CONTROLLED-NOT FUNCTION CAN PROVOKE BIASED INTERPRETATION FROM BELL'S TEST EXPERIMENTS

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Abstract. Recently, we showed that the controlled-NOT function is a permutation that cannot be inverted in subexponential time in the worst case [Quantum Information Processing. 16:149 (2017)]. Here, we show that such a condition can provoke biased interpretations from Bell's test experiments.

Let $CNOT$ be the canonical two-qubit entangling gate in quantum key distribution (QKD) cryptographic protocols, where $CNOT|a,x\rangle = |a, a+x\rangle$, so that the control parameter a and the target variable $x \in F_2 = \{0,1\}$.

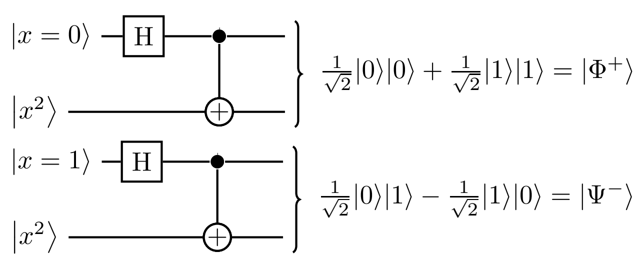
For $x = a$, $CNOT|a,x\rangle = |a, x^2+x\rangle$, since $x \wedge x = x = x^2$, and for $x \neq a$, $CNOT|a,x\rangle = |a, x^2+x+1\rangle$, since $\neg x = x+1 = x \wedge x + 1 = x^2+1$ [1]:

(i) The permutation $x^2+x = x \oplus x$ is a factorable polynomial (reducible) over a finite field of two elements, whose Hamming distance between its even inputs is equal to 0 (local model), and (ii) The permutation $x^2+x+1 = x \oplus NOT(x)$ is a nonfactorable polynomial (irreducible) over a finite field of two elements, whose Hamming distance between its odd inputs is not equal to 0 (nonlocal model). However, these models are deducible from each other because $x^2+x\ (+1) = 0\ (+1) = x^2+x+1$ and $x^2+x+1\ (+1) = 1\ (+1) = x^2+x$ [1].

Consider the Hadamard basis $\{|+\rangle, |-\rangle\}$ of a one-qubit register given by:

$$|x\rangle_{x=0,1} \xrightarrow{H} \frac{1}{\sqrt{2}}[(-1)^x|x\rangle + |1-x\rangle].$$

The circuit below takes computational basis $F_2 = \{0,1\}$ to Bell states:



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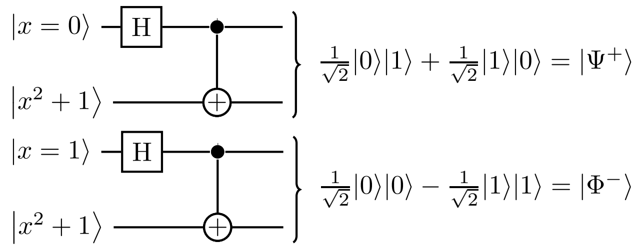

Entangled states of two qubits known as the Bell states occur in conjugate pairs. Quantum states which are conjugates of each other have the same absolute value.

Hence,
$|x^2+x| = |\frac{1}{\sqrt{2}}|0\rangle|0\rangle + \frac{1}{\sqrt{2}}|1\rangle|1\rangle| =$
$= |\frac{1}{\sqrt{2}}|0\rangle|0\rangle - \frac{1}{\sqrt{2}}|1\rangle|1\rangle| = |x^2+x+1|$ and
$|x^2+x| = |\frac{1}{\sqrt{2}}|0\rangle|1\rangle - \frac{1}{\sqrt{2}}|1\rangle|0\rangle| =$
$= |\frac{1}{\sqrt{2}}|0\rangle|1\rangle + \frac{1}{\sqrt{2}}|1\rangle|0\rangle| = |x^2+x+1|$.

Therefore, $|x^2+x| = |x^2+x+1|$, since these models are deducible from each other. Notice that we can map the elements of the Hadamard basis to the computational basis using the group homomorphism $\{+1,-1,\times\} \mapsto \{0,1,+\}$ so that its inverse is also a group homomorphism.

Then, the exclusive disjunction x^2+x+1 over F_2 can be rewritten as $x + NOT(x) := X' \wedge \neg X''$, once the field's multiplication operation corresponds to the logical AND operation over the field of two elements. It is not difficult to see that for $X' = X'' = X'''$, $X' \wedge \neg X' = (X' \vee X'' \vee X''') \wedge (\neg X' \vee \neg X'' \vee \neg X''')$ can be written as a conjunctive normal form, $(X' \vee X'' \vee X''') \wedge (X' \vee X'' \vee \neg X''') \wedge (X' \vee \neg X'' \vee X''') \wedge (X' \vee \neg X'' \vee \neg X''') \wedge (\neg X' \vee X'' \vee X''') \wedge (\neg X' \vee X'' \vee \neg X''') \wedge (\neg X' \vee \neg X'' \vee X''') \wedge (\neg X' \vee \neg X'' \vee \neg X''')$ corresponding to the universal set $\{X', X'', X'''\}$ as shown in the following framework.

Suppose that we take a particle in the state X and subjected to three tests with two possible outcomes. (This is equivalent to three $spin$1/2 subsystems). We will call a first test X', a second test X'' and a third test X''', and label the outcomes *pass* and *fail* in accordance with Fig. 1 below.

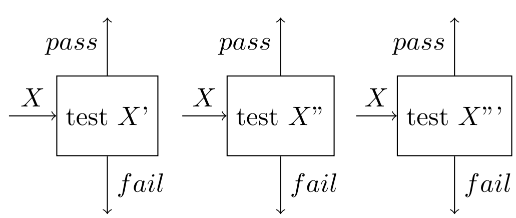

FIGURE 1. This simple experiment can also be seen as a straightforward probability problem, where we are going to flip a coin three times, so that 0 represents *tail*, and 1 represents *head*.

There are 8 possible outcomes of these three tests using 0 and 1 to represent *fail* and *pass* over a finite field of two elements.

Let Ω be the universal set $\{X', X'', X'''\}$, then all 8 possible different outcomes are represented by its subsets:
$\{\emptyset\} = \{000\}$,
$\{X'\} = \{100\}$,
$\{X''\} = \{010\}$,
$\{X'''\} = \{001\}$,
$\{X', X''\} = \{110\}$,
$\{X', X'''\} = \{101\}$,
$\{X'', X'''\} = \{011\}$,
$\{X', X'', X'''\} = \{111\}$.
The following elements shown in Table 1 are equivalent representations of the same value over a finite field of two elements [2, p. 134]:

TABLE 1. Polynomial representation $Poly(x)$ for all the mutually exclusive (8) possibilities of experiment. Set theory is isomorphic to Boolean Algebra.

Tests X',X$''$, X'''	Poly(x)	Probability
111	$x^2 + x + 1$	$\mathcal{P}r_1$
110	$x^2 + x$	$\mathcal{P}r_2$
101	$x^2 + 1$	$\mathcal{P}r_3$
100	x^2	$\mathcal{P}r_4$
011	$x + 1$	$\mathcal{P}r_5$
010	x	$\mathcal{P}r_6$
001	1	$\mathcal{P}r_7$
000	0	$\mathcal{P}r_8$

In third column of Table 1, $\mathcal{P}r_i$, with $i = 1, ...8$, is the probability of a specific outcome occurring in the sample space including all possible outcomes.

The probabilities $\mathcal{P}r_i$ are nonnegative, and therefore $\mathcal{P}r_3 + \mathcal{P}r_4 \leq \mathcal{P}r_3 + \mathcal{P}r_4 + \mathcal{P}r_2 + \mathcal{P}r_7$ within the framework conceived by Wigner [3, 4, 5], as described in detail in [6, p. 227-228]. (If we assume, with Wigner, the existence of these probabilities, his inequality must be true, because the existence of these probabilities corresponds in essence to Kolmogorov's consistency conditions).

Let an event E_i be a set of the outcomes of experiment, i.e, a subset of the sample space Ω. If each outcome in the sample space Ω is equally likely, then the probability that event E_i occurs is $\mathcal{P}r_i = \frac{|E_i|}{|\Omega|}$, where the bars $|\cdot|$ denote the cardinality of sets. As each bit string can be written as a polynomial over a finite field of two elements, then the cardinality of Ω, and for each E_i, is the modulus of a polynomial. Hence, $|x^2 + 1| + |x^2| \leq |x^2 + 1| + |x^2| + |x^2 + x| + |1|$, because $|\Omega| = 1$, since the universal set $x^2 + x + 1 = 1$ for $x = \{0, 1\}$. Consequently, $|x^2 + 1 + x^2| \leq |x^2 + 1 + x^2 + x^2 + x + 1|$, once the all polynomials are nonnegative.

Considering that field's multiplication corresponds to the logical AND, then $x^2 = x$, since $x \wedge x = x$. Hence, $|x^2 + 1 + x| \leq |x + 1 + x + x^2 + x + 1|$.

Rearranging this inequality, we get $|x^2 + x + 1| \leq |x^2 + x|$, because the field's addition operation $x + x = 0$ corresponds to the logical XOR operation. Notice that the polynomial $x^2 + x = NOT(x^2 + x + 1)$ for $x = \{0, 1\}$. Therefore, $|x^2 + x + 1| \leq |1 - (x^2 + x + 1)|$ since, algebraically, the negation $NOT(x^2 + x + 1)$ is replaced with complement $1-(x^2+x+1)$. Hence, $|x^2+x+1| \leq 1-|x^2+x+1|$ because $0 \leq x^2+x+1 \leq 1$.

It is straightforward to see that $|x^2+x+1| \leq \frac{1}{|x^2+x+1|}$, consequently, $\frac{1}{|x^2+x+1|} \leq 1-\frac{1}{|x^2+x+1|}$, where $\frac{1}{|x^2+x+1|} = \left(\frac{1}{|x^2+x+1|}\right)^2$.

As a result,

$$\left(\frac{1}{|x^2+x+1|}\right)^2 \leq 1-\frac{1}{|x^2+x+1|} \tag{1}$$

The polynomial x^2+x+1 over a finite field with a characteristic 2 corresponds to the exclusive disjunction $x \oplus NOT(x)$, where $NOT(x) = x^2 \oplus 1$ for $x = |0\rangle$ or $x = |1\rangle$.

Therefore:

$$|+\rangle = \begin{pmatrix}1\\0\end{pmatrix} + \begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}+1\\+1\end{pmatrix}$$
$$|-\rangle = \begin{pmatrix}1\\0\end{pmatrix} - \begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}+1\\-1\end{pmatrix},$$

so that $|0\rangle = \begin{pmatrix}1\\0\end{pmatrix}$ and $|1\rangle = \begin{pmatrix}0\\1\end{pmatrix}$, where the normalizing constant $\frac{1}{\sqrt{2}}$ was omitted. This logical operation can also be regarded as the Fourier transform [7, p. 50] on the Galois field of two elements $H_2|x\rangle_{x=\{0,1\}} = |\pm\rangle$, where $H_2 = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\1 & -1\end{pmatrix}$ is the Hadamard matrix of order 2.

Fig. 2 depicts the Hadamard basis $\{|+\rangle, |-\rangle\}$ of a one-qubit register on the Hilbert space. Notice that the ratio $\frac{1}{|x^2+x+1|}$ in Ineq. 1 corresponds to $\sin 45°$ over $\mathbb{R}^2$, since the vectors with coordinates $(+1, \pm 1)$ have the same direction as the unit vectors $\frac{1}{\sqrt{2}}|0\rangle \pm \frac{1}{\sqrt{2}}|1\rangle$ that make half a right angle with the axes in the plane. Hence, Ineq. 1 stays $(\sin\theta)^2 \leq 1-\sin\theta$ for $\theta = 45°$.

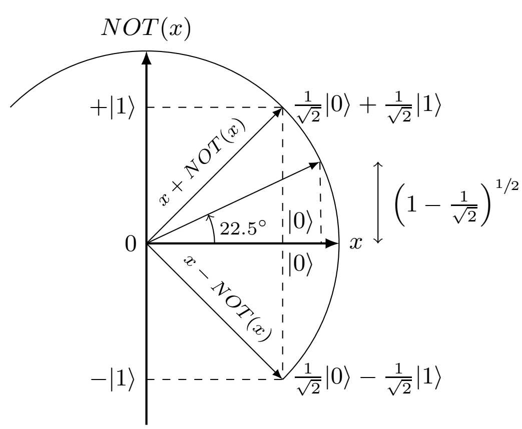

FIGURE 2. The Hadamard gate operates as a reflection around $= \frac{\pi}{8}$ that maps the x-axis to the 45° line, and the $NOT(x)$-axis to the −45° line.

Consider the trigonometric identity $|\sin\left(\frac{\theta}{2}\right)| = \left(\frac{1-\cos(\theta)}{2}\right)^{1/2}$. Then, the equality $1 - \sin\theta = 2(\sin\frac{\theta}{2})^2$ holds, since $\cos\theta = \sin\theta$ for $\theta = 45°$. Consequently, $(\sin 45°)^2 \leq 2(\sin 22.5°)^2$.

Rearranging this last inequality, we get:

$$\frac{1}{2}(\sin 45°)^2 \leq \frac{1}{2}(\sin 22.5°)^2 + \frac{1}{2}(\sin 22.5°)^2, \tag{2}$$

that is the inequality obtained by Bell is his paper [6, p. 230][8], where 45° and 22.5° are Bell test angles, these being the ones for which the quantum theory gives the greatest violation of the inequality, i.e., $0.2500 \leq 0.1464$(i).

Remember that $\{X', X''\}$ is a subset of the universal set $\{X', X'', X'''\}$, hence, the cardinality of subset $\{X', X''\}$ is less than or equal to the cardinality of set $\{X', X'', X'''\}$. Then, obviously, the inequality $|x^2 + x| \leq |x^2 + x + 1|$ holds. (If we trust standard set theory, this axiomatic inequality has to be true).

So, Ineq. 1 is reversed:

$$\frac{1}{2}(\sin 45°)^2 \geq \frac{1}{2}(\sin 22.5°)^2 + \frac{1}{2}(\sin 22.5°)^2, \tag{3}$$

as opposed to Ineq. 2. Consequently, $0.2500 \geq 0.1464$(ii).

The inequalities (i) and (ii) exist at once for Bell test angles, which shows that there is an ambiguity in axiomatic set theory on which Wigner [3] relied to derive a general form of Bell's inequalities. As a consequence, we have that $|x^2 + x| \leq |x^2 + x + 1|$ and $|x^2 + x + 1| \leq |x^2 + x|$, where $2|x^2 + x + 1|_{x=\{0,1\}} = \frac{1}{\sqrt{2}}(||01\rangle + |10\rangle| + ||00\rangle - |11\rangle|)$, so that:

$$|[x \oplus NOT(x)]_{x=0}| \mapsto \quad \frac{1}{\sqrt{2}}||0\rangle|1\rangle + |1\rangle|0\rangle|$$

$$|[x \oplus NOT(x)]_{x=1}| \mapsto \quad \frac{1}{\sqrt{2}}||0\rangle|0\rangle - |1\rangle|1\rangle|$$

As the set $x^2 + x + 1$ is a subset of itself, hence, $|x^2 + x + 1| \leq |x^2 + x + 1|$. It follows that the conditions $|x^2 + x + 1| \leq 1$ and $|x^2 + x + 1| > 1$ hold. Consequently, $\frac{1}{2\sqrt{2}}(||01\rangle + |10\rangle| + ||00\rangle - |11\rangle|) \leq 1$ and $\frac{1}{2\sqrt{2}}(||01\rangle + |10\rangle| + ||00\rangle - |11\rangle|) > 1$.

Defining $\frac{1}{\sqrt{2}}(||01\rangle + |10\rangle| + ||00\rangle - |11\rangle|)$ as a sum of correlations S, we have $S \leq 2$ and $S > 2$ at once, which shows that the number 2 cannot be used as separability criterion. As a result of this logical hole, the problem to determine whether a given state is entangled or classically correlated is undecidable via CHSH inequality [9, 10], i.e, $2 < ||00\rangle + |01\rangle + |10\rangle - |11\rangle| \leq 2$, which can provoke interpretation bias in Bell's test experiments for quantum key distribution (QKD) cryptographic protocols.

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