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On CAP-quasinormal Subgroups of Finite Groups

ZHANG Li, WU Zhenfeng, GUO Wenbin*

(*School of Mathematical Sciences, University of Science and Technology of China, Hefei, Anhui, 230026, P. R. China*)

Abstract: A subgroup A of a finite group G is called a generalized CAP-subgroup of G if for each G-chief factor H/K either A avoids H/K or the following hold: (1) if H/K is non-abelian, then $(A \cap H)K/K$ is a Hall subgroup of H/K; (2) if H/K is a p-group, then $|G : N_G((A \cap H)K)|$ is a p-number. We say that a subgroup H of G is CAP-quasinormal in G if G has a quasinormal subgroup T and a generalized CAP-subgroup A such that HT is S-quasinormal in G and $H \cap T \leq A \leq H$. In this paper, we obtain some results about the CAP-quasinormal subgroups and use them to give the condition under which a finite group belongs to a saturated formation containing supersolvable groups. Many recent results are extended.

Keywords: finite groups; CAP-quasinormal subgroups; saturated formation; $\mathcal{F}$-hypercentre; p-nilpotent group

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0 Introduction

Throughout this paper, all groups are finite. G always denotes a group and p denotes a prime. $|G|_p$ denotes the order of a Sylow p-subgroup of G. For a subgroup H of G, let H^G and H_G denote the normal closure of H and the core of H in G, respectively. That is, $H^G = \langle H^g \mid g \in G\rangle$ and $H_G = \bigcap_{g\in G} H^g$. An integer n is called a p-number if n is a power of p.

Recall that a class $\mathcal{F}$ of groups is called a formation if $\mathcal{F}$ is closed under taking homomorphic images and subdirect products. A formation $\mathcal{F}$ is said to be saturated if $G \in \mathcal{F}$ whenever $G/\Phi(G) \in \mathcal{F}$. We use $\mathcal{U}$ to denote the saturated formation of all supersolvable groups. A normal subgroup N of G is said to be $\mathcal{F}$-hypercentral in G if either $N = 1$ or every G-chief factor H/K below N is $\mathcal{F}$-central in G, that is, $H/K \rtimes G/C_G(H/K) \in \mathcal{F}$. The product of all normal $\mathcal{F}$-hypercentral subgroups is called the $\mathcal{F}$-hypercentre of G and denoted by $Z_{\mathcal{F}}(G)$.

A subgroup H of G is said to be quasinormal or permutable[5] (respectively, S-quasinormal or S-permutable[18]) in G if H permutes with every subgroup (respectively, Sylow subgroup) of G.

In recent years, in order to investigate the structure of finite groups, a number of embedding properties of subgroups were introduced by many authors, see, for example, [2, 8–9, 11, 14, 17]. Recall that a subgroup H of G is said to be *S-quasinormally embedded* (see [7, p. 132]) in G if

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E-mail: * wbguo@ustc.edu.cn

each Sylow subgroup of H is also a Sylow subgroup of some S-quasinormal subgroup of G. A subgroup H of G is called *c-normal*[19] in G if there exists a normal subgroup T of G such that $G = HT$ and $H \cap T \leq H_G$. A subgroup H of G is said to be *n-embedded*[9] in G if for some normal subgroup T of G and some S-quasinormal subgroup S of G, HT is normal in G and $H \cap T \leq S \leq H$. A subgroup H of G is called *S-embedded*[8] in G if for some normal subgroup T of G and some S-quasinormal subgroup S of G, HT is S-quasinormal in G and $H \cap T \leq S \leq H$. A subgroup H of G is said to be *sn-embedded*[17] in G if G has a normal subgroup T and an S-quasinormally embedded subgroup S such that HT is S-quasinormal in G and $H \cap T \leq S \leq H$.

Recall that a subgroup A of G is called a *CAP-subgroup* of G if A either covers or avoids each G-chief factor (see [6, p. 37]). As a generalization of CAP-subgroups and S-quasinormally embedded subgroups, the authors in [11] introduced the notion of generalized CAP-subgroup: a subgroup A of G is called a *generalized CAP-subgroup* of G if for each G-chief factor H/K either A avoids H/K or the following hold: (1) if H/K is non-abelian, then $(A \cap H)K/K$ is a Hall subgroup of H/K; (2) if H/K is a p-group, then $|G : N_G((A \cap H)K)|$ is a p-number.

Bearing in mind the above, a problem naturally arises:

Question Can we give a notion which develops and unifies all the above-mentioned concepts and the related results?

In order to resolve the above problem, we now introduce the notion of CAP-quasinormal subgroup as follows.

Definition 0.1 A subgroup H of G is called CAP-quasinormal in G if G has a quasinormal subgroup T and a generalized CAP-subgroup A such that HT is S-quasinormal in G and $H \cap T \leq A \leq H$.

It is easy to see that the quasinormal subgroups, S-quasinormally embedded subgroups, c-normal subgroups, n-embedded subgroups, S-embedded subgroups, sn-embedded subgroups, CAP-subgroups, generalized CAP-subgroups are all CAP-quasinormal. Hence the notion of CAP-quasinormal subgroup unifies all the mentioned subgroups. But the following examples show that the converse is not true.

Example 0.1 Let $L_1 = \langle a, b \,|\, a^5 = b^5 = 1, ab = ba \rangle$ and $L_2 = \langle a', b' \rangle$ be a copy of L_1. Assume that α is an automorphism of L_1 of order 3 satisfying $a^\alpha = b, b^\alpha = a^{-1}b^{-1}$. Put $G = (L_1 \times L_2) \rtimes \langle \alpha \rangle$ and $H = \langle a \rangle \times \langle a' \rangle$. Note that $T = \langle aa'b, a^{-1}b' \rangle$ is a normal subgroup of G such that $HT = L_1L_2$ and $H \cap T = 1$. Thus H is n-embedded in G and so it is CAP-quasinormal in G. However, $|G : N_G(H \cap L_1)| = |G : N_G(\langle a \rangle)| = 3$ is not a 5-number, that is, H is not a generalized CAP-subgroup of G.

Example 0.2 Let $G = P \rtimes A_5$, where A_5 is the alternating group of degree 5 and P is a simple $\mathbb{F}_3A_5$-modular which is faithful for A_5. Let $H = PB$, where B is a Sylow 5-subgroup of A_5. Following [11], H is a generalized CAP-subgroup of G. Then clearly, H is CAP-quasinormal in G. By [11] again, H is neither a CAP-subgroup of G nor S-quasinormally embedded in G. Moreover, H is not sn-embedded in G. In fact, the normal subgroup T of G satisfying that HT is S-quasinormal in G must be G. In this case, H is S-quasinormally embedded in G, which is impossible. Consequently, H is not c-normal, n-embedded or S-embedded in G.

In this paper, we obtain the following main result.

Theorem 0.1 Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$ and E, X be normal subgroups of G such that $F^*(E) \leq X \leq E$ and $G/E \in \mathcal{F}$. Then $G \in \mathcal{F}$ if for every prime $p \in \pi(X)$ and every non-cyclic Sylow p-subgroup X_p of X, X_p has a subgroup D such that $1 \leq |D| < |X_p|$ and the following hold:

(1) if X_p is a non-abelian 2-group and $|D| = 1$, then every cyclic subgroup of X_p with order 2 or 4 is CAP-quasinormal in G;

(2) otherwise, every proper subgroup of X_p with order $|D|$ or $p|D|$ is CAP-quasinormal in G.

In this theorem, $F^*(G)$ denotes the generalized Fitting subgroup of G, that is, the largest normal quasinilpotent subgroup of G. The following result can be deduced immediately from Theorem 0.1.

Corollary 0.1 A group G is supersolvable if and only if G has a normal subgroup E such that G/E is supersolvable and for every non-cyclic Sylow subgroup P of E (respectively, $F^*(E)$), one of the following holds:

(1) every cyclic subgroup of P with order p or 4 (when P is a non-abelian 2-group) is CAP-quasinormal in G;

(2) every maximal subgroup of P is CAP-quasinormal in G.

Theorem 0.1 covers and unifies the results in many papers, for example,

Corollary 0.2 Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$ and G a group with a normal subgroup E such that $G/E \in \mathcal{F}$. Then $G \in \mathcal{F}$ if one of the following holds:

(1) for every non-cyclic Sylow subgroup P of E, every maximal subgroup of P is S-embedded in G (see [8, Theorem C]);

(2) for every non-cyclic Sylow subgroup P of $F^*(E)$, every maximal subgroup of P is S-embedded in G (see [8, Theorem D]);

(3) for every non-cyclic Sylow subgroup P of E, every maximal subgroup of P is n-embedded in G (see [9, Theorem D]);

(4) for every non-cyclic Sylow subgroup P of $F^*(E)$, every maximal subgroup of P is n-embedded in G (see [9, Theorem E]);

(5) all maximal subgroups of any Sylow subgroup of $F^*(E)$ are S-quasinormally embedded in G (see [16, Theorem 1.1]);

(6) all subgroups of prime order or order 4 of $F^*(E)$ are S-quasinormally embedded in G (see [16, Theorem 1.2]);

(7) for every prime p dividing $|E|$ and every Sylow p-subgroup P of E, every cyclic subgroup of order p or 4 (when $p = 2$) of $P \cap G^{\mathcal{U}}$ is sn-embedded in G (see [17, Theorem 3.13]);

(8) all maximal subgroups of any Sylow subgroup of $F^*(E)$ are c-normal in G (see [20, Theorem 3.1]);

(9) all minimal subgroups and all cyclic subgroups of $F^*(E)$ are c-normal in G (see [20, Theorem 3.2]).

Besides, Theorem 0.1 also implies Theorem 3.1 in [1], Theorem 3.3 and Corollary 3.4 in [2], Theorems 1 and 2 in [4], Theorem A in [8], Theorems A and B in [9], Theorems 3.1 and 3.3 in [15], Theorem 3.12 in [17], Theorems 4.1 and 4.2 in [19] and so on.

All unexplained notations and terminology are standard, as in [3, 6–7, 12].

1 Preliminaries

Lemma 1.1 ([3, Chapter 1] or [7, Chapter 1, Lemmas 5.34 and 5.35]) Assume that H is a subgroup of G, $E \leq G$ and $N \trianglelefteq G$.

(1) If H is quasinormal in G, then $H^G/H_G \leq Z_\infty(G/H_G)$.

(2) If H is quasinormal (respectively, S-quasinormal) in G, then $H \cap E$ is quasinormal (respectively, S-quasinormal) in E.

(3) If H is quasinormal (respectively, S-quasinormal) in G, then HN/N is quasinormal (respectively, S-quasinormal) in G/N.

(4) Assume that H is a p-group. Then H is S-quasinormal in G if and only if $O^p(G) \leq N_G(H)$.

(5) The set of S-quasinormal subgroups of G is a sublattice of the subnormal subgroup lattice of G.

(6) If H is a π-group and H is subnormal in G, then $H \leq O_\pi(G)$.

Lemma 1.2 (1) If A is a generalized CAP-subgroup of G and $N \trianglelefteq G$, then AN/N is a generalized CAP-subgroup of G/N (see [11, Lemma 2.2 (1)]).

(2) If H is CAP-quasinormal in G and $N \trianglelefteq G$ such that either $N \leq H$ or $(|N|, |H|) = 1$, then HN/N is CAP-quasinormal in G/N.

Proof (2) Let T be a quasinormal subgroup of G and A a generalized CAP-subgroup of G such that HT is S-quasinormal in G and $H \cap T \leq A \leq H$. By Lemma 1.1 (3), TN/N is quasinormal in G/N and $HN/N \cdot TN/N = HTN/N$ is S-quasinormal in G/N. If $N \leq H$, then $H \cap TN = (H \cap T)N$. Now assume that $(|H|, |N|) = 1$. Then $(|HN \cap T : H \cap T|, |HN \cap T : N \cap T|) = (|N \cap HT|, |H \cap NT|) = 1$, which implies $HN \cap T = (H \cap T)(N \cap T)$ (see [6, Chapter A, Lemma 1.6]). Hence, $HN \cap TN = (HN \cap T)N = (H \cap T)N$. This implies that $HN/N \cap TN/N = (H \cap T)N/N \leq AN/N \leq HN/N$, where AN/N is a generalized CAP-subgroup of G/N by (1). Thus (2) holds. □

Let P be a p-group. If P is not a non-abelian 2-group, then we denote $\Omega(P) = \Omega_1(P)$. Otherwise, $\Omega(P) = \Omega_2(P)$.

Lemma 1.3[7, Chapter 3, Lemmas 2.2, 2.9 and 2.11] Let P be a normal p-subgroup of G and C a Thompson critical subgroup of P. Then $P \leq Z_{\mathcal{U}}(G)$ if either $\Omega(C) \leq Z_{\mathcal{U}}(G)$ or $P/\Phi(P) \leq Z_{\mathcal{U}}(G/\Phi(P))$.

Lemma 1.4[10, Lemma 4.3] Let C be a Thompson critical subgroup of a non-trivial p-subgroup P. Then $\Omega(C)$ is of exponent p if p is an odd prime, or exponent 4 if P is a non-abelian 2-group.

The following lemma follows directly from Burnside Theorem.

Lemma 1.5 Let P be a Sylow p-subgroup of G for a prime p satisfying $(|G|, p-1) = 1$. If P is cyclic, then G is p-nilpotent.

Lemma 1.6[7, Chapter 1, Theorem 2.8] Let $\mathcal{F}$ be any formation and $E \trianglelefteq G$. If $F^*(E) \leq Z_{\mathcal{F}}(G)$, then $E \leq Z_{\mathcal{F}}(G)$.

2 Proof of Main Theorem

The proof of Theorem 0.1 consists of many steps. The following Theorems 2.1 and 2.2 are the main stages of it. Note that Theorems 2.1 and 2.2 have independent meanings, for example, Theorem 3.1 in [2], Theorem 3.14 in [13] and Corollary 3.3 in [17] follow directly from Theorem 2.2.

Theorem 2.1 Let P be a normal p-subgroup of G. Assume that P has a subgroup D such that $1 \le |D| < |P|$ and the following hold:

(1) if P is a non-abelian 2-group and $|D| = 1$, then every cyclic subgroup of P with order 2 or 4 is CAP-quasinormal in G;

(2) otherwise, every proper subgroup of P with order $|D|$ or $p|D|$ is CAP-quasinormal in G. Then $P \le Z_{\mathcal{U}}(G)$.

Proof Suppose that the assertion is false and consider a counterexample (G, P) for which $|G| + |P|$ is minimal. We proceed via the following steps.

(1) $|D| < \frac{|P|}{p^2}$. If not, then every maximal subgroup of P is CAP-quasinormal in G. In this case, we will prove that $P \le Z_{\mathcal{U}}(G)$.

Let N be a minimal normal subgroup of G contained in P. Then by Lemma 1.2 (2), $(G/N, P/N)$ satisfies the hypothesis. The choice of (G, P) implies that $P/N \le Z_{\mathcal{U}}(G/N)$. It follows that $|N| > p$ and N is the unique minimal normal subgroup of G contained in P.

Assume that $\Phi(P) = 1$. Let B be a complement of N in P and N_1 a maximal subgroup of N such that N_1 is normal in a Sylow p-subgroup of G. Clearly, $(N_1)_G = 1 < N_1$ and $P_1 = N_1B$ is a maximal subgroup of P. By the hypothesis, G has a quasinormal subgroup T and a generalized CAP-subgroup A such that P_1T is S-quasinormal in G and $P_1 \cap T \le A \le P_1$. If $P \cap T_G = 1$ and $N \le P \cap T^G$, then $|N| = p$ by the G-isomorphism $N \cong NT_G/T_G$ and Lemma 1.1 (1). This contradiction shows that either $P \cap T^G = 1$ or $N \le T$. In the former case, by Lemma 1.1 (5), $P_1 = P_1(P \cap T) = P \cap P_1T$ is S-quasinormal in G and so is N_1 $(= P_1 \cap N)$. Consequently, $N_1 \trianglelefteq G$ by Lemma 1.1 (4) and the choice of N_1, a contradiction. In the latter case, $P_1 \cap N = P_1 \cap T \cap N \le A \cap N \le P_1 \cap N$, that is, $N_1 = P_1 \cap N = A \cap N$. Since A is a generalized CAP-subgroup of G and $N_1 > 1$, $|G : N_G(N_1)|$ is a p-number. This also implies that $N_1 \trianglelefteq G$, a contradiction. Thus $\Phi(P) > 1$ and so $N \le \Phi(P)$. Finally, by Lemma 1.3, we obtain that $P \le Z_{\mathcal{U}}(G)$. This contradiction completes the proof of (1).

(2) $|D| > p$. If $|D| \le p$, then by the hypothesis, every cyclic subgroup of P with order p or 4 (when P is a non-abelian 2-group) is CAP-quasinormal in G.

Let R be a normal subgroup of G such that P/R is a G-chief factor. Clearly, (G, R) satisfies the hypothesis. By the choice of (G, P), $R \le Z_{\mathcal{U}}(G)$, so P/R is non-cyclic. Let L be a normal subgroup of G such that $L < P$. Analogously, $L \le Z_{\mathcal{U}}(G)$. If $L \nleq R$, then $P = RL \le Z_{\mathcal{U}}(G)$. This contradiction shows that $L \le R$ for any normal subgroup L of G satisfying $L < P$. Therefore, by Lemma 1.3, $\Omega(C) = P$, where C is a Thompson critical subgroup of P.

Let K/R be a minimal subgroup of $P/R \cap Z(G_p/R)$, where G_p is a Sylow p-subgroup of G. Let $x \in K \setminus R$ and $H = \langle x \rangle$. Then $K = HR$. By Lemma 1.4, H has order p or 4. By the hypothesis, H is CAP-quasinormal in G. Hence, G has a quasinormal subgroup T and a generalized CAP-subgroup A such that HT is S-quasinormal in G and $H \cap T \le A \le H$. If $P \cap T_G \le R$ and $P \cap T^G = P$, then $PT_G/T_G \le Z_\infty(G/T_G)$ by Lemma 1.1 (1). Consequently, $PT_G/RT_G \le Z_\infty(G/RT_G)$. Thus $P/R \le Z_\infty(G/R)$ by the G-isomorphism $PT_G/RT_G \cong P/R$, which is impossible. Therefore, either $P \cap T_G = P$ or $P \cap T^G \le R$. If $P \cap T_G = P$, then H is a generalized CAP-subgroup of G. Note that $(H\cap P)R/R = K/R > 1$. We have that $|G : N_G(K)|$ is a p-number and $|G/R : N_{G/R}(K/R)|$ is also a p-number. In view of the choice of K/R, we have that $K/R \trianglelefteq G/R$. So $P/R = K/R$ is cyclic, a contradiction. Now assume that $P\cap T^G \leq R$. By Lemma 1.1 (3) (5), $K/R = H(P\cap T)R/R = P/R \cap HTR/R$ is S-quasinormal in G/R. Also, $K/R \trianglelefteq G/R$ by Lemma 1.1 (4) and the choice of K/R, which also implies that $P/R = K/R$. This contradiction completes the proof of (2).

(3) $\Phi(P) = 1$. Assume that $\Phi(P) \neq 1$. If $|\Phi(P)| > |D|$, then $(G, \Phi(P))$ satisfies the hypothesis and $\Phi(P) \leq Z_{\mathcal{U}}(G)$ by the choice of (G, P). Let N be a minimal normal subgroup of G contained in $\Phi(P)$. Then $|N| = p < |D|$ by (2), so $(G/N, P/N)$ satisfies the hypothesis by Lemma 1.2 (2). Hence, $P/N \leq Z_{\mathcal{U}}(G/N)$ and $P \leq Z_{\mathcal{U}}(G)$, a contradiction. Note that $P/\Phi(P)$ is abelian. If $|\Phi(P)| \leq |D|$, then $(G/\Phi(P), P/\Phi(P))$ satisfies the hypothesis by Lemma 1.2 (2), and so $P/\Phi(P) \leq Z_{\mathcal{U}}(G/\Phi(P))$ by the choice of (G, P). Therefore, $P \leq Z_{\mathcal{U}}(G)$ by Lemma 1.3, a contradiction.

(4) Let N be a minimal normal subgroup of G contained in P. Then $p < |N| \leq |D|$ and $P/N \leq Z_{\mathcal{U}}(G/N)$. Assume that $N = P$, that is, P is a minimal normal subgroup of G. Let P_0 be a subgroup of P such that P_0 is normal in a Sylow p-subgroup of G and P_0 has order $|D|$ or $p|D|$. Obviously, $(P_0)_G = 1$. By the hypothesis, G has a quasinormal subgroup T and a generalized CAP-subgroup A such that P_0T is S-quasinormal in G and $P_0 \cap T \leq A \leq P_0$. Similarly, if $P\cap T_G = 1$ and $P\cap T^G = P$, then $PT_G/T_G \leq Z_\infty(G/T_G)$ by Lemma 1.1 (1). Hence, $P \leq Z_\infty(G)$ by the G-isomorphism $PT_G/T_G \cong P$, a contradiction. We can, therefore, assume that either $P \leq T$ or $P \cap T^G = 1$. If $P \leq T$, then P_0 is a generalized CAP-subgroup of G and $|G : N_G(P_0\cap P)| = |G : N_G(P_0)|$ is a p-number. Hence, $P_0 \trianglelefteq G$ by the choice of P_0, a contradiction as well. Now assume that $P \cap T^G = 1$. Then by Lemma 1.1 (5), $P_0 = P_0(P \cap T) = P \cap P_0T$ is S-quasinormal in G. Therefore, by Lemma 1.1 (4) and the choice of P_0, we also have $P_0 \trianglelefteq G$. This contradiction shows that $N < P$.

If $|N| > |D|$, then (G, N) satisfies the hypothesis and $N \leq Z_{\mathcal{U}}(G)$ by the choice of (G, P). In this case, $|D| = 1$, which contradicts (2). Hence, $|N| \leq |D|$. Note that P is abelian by (3). $(G/N, P/N)$ satisfies the hypothesis by Lemma 1.2 (2). The choice of (G, P) shows that (4) holds.

(5) Final contradiction. By (1) and (4), there exists a normal subgroup K of G such that $N < K < P$ and $|K| = p|D|$. By the hypothesis, every maximal subgroup of K is CAP-quasinormal in G. From the proof of (1), it follows that $K \leq Z_{\mathcal{U}}(G)$. So $|N| = p$. This contradiction completes the proof. □

Theorem 2.2 Let E be a normal subgroup of G and P a Sylow p-subgroup of E with $(|E|, p-1) = 1$. Assume that P has a subgroup D such that $1 \leq |D| < |P|$ and the following hold:

(1) if P is a non-abelian 2-group and $|D| = 1$, then every cyclic subgroup of P with order 2 or 4 is CAP-quasinormal in G;

(2) otherwise, every proper subgroup of P with order $|D|$ or $p|D|$ is CAP-quasinormal in G. Then E is p-nilpotent.

Proof Suppose that the assertion is false and let (G, E) be a counterexample for which $|G| + |E|$ is minimal. Then $|P| \leq p^2$ by Lemma 1.5. We proceed via the following steps.

(1) $O_{p'}(E) = 1$ (It follows directly from Lemma 1.2 (2) and the choice of (G, E)).

(2) $|D| < \frac{|P|}{p^2}$. If not, then every maximal subgroup of P is CAP-quasinormal in G.

(i) Let N be a minimal normal subgroup of G contained in E. Then E/N is p-nilpotent and N is the unique minimal normal subgroup of G contained in E. Clearly, PN/N is a Sylow p-subgroup of E/N. Let M/N be a maximal subgroup of PN/N. Then $M = N(P \cap M)$. Let $P_1 = P \cap M$. Then P_1 is a maximal subgroup of P and so it is CAP-quasinormal in G. Assume that T is a quasinormal subgroup of G and A is a generalized CAP-subgroup of G such that P_1T is S-quasinormal in G and $P_1 \cap T \le A \le P_1$. Clearly, TN/N is quasinormal in G/N and $MT/N = P_1TN/N$ is S-quasinormal in G/N by Lemma 1.1 (3). Since $P_1 \cap N = P \cap N$ is a Sylow p-subgroup of N and $|P_1T \cap N : T \cap N| = |P_1 \cap NT : P_1 \cap T|$ is a p-number, we have $P_1T \cap N = (P_1 \cap N)(T \cap N)$. Moreover, $M \cap TN = (P_1 \cap T)N$ (see [6, Chapter A, Lemma 1.2]). Hence, $M/N \cap TN/N = (P_1 \cap T)N/N \le AN/N \le M/N$, where AN/N is a generalized CAP-subgroup of G/N by Lemma 1.2 (1). This shows that M/N is CAP-quasinormal in G/N. Hence, $(G/N, E/N)$ satisfies the hypothesis. The choice of (G, E) implies that (i) holds.

(ii) $N = O_p(E)$ *and* $|N| > p$. Suppose that $O_p(E) = 1$. In view of (1), N is non-abelian. Clearly, $1 < N \cap P \nleq \Phi(P)$ (see [12, Chapter IV, Theorem 4.7]). So $P = (P \cap N)P_1$ for some maximal subgroup P_1 of P. By the hypothesis, G has a quasinormal subgroup T and a generalized CAP-subgroup A such that P_1T is S-quasinormal in G and $P_1 \cap T \le A \le P_1$. If $E \cap T_G = 1$ and $N \le E \cap T^G$, then $N \le Z_\infty(G)$ by the G-isomorphism $N \cong NT_G/T_G$ and Lemma 1.1 (1), a contradiction. By (i), either $N \le T$ or $E \cap T^G = 1$. In the former case, $P_1 \cap N = P_1 \cap T \cap N \le A \cap N \le P_1 \cap N$, that is, $P_1 \cap N = A \cap N$. If $A \cap N = 1$, then $|P \cap N| = |P : P_1| = p$ and N is p-nilpotent by Lemma 1.5, a contradiction. Hence A does not avoid $N/1$. Since A is a generalized CAP-subgroup of G, $P_1 \cap N = A \cap N$ is a Sylow p-subgroup of N. Then $P \cap N = P_1 \cap N \le P_1$, which contradicts the choice of P_1. In the latter case, by Lemma 1.1 (2), $P_1 = P_1(E \cap T) = E \cap P_1T$ is an S-quasinormal subgroup of E. Moreover, by Lemma 1.1 (5)–(6), $P_1 \le O_p(E) = 1$, a contradiction. Hence, $O_p(E) \neq 1$, and so $N \le O_p(E)$. Clearly, $N \nleq \Phi(G)$, hence $G = N \rtimes M$ for some maximal subgroup M of G. Note that $O_p(E) \cap M$ is normal in G. By (i), $O_p(E) \cap M = 1$, and so $O_p(E) = N(O_p(E) \cap M) = N$. If $|N| = p$, then $N \le Z(E)$. It follows from (i) that E is p-nilpotent. This contradiction shows that (ii) holds.

(iii) Final contradiction for (2). By (ii), $G = N \rtimes M$, so $P = N \rtimes (P \cap M)$. Let N_1 be a maximal subgroup of N such that N_1 is normal in a Sylow p-subgroup of G containing P. Obviously, $(N_1)_G = 1 < N_1$. Let $P_1 = N_1(P \cap M)$. Then P_1 is a maximal subgroup of P. By the hypothesis, P_1 is CAP-quasinormal in G. Let T be a quasinormal subgroup of G and A a generalized CAP-subgroup of G such that P_1T is S-quasinormal in G and $P_1 \cap T \le A \le P_1$. Similarly as the proof in (ii), $E \cap T_G = 1$ and $N \le E \cap T^G$ imply that $N \le Z_\infty(G)$, which is impossible. Therefore, we have either $N \le E \cap T_G$ or $E \cap T^G = 1$. If $N \le T$, then $P_1 \cap N = P_1 \cap T \cap N \le A \cap N \le P_1 \cap N$, that is, $A \cap N = P_1 \cap N = N_1 > 1$. Since A is a generalized CAP-subgroup of G, $|G : N_G(N_1)|$ is a p-number. Consequently, $N_1 \trianglelefteq G$ by the choice of N_1, a contradiction. We may, therefore, assume that $E \cap T^G = 1$. By Lemma 1.1 (5), $P_1 = P_1(E \cap T) = E \cap P_1T$ is S-quasinormal in G and so is N_1 $(= P_1 \cap N)$. From Lemma 1.1 (4) and the choice of N_1, it follows that $N_1 \trianglelefteq G$, a contradiction. Thus (2) holds.

(3) $|D| > p$. If not, then cyclic subgroup of P with order p or 4 (when P is a non-abelian

2-group) is CAP-quasinormal in G.

We first prove that $O_p(E) = 1$. Assume that $O_p(E) > 1$. Since $(G, O_p(E))$ satisfies the hypothesis of Theorem 2.1, $O_p(E) \leq Z_{\mathcal{U}}(G)$. Moreover, $O_p(E) \leq Z_\infty(E)$. Let M be a normal subgroup of G such that $O_p(E) < M \leq E$ and $M/O_p(E)$ is a G-chief factor. Clearly, $O_p(E) \leq Z_\infty(M)$. If $M/O_p(E)$ is p-nilpotent, then M is p-nilpotent and $M \leq O_p(E)$ by (1), a contradiction. Therefore, $M/O_p(E)$ is a non-abelian pd-G-chief factor and there exists a minimal non-p-nilpotent subgroup K of E contained in M. From [7, Chapter 1, Propositions 1.9 and 1.10], it follows that: (a) $K = K_p \rtimes K_q$, where K_p is the Sylow p-subgroup of K and K_q a cyclic Sylow q-subgroup of K; (b) $K_p/\Phi(K_p)$ is a non-cyclic K-chief factor and the exponent of K_p is p or 4 (when K_p is a non-abelian 2-group).

Let $x \in K_p \backslash \Phi(K_p)$ and $H = \langle x\rangle$. Then H has order p or 4 by (b). So H is CAP-quasinormal in G. Let T be a quasinormal subgroup of G and A a generalized CAP-subgroup of G such that HT is S-quasinormal in G and $H \cap T \leq A \leq H$. Let $T_0 = T \cap K$. Then T_0 is quasinormal in K by Lemma 1.1 (2). We divide the proof into three cases: (α) $K_p \cap (T_0)_K \leq \Phi(K_p)$ and $K_p \cap (T_0)^K = K_p$; (β) $K_p \leq T_0$; (γ) $K_p \cap (T_0)^K \leq \Phi(K_p)$. In Case ($\alpha$), by Lemma 1.1 (1), $K_p(T_0)_K/(T_0)_K \leq Z_\infty(K/(T_0)_K)$, and so $K_p(T_0)_K/\Phi(K_p)(T_0)_K \leq Z_\infty(K/\Phi(K_p)(T_0)_K)$. Then $K_p/\Phi(K_p) \leq Z_\infty(K/\Phi(K_p))$ by the G-isomorphism $K_p(T_0)_K/\Phi(K_p)(T_0)_K \cong K_p/\Phi(K_p)$, which contradicts (b). In Case (β), H is a generalized CAP-subgroup of G. Note that $M/O_p(E)$ is a non-abelian G-chief factor. If $H \leq O_p(E)$, then $H \leq Z_\infty(M) \cap K \cap K_p \leq Z_\infty(K) \cap K_p = \Phi(K) \cap K_p = \Phi(K_p)$ (see [7, Chapter 1, Proposition 1.8]), which contradicts the choice of x. Therefore, $HO_p(E)/O_p(E)$ is a Sylow p-subgroup of $M/O_p(E)$, which shows that $M/O_p(E)$ is p-nilpotent by Lemma 1.5, a contradiction. Now, we consider case (γ). By Lemma 1.1 (2)–(3) and (5), $H\Phi(K_p)/\Phi(K_p) = H(K_p \cap T_0)\Phi(K_p)/\Phi(K_p) = K_p/\Phi(K_p) \cap HT_0\Phi(K_p)/\Phi(K_p)$ is an S-quasinormal p-subgroup of $K/\Phi(K_p)$. Note that $K_p/\Phi(K_p)$ is abelian. In view of Lemma 1.1 (4), we have $H\Phi(K_p)/\Phi(K_p) \trianglelefteq K/\Phi(K_p)$. Consequently, $K_p/\Phi(K_p) = H\Phi(K_p)/\Phi(K_p)$, which contradicts (b). Thus $O_p(E) = 1$.

Now let N be a minimal normal subgroup of G contained in E. If $N < E$, then (G, N) satisfies the hypothesis and N is p-nilpotent by the choice of (G, E), which contradicts (1) since $O_p(E) = 1$. Hence, E is a minimal normal subgroup of G. Let $x \in E$ be an element of order p and $H = \langle x\rangle$. By the hypothesis, G has a quasinormal subgroup T and a generalized CAP-subgroup A such that HT is S-quasinormal in G and $H \cap T \leq A \leq H$. With a similar discussion as above, we can assume that either $E \leq T$ or $E \cap T^G = 1$. In the former case, H is a generalized CAP-subgroup of G, so H is a Sylow p-subgroup of E, which contradicts Lemma 1.5. In the latter case, $H = H(E \cap T) = E \cap HT$ is an S-quasinormal p-subgroup of E by Lemma 1.1 (2). It follows from Lemma 1.1 (5)–(6) that $H \leq O_p(E) = 1$. This contradiction completes the proof of (3).

(4) $O_p(E) = 1$. Assume that $O_p(E) \neq 1$. Then G has a minimal normal subgroup N contained in $O_p(E)$. We consider three possible cases:

Case (a): $|N| < |D|$. In view of Lemma 1.2 (2), $(G/N, E/N)$ satisfies the hypothesis. So E/N is p-nilpotent by the choice of (G, E). Let L/N be the normal p'-Hall subgroup of E/N. Since E/L is a p-subgroup, we have $E/L = PL/L$. Let H/L be a subgroup of E/L with order $|D|/|N|$ or $p|D|/|N|$. Then $H = L(H \cap P)$ and $H_0 = H \cap P$ has order $|D|$ or $p|D|$. By the hypothesis, G has a quasinormal subgroup T and a generalized CAP-subgroup A such that H_0T is S-quasinormal in G and $H_0 \cap T \leq A \leq H_0$. By Lemma 1.1 (3), TL/L is quasinormal in G/L and $HT/L = H_0TL/L$ is S-quasinormal in G/L. Since $H_0 \cap L = P \cap L$ is a Sylow p-subgroup of L and $|H_0T \cap L : T \cap L| = |H_0 \cap LT : H_0 \cap T|$ is a p-number, we have $H_0T \cap L = (H_0 \cap L)(T \cap L)$. Hence, by [6, Chapter A, Lemma 1.2], $(H/L) \cap (TL/L) = (H_0 \cap T)L/L \leq AL/L \leq H/L$, where AL/L is a generalized CAP-subgroup of G/L by Lemma 1.2 (1). Generally speaking, every subgroup of E/L with order $|D|/|N|$ or $p|D|/|N|$ is CAP-quasinormal in G/L. So $E/L \leq Z_{\mathcal{U}}(G/L)$ by Theorem 2.1. As a result, G has a normal subgroup E^* such that $L < E^* < E$ and $|E^*|_p = p|D|$. Moreover, every maximal subgroup of $P \cap E^*$, where $P \cap E^*$ is a Sylow p-subgroup of E^*, is CAP-quasinormal in G. Using the proof of (2), E^* is p-nilpotent. Then by (1), E^* is a p-group and so E is a p-group, a contradiction.

Case (b): $|N| = |D|$. Let B be a normal subgroup of G such that $N \leq B$ and E/B is a G-chief factor. If $|N| < |B|_p$, then (G, B) satisfies the hypothesis and B is p-nilpotent by the choice of (G, E). Moreover, B is a p-group by (1). Hence, $B \leq Z_{\mathcal{U}}(G)$ by Theorem 2.1. Consequently, $|N| = p$, which contradicts (3). So $|N| = |B|_p$. Let L/B be a subgroup of PB/B with order p. Then $L = (P \cap L)B$. Denote $P_0 = P \cap L$. Then $|P_0| = p|D|$. By the hypothesis, G has a quasinormal subgroup T and a generalized CAP-subgroup A such that P_0T is S-quasinormal in G and $P_0 \cap T \leq A \leq P_0$. By Lemma 1.1 (3), TB/B is a quasinormal subgroup of G/B and $LT/B = P_0TB/B$ is an S-quasinormal subgroup of G/B. Since $P_0 \cap B = P \cap B$ is a Sylow p-subgroup of B and $|P_0T \cap B : T \cap B| = |P_0 \cap BT : P_0 \cap T|$ is a p-number, we have that $P_0T \cap B = (P_0 \cap B)(T \cap B)$. So $L \cap TB = (P_0 \cap T)B$ by [6, Chapter A, Lemma 1.2]. Therefore, $L/B \cap TB/B = (P_0 \cap T)B/B \leq AB/B \leq L/B$, where AB/B is a generalized CAP-subgroup of G/B by Lemma 1.2 (1). The above shows that L/B is CAP-quasinormal in G/B. If E/B is a p-group, then $E/B = PB/B$ is elementary abelian and every subgroup of E/B with order p is CAP-quasinormal in G/B. Thus by Theorem 2.1, $E/B \leq Z_{\mathcal{U}}(G/B)$. It follows that $|E/B| = p$ and $|P| = p|B|_p = p|D|$, which contradicts (2). Hence, E/B is non-abelian. First assume that $E/B \cap (TB/B)_{G/B} = 1$ and $E/B \cap (TB/B)^{G/B} = E/B$. From Lemma 1.1 (1) and the G-isomorphism $E/B \cong (E/B)(TB/B)_{G/B}/(TB/B)_{G/B}$, it follows that $E/B \leq Z_\infty(G/B)$, a contradiction. So either $E/B \leq TB/B$ or $E/B \cap (TB/B)^{G/B} = 1$. In the former case, L/B is a generalized CAP-subgroup of G/B. Hence, L/B is a Sylow p-subgroup of E/B and E/B is p-nilpotent by Lemma 1.5, a contradiction. In the latter case, by Lemma 1.1 (5), $L/B = P_0(E \cap T)B/B = E/B \cap P_0TB/B$ is S-quasinormal in G/B. Then by Lemma 1.1 (4), $O^p(G/B) \leq N_{G/B}(L/B)$. However, $E/B \leq (L/B)^{G/B} = (L/B)^{G_pB/B} \leq G_pB/B$ is a p-group, where G_p is a Sylow p-subgroup of G containing P. This contradiction completes the proof of case (b).

Case (c): $|N| > |D|$. By Theorem 2.1, $N \leq Z_{\mathcal{U}}(G)$, and so $|D| = 1$, which contradicts (3).

The above arguments show that $O_p(E) = 1$.

(5) Final contradiction. Let N be a minimal normal subgroup of G contained in E. By (1) and (4), N is non-abelian.

First, assume that $|N|_p < |D|$. Let H/N be a subgroup of PN/N with order $|D|/|N|_p$ or $p|D|/|N|_p$. Then $H = (P \cap H)N$, where $P_0 = P \cap H$ has order $|D|$ or $p|D|$ and is CAP-quasinormal in G. With a similar discussion as in Case (b) of (4), we can obtain that H/N is

CAP-quasinormal in G/N. Hence, $(G/N, E/N)$ satisfies the hypothesis. So E/N is p-nilpotent by the choice of (G, E). Let K/N be the normal p'-Hall subgroup of E/N. Then with a similar discussion as in Case (a) of (4), E/K is a normal p-subgroup of G/K such that every subgroup of E/K with order $|D|/|N|_p$ or $p|D|/|N|_p$ is CAP-quasinormal in G/N. Hence, $E/K \leq Z_{\mathcal{U}}(G/K)$ by Theorem 2.1. This implies that G has a normal subgroup E^* such that $K < E^* < E$ and $|E^*|_p = p|D|$. Moreover, $P \cap E^*$ is a Sylow p-subgroup of E^* such that every maximal subgroup of $P \cap E^*$ is CAP-quasinormal in G. Hence, (G, E^*) satisfies the hypothesis, and so E^* is p-nilpotent, which contradicts (1) and (4).

Now assume that $|N|_p = |D|$. Let B be a normal subgroup of G such that $N \leq B < E$ and E/B is a G-chief factor. If $|D| < |B|_p$, then (G, B) satisfies the hypothesis and B is p-nilpotent by the choice of (G, E), which contradicts (1) and (4). Hence, $|D| = |B|_p$. With a similar discussion as in Case (b) of (4), we see that this is impossible.

Finally, assume that $|N|_p > |D|$. Then (G, N) satisfies the hypothesis. Assume that $N = E$. Let H be a subgroup of P with order $|D|$ or $p|D|$. Then G has a quasinormal subgroup T and a generalized CAP-subgroup A such that HT is S-quasinormal in G and $H \cap T \leq A \leq H$. Suppose that $E \cap T_G = 1$ and $E \cap T^G = E$. In view of Lemma 1.1 (1) and the G-isomorphism $E \cong ET_G/T_G$, $E \leq Z_\infty(G)$, a contradiction. So either $E \leq T$ or $E \cap T^G = 1$. In the former case, H is a generalized CAP-subgroup of G. Consequently, H is a Sylow p-subgroup of E, which contradicts (2). In the latter case, $H = H(E \cap T) = E \cap HT$ is an S-quasinormal p-subgroup of E by Lemma 1.1 (2). From Lemma 1.1 (5)–(6), it follows that $H \leq O_p(E)$, which contradicts (4). Therefore, $N < E$, and so N is p-nilpotent by the choice of (G, E), which contradicts (1) and (4). This contradiction completes the proof. □

Proof of Theorem 0.1 Let p be the smallest prime divisor of $|X|$ and X_p a Sylow p-subgroup of X. If X_p is cyclic, then X is p-nilpotent by Lemma 1.5. Now assume that X_p is non-cyclic. Then by Theorem 2.2, we also have that X is p-nilpotent. Let $X_{p'}$ be the normal p'-Hall subgroup of X. With a similar discussion as above, $X_{p'}$ is q-nilpotent, where q is the second smallest prime divisor of $|X|$. Continuing the steps, we finally obtain that X is dispersive (see [7, p. 6]).

Let r be the largest prime divisor of $|X|$ and X_r a Sylow r-subgroup of X. Then X_r is normal in G. If X_r is cyclic, then obviously, $X_r \leq Z_{\mathcal{U}}(G)$. Otherwise, by Theorem 2.1, we also have that $X_r \leq Z_{\mathcal{U}}(G)$. Let t be the second largest prime divisor of $|X|$ and $X_{\{r,t\}}$ an $\{r,t\}$-Hall subgroup of X. Then $X_{\{r,t\}} \trianglelefteq G$ and $X_{\{r,t\}} = X_r \rtimes X_t$ by Schur-Zassenhaus Theorem, where X_t is a Sylow t-subgroup of X. If X_t is cyclic, then $X_{\{r,t\}}/X_r \cong X_t$ is cyclic, and so $X_{\{r,t\}}/X_r \leq Z_{\mathcal{U}}(G/X_r)$. Suppose that X_t is non-cyclic. From the hypothesis, Lemma 1.2 (2) and Theorem 2.1, it follows that $X_{\{r,t\}}/X_r \leq Z_{\mathcal{U}}(G/X_r)$. Thus $X_{\{r,t\}} \leq Z_{\mathcal{U}}(G)$ in any case. The rest could be deduced by analogy. Thus $X \leq Z_{\mathcal{U}}(G)$, and so $F^*(E) \leq Z_{\mathcal{U}}(G)$. Then by Lemma 1.6, $E \leq Z_{\mathcal{U}}(G)$. Therefore, $E \leq Z_{\mathcal{U}}(G) \leq Z_{\mathcal{F}}(G)$. Consequently, $G \in \mathcal{F}$. □

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有限群的 CAP- 拟正规子群

张　丽,　吴珍凤,　郭文彬

(中国科学技术大学数学科学学院, 合肥, 安徽, 230026)

摘要：有限群 G 的一个子群 A 称为 G 的广义 CAP- 子群, 如果对于任一 G- 主因子 H/K, 要么 A 避免 H/K, 要么下述成立: (1) 如果 H/K 非交换, 那么 $(A\cap H)K/K$ 是 H/K 的一个 Hall 子群; (2) 如果 H/K 是一个 p- 群, 那么 $|G:N_G((A\cap H)K)|$ 是一个 p- 数. G 的一个子群 H 称为在 G 中是 CAP- 拟正规的, 如果 G 有一个拟正规子群 T 和一个广义 CAP- 子群 A 满足 HT 在 G 中是 S- 拟正规的并且 $H\cap T\leq A\leq H$. 本文得到了 CAP- 拟正规子群的一些结果并用它们给出一个有限群属于某个包含超可解群的饱和群系的条件. 文章推广了很多最近的结果.

关键词：有限群; CAP- 拟正规子群; 饱和群系; $\mathcal{F}$- 超中心; p- 幂零群