC fails, and with it the kind of argument in (4) – hence the possibility of weak discernibility, which Muller and Saunders attempt to exploit.

Their putative proof of weak discernibility develops the intuition about electrons in the Bell-Bohm state in the following way: first, let $\{\varphi_1, \ldots \varphi_d\}$ be an eigenbasis for $\mathcal{H}_1$. Let P_i be the projector onto $c\varphi_i$; let

$$P_{ij} \equiv P_i - P_j; \tag{6}$$

and let

$$P_{ij}^{(a)} \equiv I \otimes \cdots \otimes I \otimes P_{ij} \otimes I \otimes \cdots \otimes I, \tag{7}$$

where P_{ij} appears in the a^{th} slot of the tensor product space. The P_{ij} are natural generalizations of the spin-$\frac{1}{2}$ observable, σ, appealed to in Saunders' intuition: after all, $\sigma = P_{\uparrow} - P_{\downarrow}$ (taking the spin eigenvalues to be ± 1 for convenience). So, according to (2), the $P_{ij}^{(a)}$ are the generalizations of single particle spins.

Next Muller and Saunders define the following relation,

$$\mathcal{R}_t(a, b) \text{ iff } \sum_{i,j=1}^{d} P_{ij}^{(a)} P_{ij}^{(b)} \Psi = t\Psi, \tag{8}$$

and prove the following theorem (see their paper for the proof).

Theorem: suppose

$$\Psi = 1/\sqrt{n!} \sum_{p \in \mathbb{P}} c^s \alpha_1 \otimes \alpha_2 \otimes \ldots \otimes \alpha_n, \quad (\langle \alpha_i | \alpha_j \rangle = \delta_{i,j}), \tag{9}$$

where the sum is over permutations $\mathbb{P}$, $c = \pm 1$, and $s = \{0, 1\}$ depending on whether the permutation is even or odd, respectively. All fermionic states are superpositions of states of this form, with $c = -1$; some bosonic states are of this form, with $c = +1$. Then

$$\sum_{i,j=1}^{d} P_{ij}^{(a)} P_{ij}^{(b)} \Psi \quad = -2\Psi \qquad (a \neq b) \tag{10}$$

and

$$\sum_{i,j=1}^{d} P_{ij}^{(a)} P_{ij}^{(a)} \Psi \quad = 2(d-1)\Psi \neq -2\Psi. \tag{11}$$

It follows immediately from this theorem that in such states, $\mathcal{R}_{-2}(a,b)$ iff $a \neq b$; hence quanta in such states are weakly discerned by $\mathcal{R}_{-2}(a,b)$. Moreover, by linearity, the same conclusions hold for any superposition of such states (or indeed any mixture of such states), which is to say that fermions are always weakly discernible. Finally, even bosons will be weakly discernible, when in such states or their superpositions.

A couple of comments are in order. First, the operators appearing in (8) involve products of single particle observables, and so are correlations between values of the corresponding quantities for pairs of quanta. That's just what you expect for a relation between quanta: for instance, $\sigma \otimes \sigma$ has eigenvalues ± 1 representing same and opposite spin – a special case of (8) (up to a factor of 2).

Second, previous treatments, like (4), demonstrated 'probabilistic' indiscernibility, equating property bearing with possession of a given expectation value of an observable. A narrower notion of property-bearing is that of being in an eigenstate of the observable: 'categorical' property-bearing. In these terms, the earlier proofs show that quanta are not *even* probabilistically strongly or relatively discernible; while Muller and Saunders show that in the appropriate states they are probabilistically *and* categorically weakly discernible. (Moreover, their results are more comprehensive than those presented here because they also include mixed states.)

2. Criticisms

Muller and Saunders' formal results are unimpeachable; it is their interpretation that we question. In particular, we will explain why it is tendencious to claim that the operators appealed to in (8) correspond to interesting physical quantities. If they don't then the relation, $\mathcal{R}_t$, defined in terms of them is not itself a physical relation, and Muller and Saunders have not shown that quanta are weakly discernible in QM. That is the work of this section, and we will divide the task into two parts, since different considerations apply to bosons than fermions. In the following section we will offer an alternative demonstration of weak discernibility that avoids the pitfalls of Muller and Saunders'.

2.1. **Bosons.** Despite the title of their paper, Muller and Saunders also discuss the weak discernibility of bosons. Especially, they note (as we did above) that it follows from their theorem that: 'in every state [of form (8) with $c = +1$] the N similar bosons can be discerned weakly and categorically for every finite-dimensional Hilbert space, by exactly the same relations that we used to discern the N fermions!' (510)

However, the definition in (8) is unsatisfactory when applied to bosons. If the number of quanta, $n \geq 3$, then the operators used to define $\mathcal{R}_t$ in (8) violate the IP, and so are unphysical (contrary to Muller and Saunders, 537). For example, for $n = 3$, $\mathcal{R}_t(1,2)$ holds iff the system is in an eigenstate (with eigenvalue t) of $\sum P_{ij}^{(1)} \otimes P_{ij}^{(2)} \otimes I$, This operator is manifestly not invariant under permutations, from which it follows that $\mathcal{H}_+$ is not invariant under its action.[5] For instance,

$$
\begin{aligned}
(12) \qquad \sum_{i,j=1}^{d} P_{ij}^{(1)} \otimes P_{ij}^{(2)} \otimes I \quad &\cdot \quad 1/\sqrt{3}[\uparrow\uparrow\downarrow + \uparrow\downarrow\uparrow + \downarrow\uparrow\uparrow] \\
&= \quad 1/\sqrt{3}[2(d-1)\uparrow\uparrow\downarrow - 2(\uparrow\downarrow\uparrow + \downarrow\uparrow\uparrow)] \quad \notin \mathcal{H}_+,
\end{aligned}
$$

where $d = dim[\mathcal{H}_1] \neq 0$. In other words, the 'operators' involved in the definition are not well-defined on the space of bosonic states: the scare quotes indicate that they aren't operators at all on $\mathcal{H}_+$, only on the full tensor product space $\mathcal{H}_n$. Thus, according to the usual principles of QM, they do not correspond to physical quantities.

In the context of philosophical discussions of discernibility this criticism may seem unfair, since, as we saw at the start, in (2), the whole debate has been couched in terms of operators that do not satisfy the IP – that, in fact, are not operators on $\mathcal{H}_+$ (or $\mathcal{H}_-$). This approach has been justified, at least since French and Redhead ([1988]), by the idea that although quanta are only ever found in symmetrized states, in accord with the SP, in fact they somehow 'live' in the full tensor product space, $\mathcal{H}_n$. According to French and Redhead, to think otherwise (235-8) is to deny that they are 'individuals' at all, in which case the question of PII does not even get off the ground. To our knowledge, no one has ever emphasized how much this move flies in the face of the standard principles of QM (though French and Redhead do note it). But (12) makes clear what the problem is: the operators in question do not have an eigenbasis in $\mathcal{H}_+$ – otherwise every state in $\mathcal{H}_+$ would remain a superposition of the basis states under the action of the operator. But conventional measurement theory says that measurement will leave a system in an eigenstate with a probability given by the Born rule; according to this prescription IP-violating operators will give non-zero probabilities for measurements to produce non-symmetric bosons! Or, put the other way, if we insist on the SP, such operators can only be 'observables' if standard measurement theory does not apply to them. (And note that the same remarks apply to earlier treatments of fermions as well as bosons; though not to Muller and Saunder's treatment of fermions, as we shall see.)

Recall that the methodology of this paper is to establish weak discernibility without controversial interpretative assumptions. Even if one were to argue for an interpretation of bosons without the IP, one would not achieve that goal – a point that Muller and Saunders fully accept. So we now have good reason to seek an alternative approach for bosons; next we'll motivate the search in the case of fermions as well.

2.2. **Fermions.** Let's rewrite the relation by which fermions are weakly discerned, to make clearer its logical structure. Instead of equation (8) we'll write (for $t = -2$):

$$\mathcal{R}_{-2}(a,b) \quad \text{iff} \tag{13}$$

$$\big(a = b \ \wedge\ \sum P_{ij}^{(a)} P_{ij}^{(b)} \Psi = -2\Psi\big) \quad \vee \quad \big(a \neq b \ \wedge\ \sum P_{ij}^{(a)} P_{ij}^{(b)} \Psi = -2\Psi\big).$$

(13) simply makes explicit that Muller and Saunders' definition relies on different operators depending on whether there are two fermions or just one. We see no reason to doubt that this is the meaning intended by (8) – nor do we see any problem with constructing sensible relations in this way. First, it's a feature of QM that *different operators represent the same physical magnitude* for different systems or subsystems, so in particular the (putative) discerning relation is represented by a disjunction. Second, we shall see immediately below (14-15) that the operators are independent of the specific values of a and b, so (13) does not assume that fermions can be labelled, only that the cases of two entities versus one are distinct. But posing the very question of weak discernibility presupposes that logical distinction: does some relation hold in the former case but not the latter? Thus a and b are simply a book-keeping device to keep track of which operator is relevant: if there are two quanta, $a \neq b$ and one operator represents the physical quantity at stake, if there is one quantum then $a = b$ and another operator represents the same quantity.[6]

However, on closer inspection Muller and Saunders' operator turns out to be, at best, a very unfortunate choice. Their theorem, equations (10-11), says that for either of the operators involved in the definition of $\mathcal{R}_{-2}(a, b)$, every state in $\mathcal{H}_-$ is an eigenstate with the same eigenvalue: $2(d-1)$ for $a = b$, -2 for $a \neq b$. In other words *as operators on the space of fermionic states*, these observables are multiples of the identity. That is, for fermions the theorem can be expressed: $\forall \Psi \in \mathcal{H}_-$

$$\sum P_{ij}^{(a)} P_{ij}^{(b)} = -2 \cdot I \quad (a \neq b) \tag{14}$$

$$\sum P_{ij}^{(a)} P_{ij}^{(a)} = 2(d-1) \cdot I. \tag{15}$$

These operators are not multiples of the identity on the full tensor product space, but (in line with early remarks) we want to take seriously the standard assumption that the SP holds for fermions, so that they really only live in $\mathcal{H}_-$. Then physical quantities are represented by operators on that space, and indeed Muller and Saunders' operators *are* multiples of the identity. Putting this formulation together with equation (13) means that in the mathematical representation of fermions, Muller and Saunders' definition is simply

$$\mathcal{R}_{-2}(a,b) \quad \text{iff} \tag{16}$$

$$\big(a = b \ \wedge \ 2(d-1)\cdot I\Psi = -2\Psi\big) \quad \vee \quad \big(a \neq b \ \wedge \ -2\cdot I\Psi = -2\Psi\big).$$

For fermions, this form is mathematically equivalent to the one they gave, via the operator identity. It's also much simpler, and it's immediately clear that $\mathcal{R}_{-2}$ weakly discerns. But, of course, if they had presented things this way, one would be very skeptical that they had shown that quanta are discerned *by an interesting physical relation!* Instead, one would note that the disjuncts contain trivially true and trivially false conjuncts, making the satisfaction of $\mathcal{R}_{-2}(a,b)$ depend entirely on the (non-)identity of the quanta. (If that's not immediately clear, just consider the $d = 2$ case, for example.)

Of course Muller and Saunders didn't intend $\mathcal{R}_{-2}$ to be trivial, so we will discuss the two ways to argue that (16) is physically significant, despite appearances. We believe that Muller and Saunders' definition is as trivial as it appears in the reformulation – that they have not shown that fermions are weakly discerned by a physically interesting relation. However, since our goal here is only to show the need for an alternative, uncontroversial implementation of discernibility, we will just argue that both possible defenses of their approach again require controversial assumptions.

First, (14-15) only hold in $\mathcal{H}_-$; if instead $\mathcal{H}_n$ is the correct statespace for fermions, then the observables on the LHS are not trivial. But $\mathcal{H}_-$ is the standard assumption: there's no physical motivation for extending a statespace to include inaccessible states (above we rejected a metaphysical motivation). Moreover, there are uncountably many other observables whose action on $\mathcal{H}_-$ agrees with (any multiple of) I. Do they correspond to different physical magnitudes for fermions? It seems unmotivated profligacy to suggest so.

Second, one might argue that the identity can represent a physically significant property. Since it is a projection operator, in conventional thinking it represents a yes-no question along the lines of 'is the system in an allowed state?'. If that were the interpretation of the operators in Muller and Saunders' relation, then certainly discernibility by $\mathcal{R}_{-2}$ is trivial. But then again, non-trivial quantities like (conserved) mass or charge seem to be represented as multiples of the identity. Perhaps the operators in (16) are similar.

Such seems to be Muller and Saunders' line of thought: they take their operators to have physical significance because they can be decomposed into projection operators (534).[7] Two problems: (14)-(15) decompose the observables into $P_{ij}^{(a)}$, yet these observables manifestly violate the IP (see 7), and so are unphysical. Plus, this line of thought suggests that different operator decompositions reveal different physical magnitudes. That seems perverse[8]: for instance, are there two quantities, angular momentum *and* position $\times$ linear momentum? Moreover, there are infinitely many ways to decompose any operator. Do they all correspond to distinct physical magnitudes? All in all, there are good reasons for the orthodoxy of (at most) one physical quantity per operator.

Perhaps the reader can see ways to argue against the problems we have raised, and to try to show that Muller and Saunders' relation is non-trivial (and perhaps to learn something interesting about how operators represent magnitudes). But it seems very unlikely that such arguments would be uncontroversial, and so it would still be the case that Muller and Saunders have not shown *uncontroversially* that fermions are physically discernible.

Even if you set aside all these objections, another point remains: all Muller and Saunders have shown is that fermions are distinguished by c-number properties. These can be represented by multiples of I to bring them within the quantum formalism, but so doing doesn't erase their fundamentally classical character (they don't superpose). Neither does the IP have any bite on them, nor is it really a surprise that a collection of non-symmetric, relational operators can be used to form I. Thus, even if one were to accept Muller and Saunders' results for fermions, an important, unanswered question remains: *are fermions discernible by fully quantum observables?* The answer is 'yes', as we will now show.

3. Reformulating the Insight

3.1. **What Weakly Discerns?** While we have rejected Muller and Saunders' formulation, we acknowledge the strong pull of their intuition – electrons in a singlet state have opposite spins to each other but not themselves! So it is time to show that there is another way to formalize the intuition.

First, the observables invoked to weakly discern must satisfy the IP. Without the IP, for an n-particle state a single-particle observable is generally taken to have the form (2); if the IP is imposed then the natural way to reformulate the observable is to symmetrize it:

$$I \otimes I \otimes \ldots I \otimes Q \otimes I \otimes \ldots \otimes I \rightarrow \frac{1}{n} \sum_{p \in \mathbb{P}} Q \otimes I \otimes \ldots \otimes I \equiv Q_S. \tag{17}$$

(Note that when we symmetrize we must normalize by the reciprocal of the number of distinct permutations.)

Now, the result is the same whichever slot Q appears in, so symmetrization means that there is only one such single-particle observable associated with any $Q \in Herm[\mathcal{H}_1]$. So there are two things we might say (given our general approach): either such an observable represents the quantity *for each particle*, or it represents a quantity that should only be thought of as pertaining to the whole system – there aren't really any single-particle observables at all. The kind of reasoning we gave above against treating I as representing anything but a trivial property might seem to push towards the latter: one operator means one quantity, and democracy means it can't pertain to any one particle more than the others, so it pertains to none. But even putting the argument that way shows the fallacy:

there is indeed only one quantity represented by Q_S, but it pertains to each of the particles individually. Similarly, the single quantity momentum pertains to each body individually. Given that the particles are 'indistinguishable', they of course must take on equal values for any quantity in every state, and hence, given the way in which values are assigned through expectation values or eigenvalues, any quantity must be represented by the same observable for each particle.[9]

The questions of whether an observable represents many quantities, and whether it represents a quantity for many particles are not the same; there is no inconsistency in saying that it represents a single quantity, but as it pertains to many particles. As we've just seen, the alternative is to say that indistinguishable particles have no single particle properties. So we will symmetrize to obtain observables that satisfy the IP, avoiding the problem faced by Muller and Saunders' treatment of bosons; and we will accept the fact that the same observable represents a quantity with respect to many particles, as nothing more than a reflection of their indistinguishability.

The second problem is that we don't want the observables to be multiples of the identity. Interestingly, in the very case in which we first observed weak discernibility it is unavoidable! If there are only two states, such as 'spin up' and 'spin down', and only two fermions, such as a pair of electrons, then $\mathcal{H}_-$ is 1-dimensional: $1/\sqrt{2}(|\uparrow\rangle|\downarrow\rangle - |\downarrow\rangle|\uparrow\rangle)$ is the only anti-symmetric state. Then trivially all operators on $\mathcal{H}_-$ are multiples of the identity. Fortunately, such cases do not arise in our world, for particles always have spatial degrees of freedom as well as internal ones like spin. For instance, this state may represent the entanglement of spatially separated particles, but it only represents the spin part of the state, not the locations of the particles.[10] So we will assume that the dimension of $\mathcal{H}_1$ is greater than 2.

Before we give our general construction, it will be instructive to work through a concrete example: a pair of electrons with two spin states, and two possible locations, left and right.

That is (in a small abuse of the Dirac notation), let $\mathcal{H}_\sigma$ and $\mathcal{H}_x$ be spanned by $\{\uparrow, \downarrow\}$ and $\{\psi_L, \psi_R\}$, respectively; and let $\mathcal{H}_1 = \mathcal{H}_\sigma \otimes \mathcal{H}_x$. Then $\mathcal{H}_1$ is 4-dimensional, $\mathcal{H}_2$ 16-dimensional and $\mathcal{H}_-$ 6-dimensional (so there are non-trivial operators).

Concretely, if ψ_L and ψ_R represent the S-orbitals of two identical atoms, then the anti-symmetric state $\Phi = \psi_L \otimes \uparrow \otimes \psi_L \otimes \downarrow - \psi_L \otimes \downarrow \otimes \psi_L \otimes \uparrow$ represents the $1S^2$ state of the first. According to the intuition, though the electrons in Φ are in the same position, they should still be weakly discerned by having opposite spins. And they are. Again, the relation rests on spin correlations, represented by spin-product operators, such as $\sigma \otimes \sigma$, but adapted for this space, and symmetrized to satisfy the IP. We define (noting the discussion following (13)),

(18)
$$
\begin{aligned}
&\mathcal{R}_{\sigma\cdot\sigma,-1}(a,b) \quad \text{iff} \\
&\big(a = b \quad \wedge \quad \frac{1}{2}(I \otimes \sigma^2 \otimes I \otimes I + I \otimes I \otimes I \otimes \sigma^2)\Psi = -\Psi\big) \\
&\qquad \vee \\
&\big(a \neq b \quad \wedge \quad (I \otimes \sigma \otimes I \otimes \sigma)\Psi = -\Psi\big),
\end{aligned}
$$

then readily check that in Φ, $\neg\mathcal{R}_{\sigma\cdot\sigma,-1}(a,a), \mathcal{R}_{\sigma\cdot\sigma,-1}(1,2), \mathcal{R}_{\sigma\cdot\sigma,-1}(2,1)$, so 'opposite spin' weakly discerns. Indeed, we say that this is the correct way to formalize the intuition about a pair of electrons.[11]

Generalizing (18) we now define the following relation schema:

$$\begin{aligned}(19)\qquad & \mathcal{R}_{A\cdot B,t}(a,b) \quad \text{iff} \\ & \big(a=b \quad \wedge \quad \frac{1}{n}\sum_{p\,\in\,\mathbb{P}}(A\cdot B\otimes I\otimes\ldots\otimes I)\Psi = t\Psi\big) \\ & \qquad\qquad \vee \\ & \big(a\neq b \quad \wedge \quad \frac{1}{2\ {}^{n}C_{2}}\sum_{p\,\in\,\mathbb{P}}(A\otimes B\otimes I\otimes\ldots\otimes I)\Psi = t\Psi\big).\end{aligned}$$

Unless $A = B = I$ these operators will not be multiples of the identity (more carefully, the first may be if $A \cdot B = I$), avoiding the problem for Muller and Saunders' treatment of fermions. We also note that if particles are discerned by such a relation then they are categorically discerned, because the definition again depends on the state being a suitable eigenstate, not on operators taking certain expectation values.

3.2. **General Results.** In this section we will demonstrate a weak discernibility result using relations that fit our schema – vindicating the underlying intuition while using only operators that satisfy the IP and that are not multiples of the identity. Then we shall show that particles which are all in the same state are not weakly discernible (at least in the case $A = B$). We will also give some discussion of these results.

• First, categorial weak discernibility. Suppose again that the system is in the state:

$$(9)\qquad \Psi = 1/\sqrt{n!}\sum_{p\,\in\,\mathbb{P}} c^{s}\alpha_1\otimes\alpha_2\otimes\ldots\otimes\alpha_n, \quad (\langle\alpha_i|\alpha_j\rangle = \delta_{i,j}),$$

Then $\exists A \in Herm[\mathcal{H}_1]$ such that:

$$\frac{1}{{}^{n}\mathrm{C}_2}\sum_{p\,\in\,\mathbb{P}} A\otimes A\otimes I\otimes\cdots\otimes I\cdot\Psi = t\Psi, \tag{20}$$

$$\frac{1}{n}\sum_{p\,\in\,\mathbb{P}} A^2\otimes I\otimes\cdots\otimes I\cdot\Psi = u\Psi, \tag{21}$$

where $t \neq u$. That is, when the condition is met there is an A such that $\mathcal{R}_{A\cdot A,t}(a,b)$ for $a \neq b$, but $\neg\mathcal{R}_{A\cdot A,t}(a,a)$, hence $\mathcal{R}_{A\cdot A,t}(a,b)$ weakly discerns all the particles.[12]

Specifically, let

$$A = \sum_{j=1}^{n} j\cdot|\alpha_j\rangle\langle\alpha_j| = \sum_{j=1}^{n} j\cdot P_j, \tag{22}$$

a weighted sum over projectors onto the $|\alpha_j\rangle$.

We proceed by simply substituting A and Ψ into the LHSs of (20-21).

First (20):

$$= \frac{1}{{}^{n}\mathrm{C}_2}\sum_{p\,\in\,\mathbb{P}}\big(\sum_{j=1}^{n} jP_j\big)\otimes\big(\sum_{j=1}^{n} jP_j\big)\otimes I\otimes\cdots\otimes I\cdot\big(\sum_{p\,\in\,\mathbb{P}}\frac{c^s}{\sqrt{\mathrm{n!}}}\alpha_1\otimes\cdots\otimes\alpha_n\big). \tag{23}$$

This expression simplifies nicely because our operator A is built out of projectors: $P_i\alpha_j = \alpha_j\delta_{i,j}$, hence $\sum_j jP_j\alpha_k = k\alpha_k$. Thus, for instance, in the first sum over permutations, the term in which the As are located within the first and second slots contributes:

$$\frac{1}{{}^{n}\mathrm{C}_2}\sum_{p\,\in\,\mathbb{P}}\frac{c^s}{\sqrt{\mathrm{n!}}}(1\alpha_1\otimes 2\alpha_2\otimes\alpha_3\cdots\otimes\alpha_n). \tag{24}$$

The complete sum is thus

$$(25) = \frac{1}{{}^{n}\mathrm{C}_2} \sum_{p \in \mathbb{P}} \frac{c^s}{\sqrt{\mathrm{n}!}} \big[(1\alpha_1 \otimes 2\alpha_2 \otimes \alpha_3 \cdots \otimes \alpha_n) + (1\alpha_1 \otimes \alpha_2 \otimes 3\alpha_3 \otimes \cdots \otimes \alpha_n) + \dots$$
$$+(1\alpha_1 \otimes \alpha_2 \otimes \cdots \otimes n\alpha_n) + \cdots + (\alpha_1 \otimes \cdots \otimes i\alpha_i \cdots \otimes j\alpha_j \cdots \otimes \alpha_n) + \dots\big]$$

Pulling the coefficients out, each of the terms in the summation contributes a term $i \cdot j(\alpha_1 \otimes \cdots \otimes \alpha_i \cdots \otimes \alpha_j \cdots \otimes \alpha_n)$, hence:

$$= \frac{\sum_{i \neq j}^{n} i \cdot j}{2 \; {}^{n}\mathrm{C}_2} \Psi \equiv t\Psi. \tag{26}$$

Which of course implies that Ψ is an eigenstate, with eigenvalue t (the factor of 2 takes care of double counting).

Now (21):

$$= \frac{1}{n} \sum_{p \in \mathbb{P}} \frac{c^s}{\sqrt{\mathrm{n}!}} [(1^2\alpha_1 \otimes \cdots \otimes \alpha_n) + (\alpha_1 \otimes 2^2\alpha_2 \otimes \cdots \otimes \alpha_n) + \dots \tag{27}$$
$$\cdots + \; (\alpha_1 \otimes +\alpha_2 \otimes + \dots n^2\alpha_n)]$$
$$= \frac{\sum^n i^2}{n} \Psi \equiv u\Psi \tag{28}$$

Finally, $t \neq u$. First, for $n = 2$, $u = \frac{5}{2}$ and $t = 2$. For $n \geq 3$, sum the squares of all the pairwise differences from $1, 2, \dots n$,

$$(1-2)^2 + (1-3)^2 + \ldots (1-n)^2 + (2-3)^2 + \ldots (n-1-n)^2 \quad > \quad 0 \tag{29}$$

$$2 \cdot (1^2 + 2^2 + \ldots n^2) - 2 \cdot (1 \cdot 2 + 2 \cdot 3 + \ldots (n-1) \cdot n) \quad > \quad 0 \tag{30}$$

$$\text{hence} \quad \sum^n i^2 \quad > \quad \frac{1}{2} \sum_{i \neq j}^{n} i \cdot j. \tag{31}$$

But for $n \geq 3$, $(n-1) \geq 2$, so

$$u = \frac{\sum^n i^2}{n} > \frac{\sum_{i \neq j}^n i \cdot j}{n \cdot (n-1)} = \frac{\sum_{i \neq j}^n i \cdot j}{2\ {}^n\mathrm{C}_2} = t. \tag{32}$$

Therefore, for all n, $t \neq u$. QED

In short then, the state (9) is a simultaneous eigenstate, with different eigenvalues, of the observables appearing on the LHS of (20-21) – hence particles in such a state are categorically weakly discerned by $\mathcal{R}_{A \cdot A,t}(a, b)$ as defined by (19).

Before demonstrating a case of weak indiscernibility, we want to address two issues with this result. First, both our proof and Muller and Saunders theorem (10-11) require that the α_i be orthogonal (9). All such states are eigenstates of the same pair of observables proposed by Muller and Saunders, and all such states have the same eigenvalues with respect to each so, as we noted, linearity extends their result to all fermi states (and hence the observables are multiples of the identity). In our proof the pair of observables are determined by the state of the quanta, and so we cannot similarly extend our result to superpositions of (9) using linearity. For example, even for two fermions our result does not apply to the state $\Psi = \frac{1}{2}(\alpha_1 \otimes \alpha_2 - \alpha_2 \otimes \alpha_1 + \alpha_3 \otimes \alpha_4 - \alpha_4 \otimes \alpha_3)$ ($\langle \alpha_i | \alpha_j \rangle = \delta_{i,j}$) since this cannot be expressed in the form (9). However, note that our proof applies to both $1/\sqrt{2}(\alpha_1 \otimes \alpha_2 - \alpha_2 \otimes \alpha_1)$ and $1/\sqrt{2}(\alpha_3 \otimes \alpha_4 - \alpha_3 \otimes \alpha_4)$, with $A = 1 \cdot P_1 + 2 \cdot P_2$

and $A = 1 \cdot P_3 + 2 \cdot P_4$ in (20-21) respectively, and shows that their respective eigenvalues for (20-21) are the same (because A has the same eigenvalues in both cases). Thus, using the orthogonality of α_1-α_4, it immediately follows that if we set $A = 1 \cdot (P_1 + P_3) + 2 \cdot (P_2 + P_4)$ then Ψ will be an eigenstate of (20) and (21) with distinct eigenvalues, so that the fermions in Ψ are categorically weakly discerned by $\mathcal{R}_{A \cdot A,t}(a, b)$. This construction is readily extended to any number of particles and any superpositions of (9), and as long as the α_i are orthogonal *across* terms in the superposition (as well as within terms), our results will go through, showing that fermions in such states are also weakly discernible.

However, not all fermionic states satisfies these conditions: some states are superpositions of terms of form (9) in which the same α_i appear in different terms, violating the assumption of orthogonality across terms. In some cases our strategy for extending the proof works, but it does not work in general. We thus do not know whether fermions can be categorically weakly discerned in all states, given the observables we allow. We conjecture that at least they can always be probabilistically discerned: after all, any fermionic state can be written as a superposition of states in which the particles are categorically discerned. We note finally that the same considerations apply to bosons, so they too can be weakly discerned not only in states of form (9) but also in certain superpositions of them (here our results agree, we believe, with those of Muller and Saunders).

The other issue is to demonstrate that the observables involved in our result correspond to physical properties – if not, like Muller and Saunders, we haven't shown that quanta are physically weakly discerned. First, there is the observable A of (22). Note that our discernibility result can easily be generalized to the case $A = \sum_{j=1}^{n} a_j \cdot |\alpha_j\rangle\langle\alpha_j|$, where the a_j are arbitrary eigenvalues, so there is nothing special about our choice of eigenvalues. (The proof is identical, except with $j \to a_j$ in the various steps, so a little harder to follow.) In fact, as long as A has two distinct eigenvalues the proof will work, so physically all that is assumed is that in any orthonormal basis for $\mathcal{H}_1$ there are two states that can be distinguished. If there are symmetries strong enough to violate that assumption, then we are in a very special situation, not the general one probed here in which the SP is isolated for investigation.

Then there are the observables in (20-21): the symmetrized forms of $A \otimes A \otimes I \otimes \cdots \otimes I$ and $A^2 \otimes I \otimes \cdots \otimes I$, which physically represent correlations between the A-values of the quanta. They are hermitian, they satisfy the IP, they aren't multiples of the identity, and they are distinct. These observables have all the characteristics that qualify them as representing physical quantities by the standard principles of QM. Those are not in question, so any challenge to the physicality of the observables would have to arise from some further constraints. For instance, how do we know there are dynamically possible interactions between the quanta and an external system that would constitute suitable measurements?[13] The question is not whether hermitian operators on the state space generate suitable evolutions, since that is a mathematical fact. Rather, any concern presumably questions whether there are hamiltonians featuring only nomically possible interaction terms, such as a Coulomb force, or satisfying additional symmetries, say. But to raise such concerns is to change the question under investigation, which is *whether particles satisfying the SP are physically discernible.* It would be to ask instead whether particles subject to the SP *and additional constraints* are discernible. The first question is the relevant one here, and our result answers it positively.

• Second, categorical indiscernibility. Suppose the state is

$$\Psi = \otimes^n \psi, \tag{33}$$

so we are considering a bosonic state in which all particles are in the same state.

Our strategy here is to show that if Ψ is an eigenstate of the observable appearing in (20) then it is an eigenstate *with the same eigenvalue* of the observable in (21) – hence $\mathcal{R}_{A \cdot A,t}(a,b)$ iff $\mathcal{R}_{A \cdot A,t}(a,a)$, and the relation does not weakly discern the particles. In more detail, suppose that the state is the t-valued eigenstate of the observable representing the correlation of particle pairs with respect to the operator A:

$$\frac{1}{{}^nC_2} \sum_{p \in \mathbb{P}} A \otimes A \otimes I \otimes \ldots I \otimes I \cdot \Psi = t\Psi. \tag{34}$$

Then, as we shall show below,

$$A\psi = \pm\sqrt{t}\psi \tag{35}$$

and hence

$$\frac{1}{n} \sum_{p \in \mathbb{P}} A^2 \otimes I \otimes \ldots I \otimes I \cdot \Psi = t\Psi. \tag{36}$$

QED.

The only work to do is to establish (35). Substitute for Ψ in (34):

$$\frac{1}{{}^nC_2} \sum_{p \in \mathbb{P}} A\psi \otimes A\psi \otimes \psi \otimes \cdots \otimes \psi = t \cdot \psi \otimes \psi \otimes \cdots \otimes \psi. \tag{37}$$

Break this expression into the sum of two terms according to the vector in the first slot (though the same reasoning applies to any slot):

$$\frac{1}{{}^nC_2}\{A\psi\otimes\left(\sum_{p\,\in\,\mathbb{P}}A\psi\otimes\psi\otimes\cdots\otimes\psi\right) \quad + \quad \psi\otimes\left(\sum_{p\,\in\,\mathbb{P}}A\psi\otimes A\psi\otimes\psi\otimes ...\otimes\psi\right)\}$$

$$(38) \qquad\qquad = \quad t\cdot\psi\otimes\big(\psi\otimes\cdots\otimes\psi\big),$$

where the permutations are over the remaining $n-1$ slots. Equation (38) equates two vectors, which of course therefore cannot have orthogonal components; hence the first slot of the RHS vector must equal some linear combination of the terms appearing in the first slot on the LHS.[14] Absorbing constants:

$$(39) \qquad aA\psi + b\psi = \psi,$$

so that ψ is an eigenstate: $A\psi = \lambda\psi$. Substitution into (37) quickly allows the eigenvalue to be found: $\lambda = \pm\sqrt{t}$, as we asserted in (35).

We note a lacuna in our demonstration of categorical weak indiscernibility: it does not cover the full class of relations allowed by (19), but only those for which $A = B$. We do not know whether the demonstration extends, but is seems likely that it does extend if ψ is an eigenstate of both A and B, while if not then (33) fails to be an eigenstate of (20-21); either way, the particles fail to be categorically weakly discernible.[15]

4. Conclusion

While we acknowledging the correctness of Muller and Saunder's intuitions we have given demonstrations that their formulation is either unphysical or would require the addition of controversial assumptions in direct conflict with their conservative methodology. We have shown that in the case of bosons their operator does not satisfy the IP and that as a result H_+ is not invariant under its action. In the case of fermions we showed that Muller and Saunder's operator involves multiples of the identity, of dubious physical significance.

Thus, in order to salvage Muller and Saunder's insight while respecting their aims, we have provided new observables that avoid these problems – and we have demonstrated when these observables do, and when they do not, weakly discern quanta. We will not here explore the broader consequences of these results, except to note that Muller ([2011]) assumes the categorical weak discernibility of quanta in his discussion of 'ontic structural realism', so it might seem that our conclusions undermine some of his. However, as we understand him, probabilistic weak discernibility suffices and, if anything, we have suggested that quanta are so discernible.

In sum, Saunders was quite right when he noticed that quanta are weakly discernible, but previous attempts to provide general proofs have not succeeded. Our approach shows the right way to do it.

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Notes

[1]This line of argument has been extended to all elementary particles by Muller and Seevinck ([2009]); in particular, they do not assume a finite dimensional state space and extended their results to all bosonic states. Our criticisms of Muller and Saunders apply to them too, but we will focus exclusively on the former here. (Our proofs assume a state space of countable dimensions.)

[2]While this intuition is developed and generalised by Muller and Saunders, it is actually due to Saunders ([2003]). Indeed, he developed the point even earlier: NH recalls discussing it with him in Bielefeld in 1999. Regrettably, it took NH a long time, and additional prompting by Muller (especially by a presentation in Dubrovnik in 2006), to understand its significance.

[3]We use 'boson' and 'fermion' in their statistical sense in this paper, not in the sense of integer versus half-integer spin: the two conceptions are linked by the spin-statistics theorem.

[4]For a discussion of what happens if the SP does not hold, see Huggett ([2004]).

[5]Strictly speaking, that's not quite right. An operator on $\mathcal{H}_n$ could violate the IP and yet have the same action as an IP satisfying operator on $\mathcal{H}_+$ – because it acted non-symmetrically only on states orthogonal to $\mathcal{H}_+$. To avoid painful qualifications in the main text we shall take it here that violation of the IP means that an operator does not act as a symmetrical operator on $\mathcal{H}_+$; then the claim just made in the text follows. (Note, however, that the operators in (8) are well-defined on $\mathcal{H}_-$; the violation of the IP is not a problem for Muller and Saunders' treatment of fermions. Something else is!)

[6]Two comments: (i) a and b of course label slots in the tensor product formalism, but it does not follow that they label quanta, because that requires the further assumption that each quanta is assigned to a specific slot. But as Huggett ([1995, §3]) and Pooley ([2006, §4 and 7]) point out, it is consistent for quanta to be countable (so the '=' relation holds) but not labelable. (In that case, one would think of the labels as 'surplus structure' in the formalism.) (ii) It is clear from its logical form that the appearance of '=' in (13) does not *reduce* $\mathcal{R}_{-2}(a,b)$ to $\neq$, nor as we have just explained does the definition introduce different physical magnitudes in the two cases, but specifies which operators represent a single magnitude. Thus (13) does not trivialize Muller and Saunders' proposal. However, if one is interested in the further metaphysical

question of whether primitive logical identity can be *eliminated* by appeal to qualitative discernibility by $\mathcal{R}_{-2}$, one might be concerned about circularity at this point. Happily, such a metaphysical program is beyond the scope of this paper; our ambition is simply to settle the issue of weak discernibility.

[7]Muller and Seevinck ([2009, 8-10]) have something similar in mind when they decompose the identity into a multiple of $[p, q]$; similar remarks apply.

[8]Though not entirely unprecedented: see ([Arntzenius 2000, section 4]).

[9]Or to put it another way, the IP plus CC (3) imply that every operator in a family of single-particle observables is the same.

[10]Indeed, if the first slot of the tensor product represents the particle on one side and the second slot the particle on the other then the slots correspond to particles strongly discerned by position! To represent the full, SP satisfying, fermonic state, one would also need to anti-symmetrize the spatial degrees of freedom.

[11]Once again (see after 13) we do not assume that particles can be labelled, only, as always, that the cases of two versus one are distinct.

[12]Note that the normalization in (20) differs by a factor of two from (19), because we have the case $A = B$, so half the number of distinct permutations.

[13]A measurement is an interaction between a measuring apparatus and measured subsystem such that the joint system of apparatus in its 'ready state' and subsystem in any eigenstate of the measured observable will evolve unitarily into a state in which the apparatus is in a suitable 'pointer state'.

[14]We note a slightly stronger result than that used at this step: if $\alpha \otimes \beta + \gamma \otimes \delta = \phi \otimes \psi$ then *either* $\phi = a\alpha + b\gamma$ and $\psi \propto \beta \propto \delta$, *or* $\psi = a\beta + b\delta$ and $\phi \propto \alpha \propto \gamma$.

[15]Notwithstanding Muller and Saunders (538) and Muller and Seevinck ([2009, 13-4]), who discuss ways in which their results might be extended to all bosonic states. We think that all they have really noticed is

that the existence of some states in which bosons *are* weakly discerned immediately implies that in any state bosons *could* have been be weakly discerned (had the state been different) or even *will* be weakly discerned (if it evolves into an appropriate state). We don't have a beef with saying in that broad way that bosons can always be weakly discerned, but we do think that these senses should be distinguished.

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