A Basis of the q-Schur Module

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Abstract

In this paper, we construct the q-Schur modules as left principle ideals of the cyclotomic q-Schur algebras, and prove that they are isomorphic to those cell modules defined in [3] and [10] at any level r. Then we prove that these q-Schur modules are free modules and construct their bases. This result gives us new versions of several results about the standard basis and the branching theorem. With the help of such realizations and the new bases, we re-prove the Branch rule of Weyl modules which was first discovered and proved by Wada in [20].

Keywords: q-Schur module, cyclotomic q-Schur algebra, branching theorem

2010 Mathematics Subject Classification: 20G43

1. **Introduction**

Weyl modules for a cyclotomic q-Schur algebra $\mathscr{S}_{n,r}$ have been investigated recently in the context of cellular algebras (see [3]). These modules are defined as quotient modules of certain *permutation* modules, that is, as *cell modules* via cellular bases. Such cellular bases play a decisive role in the study.

However, the classical theory [1] and the work [4] [5] in the case when $m = 1, 2$ suggested that a construction as *submodules* without using cellular bases should exist in the case of Iwahori-Hecke algebra. Following Dipper and James' work [2], when the level l equals to one, the *basis* and *structure* appearing in Hecke algebras can still be constructed in q-Schur algebras with totally different proof.

This phenomena needs great change to stay valid in the case of cyclotomic q-Schur algebras of arbitrary level, which is the major motivation of this paper. We can solve this difficulty by constructing a series of principle left ideals in the cyclotomic q-Schur algebras generated by a single element z_λ, which we construct as $\varphi^1_{\lambda w} \cdot T_{w_\lambda} \cdot y_{\lambda'}$ by the right Ariki-Koike algebra $\mathcal{H}_{n,r}$-module structure, where the element $y_{\lambda'}$ and morphism $\varphi^d_{\lambda w}$ are defined in 2.3 and 2.4 respectively. The *q-Schur module* $\mathcal{A}^\lambda$ is defined as $\mathscr{S}_{n,r} \cdot \varphi^1_{\lambda w} T_{w_\lambda} y_{\lambda'}$ as given in Definition 2.4. Then in Theorem 3.1, we prove that the $\mathcal{A}^\mu$ as $\mathscr{S}_{n,r} \cdot z_\mu$ is exactly a realization of the Weyl modules in the category of modules over cyclotomic q-Schur algebras which is a generalization of Dipper and James' work [2]. After that, we construct an R-linear basis of q-Schur module $\mathcal{A}^\mu$ and prove the following theorem,

Main Theorem: *Suppose that $\lambda \in \Lambda^+_{n,r}(\boldsymbol{m})$. Then the q-Schur module $\mathcal{A}^\lambda$ is free as an R-module and $\{\varphi^{1_A}_{\mu\lambda} \cdot z_\lambda | A \in \mathcal{T}^{ss}_\mu(\lambda)$ and $\mu \in \Lambda_{n,r}(\boldsymbol{m})\} \subseteq \mathcal{A}^\lambda$ is a basis.*

Here μ is any multipartition as defined in Section 2.1 and A is its semi-standard tableau as defined in Remark 3.3, which lies in between the semi-standard basis that appeared in [3] and the definition of φ_μ. With the help of this basis, we can show a new version of the branch rule which appeares in the category of modules over a cyclotomic q-Schur algebra.

The paper is organized as follows. In Section 3, we construct the left ideals $\mathcal{A}^\mu$ called *q-Schur modules* over the cyclotomic q-Schur algebra ${}_R\mathscr{S}_{n,r}$, and prove that these q-Schur modules are the same as the Weyl modules in [3]. After that, we construct the natural bases $\{\varphi^{1_A}_{\mu\lambda} \cdot z_\lambda | \mu \in \Lambda_{n,r}(\mathbf{m})$ and A$\in \mathcal{T}^{ss}_\mu(\lambda)\}$ in these ideals, following the work of Dipper and James obtained in [4] in case of Iwahori-Hecke algebras. In the final section, by using these new bases in the q-Schur modules, we construct their filtrations, which gives a new point of view to the branch rule in Wada's work [20].

2. Prelimilaries

2.1. **Some notations about tableaux.** A *composition* λ of n is a finite sequence of non-negative integers $(\lambda_1, \lambda_2, \ldots, \lambda_m)$ such that $|\lambda| = \sum_i \lambda_i = n$. Moreover, there is a partial order $\trianglelefteq$ (resp. $\trianglerighteq$) within compositions of n defined as follows. We denote $\lambda \trianglelefteq \mu$ when $\sum_{i=1}^k \lambda_i \leq \sum_{i=1}^k \mu_i$ (resp. $\sum_{i=1}^k \lambda_i \geq \sum_{i=1}^k \mu_i$) for all $1 \leq k \leq m$. Moreover, if a composition λ satisfies that $\lambda_1 \geq \lambda_2 \geq \cdots \lambda_m$, we call it a *partition*.

Let $\mathfrak{S}_n$ denote the symmetric group of all permutations of $1, \ldots, n$ with Coxeter generators $s_i := (i, i+1)$, and $\mathfrak{S}_\lambda$ the Young subgroup corresponding to the composition λ of n. Thus, we have

$$\mathfrak{S}_\lambda = \mathfrak{S}_{\mathbf{a}} = \mathfrak{S}_{\{1,\ldots,a_1\}} \times \mathfrak{S}_{\{a_1+1,\ldots,a_2\}} \times \cdots \times \mathfrak{S}_{\{a_{n-1}+1,\ldots,a_n\}},$$

where $\mathbf{a} = [a_0, a_1, \ldots, a_n]$ with $a_0 = 0$ and $a_i = \lambda_1 + \cdots + \lambda_i$ for all $i = 1, \ldots, m$. We denote by $\mathscr{D}_\lambda$ the set of distinguished representatives of the right $\mathfrak{S}_\lambda$-cosets and write $\mathscr{D}_{\lambda\mu} := \mathscr{D}_\lambda \cap \mathscr{D}_\mu^{-1}$, which is the set of distinguished representatives of the double cosets $\mathfrak{S}_\lambda \setminus \mathfrak{S}_n / \mathfrak{S}_\mu$.

As usual one identifies a composition λ to its *Young diagram* and we say that λ is the *shape* of the corresponding Young diagram. A λ-*tableau* is a filling of the n boxes of the Young diagram of λ of the numbers $1, 2, \ldots, n$. We denote the set of λ-tableaux by $\mathcal{T}(\lambda)$ and usually denote $\mathfrak{t}$ as an element of $\mathcal{T}(\lambda)$.

For later use, let $\Lambda(n)$ (resp. $\Lambda^+(n)$) denote the set of all compositions (resp. all partitions) of r. For $\lambda \in \Lambda(n)$, let λ' be the dual partition of λ, i.e., $\lambda'_i := \#\{j; \lambda_j \geq i\}$. There is a unique element $w_\lambda \in \mathfrak{S}_n$ with the trivial intersection property [4](4.1):

$$(2.1) \qquad {}^{w_\lambda}\mathfrak{S}_\lambda \cap \mathfrak{S}_{\lambda'} = w_\lambda^{-1}\mathfrak{S}_\lambda w_\lambda \cap \mathfrak{S}_{\lambda'} = \{1\}.$$

We can represent w_λ with the help of Young diagrams. For example, $\yng(3,2)$ represents $\lambda = (3,2)$, then $w_\lambda \in \mathfrak{S}_n$ is defined by the equation $\mathfrak{t}^\lambda w_\lambda = \mathfrak{t}_\lambda$, where $\mathfrak{t}^\lambda$ (resp. $\mathfrak{t}_\lambda$) is the λ-tableau obtained by putting the number $1, 2, \ldots, n$ in order into the boxes from left to right down successive rows (resp. columns). Thus, in the example, $\mathfrak{t}^{(3,2)} = \young(123,45)$, and $\mathfrak{t}_{(3,2)} = \young(135,24)$.

We quote the following definition as in [2].

Definition 2.1. *Suppose that $\mathfrak{t}_1$ is a λ-tableau and $\mathfrak{t}_2$ is a μ-tableau for λ, $\mu \in \Lambda^+(n)$. Let $\chi(\mathfrak{t}_1, \mathfrak{t}_2)$ be the n-by-n matrix whose entry in row i and column j is the cardinality of*

$$\{\textit{entries in the first } i \textit{ rows of } \mathfrak{t}_1\} \cap \{\textit{entries in the first } j \textit{ columns of } \mathfrak{t}_2\}.$$

Also from [2] we have the following remark,

Remark 2.2. *If $\mathfrak{t}_1$ and $\mathfrak{t}'_1$ are λ-tableaux and $\mathfrak{t}_2$ and $\mathfrak{t}'_2$ are μ-tableaux for λ and $\mu \in \Lambda^+(n)$, then write $\chi(\mathfrak{t}_1, \mathfrak{t}_2) \geq \chi(\mathfrak{t}'_1, \mathfrak{t}'_2)$ if each entry in $\chi(\mathfrak{t}_1, \mathfrak{t}_2)$ is at least as big as the corresponding entry in $\chi(\mathfrak{t}'_1, \mathfrak{t}'_2)$. Write $\chi(\mathfrak{t}_1, \mathfrak{t}_2) > \chi(\mathfrak{t}'_1, \mathfrak{t}'_2)$ if, in addition, $\chi(\mathfrak{t}_1, \mathfrak{t}_2) \neq \chi(\mathfrak{t}'_1, \mathfrak{t}'_2)$.*

The following properties are immediate from the definitions.

$$\chi(\mathfrak{t}_1 w, \mathfrak{t}_2 w) = \chi(\mathfrak{t}_1, \mathfrak{t}_2) \quad for\ all\ w \in \mathfrak{S}_r. \tag{2.2}$$

$$\chi(\mathfrak{t}_1 w, \mathfrak{t}_2) = \chi(\mathfrak{t}_1, \mathfrak{t}_2) \quad if\ w \in \mathfrak{S}_\lambda. \tag{2.3}$$

$$\chi(\mathfrak{t}_1, \mathfrak{t}_2 w) = \chi(\mathfrak{t}_1, \mathfrak{t}_2) \quad if\ w \in \mathfrak{S}_{\mu'}. \tag{2.4}$$

Let $\mathbf{m} = (m_1, \cdots, m_r) \in \mathbb{Z}^r_{>0}$ be an r-tuple of positive integers. Define a subset of r-composition of n as:

$$\Lambda_{n,r}(\mathbf{m}) = \left\{ \mu = (\mu^{(1)}, \cdots, \mu^{(r)}) \;\middle|\; \begin{array}{c} \mu^{(k)} = (\mu^{(k)}_1, \cdots, \mu^{(k)}_{m_k}) \in \mathbb{Z}^{m_k}_{\geq 0} \\ \sum_{k=1}^{r} \sum_{i=1}^{m_k} \mu^{(k)}_i = n \end{array} \right\}.$$

We denote by $|\mu^{(k)}| = \sum_{i=1}^{m_k} \mu^{(k)}_i$ (resp. $|\mu| = \sum_{k=1}^{r} |\mu^{(k)}|$) the size of $\mu^{(k)}$ (resp. the size of μ). We define the map $\zeta : \Lambda_{n,r}(\mathbf{m}) \to \mathbb{Z}^r_{\geq 0}$ by $\zeta(\mu) = (|\mu^{(1)}|, |\mu^{(2)}|, \cdots, |\mu^{(r)}|)$ for $\mu \in \Lambda_{n,r}(\mathbf{m})$. Put $\Lambda^+_{n,r}(\mathbf{m}) = \{\lambda \in \Lambda_{n,r}(\mathbf{m}) | \lambda^{(k)}_1 \geq \lambda^{(2)}_2 \geq \cdots \geq \lambda^{(k)}_{m_k} \text{ for any } k = 1, \cdots, r\}$.

Let $\lambda' := (\lambda^{(r)\prime}, \ldots, \lambda^{(1)\prime})$ denote the m-composition dual to λ. By concatenating the components of λ, the resulting composition of r will be denoted by

$$\overline{\lambda} = \lambda^{(1)} \vee \cdots \vee \lambda^{(r)}.$$

We can identify λ with its Young diagram. For example, $\lambda = ((31), (22), (1))$ is identified with

$$(\yng(3,1), \yng(2,2), \yng(1)).$$

Let $\mathfrak{t}^\lambda$ be the λ-tableau obtained by putting the number $1, \ldots, r$ in order into the boxes down successive rows in the first diagram of λ, then in the second diagram and so on. From the example above, we have

$$\mathfrak{t}^\lambda = \left(\begin{smallmatrix} 1 & 2 & 3 \\ 4 & & \end{smallmatrix}, \begin{smallmatrix} 5 & 6 \\ 7 & 8 \end{smallmatrix}, \begin{smallmatrix} 9 \end{smallmatrix}\right).$$

We also define the λ-tableau $\mathfrak{t}_\lambda$ by putting the numbers in the order down successive columns in the last diagram of λ, then in the second last diagram, and so on. For the above example, we have

$$\mathfrak{t}_\lambda = \left(\begin{smallmatrix} 6 & 8 & 9 \\ 7 & & \end{smallmatrix}, \begin{smallmatrix} 2 & 4 \\ 3 & 5 \end{smallmatrix}, \begin{smallmatrix} 1 \end{smallmatrix}\right).$$

Now, associated to a r-partition $\lambda = (\lambda^{(1)}, \dots, \lambda^{(r)})$ of n, we define the element $w_\lambda \in \mathfrak{S}_n$ by $\mathfrak{t}^\lambda w_\lambda = \mathfrak{t}_\lambda$. More precisely, if $\mathfrak{t}^i$ (resp. $\mathfrak{t}_i$) denotes the i-th subtableau of $\mathfrak{t}^\lambda$ (resp. $\mathfrak{t}_\lambda w_{[\lambda]}^{-1}$) and define $w_{(i)}$ by $\mathfrak{t}^i w_{(i)} = \mathfrak{t}_i$, then $\mathfrak{t}^\lambda w_{(1)} \cdots w_{(r)} w_{[\lambda]}$. Likewise, if we define $\tilde{\mathfrak{t}}^i$ (resp. $\tilde{\mathfrak{t}}_i$) the i-th subtableau of $\mathfrak{t}^\lambda w_{[\lambda]}$ (resp. $\mathfrak{t}_\lambda$) and $\tilde{w}_{(1)}$ with $\tilde{\mathfrak{t}}^i \tilde{w}_{(i)} = \tilde{\mathfrak{t}}_i$, then $\mathfrak{t}^\lambda w_{[\lambda]} \tilde{w}_{(1)} \cdots \tilde{w}_{(r)} = \mathfrak{t}_\lambda$. We have, therefore

$$w_\lambda = w_{(1)} \cdots w_{(r)} w_{[\lambda]} = w_{[\lambda]} \tilde{w}_{(r)} \cdots \tilde{w}_{(1)}, \qquad w_{[\lambda]}^{-1} w_{(i)} w_{[\lambda]} = \tilde{w}_{(r-i+1)}. \tag{2.5}$$

Note that $w_{(i)} w_{(j)} = w_{(j)} w_{(i)}$ and $\tilde{w}_{(i)} \tilde{w}_{(j)} = \tilde{w}_{(j)} \tilde{w}_{(i)}$ for $i, j = 1, 2, \dots, r$.

2.2. **Ariki-Koike algebras and cyclotomic q-Schur algebras.** In this subsection, we recall the definition of the cyclotomic q-Schur algebra $\mathscr{S}_{n,r}$ introduced by [3], and review the presentations of $\mathscr{S}_{n,r}$ by generators and fundamental relations given by [21].

Let R be a commutative ring, and we take parameters $q, Q_1, \cdots, Q_r \in R$ such that q is invertible in R. The Ariki-Koike algebra $\mathcal{H}_{n,r}$ associated to the complex group $\mathfrak{S}_n \ltimes (\mathbb{Z}/r\mathbb{Z})^n$ is the associative algebra with 1 over R generated by $T_0, T_1, \dots, T_{n-1}$ with the following defining relations:

$$\begin{aligned}
&(T_0 - Q_1)(T_0 - Q_2) \cdots (T_0 - Q_r) = 0, && \\
&(T_i - q)(T_i + q^{-1}) = 0 && (1 \le i \le n-1), \\
&T_0 T_1 T_0 T_1 = T_1 T_0 T_1 T_0, && \\
&T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} && (1 \le i \le n-2), \\
&T_i T_j = T_j T_i && (|i-j| \ge 2).
\end{aligned}$$

The subalgebra of $\mathcal{H}_{n,r}$ generated by $T_1, \cdots, T_{n-1}$ is isomorphic to the *Iwahori-Hecke algebra* associated to the symmetric group $\mathfrak{S}_n$ which is discussed in [16]. For $w \in \mathfrak{S}_n$, denote by $\ell(w)$ the length of w and by T_w the standard basis of $\mathcal{H}_{n,r}$ corresponding to w.

For each r-composition $\lambda = (\lambda^{(1)}, \dots, \lambda^{(r)})$, define $[\lambda] := [a_0, a_1, \dots, a_r]$ such that $a_0 := 0$ and $a_i := \sum_{j=1}^{i} |\lambda^{(j)}|$. In the case of Iwahori-Hecke algebras, we can define an element $m_\lambda \in \mathcal{H}_n$ as $m_\lambda := \sum\limits_{w \in \mathfrak{S}_\lambda} T_w$. Here $w_\lambda \in \mathfrak{S}_n$ is defined in last subsection.

Definition 2.3. *Let $\mathcal{H}_{n,r}$ be a cyclotomic Hecke algebra with generators $\{T_0, T_1, \dots, T_{n-1}\}$, and elements $L_1 = T_0$, $L_i = q^{-1} T_{i-1} L_{i-1} T_{i-1}$ for $i = 2, \cdots, n$, and put $\pi_0 = 1$, $\pi_a(x) = \Pi_{j=1}^{a} (L_j - x)$ for any $x \in R$ and any positive integer a. Following* [3], *for $\boldsymbol{a} = [\lambda] = [a_0, a_1, \dots, a_r] \in \Lambda[m, r]$ for some m, we define that*

$$u_{\boldsymbol{a}}^{+} = \pi_{a_1}(Q_2) \cdots \pi_{a_{r-1}}(Q_r) \quad \textit{and} \quad u_{\boldsymbol{a}}^{-} = \pi_{a_1}(Q_{r-1}) \cdots \pi_{a_{r-1}}(Q_1),$$

and, for $\lambda \in \Lambda_{n,r}(\boldsymbol{m})$, we define that

$$x_\lambda := u^+_{[\lambda]} m_{\overline{\lambda}} = m_{\overline{\lambda}} u^+_{[\lambda]} \text{ and } y_\lambda := u^-_{[\lambda]} m_{\overline{\lambda}} = m_{\overline{\lambda}} u^-_{[\lambda]}.$$

Define the right ideal as $M^\lambda := x_\lambda \mathcal{H}_{n,r}$ which is called a permutation module*.*

The cyclotomic q-Schur algebra $\mathscr{S}_{n,r}$ associated to $\mathcal{H}_{n,r}$ is defined by

$${}_R\mathscr{S}_{n,r} = {}_R\mathscr{S}_{\Lambda_{n,r}}(\mathbf{m}) = \mathrm{End}_{\mathcal{H}_{n,r}}\Big(\bigoplus_{\mu \in \Lambda_{n,r}(\mathbf{m})} M^\mu\Big).$$

In order to describe a presentation of ${}_R\mathscr{S}_{n,r}$, we need some notations. Put $m = \sum_{k=1}^r m_k$, and let $P = \bigoplus_{i=1}^m \mathbb{Z}\varepsilon_i$ be the weight lattice of $\mathfrak{gl}_m$. Set $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ for $i = 1, \cdots, m-1$, then $\Pi = \{\alpha_i | 1 \le i \le m-1\}$ is the set of simple roots, and $Q = \bigoplus_{i=1}^{m-1} \mathbb{Z}\alpha_i$ is the root lattice of $\mathfrak{gl}_m$. Put $Q^+ = \bigoplus_{i=1}^{m-1} \mathbb{Z}_{\ge 0}\alpha_i$. We define a partial order "$\geq$" on P, so called dominance order, by $\lambda \ge \mu$ if $\lambda - \mu \in Q^+$. It is the alternative definition of "dominant order in multipartitions" when $\lambda, \mu \in \Lambda_{n,r}(\mathbf{m})$, i.e., $\lambda \trianglerighteq \mu$ if $\sum_{i=1}^{l-1} |\lambda^{(i)}| + \sum_{k=1}^{j} \lambda_k^{(l)} \ge \sum_{i=1}^{l-1} |\mu^{(i)}| + \sum_{k=1}^{j} \mu_k^{(l)}$ for any $1 \le l \le r$, $1 \le j \le m_l$.

For $(i,k) \in \Gamma'(\mathbf{m})$, we define the elements $E_{(i,k)}$, $F_{(i,k)} \in {}_R\mathscr{S}_{n,r}$ by

$$E_{(i,k)}(m_\mu \cdot h) = \begin{cases} q^{-\mu_{i+1}^{(k)}+1}\Big(\sum_{x \in X_\mu^{\mu+\alpha_{(i,k)}}} q^{\ell(x)} T_x^*\Big) h^\mu_{+(i,k)} m_\mu \cdot h & if \quad \mu + \alpha_{(i,k)} \in \Lambda_{n,r}(\mathbf{m}), \\ 0 & if \quad \mu + \alpha_{(i,k)} \notin \Lambda_{n,r}(\mathbf{m}), \end{cases}$$

$$F_{(i,k)}(m_\mu \cdot h) = \begin{cases} q^{-\mu_{i}^{(k)}+1}\Big(\sum_{y \in X_\mu^{\mu-\alpha_{(i,k)}}} q^{\ell(x)} T_y^*\Big) m_\mu \cdot h & if \quad \mu - \alpha_{(i,k)} \in \Lambda_{n,r}(\mathbf{m}), \\ 0 & if \quad \mu - \alpha_{(i,k)} \notin \Lambda_{n,r}(\mathbf{m}), \end{cases}$$

for $\mu \in \Lambda_{n,r}(\mathbf{m})$ and $h \in {}_R\mathcal{H}_{n,r}$, where $h^\mu_{+(i,k)} = \begin{cases} 1 & (i \neq m_k), \\ L_{N+1} - Q_{k+1} & (i = m_k). \end{cases}$

For $\lambda \in \Lambda_{n,r}(\mathbf{m})$, we define the element $1_\lambda \in {}_R\mathscr{S}_{n,r}$ by

$$1_\lambda(m_\mu \cdot h) = \delta_{\lambda\mu} m_\lambda \cdot h$$

for $\mu \Lambda_{n,r}(\mathbf{m})$ and $h \in {}_R\mathcal{H}_{n,r}$. For this definition, we see that $\{1_\lambda | \lambda \in \Lambda_{n,r}(\mathbf{m})\}$ is a set of pairwise orthogonal idempotents, and we have $1 = \sum_{\lambda \in \Lambda_{n,r}(\mathbf{m})} 1_\lambda$.

Definition 2.4. *For any $\mu \in \Lambda_{n,r}(\boldsymbol{m})$, we can define a left principle ideal of cyclotomic q-Schur algebra as a submodule as in* [2] *with $m = 1$:*

$\mathcal{A}^\mu \triangleq \mathscr{S}_{n,r}\varphi^1_{\mu\omega} T_{w_\mu} y_{\mu'}$ with $\varphi^1_{\mu\omega} \in Hom_{\mathcal{H}_{n,r}}(\mathcal{H}_{n,r}, M^\mu) = M^\mu$ defined as $\varphi^1_{\mu\omega}(h) := x_\mu h$ for any $h \in \mathscr{H}_{n,r}$ and element $T_{w_\mu} y_{\mu'}$ acts on $\varphi^1_{\mu\omega}$ by the right $\mathscr{H}_{n,r}$-module structure of M^μ. From now on, the module $\mathcal{A}^\mu$ is called a q-Schur module, and denote the element $\varphi^1_{\mu\omega} T_{w_\mu} y_{\mu'} \in \mathscr{S}_{n,r}$ by z_μ.

Recall in [6] that the set of all $[\lambda]$ form a poset $\Lambda[m,r]$ with $m = \sum_i a_i$, which is isomorphic to the poset $\Lambda(m,r)$ of all compositions of m with at most r parts as set but with different order. Here the partial ordering on $\Lambda[m,r]$ is given by $\preceq$: $[a_i] \preceq [b_i]$ if $a_i \le b_i$ for all $i = 1, \ldots, r$. While $\Lambda(m,r)$ has the usual dominance order $\trianglelefteq$.

The following results will be useful in the sequel. See (2.8), (3.1), (3.4) in [6].

Lemma 2.5. [6] *Let $\boldsymbol{a}, \boldsymbol{b} \in \Lambda[m,r]$, and also note $\mathcal{H}(\mathfrak{S}_n)$ as the Iwahori-Hecke algebra associated with $\mathfrak{S}_n$.*

$$u_{\boldsymbol{a}}^{+}\mathcal{H}_{n,r}u_{\boldsymbol{b}'}^{-} = 0 \text{ unless } \boldsymbol{a} \preceq \boldsymbol{b}. \tag{2.6 (a)}$$

$$u_{\boldsymbol{a}}^{+}\mathcal{H}(\mathfrak{S}_n)u_{\boldsymbol{a}'}^{-} = v_{\boldsymbol{a}}\mathcal{H}(\mathfrak{S}_{\boldsymbol{a}'}) = \mathcal{H}(\mathfrak{S}_{\boldsymbol{a}})v_{\boldsymbol{a}}, \text{where } v_{\boldsymbol{a}} = u_{\boldsymbol{a}}^{+}T_{w_{\boldsymbol{a}}}u_{\boldsymbol{a}'}^{-}. \tag{2.7 (b)}$$

$$u_{\boldsymbol{a}}^{+}\mathcal{H}_{n,r}u_{\boldsymbol{a}'}^{-} = u_{\boldsymbol{a}}^{+}\mathcal{H}(\mathfrak{S}_n)u_{\boldsymbol{a}'}^{-}. \tag{2.8 (c)}$$

$$v_{\boldsymbol{a}}\mathcal{H}_{n,r} \text{ is a free } R\text{-submodule with basis } \{v_{\boldsymbol{a}}T_w | w \in \mathfrak{S}_r\}. \tag{2.9 (d)}$$

Definition 2.6. [17] *For $\lambda \in \Lambda_{n,r}^{+}(\boldsymbol{m})$ and $\mu \in \Lambda_{n,r}(\boldsymbol{m})$, a λ-tableau of type μ denoted as T is said to be* **semistandard** *if*

(i) the entries in each row of each component of $T^{(k)}$ of T are non-decreasing;

(ii) the entries in each column of each component $T^{(k)}$ of T are strictly increasing;

(iii) if $(a,b,c) \in \lambda$, and $T(a,b,c) = (i,s)$ then $s \geq c$.

Let $\mathcal{T}_{\mu}^{ss}(\lambda)$ be the set of semistandard λ-tableau of type μ and denote $\mathcal{T}_{\Lambda}^{ss}(\lambda) = \cup_{\mu\in\Lambda}\mathcal{T}_{\mu}^{ss}(\lambda)$.

The set

$$\{\Psi_{ST} | S, T \in \mathcal{T}_{\Lambda}^{ss}(\lambda),\ \lambda \in \Lambda^{+}(n,r)\}, \tag{2.10}$$

which is called the *semi-standard* basis of cyclotomic q-Schur algebras in [3], forms a cellular basis of $\mathscr{S}_{n,r}$ in the sense of [11] with the dominance order $\trianglelefteq$ on $\Lambda_{n,r}^{+}(\mathbf{m})$. Let $\mathscr{S}_{n,r}^{\rhd\lambda}$ be the two sides ideal of $\mathscr{S}_{n,r}$ spanned by all Ψ_{ST} with $S, T \in \mathcal{T}_{\Lambda}^{ss}(\mu)$ and $\mu \rhd \lambda$ (i.e., shape(S) = shape$(T) \rhd \lambda$), where shape(T) means the partition associated with tableaux T.

In particular, let $\lambda \in \Lambda^{+}(n,r)$ be a partition and recall that $T^{\lambda} = \lambda(t^{\lambda})$, as in [3] and [16], is the unique semistandard λ-tableau of type λ. From the definitions one sees that $\Psi_{T^{\lambda}T^{\lambda}}$ restricts to the identity map on M_{λ}, and sometimes we denote this element by Ψ_{λ} . Then, we can define the "cell module" as a submodule of $\mathscr{S}_{n,r}/\mathscr{S}_{n,r}^{\rhd\lambda}$:

$$W^{\lambda} = \mathscr{S}_{n,r}\bar{\Psi}_{\lambda}, \qquad \text{where } \bar{\Psi}_{\lambda} := (\mathscr{S}_{n,r}^{\rhd\lambda} + \Psi_{\lambda})/\mathscr{S}_{n,r}^{\rhd\lambda}. \tag{2.11}$$

The module W^{λ} is called a *Weyl module* in [3].

3. Main theorem and its proof

We now prove that the q-Schur modules we defined above are isomorphic to those in [3] as "cell modules" when $\lambda \in \Lambda_{n,r}^{+}(\mathbf{m})$. Recall the definitions given in 2.6.

Theorem 3.1. *For each $\lambda \in \Lambda_{n,r}^{+}(\boldsymbol{m})$, we have the following $\mathscr{S}_{n,r}$-module isomorphism:*

$$\mathcal{A}^{\lambda} \cong W^{\lambda}.$$

Proof. Consider the epimorphism:

$$\theta : \mathscr{S}_{n,r}\Psi_{\lambda} \longrightarrow \mathscr{S}_{n,r}z_{\lambda}; \quad h\Psi_{\lambda} \mapsto hz_{\lambda} = h\varphi_{\lambda\omega}^{1}T_{w_{\lambda}}y_{\lambda'} = h\varphi_{\lambda\omega}^{1} \cdot T_{w_{(1)}\cdots w_{(r)}}y_{\mu^{(1)\prime}\vee\cdots\vee\mu^{(r)\prime}} \cdot v_{[\mu]}.$$

Suppose that $T \in \mathcal{T}_{\lambda}^{ss}(\mu)$ and $S \in \mathcal{T}_{\nu}^{ss}(\mu)$ with $\mu \in \Lambda_{n,r}(\mathbf{m})$ and $\nu \in \Lambda_{n,r}(\mathbf{m})$. By the definition of Ψ_{ST} in [3] and semistandard basis theorem [3] (**6.6**), we easily find that the set $\{\Psi_{ST} | T \in \mathcal{T}^{ss}_{\lambda}(\mu),\ S \in \mathcal{T}^{ss}_{\nu}(\mu)\ with\ \mu \trianglerighteq \lambda$ and $\mu \in \Lambda^+_{n,r}(\mathbf{m}), \nu \in \Lambda_{n,r}(\mathbf{m})\}$ is an R-basis of $\mathscr{S}_{n,r}\Psi_{\lambda}$. More precisely, we can write this basis as

(3.1) $\quad \{\Psi_{TT^{\lambda}} | T \in \mathcal{T}^{ss}_{\nu}(\lambda)\} \cup \{\Psi_{ST} | T \in \mathcal{T}^{ss}_{\lambda}(\mu)\ and\ S \in \mathcal{T}^{ss}_{\nu}(\mu)\ with\ \mu \triangleright \lambda\}.$

Then we obviously have that

$$W^{\lambda} \cong \mathscr{S}_{n,r}\Psi_{\lambda}/(\mathscr{S}_{n,r}\Psi_{\lambda} \cap \mathscr{S}^{\triangleright\lambda}_{n,r}).$$

We claim that, with $\mu \trianglerighteq \lambda$ and $\lambda \in \Lambda^+_{n,r}(\mathbf{m})$, $\nu \in \Lambda_{n,r}(\mathbf{m})$, if $\theta(\Psi_{ST}) = \theta(\Psi_{ST}\Psi_{T^{\lambda}T^{\lambda}}) = \Psi_{ST}\varphi^1_{\lambda\omega}T_{w_{\lambda}}y_{\lambda'} \neq 0$,then $\mu = \lambda$.

Consider the action on the unit of $\mathcal{H}_{n,r}$:

$$\begin{aligned}
\Psi_{ST}\varphi^1_{\lambda\omega}T_{w_{\lambda}}y_{\lambda'}(1) &= m_{ST}T_{w_{\lambda}}y_{\lambda'} \\
&= \sum_{\substack{\mathfrak{t}\in Std(\mu)\\ \lambda(\mathfrak{t})=T}} m_{S\mathfrak{t}}T_{w_{\lambda}}y_{\lambda'} = \sum_{\substack{\mathfrak{t}\in Std(\mu)\\ \lambda(\mathfrak{t})=T}} \sum_{\substack{\mathfrak{s}\in Std(\mu)\\ \nu(\mathfrak{s})=S}} m_{\mathfrak{s}\mathfrak{t}}T_{w_{\lambda}}y_{\lambda'} \\
&= \sum_{\mathfrak{s},\mathfrak{t}} T_{d(\mathfrak{s})}x_{\mu}T_{d(\mathfrak{t})}T_{w_{\lambda}}y_{\lambda'} = \sum_{\mathfrak{s},\mathfrak{t}} T_{d(\mathfrak{s})}x_{\bar{\mu}}u^+_{[\mu]}T_{d(\mathfrak{t})}T_{w_{\lambda}}u^-_{[\lambda']}y_{\bar{\lambda}'} \\
&= (*).
\end{aligned}$$

Recall that by Lemma 2.5, $u^+_{\mathbf{a}}\mathcal{H}_{n,r}u^-_{\mathbf{b}'} = 0$ unless $\mathbf{a} \preceq \mathbf{b}$. $\Psi_{ST}\varphi^1_{\lambda\omega}T_{w_{\lambda}}y_{\lambda'} \neq 0$ implies that for some $\mathfrak{s}$ and $\mathfrak{t}$ above, $T_{d(\mathfrak{s})}x_{\bar{\mu}}u^+_{[\mu]}T_{d(\mathfrak{t})}T_{w_{\lambda}}u^-_{[\lambda']}y_{\bar{\lambda}'} \neq 0$. Thus, this condition shows that $[\mu] \preceq [\lambda]$. On the other hand, with the assumption in the above claim, i.e., $\mu \trianglerighteq \lambda$, it is obvious that $[\mu] \succeq [\lambda]$ by the definition of $[\mu]$, $[\lambda]$ and $\trianglerighteq$, $\succeq$. So $[\mu] = [\lambda]$. Then we find

$$\begin{aligned}
(*) &= \sum_{\substack{\mathfrak{s},\mathfrak{t}\\ [\mu]=[\lambda]}} T_{d(\mathfrak{s})}x_{\bar{\mu}}u^+_{[\mu]}T_{d(\mathfrak{t})}T_{w_{\lambda}}u^-_{[\mu]'}y_{\bar{\lambda}'} \\
&= \sum_{\substack{\mathfrak{s},\mathfrak{t}\\ [\mu]=[\lambda]\\ h'\in\mathfrak{S}_{[\mu]}}} T_{d(\mathfrak{s})}x_{\bar{\mu}}h'v_{[\mu]}y_{\bar{\lambda}'} \qquad \text{by (b), (c) in Lemma 2.5} \\
&= \sum_{\substack{\mathfrak{s},\mathfrak{t}\\ [\mu]=[\lambda]\\ h'_i\in\mathfrak{S}_{\{|\lambda_{i-1}|+1,\cdots,|\lambda_i|\}}}} T_{d(\mathfrak{s})}x_{\mu^{(1)}\vee\cdots\vee\mu^{(r)}}h'_1\cdots h'_m y_{\lambda^{(1)\prime}\vee\cdots\vee\lambda^{(r)\prime}}v_{[\mu]} \qquad \text{by [8]} \\
&= \sum_{\substack{\mathfrak{s},\mathfrak{t}\\ [\mu]=[\lambda]\\ h'_i\in\mathfrak{S}_{\{|\lambda_{i-1}|+1,\cdots,|\lambda_i|\}}}} T_{d(\mathfrak{s})}(x_{\mu^{(1)}}h'_1y_{\lambda^{(1)\prime}})\cdots(x_{\mu^{(r)}}h'_my_{\lambda^{(r)\prime}})v_{[\mu]}.
\end{aligned}$$

Since $[\lambda] = [\mu]$, the fact that this is non-zero implies, by [4] (**4.1**), that $\lambda^{(i)} \trianglerighteq \mu^{(i)}$ for all $i = 1, \ldots, r$. On the other hand, by [8] (**1.6**), $\mu \trianglerighteq \lambda$ and $[\mu] = [\lambda]$ implies $\mu^{(i)} \trianglerighteq \lambda^{(i)}$, with $1 \leq i \leq r$. Hence $\mu^{(i)} = \lambda^{(i)}$ for all i, and therefore, $\mu = \lambda$. This completes the proof of the above claim.

By the claim and (3.1), one see that

$$ker\theta = \{\Psi_{ST} \mid T \in \mathcal{T}^{ss}_{\lambda}(\mu) \text{ and } S \in \mathcal{T}^{ss}_{\nu}(\mu) \text{ with } \mu \triangleright \lambda\} = \mathscr{S}_{n,r}\Psi_{\lambda} \cap \mathscr{S}^{\triangleright\lambda}_{n,r}.$$

Therefore, $\mathcal{A}^{\lambda} \cong W^{\lambda}$. $\square$

Definition 3.2. [4] *For $w \in \mathfrak{S}_n$ and $S \in \mathcal{T}_\lambda(\mu)$ with $\lambda, \mu \in \Lambda(n, r)$, define a map*

$$\begin{aligned} \mathfrak{S}_n \times \mathcal{T}_\lambda(\mu) &\longrightarrow \mathscr{D}_\lambda && (3.2)\\ (w, S) &\longmapsto w_S && (3.3) \end{aligned}$$

where the element w_S is defined by the row-standard λ-tableau $\mathfrak{t}^\lambda w_S$ for which i belongs to the row a if the place occupied by i in $\mathfrak{t}^\mu w$ is occupied by a.

For example, $S = \begin{array}{|c|c|c|}\hline 1&2&3\\ \hline 1&2\\ \cline{1-2}\end{array}$ and $\mathfrak{t}^\mu w = \begin{array}{|c|c|c|}\hline 1&2&4\\ \hline 3&5\\ \cline{1-2}\end{array}$ with $\mu = (3, 2)$ and $\lambda = (2, 2, 1)$, then $\mathfrak{t}^\lambda w_S = \begin{array}{|c|c|}\hline 1&3\\ \hline 2&5\\ \hline 4\\ \cline{1-1}\end{array}$.

Remark 3.3. *Let $\mathcal{T}^{ss}_\lambda(\mu)$ be the set of all semi-standard μ-tableaux of type λ, with λ and $\mu \in \Lambda_{n,r}(\boldsymbol{m})$. For any $S \in \mathcal{T}^{ss}_\lambda(\mu)$, we define $1_S := 1_{\bar{S}}$. Since S is a semi-standard μ-tableau of type λ, it implies that $\bar{S}$ is a row-standard $\bar{\mu}$-tableau of type $\bar{\lambda}$, as in* [7].

We compare the definition of semi-standard tableaux in [3] with that in [7]. Note that every entry in S is written as the symbol (i, j) and is replaced by $i + \sum_{k=1}^{j-1} m_k$, for $1 \le i \le m_j$, $1 \le j \le n$.

Then, by the above definition, we obtain the following consequence:

Lemma 3.4. *Suppose that $u \in \mathfrak{S}_r$ and $w \in \mathfrak{S}_{\mu^{(1)\prime} \vee \cdots \vee \mu^{(r)\prime}}$, with $\lambda, \mu \in \Lambda_{n,r}(\boldsymbol{m})$. Then $\varphi^1_{\bar{\lambda}\omega} T_u T_w$ is a linear combination of the terms $\varphi^d_{\bar{\lambda}\omega}$ $(d \in \mathscr{D}_{\bar{\lambda}})$ for which $\chi(t^{\bar{\lambda}} d, t^{\bar{\mu}} w_{(1)} \cdots w_{(r)}) = \chi(t^{\bar{\lambda}} u, t^{\bar{\mu}} w_{(1)} \cdots w_{(r)})$.*

Proof. The conclusion is ture when $w = 1$ since $\varphi^1_{\bar{\lambda}\omega} T_u = \varphi^u_{\bar{\lambda}\omega}$ for some $u \in \mathfrak{S}_n$. Below we assume that $w \neq 1$.

For some $w' \in \mathfrak{S}_n$ and some $a = (i, i+1) \in \mathfrak{S}_{\mu^{(1)\prime} \vee \cdots \vee \mu^{(r)\prime}}$, we have that $w = w'a$, and without losing generality, we can set $(i, i+1) \in \mathfrak{S}_{\mu^{(1)\prime}}$ satisfying:

$$w' = w'_1 \cdots w'_r,\ w = w_1 \cdots w_r \quad \text{with} \quad \begin{aligned} w'_1(i, i+1) &= w_1, \\ w_i &= w'_i \quad for\ i = 2, \cdots, r. \end{aligned}$$

By induction on the length $\ell(w)$, we have $\varphi^1_{\bar{\lambda}\omega} T_u T_{w'}$ as a linear combination of the terms $\varphi^d_{\bar{\lambda}\omega}$ $(d \in \mathscr{D}_{\bar{\lambda}})$ for which $\chi(t^{\bar{\lambda}} d, t^{\bar{\mu}} w_{(1)} \cdots w_{(r)}) = \chi(t^{\bar{\lambda}} u, t^{\bar{\mu}} w_{(1)} \cdots w_{(r)})$.

Consider

$$\varphi^1_{\bar{\lambda}\omega} T_u T_w = \varphi^1_{\bar{\lambda}\omega} T_u T_w T_a = \sum_{\chi(t^{\bar{\lambda}} d, t^{\bar{\mu}} w_{(1)} \cdots w_{(r)}) = \chi(t^{\bar{\lambda}} u, t^{\bar{\mu}} w_{(1)} \cdots w_{(r)})} C_d \varphi^d_{\bar{\lambda}\omega} T_a.$$

By [2] or [4], we have

(3.4)

$$\varphi^d_{\bar{\lambda}\omega} T_a = \begin{cases} q\varphi^d_{\bar{\lambda}\omega} & \text{if } i, i+1 \text{ belong to the same row of } t^{\bar{\lambda}} d; \\ \varphi^{da}_{\bar{\lambda}\omega} & \text{if the row index of } i \text{ in } t^{\bar{\lambda}} \text{ is less than that of } i+1; \\ q\varphi^{da}_{\bar{\lambda}\omega} + (q-1)\varphi_{\bar{\lambda}} \varphi^d_{\bar{\lambda}\omega} & \text{otherwise.} \end{cases}$$

Then the proof is completed through checking the formula above case by case. □

By the definition in Remark 3.3, we can show the following theorem about the bases, which is the main result in this paper.

Theorem 3.5. *Suppose that* $\lambda \in \Lambda^+_{n,r}(\boldsymbol{m})$. *Then the q-Schur module* $\mathcal{A}^\lambda$ *is free as an R-module and* $\{\varphi^{1_A}_{\mu\lambda} \cdot z_\lambda | A \in \mathcal{T}^{ss}_\mu(\lambda)$ *and* $\mu \in \Lambda_{n,r}(\boldsymbol{m})\} \subseteq \mathcal{A}^\lambda$ *is a basis.*

Proof. With the help of Theorem 3.1, it is enough to show that $\{\varphi^{1_A}_{\mu\lambda} z_\lambda | A \in \mathcal{T}^{ss}_\mu(\lambda)$ *and* $\mu \in \Lambda_{n,r}(\mathbf{m})\} \subseteq \mathcal{A}^\lambda$ is R-linearly independent. We calculate the action of the element $\varphi^{1_A}_{\lambda\mu} \cdot z_\mu$ on the unit of $\mathcal{H}_{n,r}$,

$$\begin{aligned}
\varphi^{1_A}_{\lambda\mu} \cdot z_\mu(1) = \varphi^{1_A}_{\lambda\mu}\varphi^{1}_{\mu\omega} T_{w_\mu} y_{\mu'}(1) &= \varphi^{1_A}_{\lambda\mu}(x_\mu) T_{w_\mu} y_{\mu'} \\
&= (\sum_{d \in \mathfrak{S}_{\bar\lambda} 1_A \mathfrak{S}_{\bar\mu}} T_d) \cdot u^+_{[\mu]} T_{w_\mu} y_{\bar\mu'} u^-_{[\mu']} && \text{by [7]} \\
&= (\sum_{d \in \mathfrak{S}_{\bar\lambda} 1_A \mathfrak{S}_{\bar\mu}} T_d) \cdot T_{w_{(1)}\cdots w_{(m)}} u^+_{[\mu]} T_{w_{[\mu]}} u^-_{[\mu']} y_{\bar\mu'} \\
&= \varphi^{1_A}_{\bar\lambda\bar\mu}(x_{\bar\mu}) \cdot T_{w_{(1)}\cdots w_{(r)}} v_{[\mu]} y_{\mu^{(r)\prime} \vee \cdots \vee \mu^{(1)\prime}} && \text{by Lemma 2.5} \\
&= \varphi^{1_A}_{\bar\lambda\bar\mu}(x_{\bar\mu}) \cdot T_{w_{(1)}\cdots w_{(r)}} \cdot y_{\mu^{(1)\prime} \vee \cdots \vee \mu^{(r)\prime}} \cdot v_{[\mu]} && \text{by [6]} \\
&= \varphi^{1_A}_{\bar\lambda\bar\mu}(x_{\mu^{(1)} \vee \cdots \vee \mu^{(r)}} T_{w_{(1)}\cdots w_{(r)}} y_{\mu^{(1)\prime} \vee \cdots \vee \mu^{(r)\prime}}) \cdot v_{[\mu]} \\
&= \varphi^{1_A}_{\bar\lambda\bar\mu}\varphi^{1}_{\bar\mu\omega} \cdot T_{w_{(1)}\cdots w_{(r)}} y_{\mu^{(1)\prime} \vee \cdots \vee \mu^{(r)\prime}}(1) \cdot v_{[\mu]}
\end{aligned}$$

Then, following from the calculation in [2], for $A, B \in \mathcal{T}_{\bar\lambda}(\bar\mu)$, we write $A \sim B$ if A and B are *row equivalent*, which as defined in [3], i.e. if one tableau A can be changed to B by a sequence of elementary row permutations. Then, $\mathfrak{S}_{\bar\lambda} 1_A \mathfrak{S}_{\bar\mu} = \bigcup_{B \sim A} \mathfrak{S}_{\bar\lambda} 1_B$. In addition, if $w \in \mathfrak{S}_n$, we denote by $\overline{w}$ the unique element of $\mathfrak{S}_\lambda w \cap \mathscr{D}_\lambda$ for some $\lambda \in \Lambda(n,r)$, i.e. the shortest element in $\mathfrak{S}_\lambda w$. We have

$$\begin{aligned}
&\varphi^{1_A}_{\bar\lambda\bar\mu}\varphi^{1}_{\bar\mu\omega} \cdot T_{w_{(1)}\cdots w_{(r)}} y_{\mu^{(1)\prime} \vee \cdots \vee \mu^{(r)\prime}} \\
=\ & (\sum_{B \sim A} \varphi^{1_B}_{\bar\lambda\omega} T_{w_{(1)}\cdots w_{(r)}}) y_{\mu^{(1)\prime} \vee \cdots \vee \mu^{(r)\prime}} \\
=\ & (\sum_{B \sim A} \varphi^{1}_{\bar\lambda\omega} T_{1_B} T_{w_{(1)}\cdots w_{(r)}}) y_{\mu^{(1)\prime} \vee \cdots \vee \mu^{(r)\prime}} \\
=\ & (\sum_{B \sim A} q^{K_B} \varphi^{1}_{\bar\lambda\omega} T_{\overline{1_B w_{(1)}\cdots w_{(r)}}} + s_B) \cdot y_{\mu^{(1)\prime} \vee \cdots \vee \mu^{(r)\prime}} \qquad \text{by [2]}
\end{aligned}$$

where K_B is an integer and s_B is a linear combination of terms $\varphi^{d}_{\bar\lambda\omega}$ for which

$$\chi(t^{\bar\lambda} 1_B, t^{\bar\mu}) > \chi(t^{\bar\lambda} d, t^{\bar\mu} w_{(1)} \cdots w_{(r)}).$$

Moreover, $\chi(t^{\bar\lambda} 1_A, t^{\bar\mu}) > \chi(t^{\bar\lambda} 1_B, t^{\bar\mu}) = \chi(t^{\bar\lambda} \overline{1_B w_{(1)} \cdots w_{(r)}}, t^{\bar\mu} 1_B w_{(1)} \cdots w_{(r)})$ if $B \sim A$ but $B \neq A$. Hence

$$\varphi^{1_A}_{\bar\lambda\bar\mu}\varphi^{1}_{\bar\mu\omega} \cdot T_{w_{(1)}\cdots w_{(r)}} y_{\mu^{(1)\prime} \vee \cdots \vee \mu^{(r)\prime}} = (q^K \varphi^{1}_{\bar\lambda\omega} T_{\overline{1_A w_{(1)}\cdots w_{(r)}}} + s) \cdot y_{\mu^{(1)\prime} \vee \cdots \vee \mu^{(r)\prime}} \tag{3.5}$$

where K is an integer and s is a linear combination of terms $\varphi^{d}_{\bar\lambda\omega}$ with

$$\chi(t^{\bar\lambda} 1_A, t^{\bar\mu}) > \chi(t^{\bar\lambda} d, t^{\bar\mu} w_{(1)} \cdots w_{(r)}).$$

Now suppose that $\sum_A c_A \varphi^{1_A}_{\bar\lambda\bar\mu}\varphi^1_{\bar\mu\omega}\cdot T_{w_{(1)}\cdots w_{(r)}} y_{\mu^{(1)\prime}\vee\cdots\vee\mu^{(r)\prime}} = 0$, where $c_A \in R$ and the sum is over $A \in \mathcal{T}^{ss}_\lambda(\mu)$. Choose $D \in \mathcal{T}^{ss}_\lambda(\mu)$ such that $c_A = 0$ for all A with $\chi(t^{\bar\lambda}1_A, t^{\bar\mu}) > \chi(t^{\bar\lambda}1_D, t^{\bar\mu})$. If we can prove that $c_D = 0$, it will follow that every coefficient $c_A = 0$, and then the proof is completed.

By (3.5), there exists an integer K and $s \in M^\lambda$ such that

$$\sum_A c_A \varphi^{1_A}_{\bar\lambda\bar\mu}\varphi^1_{\bar\mu\omega}\cdot T_{w_{(1)}\cdots w_{(r)}} y_{\mu^{(1)\prime}\vee\cdots\vee\mu^{(r)\prime}} = c_D q^K \varphi^1_{\bar\lambda\omega} T_{\overline{1_D w_{(1)}\cdots w_{(r)}}} y_{\mu^{(1)\prime}\vee\cdots\vee\mu^{(r)\prime}} + s y_{\mu^{(1)\prime}\vee\cdots\vee\mu^{(r)\prime}}$$

where s is a linear combination of terms $\varphi^d_{\bar\lambda\omega}(d \in \mathscr{D}_{\bar\lambda})$ for which

$$\chi(t^{\bar\lambda}d, t^{\bar\mu}w_{(1)}\cdots w_{(r)}) \ngeq \chi(t^{\bar\lambda}1_D, t^{\bar\mu}). \tag{3.6}$$

Now, suppose

$$c_D q^K \varphi^1_{\bar\lambda\omega} T_{\overline{1_D w_{(1)}\cdots w_{(r)}}} y_{\mu^{(1)\prime}\vee\cdots\vee\mu^{(r)\prime}} + s y_{\mu^{(1)\prime}\vee\cdots\vee\mu^{(r)\prime}} = 0$$

and by Lemma 3.4, $\varphi^1_{\bar\lambda\omega} T_{\overline{1_D w_{(1)}\cdots w_{(r)}}} y_{\mu^{(1)\prime}\vee\cdots\vee\mu^{(r)\prime}}$ is the linear combination of the terms $\varphi^d_{\bar\lambda\omega}$ $(d \in \mathscr{D}_{\bar\lambda})$ for which $\chi(t^{\bar\lambda}d, t^{\bar\mu}w_{(1)}\cdots w_{(r)}) = \chi(t^{\bar\lambda}\overline{1_D w_{(1)}\cdots w_{(r)}}, t^{\bar\mu}w_{(1)}\cdots w_{(r)}) = \chi(t^{\bar\lambda}1_D, t^{\bar\mu})$, while $s y_{\mu^{(1)\prime}\vee\cdots\vee\mu^{(r)\prime}}$ is a linear combination of the terms $\varphi^1_{\bar\lambda\omega}(d \in \mathscr{D}_\lambda)$ for which $\chi(t^{\bar\lambda}, t^{\bar\mu}) \neq \chi(t^{\bar\lambda}1_D, t^{\bar\mu})$ by (3.6). Therefore,

$$c_D q^K \varphi^1_{\bar\lambda\omega} T_{\overline{1_D w_{(1)}\cdots w_{(r)}}} y_{\mu^{(1)\prime}\vee\cdots\vee\mu^{(r)\prime}} = 0.$$

But $\varphi^1_{\bar\lambda\omega} T_{\overline{1_D w_{(1)}\cdots w_{(r)}}} y_{\mu^{(1)\prime}\vee\cdots\vee\mu^{(r)\prime}} \neq 0$, since the numbers strictly increase down the columns for every component of D. Therefore, $c_D = 0$, as we claimed.

Now, we have already known that the elements $\varphi^{1_A}_{\bar\lambda\bar\mu}\varphi^1_{\bar\mu\omega}\cdot T_{w_{(1)}\cdots w_{(r)}} y_{\mu^{(1)\prime}\vee\cdots\vee\mu^{(r)\prime}}$ is linearly independent. It implies that $\varphi^{1_A}_{\lambda\mu}\varphi^1_{\mu\omega}T_{w_\mu}y_{\mu'} = \varphi^{1_A}_{\bar\lambda\bar\mu}\varphi^1_{\bar\mu\omega}\cdot T_{w_{(1)}\cdots w_{(r)}} y_{\mu^{(1)\prime}\vee\cdots\vee\mu^{(r)\prime}}\cdot v_{[\mu]}$ are R-linearly independent, since by Lemma 2.5 it is trivial that $a \cdot v_{[\mu]} = 0$ if and only if $a = 0$ for any $a \in \mathcal{H}(\mathfrak{S}_r)$. □

4. Application to a new proof of The Branch rule

In this section, by using this embedding and restriction functors described in [20], we give a new proof of the branch rule in a cyclotomic q-Schur algebra of rank n to the one of rank $n+1$.

From now on, throughout this paper, we argue under the following setting:

$$\begin{aligned}
&\mathbf{m} = (m_1, \cdots, m_r) \text{ such that } m_k \geq n+1 \text{ for all } k = 1, \cdots, r,\\
&\mathbf{m}' = (m_1, \cdots, m_{r-1}, m_r - 1),\\
&\mathscr{S}_{n+1,r} = {}_R\mathscr{S}_{n+1,r}(\Lambda_{n+1,r}(\mathbf{m})),\\
&\mathscr{S}_{n,r} = {}_R\mathscr{S}_{n,r}(\Lambda_{n,r}(\mathbf{m}')).
\end{aligned}$$

We will omit the subscript R when there is no risk of confusion.

We define the injective map

$$\gamma : \Lambda_{n,r}(\mathbf{m}') \to \Lambda_{n+1,r}(\mathbf{m}), \qquad (\lambda^{(1)}, \cdots, \lambda^{(r-1)}, \lambda^{(r)}) \mapsto (\lambda^{(1)}, \cdots, \lambda^{(r-1)}, \widehat{\lambda}^{(r)}),$$

where $\widehat{\lambda}^{(r)} = (\lambda_1^{(1)}, \cdots, \lambda_{m_r-1}^{(r)}, 1)$. Put $\Lambda_{n+1,r}^{\gamma}(\mathbf{m}) = \mathrm{Im}\gamma$, we have

$$\Lambda_{n+1,r}^{\gamma}(\mathbf{m}) = \{\mu = (\mu^{(1)}, \cdots, \mu^{(r)}) \in \Lambda_{n+1,r}(\mathbf{m}) | \mu_{m_r}^{(r)} = 1\},$$

where we define $\mu^{(i)} = (\mu_1^{(i)}, \cdots, \mu_{m_i}^{(r)}) \in \mathbb{Z}_{>0}^{m_i}$ for $1 \leq i \leq r$.

For $\lambda \in \Lambda_{n+1,r}^{+}$, and $T \in \mathcal{T}_{\Lambda}^{ss}(\lambda)$, let $T \setminus (n+1)$ be the standard tableau obtained by removing the node x such that $T(x) = n+1$, and denote the shape of $T \setminus (n+1)$ by $\mathrm{Shape}(T \setminus (n+1))$. Note that x here is a removable node of λ, and that $\mathrm{Shape}(T \setminus (n+1)) = \lambda \setminus x$.

Proposition 4.1. [20]*(**Wada inclusion**) There exists an algebra homomorphism $\iota : \mathscr{S}_{n,r} \to \mathscr{S}_{n+1,r}$ such that*

$$E_{(i,k)}^{(l)} \mapsto E_{(i,k)}^{(l)}\xi, \qquad F_{(i,k)}^{(l)} \mapsto F_{(i,k)}^{(l)}\xi, \qquad 1_{\lambda} \mapsto 1_{\gamma(\lambda)} \tag{4.1}$$

for $(i,k) \in \Gamma'(\boldsymbol{m}')$, $l \geq 1$, $\lambda \in \Lambda_{n,r}(\boldsymbol{m}')$, where $\xi = \sum_{\lambda \in \Lambda_{n+1,r}^{\gamma}(\boldsymbol{m})} 1_{\lambda}$ is an idempotent of $\mathscr{S}_{n+1,r}$. In particular, we have that $\iota(1_{\mathscr{S}_{n,r}}) = \xi$, and that $\iota(\mathscr{S}_{n,r}) \subsetneq \xi \mathscr{S}_{n+1,r} \xi$, where $1_{\mathscr{S}_{n,r}}$ is the unit element of $\mathscr{S}_{n,r}$. Moreover, ι is injective.

We define a restriction functor $\mathrm{Res}_n^{n+1} : \mathscr{S}_{n+1,r}$-mod$\to \mathscr{S}_{n,r}$-mod by

$$\mathrm{Res}_n^{n+1} = \mathrm{Hom}_{\mathscr{S}_{n+1,r}}(\mathscr{S}_{n+1,r}\xi, ?) \cong \xi \mathscr{S}_{n+1,r} \otimes_{\mathscr{S}_{n+1,r}} ?.$$

We recall that, for $\lambda \in \Lambda_{n+1,r}^{+}$, the q-Schur module $\mathcal{A}^{\lambda}$ of $\mathscr{S}_{n+1,r}$ has an R-free basis $\{\varphi_{\mu\lambda}^{1_A} z_{\lambda} | A \in \mathcal{T}_{\mu}^{ss}(\lambda), \mu \in \Lambda_{n+1,r}(\mathbf{m})\}$. From the definition, we have that

$$\mathrm{Res}_n^{n+1}(\mathcal{A}^{\lambda}) = \xi \mathcal{A}^{\lambda}.$$

Thus, $\mathrm{Res}_n^{n+1}(\mathcal{A}^{\lambda})$ has an R-free basis $\{\varphi_{\mu\lambda}^{1_A} z_{\lambda} | A \in \mathcal{T}_{\mu}^{ss}(\lambda), \mu \in \Lambda_{n+1,r}^{\gamma}(\mathbf{m})\}$.

For a partition $\lambda = (\lambda_1, \cdots, \lambda_m)$ of n, we identify the boxes in the Young diagram $\mathcal{Y}(\lambda)$ with its position coordinates. Thus,

$$\mathcal{Y}(\lambda) = \{(i,j) \in \mathbb{Z}^{+} \times \mathbb{Z}^{+} | j \leq \lambda_i\}.$$

The elements of $\mathcal{Y}(\lambda)$ will be called *nodes*. A node of the form (i, λ_i) (resp. $(i, \lambda_i + 1)$) is called *removable* (resp. *addable*) if $i = m$ or $\lambda_i > \lambda_{i+1}$ for $i \neq m$ (resp. $(i, \lambda_i) = (0,1)$ for $\lambda_1 = \cdots = \lambda_m = 1$ or $i = 1$ or $\lambda_{i-1} > \lambda_i$ if $i \neq 1$). Let $\lambda = (\lambda^{(1)}, \cdots, \lambda^{(r)})$ be an r-partition. Then its Young diagram $\mathcal{Y}(\lambda)$ is the union of the Young diagram $\mathcal{Y}(\lambda^{(k)})$, $1 \leq k \leq r$. Thus, a set of nodes is as follows,

$$\mathcal{Y}(\lambda) = \{(i,j,k) | i,j \in \mathbb{Z}^{+}, j \leq \lambda_i^{(k)},\ 1 \leq k \leq m\}.$$

A node of $\mathcal{Y}(\lambda)$ is said to be *removable* (resp. *addable*) if it is a removable (resp. addable) node of $\mathcal{Y}(\lambda^{(k)})$ for some k. Denote by $\mathcal{R}_{\lambda}$ the set of all removable nodes of $\mathcal{Y}(\lambda)$. Then $N = \#\mathcal{R}_{\lambda} = \sum_{i=1}^{r} \#\mathcal{R}_{\lambda^{(i)}}$.

A partial ordering " $\succ$ " on $\mathcal{R}_{\lambda}$ will be fixed from top to bottom and from left to right, that is, it satisfies that

$$(i,j,k) \succ (i',j',k') \text{ if } k < k', \text{ or if } k = k' \text{ and } i < i'.$$

Then, we have $\mathcal{R}_\lambda = \{\mathfrak{n}_1, \cdots, \mathfrak{n}_N\}$, with the property that $\mathfrak{n}_i \succ \mathfrak{n}_j$ for $i > j$. Let $j_\mathfrak{n}$, $\mathfrak{n} \in \mathcal{R}_\lambda$, be the number at the node $\mathfrak{n}$ in $\mathfrak{t}_\lambda$. For example, for $\lambda = \big((31),(22),(1)\big)$, $\mathcal{R}_\lambda = \{(1,3,1),(2,1,1),(1,1,3)\}$.

Also, we define a partial order $\succeq$ on $\mathbb{Z}_{>0} \times \{1,\dots,r\}$ by

$$(i,k) \succ (i',k') \text{ if } (i,1,k) \succ (i',1,k').$$

Proposition 4.2. *Let $\lambda \in \Lambda^+_{n+1,r}$, $\mu \in \Lambda^\gamma_{n+1,r}(\boldsymbol{m})$, $A \in \mathcal{T}^{ss}_\mu(\lambda)$. For $(i,k) \in \Gamma'(\boldsymbol{m}')$, we have the following*

$$E_{(i,k)} \cdot \varphi^{1_A}_{\mu\lambda} z_\lambda = \sum_{\substack{B \in \mathcal{T}^{ss}_{\mu+\alpha_{(i,k)}}(\lambda) \\ shape(B\backslash(m_r,r)) \trianglerighteq shape(A\backslash(m_r,r))}} r_B \varphi^{1_B}_{\mu+\alpha_{(i,k)},\lambda} z_\lambda \ (r_B \in R); \tag{4.2}$$

$$F_{(i,k)} \cdot \varphi^{1_A}_{\mu\lambda} z_\lambda = \sum_{\substack{B \in \mathcal{T}^{ss}_{\mu-\alpha_{(i,k)}}(\lambda) \\ shape(B\backslash(m_r,r)) \trianglerighteq shape(A\backslash(m_r,r))}} r_B \varphi^{1_B}_{\mu-\alpha_{(i,k)},\lambda} z_\lambda \ (r_B \in R). \tag{4.3}$$

Proof. Following from (5.8), (5.9)'s notations in [7], one shows that $\varphi^{1_A}_{\mu\lambda} = \Psi_{AT^\lambda}$. On the other hand, by a general theory of cellular algebras together with Proposition 3.3 in [20], we have that, for $(i,k) \in \Gamma'(\mathbf{m}')$,

$$E_{(i,k)} \cdot \varphi^{1_A}_{\mu\lambda} \equiv \sum_{\substack{B \in \mathcal{T}^{ss}_{\mu+\alpha_{(i,k)}}(\lambda) \\ \text{shape}(B\backslash(m_r,r)) \trianglerighteq \text{shape}(A\backslash(m_r,r))}} r_B \varphi^{1_B}_{\mu+\alpha_{(i,k)},\lambda} \qquad \text{mod } \mathscr{S}^{\rhd\lambda}_{n+1,r}, \tag{4.4}$$

where $r_B \in R$.

By definitions, $z_\lambda := \varphi^1_{\lambda\omega} T_w y_{\lambda'}$ and $\mathscr{S}^{\rhd\lambda}_{n+1,r}$ is linearly generated by Ψ_{ST} for $S,T \in \mathcal{T}_\Lambda(\nu)$ with $\nu \rhd \lambda$, it follows that $\mathscr{S}^{\rhd\lambda}_{n+1,r} \cdot z_\lambda = 0$. On the other hand, we suppose that there exists some $S,T \in \mathcal{T}^{ss}_\Lambda(\nu)$, such that $\Psi_{ST} z_\lambda \neq 0$, which means $\lambda = \nu$ from the proof of Theorem 3.1. This consequence is contradict to the fact $\nu \rhd \lambda$. Finally, we reach the consequence of the first statement after multiplying the element z_λ on both sides of (4.4).

The case for $F_{(i,k)}$ with $(i,k) \in \Gamma'(\mathbf{m}')$ can be proved in the same way as the above proof for the case of $E_{(i,k)}$. □

By Theorem 3.5, let ${}_RM_i$ be an R-submodule of $\mathrm{Res}^{n+1}_n(\mathcal{A}^\lambda)$ spanned by

$$\{\varphi^{1_A}_{\mu\lambda} z_\lambda | A \in \mathcal{T}^\gamma_\Lambda(\lambda) \cap \mathcal{T}^{ss}_\Lambda(\lambda) \text{ such that } A(\mathfrak{n}_j) = (m_r, r) \text{ for some } j \geq i\},$$

where we put $\mathcal{T}^\gamma_\Lambda(\lambda) := \bigcup_{\mu \in \Lambda^\gamma_{n+1,r}(\mathbf{m})} \mathcal{T}_\mu(\lambda)$. When there is no confusion about R, we also denote it as M_i (i.e., delete the subscript.).

Then we have a filtration of R-modules

$$\mathrm{Res}^{n+1}_n(\mathcal{A}^\lambda) = M_1 \supset M_2 \supset \cdots \supset M_k \supset M_{k+1} = 0.$$

For $\lambda \in \Lambda^+_{n+1,r}$ and a removable node x of λ, we define the semi-standard tableau $T^\lambda_x \in \mathcal{T}^{ss}_\Lambda(\lambda)$ by

$$T^\lambda_x(a,b,c) = \begin{cases} (a.c) & \text{if } (a,b,c) \neq x, \\ (m_r, r) & \text{if } (a,b,c) = x. \end{cases} \tag{4.5}$$

We see that $T_x^\lambda \in \mathcal{T}_\Lambda^\gamma(\lambda) \cap \mathcal{T}_\Lambda^{ss}(\lambda)$, and $T_x^\lambda \setminus (m_r, r) = T^{\lambda\setminus x}$, where the tableau $T^{\lambda\setminus x}$ denotes the unique element in set $\mathcal{T}_{\lambda\setminus x}^{ss}(\lambda \setminus x)$.

From the definition, M_i/M_{i+1} has an R-free basis

$$\{\varphi_{\gamma(\mu)\lambda}^{1_A} z_\lambda + M_{i+1} | A \in \mathcal{T}_\Lambda^\gamma(\lambda) \cap \mathcal{T}_\Lambda^{ss}(\lambda) \text{ such that } A(\mathfrak{n}_i) = (m_r, r) \text{ and } \mu \in \Lambda_{n,r}(\mathbf{m})\}.$$

For $A \in \mathcal{T}_\Lambda^\gamma(\lambda) \cap \mathcal{T}_\Lambda^{ss}(\lambda)$ such that $A(\mathfrak{n}_i) = (m_r, r)$, we have $\text{Shape}(A \setminus (m_r, r)) = \lambda \setminus \mathfrak{n}_i$ by definition. Note that $\lambda \setminus \mathfrak{n}_j \rhd \lambda \setminus \mathfrak{n}_i$ if and only if $\mathfrak{n}_j \prec \mathfrak{n}_i$ (i.e., $j > i$). Then, by Proposition 4.2, we see that $\{M_i\}$ is a filtration of $\mathscr{S}_{n,r}$-modules.

Now, using the main result in Section 3 we give a new version of the branch rule of Weyl modules in [20].

Theorem 4.3. [20] *Assume that R is a field. For any $\lambda \in \Lambda_{n+1,r}^+(\boldsymbol{m})$, let $\mathfrak{n}_1, \cdots, \mathfrak{n}_k$ be the removable nodes of $\mathcal{Y}(\lambda)$ counted from top to bottom, and define M_t as above for $1 \le t \le k$. Then, we have a filtration of $\mathscr{S}_{n,1}$-submodule for $\mathcal{A}^\lambda$:*

$$0 = M_{k+1} \subset M_k \subset \cdots \subset M_1 = \mathcal{A}^\lambda$$

with the sections of Weyl modules (or q-Schur modules): $M_t/M_{t-1} \cong W^{\lambda\setminus\mathfrak{n}_t}$.

Proof. First of all we set $\widehat{\mu} := \gamma(\mu)$, and consider the weight decomposition of the $\mathscr{S}_{n,r}$-module $M_i/M_{i+1} = \bigoplus_{\mu\in\Lambda_{n,r}(\mathbf{m})} {}_\mu(M_i/M_{i+1}) = \bigoplus_{\mu\in\Lambda_{n,r}(\mathbf{m})} 1_\mu \cdot M_i/M_{i+1} = \bigoplus_{\mu\in\Lambda_{n,r}(\mathbf{m})} 1_{\widehat{\mu}}(M_i/M_{i+1})$, where $1_{\widehat{\mu}}(M_i/M_{i+1})$ is generated by

$$\{\varphi_{\widehat{\mu}\lambda}^{1_A} z_\lambda + M_{i+1} | A \in \mathcal{T}_\Lambda^\gamma(\lambda) \cap \mathcal{T}_\Lambda^{ss}(\lambda) \text{ such that } A(\mathfrak{n}_i) = (m_r, r)\}.$$

Since $A \setminus (m_r, r) \in \mathcal{T}_\mu^{ss}(\lambda \setminus \mathfrak{n}_i)$, we can find that ${}_\mu(M_i/M_{i+1}) \neq 0$ only if $\lambda \trianglerighteq \widehat{\mu}$, which implies that $\lambda \setminus \mathfrak{n}_i \trianglerighteq \mu$.

Let $\mathfrak{n}_i = (a, b, c)$. Note that $E_{(j,l)} \cdot \varphi_{\widehat{\mu}\lambda}^{1_A} z_\lambda$ is a linear combination of $\{\varphi_{\widehat{\mu}+\alpha_{(j,l)},\lambda}^{1_B} z_\lambda | B \in \mathcal{T}_{\widehat{\mu}+\alpha_{(j,l)}}^{ss}(\lambda)\}$ and that $\mathcal{T}_{\widehat{\mu}+\alpha_{(j,l)}}^{ss}(\lambda) = \emptyset$ unless $\lambda \trianglerighteq \widehat{\mu} + \alpha_{(j,l)}$.

We have $T_{\mathfrak{n}_i}^\lambda \in \mathcal{T}_\tau^{ss}(\lambda)$ in the case of $\tau := \widehat{\lambda \setminus \mathfrak{n}_i}$, i.e., $\tau = \lambda - (\alpha_{(a,c)} + \alpha_{(a+1,c)} + \cdots + \alpha_{(m_r-1,r)})$.

If $(j, l) \succ (a, c)$, we have $E_{(j,l)} \cdot \varphi_{\tau\lambda}^{1_A} z_\lambda = 0$ since $\lambda \not\trianglerighteq \tau + \alpha_{(j,l)}$ for any $A \in \mathcal{T}_\tau^{ss}(\lambda)$.

If $(j, l) \preceq (a, c)$, for any $S \in \mathcal{T}_{\tau+\alpha_{(j,l)}}^{ss}(\lambda)$ together with the definition of semi-standard tableaux, we can easily check that $S\big((a', b', c')\big) \succeq (j, l)$ for any $(a', b', c') \in \lambda$ satisfying $(a', c') \succeq (j, l)$. This implies that

$$|S \setminus (m_r, r)| \neq |\lambda \setminus \mathfrak{n}_i| \text{ for any } S \in \mathcal{T}_{\tau+\alpha_{(j,l)}}^{ss}(\lambda), \tag{4.6}$$

since $(a, c) \succeq (j, l)$ and $T_{\mathfrak{n}_i}^\lambda\big((a, b, c)\big) = (m_r, r) \preceq (j, l)$. From now on, we denote the tableau $T_{\mathfrak{n}_i}^\lambda$ as X.

Thus, Proposition 4.2 together with (4.6) implies that

$$E_{(j,l)} \cdot \varphi_{\tau\lambda}^{1_X} \cdot z_\lambda = 0 \in M_{i+1} \text{ for any } (j, l) \in \Gamma'(\mathbf{m}').$$

Thus, $\varphi_{\tau\lambda}^{1_X} \cdot z_\lambda + M_{i+1}$ is a highest weight vector of weight $\lambda \setminus \mathfrak{n}_i$ of $\mathscr{S}_{n,r}$-module in the sense of [21]. Moreover, since the Weyl modules are simple modules in the category of

${}_{\mathcal{K}}\mathscr{S}_{n,r}$-modules, due to the universality of the Weyl modules in [21], we have an ${}_{\mathcal{K}}\mathscr{S}_{n,r}$-isomorphism:

$$\theta_{\mathcal{K}}^{\lambda\backslash\mathfrak{n}_i} : \qquad {}_{\mathcal{K}}\mathcal{A}^{\lambda\backslash\mathfrak{n}_i} \to {}_{\mathcal{K}}\mathscr{S}_{n,r} \cdot (\varphi_{\tau\lambda}^{1x} \cdot z_\lambda) + {}_{\mathcal{K}}M_{i+1}. \tag{4.7}$$

Note that $\theta_{\mathcal{K}}^{\lambda\backslash\mathfrak{n}_i}$ is determined by $\theta_{\mathcal{K}}^{\lambda\backslash\mathfrak{n}_i}(\varphi^1_{\lambda\backslash\mathfrak{n}_i\lambda\backslash\mathfrak{n}_i} \cdot z_{\lambda\backslash\mathfrak{n}_i}) = \varphi_{\tau\lambda}^{1x} \cdot z_\lambda + {}_{\mathcal{K}}M_{i+1}$. We see that $\theta_{\mathcal{A}}^{\lambda\backslash\mathfrak{n}_i}$ is a restriction of $\theta_{\mathcal{K}}^{\lambda\backslash\mathfrak{n}_i}$ which assigns the submodule ${}_{\mathcal{A}}\mathcal{A}^{\lambda\backslash\mathfrak{n}_i}$ onto the submodule ${}_{\mathcal{A}}\mathscr{S}_{n,r} \cdot (\varphi_{\tau\lambda}^{1x} \cdot z_\lambda) + {}_{\mathcal{A}}M_{i+1}$. Then, we find that $\theta_{\mathcal{A}}^{\lambda\backslash\mathfrak{n}_i}$ is a ${}_{\mathcal{A}}\mathscr{S}_{n,r}$-mod isomorphism. Furthermore, by the argument of specialization to any arbitrary commutative ring, it follows that $\theta_R^{\lambda\backslash\mathfrak{n}_i} := \theta_{\mathcal{A}}^{\lambda\backslash\mathfrak{n}_i} \otimes_{\mathcal{A}} R$ is an isomorphism for the algebra ${}_R\mathscr{S}_{n,r}$.

Assume that R is a field. Since $W^{\lambda\backslash\mathfrak{n}_i} \cong \mathcal{A}^{\lambda\backslash\mathfrak{n}_i} \cong {}_R\mathscr{S}_{n,r} \cdot (\varphi_{\tau\lambda}^{1x} \cdot z_\lambda) + {}_R M_{i+1}$, which is a ${}_R\mathscr{S}_{n,r}$-submodule of M_i/M_{i+1}, we finally reach the consequence by comparing the dimensions of $\mathcal{A}^{\lambda\backslash\mathfrak{n}_i}$ and M_i/M_{i+1}. □

Acknowledgements: *The authors thank the support from the projects of the National Natural Science Foundation of China (No.11271318 and No.11171296) and the Specialized Research Fund for the Doctoral Program of Higher Education of China (No.20110101110010).*

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