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BOLTJE-MAISCH RESOLUTIONS OF SPECHT MODULES

XINGYU DAI, FANG LI, KEFENG LIU

Abstract. In [5], Boltje and Maisch found a permutation complex of Specht modules in representation theory of Hecke algebras, which is the same as the Boltje-Hartmann complex appeared in the representation theory of symmetric groups and general linear groups. In this paper we prove the exactness of Boltje-Maisch complex in the dominant weight case as conjectured in [5].

1. Introduction

In [12], Hartmann and Boltje constructed, for any composition λ of a positive integer r, a finite chain complex of modules for the group algebra $R\mathfrak{S}_r$ of the symmetric group $\mathfrak{S}_r$ on r letters over an arbitrary commutative ring R. The last module in this complex is the dual of the Specht module S^λ and the other modules are permutation modules with point stabilizers given by Young subgroups of $\mathfrak{S}_r$. It was conjectured that this chain complex is exact whenever λ is a partition. In other words the chain complex is conjectured to be a resolution of the (dual) of the Specht module by permutation modules. Partial exactness results were already established in [12] and a full proof of the exactness was given recently by Yudin and Santana (see [25]) by translating the construction via the Schur functor. Other permutation resolutions of Specht modules have been considered by Donkin in [8], Akin in [1], Zelevinskii in [32], Akin-Buchsbaum in [2] and [3], Santana in [24], Woodcock in [27] and [28], Doty in [9] and Yudin in [31]. See ([12], Section 6) for a more detailed comparison of these constructions with the construction in [12].

In [5], Robert Boltje and Filix Maisch lifted the construction of the chain complex in [4] for the group algebra $R\mathfrak{S}_r$ to a chain complex for modules of the Iwahori-Hecke algebra $\mathcal{H}_r$ for any integral domain R such that the specialization $q = 1$ reproduces the original chain complex; and lifted the partial exactness proofs from [4] to the new construction.

The construction of the chain complex was completely combinatorial and characteristic free. It was conjectured that this chain complex is exact whenever λ is a partition. In other words the chain complex is conjectured to be a resolution of the (dual) of the Specht module by permutation modules. In this paper, we find a way to prove this conjecture. In order to prove its exactness in the dominant weight case, we follow the method of [25], which constructs a bar resolution $B_*(A, S, M)$ in the Borel subalgebra case and transformed it into the module category of q-Schur algebras as $S_R \otimes_{S_R^+} B_*(A, S, M)$.

The paper is organised as follows. In section 2, we find an ideal sequence $J_0 \supseteq J_1 \supseteq \cdots \supseteq 0$, and use J_0 and J_1 to construct a bar resolution of the module in

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the representations of Borel subalgebra $S_R^+(n,r)$, just as Ana Paula Santana and Ivan Yudin did in case of symmetric groups, as in [25]. However, since the proof of vanishing theorem in [28] failed for the case of Iwahori Hecke algebra's. In section 3, we use different tools to prove the module R_λ is $S_R(n,r)\otimes_{S_R^+(n,r)}$-acyclic, which was introduced in [30]. After that, we reach the main results Theorem 5.10 and Theorem 6.2 which give the positive answer of the conjecture given in [5].

2. Notations and Quoted Results

2.1. Combinatorics. In this section we collect the combinatorial notations used in the paper. We will give general definitions, which include as special cases the usual tools in the subject, such as multi-indices, compositions, etc.

Let R be a commutative ring with identity e, and n and r two arbitrary fixed positive integers. For any natural number s we denote by $\bar{s}$ the set $\{1,\dots,s\}$ and by $\mathfrak{S}_s$ the symmetric group on $\bar{s}$. Given a finite set X, we write:

- $\mu=(\mu_x)_{x\in X}$ *and* $|\mu|=\sum_{x\in X}$, *for each map* $\mu: X\to\mathbb{N}_0$ *given by* $x\mapsto\mu_x$;
- $\Lambda(X;r):=\{\mu: X\to\mathbb{N}_0 || \mu|=r\}$;
- *the weight map* $wt: X^r\to\Lambda(X;r)$ *defined by*
$$wt(u)_x=\#\{s|u_x=x, s=1,\dots,r\}.$$

Next we consider several particular cases of the definitions given above. We write $I(n,r)$ for $\bar{n}^r$. The elements of $I(n,r)$ are called *multi-indices* and will be usually denoted by the letter $\boldsymbol{i}$, $\boldsymbol{j}$. Denote $\bar{n}\times\bar{n}=\{(i,j)|1\le i,j\le n\}$. We identify the set $(\bar{n}\times\bar{n})^r$ and $I(n,r)\times I(n,r)$ via the map

$$((i_1,j_1),\dots,(i_r,j_r))\mapsto((i_1,\dots,i_r),(j_1,\dots,j_r)).$$

The sets $\Lambda(\bar{n};r)$ and $\Lambda(\bar{n}\times\bar{n};r)$ are denoted by $\Lambda(n,r)$ and $\Lambda(n,n;r)$, respectively. We can think of the elements of $\Lambda(n,r)$ as the compositions of r into at most n parts, and we will write $\Lambda^+(n,r)$ for those $(\lambda_1,\dots,\lambda_n)\in\Lambda(n,r)$ that satisfies $\lambda_1\ge\cdots\ge\lambda_n$(the partitions of r into at most n parts).

On the set $I(n,r)$ we defined the ordering $\le$ by

$$\boldsymbol{i}\le\boldsymbol{j}\iff i_1\le j_1,\ i_2\le j_2,\dots,\ i_r\le j_r.$$

We write $\boldsymbol{i}<\boldsymbol{j}$ if $\boldsymbol{i}\le\boldsymbol{j}$ and $\boldsymbol{i}\ne\boldsymbol{j}$.

2.2. Symmetric groups and Iwahori-Hecke algebras. A *composition* λ of r is a finite sequence of non-negative integers $(\lambda_1,\lambda_2,\dots,\lambda_n)$ such that $|\lambda|=\sum_i\lambda_i=r$. Moreover, there is a partial order $\trianglelefteq$(resp. $\trianglerighteq$) within compositions of r as: we denote $\lambda\trianglelefteq\mu$ when $\sum_{i=1}^k\lambda_i\le\sum_{i=1}^k\mu_i$(resp. $\sum_{i=1}^k\lambda_i\ge\sum_{i=1}^k\mu_i$) for all $1\le k\le n$.

Let $\mathfrak{S}_r$ denote the symmetric group of all permutations of $1,\dots,r$ with Coxeter generators $s_i:=(i,i+1)$, and $\mathfrak{S}_\lambda$ the Young subgroup corresponding to the composition λ of r. Thus, we have

$$\mathfrak{S}_\lambda=\mathfrak{S}_{\mathbf{a}}=\mathfrak{S}_{\{1,\dots,a_1\}}\times\mathfrak{S}_{\{a_1+1,\dots,a_2\}}\times\cdots\times\mathfrak{S}_{\{a_{n-1},\dots,a_n\}},$$

where $\mathbf{a}=[a_0,a_1,\dots,a_n]$ with $a_0=0$ and $a_i=\lambda_1+\cdots+\lambda_i$ for all $i=1,\dots,n$. We denote by $\mathscr{D}_\lambda$ the set of distinguished representatives of right $\mathfrak{S}_\lambda$-cosets and write $\mathscr{D}_{\lambda\mu}:=\mathscr{D}_\lambda\cap\mathscr{D}_\mu^{-1}$, which is the set of distinguished representatives of double cosets $\mathfrak{S}_\lambda\setminus\mathfrak{S}_r/\mathfrak{S}_\mu$.

As usual one identifies composition λ with *Young diagram* and we say that λ is the *shape* of the corresponding Young diagram. For example, we can represent the partition $(3, 2)$ as $\begin{array}{|c|c|c|}\hline \ & \ & \ \\ \hline \ & \ \\ \cline{1-2}\end{array}$. A λ-tableau is a filling of the r boxes of the Young diagram of λ of the numbers $1, 2, \dots, r$. We denote the set of λ-tableaux by $\mathcal{T}(\lambda)$ and usually denote $\mathfrak{t}$ as an element of $\mathcal{T}(\lambda)$.

For later use, let $\Lambda(r)$ (resp. $\Lambda^+(r)$) denote the set of all compositions (resp. all partitions) of r. For $\lambda \in \Lambda(r)$, let λ' be the dual partition of λ, i.e., $\lambda'_i := \#\{j; \lambda_j \geq i\}$. There is a unique element $w_\lambda \in \mathscr{D}_{\lambda\lambda'}$ with the trivial intersection property [6](4.1):

$$ {}^{w_\lambda}\mathfrak{S}_\lambda \cap \mathfrak{S}_{\lambda'} = w_\lambda^{-1}\mathfrak{S}_\lambda w_\lambda \cap \mathfrak{S}_{\lambda'} = \{1\}. \tag{2.1} $$

We can represent w_λ with help of Young diagrams. For example, $\begin{array}{|c|c|c|}\hline \ & \ & \ \\ \hline \ & \ \\ \cline{1-2}\end{array}$ represents $\lambda = (3, 2)$, then $w_\lambda \in \mathfrak{S}_r$ is defined by the equation $\mathfrak{t}^\lambda w_\lambda = \mathfrak{t}_\lambda$, where $\mathfrak{t}^\lambda$ (resp. $\mathfrak{t}_\lambda$) is the λ-tableau obtained by putting the number $1, 2, \dots, r$ in order into the boxes from left to right down successive rows (resp. columns). Thus, in the example, $\mathfrak{t}^{(3,2)} = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 \\ \cline{1-2}\end{array}$, and $\mathfrak{t}_{(3,2)} = \begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 2 & 4 \\ \cline{1-2}\end{array}$.

The group $\mathfrak{S}_r$ acts from the right on $\mathcal{T}(\lambda)$ by simply applying an element $w \in \mathfrak{S}_r$ to the entries of the tableau $\mathfrak{t} \in \mathcal{T}(\lambda)$. This action is free and transitive, and it yields a bijection

$$ \mathfrak{S}_r \xrightarrow{\sim} \mathcal{T}(\lambda), \qquad w \mapsto t^\lambda w. $$

A λ-tableau $\mathfrak{t}$ is called *row-standard* if its entries are increasing in each row form left to right. The row-standard tableaux form a subset $\mathcal{T}^{rs}(\lambda)$ of $\mathcal{T}(\lambda)$. Two λ-tableaux $\mathfrak{t}_1$ and $\mathfrak{t}_2$ are called *row-equivalent* if $\mathfrak{t}_1$ and $\mathfrak{t}_2$ can arise from each other by rearranging elements within each row. We denote the row-equivalent class of $\mathfrak{t}$ by $\{\mathfrak{t}\}$ and the set of row equivalent classes by $\overline{\mathcal{T}}(\lambda)$. One has the canonical bijections

$$ \mathscr{D}_\lambda \xrightarrow{\sim} \mathcal{T}^{rs}(\lambda) \xrightarrow{\sim} \overline{\mathcal{T}}(\lambda) \tag{2.2} $$

given by $d \mapsto \mathfrak{t}^\lambda d$ and $\mathfrak{t} \mapsto \{\mathfrak{t}\}$.

Definition 2.1. Let R be a commutative domain with 1 and let q be an arbitrary element of R. The **Iwahori-Hecke algebra** $\mathcal{H} = \mathcal{H}_{R,q}(\mathfrak{S}_r)$ of $\mathfrak{S}_r$ is the unital associative R-algebra with generators $T_1, T_2, \dots, T_{r-1}$ and relations:

$$ \begin{aligned} (T_i - q)(T_i + 1) &= 0, && \text{for } i = 1, 2, \dots, r-1, \\ T_i T_j &= T_j T_i, && \text{for } 1 \leq i < j - 1 \leq r - 2, \\ T_i T_{i+1} T_i &= T_{i+1} T_i T_{i+1}, && \text{for } i = 1, 2, \dots, r-2. \end{aligned} $$

It is well-known that $\mathcal{H}$ is a free R-module with a finite R-basis $\{T_w | w \in \mathfrak{S}_r\}$. T_w is defined as $T_{i_1} T_{i_2} \cdots T_{i_s}$ if w has a reduced presentation $w = s_{i_1} s_{i_2} \cdots s_{i_s}$. Then, with the definition of young subgroups in $\mathfrak{S}_r$ and T_w in Iwahori-Hecke algebra, we can define $x_\mu = \sum_{w \in \mathfrak{S}_\mu} T_w$, which is an element in $\mathcal{H}_r$, and define M^μ to be the right $\mathfrak{S}_r$-module $x_\mu \mathcal{H}$.

Definition 2.2. Fix a non-negative integer r and let $\Lambda(n, r)$ be the set of compositions of r with at most n non-zero parts; more precisely, $\Lambda(n, r) = \{\mu = (\mu_1, \dots, \mu_n) \in \mathbb{Z}^n_{\geq 0} | \sum_i \mu_i = r\}$. The *q-Schur algebra* is the endomorphism algebra

$$ S_R(n, r) = End_{\mathcal{H}}\Big(\bigoplus_{\mu \in \Lambda(n,r)} M^\mu\Big) = \bigoplus_{\lambda, \mu \in \Lambda(n,r)} Hom_{\mathcal{H}}(M^\mu, M^\lambda). $$

The homomorphisms $\psi^d_{\lambda\mu} : M^\mu \to M^\lambda$, with $\lambda, \mu \in \Lambda(n,r)$. Then the elements of $\mathscr{D}_{\lambda\mu} := \mathscr{D}_\lambda \cap \mathscr{D}_\mu^{-1}$ form a set of representatives of the double cosets $\mathfrak{S}_\lambda \setminus \mathfrak{S}_r / \mathfrak{S}_\mu$, and each element $d \in \mathscr{D}_{\lambda\mu}$ is the unique element of shortest length in its double coset $\mathfrak{S}_\lambda d \mathfrak{S}_\mu$. By [[6], Theorem 3.4], the set $\mathscr{D}_{\lambda\mu}$ parametrizes an R-basis $\psi^d_{\lambda\mu}$, $d \in \mathscr{D}_{\lambda\mu}$ of $Hom_{\mathcal{H}}(M^\mu, M^\lambda)$ given by

$$\psi^d_{\lambda\mu}(x_\mu) = \sum_{w \in \mathfrak{S}_\lambda d \mathfrak{S}_\mu} T_w = x_\lambda \sum_{e \in \mathcal{D}_\nu \cap \mathfrak{S}_\mu} T_{de} = x_\lambda T_d \sum_{e \in \mathcal{D}_\nu \cap \mathfrak{S}_\mu} T_e, \tag{2.3}$$

where $\nu \in \Lambda(n,r)$ is determined by $\mathfrak{S}_\nu = d^{-1}\mathfrak{S}_\lambda d \cap \mathfrak{S}_\mu$.

We will use a combinatorial description of the set $\mathcal{D}_{\lambda\mu}$ in terms of the set $\mathcal{T}(\lambda,\mu)$, which is a set of generalized tableaux of λ with content μ. Such a $T \in \mathcal{T}(\lambda,\mu)$ is a filling of the r boxes of the Young diagram of composition λ with μ_1 entries equal to 1, μ_2 entries equal to 2, etc. We may view $\mathcal{T}(\lambda,\mu)$ as the set of function $T : \{1,\ldots,r\} \to \mathbb{N}$ with the property $|T^{-1}(i)| = \mu_i$ for all $i \in \mathbb{N}$. The group $\mathfrak{S}_r$ acts transitively on $\mathcal{T}(\lambda,\mu)$ form the left by $(wT)(i) := T(iw)$. The stabilizer of T^λ_μ is equal to $\mathfrak{S}_\mu$, with denoting that T^λ_μ is the semi-standard which has its boxes filled in the natural order. Then it can determine a bijection $\mathfrak{S}_r/\mathfrak{S}_\mu \xrightarrow{\sim} \mathcal{T}(\lambda,\mu)$, with $w\mathfrak{S}_\mu \mapsto wT^\lambda_\mu$. We call two generalized tableaux $T_1, T_2 \in \mathfrak{T}(\lambda,\mu)$ *row-equivalent* if they arise form each other by rearranging the entries within the rows, i.e., if $T_2 = T_1 w$ for some $w \in \mathfrak{S}_\lambda$ and $\{T\}$ denotes T's equivalent class. If we denote the set of row equivalent classes by $\overline{\mathcal{T}}(\lambda.\mu)$, then we can achieve a bijection $\mathfrak{S}_\lambda \setminus \mathfrak{S}_r/\mathfrak{S}_\mu \xrightarrow{\sim} \overline{\mathcal{T}}(\lambda,\mu)$, $\mathfrak{S}_\lambda w \mathfrak{S}_\mu \mapsto wT^\lambda_\mu$ with denoting T as the row-equivalent class of T in $\overline{\mathcal{T}}(\lambda,\mu)$ for any $T \in \mathcal{T}(\lambda,\mu)$.

A generalized tableau $T \in \mathcal{T}(\lambda,\mu)$ is called row-semistandard if in each of its rows the entries are in their natural order from left to right. We denote the subset of $\mathcal{T}(\lambda,\mu)$ as $\mathcal{T}^{rs}(\lambda,\mu)$. Each row-equivalent class contains a unique row-semistandard element. Thus, $\mathcal{T}^{rs} \xrightarrow{\sim} \overline{\mathcal{T}}(\lambda,\mu), T \mapsto \{T\}$, is a bijection. Altogether, we now have canonical bijection

$$\mathscr{D}_{\lambda\mu} \xrightarrow{\sim} \mathfrak{S}_\lambda \setminus \mathfrak{S}_r/\mathfrak{S}_\mu \xrightarrow{\sim} \overline{\mathcal{T}}(\lambda,\mu) \xleftarrow{\sim} \mathcal{T}^{rs}(\lambda,\mu) \tag{2.4}$$

Moreover, Theorem 4.7 of [21] shows that $Hom_{\mathcal{H}}(M^\mu, M^\lambda)$ is free as an R-module with basis $\{\psi^d_{\lambda\mu} | d \in \mathscr{D}_{\lambda\mu}\}$. The q-Schur algebra $S_R(n,r)$ also can be written as a free R-module $\bigoplus_{\substack{\lambda,\mu \in \Lambda(n,r) \\ d \in \mathscr{D}_{\lambda\mu}}} R\psi^d_{\lambda\mu}$.

3. Coordinate ring and bar resolutions

Definition 3.1. [12] For a commutative ring R, let $R[M_n(q)]$ be the associative algebra over R generated by X_{ij} with $1 \le i,j \le n$ such that

$$\begin{cases} X_{ij}X_{ik} = qX_{ik}X_{ij}, & if\ j > k, \\ X_{ji}X_{ki} = X_{ki}X_{ji}, & if\ j > k, \\ X_{ij}X_{rs} = q^{-1}X_{rs}X_{ij}, & if\ i > r, j < s, \\ X_{ij}X_{rs} - X_{rs}X_{ij} = (q^{-1} - 1)X_{is}X_{rj}, & if\ i < r, j < s \end{cases} \tag{3.1}$$

As an R-module, $R[M_n(q)]$ has a basis $\{\prod_{ij} X_{ij}^{t_{ij}} | t_{ij} \in \mathbb{Z}^+\}$, where the products are formed with respect to any fixed order of the X_{ij}'s. Let $A_q(n,r)$ be the rth homogeneous component of $R[M_n(q)]$. Then $A_q(n,r)$ has a basis

$$\{X^d_{\lambda\mu} \doteq X_{\boldsymbol{i}_\lambda d, \boldsymbol{i}_\mu} | \lambda, \mu \in \Lambda(n,r),\ d \in \mathscr{D}_{\lambda\mu}\},$$

where $\mathscr{D}_{\lambda\mu}$ denotes the set of distinguished representatives for the double cosets $\mathfrak{S}_\lambda \backslash \mathfrak{S}_r / \mathfrak{S}_\mu$, and $X_{\boldsymbol{i}\boldsymbol{j}} = X_{i_1,j_1} X_{i_2,j_2} \cdots X_{i_r,j_r}$ if $\boldsymbol{i} = (i_1, \cdots, i_r)$ and $\boldsymbol{j} = (j_1, \cdots, j_r)$. Denote by $A_q(n,r)^*$ the linear dual of $A_q(n,r)$. Then, by [11].

$$\varphi :\ End_{\mathcal{H}_R}(\bigoplus_{\lambda \in \Lambda(n,r)} x_\lambda \mathcal{H}_R) \cong A_q(n,r)^*,$$

where the natural basis for q-Schur algebra $End_{\mathcal{H}_R}(\bigoplus_{\lambda \in \Lambda(n,r)} x_\lambda \mathcal{H}_R)$ is given as follows: For $\lambda, \mu \in \Lambda(n,r), d \in \mathscr{D}_{\lambda\mu}$, let $\psi^d_{\lambda\mu}$ be defined by

$$\psi^d_{\lambda\mu}(x_\nu h) \doteq \delta_{\mu\nu} \sum_{w \in \mathfrak{S}_\lambda d \mathfrak{S}_\mu} T_w h.$$

If we identify $\psi^d_{\lambda\mu}$ with its images under the isomorphism above. The basis $\{\psi^d_{\lambda\mu}\}$ is the dual of the basis $\{X^d_{\lambda\mu} \doteq X_{\boldsymbol{i}_\lambda d, \boldsymbol{i}_\mu} | \lambda, \mu \in \Lambda(n,r),\ d \in \mathscr{D}_{\lambda\mu}\}$ for $A_q(n,r)$. Moreover by [11], we have $\varphi(\psi^d_{\lambda\mu})(X^{d_1}_{\rho\nu}) = \delta_{\lambda\rho}\delta_{\mu\nu}\delta_{d,d_1}$. Sometimes we denote the basis $\{\varphi(\psi^d_{\lambda\mu}) | \lambda, \mu \in \Lambda(n,r),\ d \in \mathscr{D}_{\lambda\mu}\} \subset A_q(n,r)^*$ as $\{\psi_{\boldsymbol{i}_\lambda d, \boldsymbol{i}_\mu} | \lambda, \mu \in \Lambda(n,r),\ d \in \mathscr{D}_{\lambda\mu}\}$.

Remark 3.2. Combination correspondence:
By using the notations of section 2.1, we can identify the following three sets:

$$\Xi : \dot{\coprod_{\lambda,\mu \in \Lambda(n;r)}} \mathscr{D}_{\lambda\mu} \ \to\ I(n,r) \times I(n,r)/\sim\ \to \Lambda(n,n;r) \tag{3.2}$$

$$d \in \mathscr{D}_{\lambda\mu} \ \mapsto\ (\boldsymbol{i}_\lambda d, \boldsymbol{i}_\mu) \ \mapsto\ wt(\boldsymbol{i}_\lambda d, \boldsymbol{i}_\mu) \tag{3.3}$$

The relations within class $I(n,r) \times I(n,r)$ is following:

$$(\boldsymbol{i}, \boldsymbol{j}) \sim (\boldsymbol{i}', \boldsymbol{j}')\ when\ wt(\boldsymbol{i}, \boldsymbol{j}) = (\boldsymbol{i}', \boldsymbol{j}')$$

Remark 3.3. The elements of $\Lambda(n,n;r)$ are functions from $\bar{n} \times \bar{n}$ to $\mathbb{N}_0$ and can be considered as $n \times n$ matrices of non-negative integers $(\omega_{st})_{s,t}$ with $1 \le s,t \le n$ such that $\sum_{s,t=1}^n \omega_{st} = r$.

Denote by $\Lambda^s(n,n;r)$ the subset of $\Lambda(n,n;r)$

$$\Lambda^s(n,n;r) = \{\omega \in \Lambda(n,n;r) | (\omega_{st})_{1 \le s,t \le n} \text{ an upper triangular matrix}, \sum_{1 \le k \le l \le n} (l-k)\omega_{kl} \ge s\}.$$

Under the identification in 3.2, we have

$$\Omega^{\succeq} \doteq \Xi^{-1}(\Lambda^0(n,n;r)) = \dot{\coprod_{\lambda,\mu \in \Lambda(n,r)}} \{d \in \mathscr{D}_{\lambda\mu} | \boldsymbol{i}_\lambda d \ge \boldsymbol{i}_\mu\} \doteq \dot{\coprod_{\lambda,\mu \in \Lambda(n,r)}} \Omega^{\succeq}_{\lambda\mu}.$$

and obviously the subset

$$\Omega^{\succeq 1} \doteq \Xi^{-1}(\Lambda^1(n,n;r)) = \{d \in \mathscr{D}_{\lambda\mu} | \lambda, \mu \in \Lambda(n,r),\ \lambda \ne \mu\ and\ d \ne 1\} \subseteq \Omega^{\succeq},$$

Similarly, we can defined a sequence of sets as $\Omega^{\succeq} \supseteq \Omega^{\succeq 1} \supseteq \Omega^{\succeq 2} \supseteq \cdots \Omega^{\succeq n} \supseteq \cdots$, and defined subspaces of $S_{R,q}(n,r)$:

$$J_n \doteq \bigoplus_{d\in\Omega^{\succeq n}} R\psi^d_{\lambda\mu} \subseteq S^+_{R,q}(n,r) \qquad \text{with } d \in \mathscr{D}_{\lambda\mu}.$$

We set $S^+_{R,q}(n,r) := J_0$, it is *Borel subalgebra* of q-Schur algebra. This definition is the same as [12]. $J_1 = \bigoplus_{\psi^d_{\lambda\mu}\neq\psi^1_{\lambda\lambda}} R\psi^d_{\lambda\mu}$ and $J_0 \supseteq J_1 \supseteq J_2 \supseteq \cdots$

Moreover, if we define subsets of $\Omega^{\succeq n}$ as $\Omega^{\succeq n}_{\lambda\mu} \doteq \Omega^{\succeq n} \cap \mathscr{D}_{\lambda\mu}$, then, we find trivially that $\dot{\coprod}_{\lambda,\mu\in\Lambda(n;r)} \Omega^{\succeq n}_{\lambda\mu} = \Omega^{\succeq n}$. Using this notations, we can state the following lemmas.

Lemma 3.4. *If $d_1 \in \Omega^{\succeq n}_{\lambda\mu}$ and $d_2 \in \Omega^{\succeq m}_{\mu\nu}$, then $\psi^{d_1}_{\lambda\mu}\psi^{d_2}_{\mu\nu} = \sum_{d\in\Omega^{\succeq m+n}_{\lambda\nu}} a_d\psi^d_{\lambda\nu}$ for some $a_d \in R$.*

Proof. Suppose $\lambda,\mu \in \Lambda(n,r), d \in \Omega^{\succeq n}_{\lambda\mu} \subseteq \mathscr{D}_{\lambda\mu}$. We claim that $\psi^d_{\lambda\mu}(X_{\mathbf{ij}}) \neq 0$ implies $(\mathbf{i},\mathbf{j}) \in I(n,r)\times I(n,r)$ satisfies $\sum_{1\le l\le k\le n}(l-k)\omega_{lk} = \sum_{k=1}^{r}(i_k - j_k) \geq n$.

Indeed, by the definition above, we have hypothesis that $\mathbf{j} = \mathbf{i}_\mu w$ for some $w \in \mathfrak{S}_r$. If $\ell(w) = 0$, i.e., $w = 1$, then $\mathbf{j} = \mathbf{i}_\mu$ and $\mathbf{i} = \mathbf{i}_\lambda d$, which trivially shows $(\mathbf{i},\mathbf{j}) \in \Omega^{\succeq n}_{\lambda\mu}$.

Assume now $\ell(w) > 0$. Write $w = w't$ with $t = (a, a+1)$ and $\ell(w) = \ell(w') + 1$. Then by definition of $\mathbf{i}_\mu$, we have $j_a > j_{a+1}$. If $i_a \leq i_{a+1}$, then by 3.1, $X_{\mathbf{ij}} = qX\mathbf{i}t, \mathbf{j}t = qX\mathbf{i}t, \mathbf{i}_{\mu w'}$. By induction on the length of elements in $\mathfrak{S}_r$, $\psi^d_{\lambda\mu}(X_{\mathbf{ij}}) = q\psi^d_{\lambda\mu}(X\mathbf{i}t, \mathbf{i}_{\mu w'}) \neq 0$ shows $(\mathbf{i}t, \mathbf{i}_\mu w')$ satisfy above claim, which trivially implies $(\mathbf{i},\mathbf{j})$ satisfy this claim too.

If $i_a > i_{a+1}$, then also by 3.1 relations.

$$\begin{aligned} X_{i_aj_a}X_{i_{a+1}j_{a+1}} &= X_{i_{a+1}j_{a+1}}X_{i_aj_a} - (q^{-1}-1)X_{i_{a+1}j_a}X_{i_aj_{a+1}} \\ &= X_{i_{a+1}j_{a+1}}X_{i_aj_a} - (1-q)X_{i_aj_{a+1}}X_{i_{a+1}j_a} \end{aligned}$$

Thus $\psi^d_{\lambda\mu}(X_{\mathbf{ij}}) = \psi^d_{\lambda\mu}(X_{\mathbf{i}t,\mathbf{j}t}) - (1-q)\psi^d_{\lambda\mu}(X_{\mathbf{i},\mathbf{j}t}) \neq 0$. We have either $\psi^d_{\lambda\mu}(X_{\mathbf{i}t,\mathbf{j}t}) \neq 0$ or $\psi^d_{\lambda\mu}(X_{\mathbf{i},\mathbf{j}t}) \neq 0$. By induction, either $(\mathbf{i}t, \mathbf{j}t)$ or $(\mathbf{i}, \mathbf{j}t)$ satisfy above claim, which trivially shows that $(\mathbf{i},\mathbf{j})$ satisfies this claim too.

Using the multiplication rules in $A_q(n,r)^*$, we have following:

$$\begin{aligned} \psi^{d_1}_{\lambda\mu}\cdot\psi^{d_2}_{\mu\nu}(X_{\mathbf{ij}}) &= <\psi^{d_1}_{\lambda\mu}\otimes\psi^{d_2}_{\mu\nu}, \triangle(X_{\mathbf{ij}})> \\ &= \sum_{\mathbf{k}\in I(n,r)} \psi^{d_1}_{\lambda\mu}(X_{\mathbf{ik}})\psi^{d_2}_{\mu\nu}(X_{\mathbf{kj}}) \neq 0. \end{aligned}$$

So there is $\mathbf{k} \in I(n,r)$ such that $\psi^{d_1}_{\lambda\mu}(X_{\mathbf{ik}}) \neq 0$ and $\psi^{d_2}_{\mu\nu}(X_{\mathbf{kj}}) \neq 0$. Thus, $\sum_{h=1}^{r}(i_h - k_h) \geq n$ and $\sum_{l=1}^{r}(k_l - j_l) \geq m$, and hence $\sum_{s=1}^{r}(i_s - j_s) = \sum(i_h - k_h) - \sum(k_l - j_l) \geq n + m$.

And we only need to show that $a_d \neq 0$ only if $\mathbf{i}_\lambda \succeq \mathbf{i}_\nu$, which has already been done in [12] by Du.

$\square$

Proposition 3.5. *The subspaces $\{J_m\}$ with $m \in \mathbb{N}_0$ are ideals of $S_R^+(n,r)$. Moreover, nilpotent ideal J_1 is actually the radical of $S_R^+(n,r)$ when R is a field, which is spanned by $\{\psi_{\lambda\mu}^d | \lambda \neq \mu,\ d \neq 1\}$.*

Proof.

$$\begin{aligned}\psi_{\lambda\mu}^{d_1} \cdot \psi_{\omega\nu}^{d_2}(X_{\mathbf{ij}}) &= <\psi_{\lambda\mu}^{d_1} \otimes \psi_{\omega\nu}^{d_2}, \triangle(X_{\mathbf{ij}})> \\ &= \sum_{k \in I(n,r)} \psi_{\lambda\mu}^{d_1}(X_{\mathbf{ik}})\psi_{\omega\nu}^{d_2}(X_{\mathbf{kj}}).\end{aligned}$$

Using the formula above, $\psi_{\lambda\mu}^{d}(X_{\rho\nu}^{d_1}) = \delta_{\lambda\rho}\delta_{\mu\nu}\delta_{d,d_1}$, we can claim that: if that $\mu \neq \omega$, then $\psi_{\lambda\mu}^{d_1} \cdot \psi_{\omega\nu}^{d_2}(X_{\mathbf{ij}}) = 0$. Then by the consequence of Lemma 3.4, we find subspace $J_n \doteq \bigoplus_{d \in \Omega^{\succeq n}} R\psi_{\lambda\mu}^d$ is ideal of $S_{R,q}^+(n,r)$, since $J_0 J_n \subseteq J_n$ with $J_0 = S_{R,q}^+(n,r)$.

Let $L_{n,r} := \bigoplus_{\lambda \in \Lambda(n,r)} R\psi_{\lambda\lambda}^1$, then $L_{n,r}$ is a commutative R-subalgebra of $S_R^+(n,r)$, and $S_R^+(n,r) = L_{n,r} \oplus J_1$. For every $\lambda \in \Lambda(n,r)$ we have a R-free module $R_\lambda := R\psi_{\lambda\lambda}^1$ of rank one. Note that $\psi_{\lambda\lambda}^1$ acts on R_λ as identity, and $\psi_{\mu\mu}^1,\ \mu \neq \lambda$, acts as zero. We will denote in the same way the module over $S_R^+(n,r)$ obtain form R_λ by the natural projection of $S_R^+(n,r)$ on $L_{n,r}$.

Note that if R is a field the algebra $L_{n,r}$ is semi-simple, and so J_1 is the radical of $S_R^+(n,r)$. In this case, $\{R_\lambda | \lambda \in \Lambda(n,r)\}$ is a complete set of pairwise non-isomorphic simple modules over $S_R^+(n,r)$. □

Remark 3.6. Suppose the base ring R is a field. There is another easier proof of this proposition. By looking $S_R^+(n,r)$ as a quasi-hereditary algebra, as Du and Rui showed in [12], we have this following proposition:

Proposition 3.7. *Let $rad(S_R^+(n,r))$ be the radical of Borel-Schur algebra $S_R^+(n,r)$. Then $rad(S_R^+(n,r))$ is spanned by $\psi_{\mu\lambda}^d \in S_R^+(n,r)$ with any $\mu, \lambda \in \Lambda(n,r)$ and $d \neq 1, \lambda \neq \mu$.*

Proof. From [12], Theorem 5.16, we know that when R is a field, all of the simple modules of $S_R^+(n,r)$ are determined by the composition set $\Lambda(n,r)$(that is the co-standard modules $\nabla(S_R^+(n,r),\lambda)$), with dimension one. Moveover, $\nabla(S_R^+(n,r),\lambda) \cong S_R^+(n,r)\psi_{\lambda\lambda}^1/rad(S_R^+(n,r)\psi_{\lambda\lambda}^1)$ where $rad(S_R^+(n,r)\psi_{\lambda\lambda}^1)$ is spanned by $\psi_{\mu\lambda}^d \in S_R^+(n,r)$ with $d \neq 1,\ \lambda \neq \mu$.

With the natural property of finite dimension algebras: $rad(M \oplus N) = rad(M) \oplus rad(N)$, then since $\{S_R^+(n,r)\psi_{\lambda\lambda}^1 | \lambda \in \Lambda(n,r)\}$ is a complete set of principal indecomposable $S_R^+(n,r)$ modules, which has $S_R^+(n,r) = \bigoplus_{\lambda \in \Lambda(n,r)} S_R^+(n,r)\psi_{\lambda\lambda}^1$. Then obviously we have that: $rad(S_R^+(n,r))$ is spanned by $\psi_{\mu\lambda}^d \in S_R^+(n,r)$ with any $\mu, \lambda \in \Lambda(n,r)$ and $d \neq 1,\ \lambda \neq \mu$. □

Remark 3.8. By the result of 3.5, in the category of rings, we can construct a splitting map p: $S_R^+(n,r) \to S_R^+(n,r)/rad(S_R^+(n,r)) = \bigoplus_{\lambda \in \Lambda(n,r)} R\psi_{\lambda\lambda}^1$ of the including map i: $\bigoplus_{\lambda \in \Lambda(n,r)} R\psi_{\lambda\lambda}^1 \hookrightarrow S_R^+(n,r)$.

From now on, denote $A \doteq S_R^+(n,r)$, $J \doteq rad(S_R^+(n,r))$ and $S \doteq \bigoplus_{\lambda \in \Lambda(n,r)} R\psi_{\lambda\lambda}^1$. Then, we define a homomorphism of S-bimodules as $\widetilde{p}$: $A \to J$ with $a \mapsto a - p(a)$. Obviously S is a commutative ring and the restriction of $\widetilde{p}$ to I is the identity map.

Definition 3.9. For every left A-module M, define a complex $B_k(A,S,M)$, $k \geq -1$, as we follow the notation of 3.8. We set the several notations:

$$\left\{\begin{array}{cc} B_{-1}(A,S,M) = M & k=-1 \\ B_0(A,S,M) = A\otimes M & k=0 \\ B_k(A,S,M) = A\otimes J^{\otimes k}\otimes M & k>1 \end{array}\right. \tag{3.4}$$

where all the tensor products are taken over S.
Next we define A-module homomorphism $d_{k,j} : B_k(A,S,M) \to B_{k-1}(A,S,M)$, $0 \leq j \leq k$, and S-module homomorphisms s_k:$B_k(A,S,M) \to B_{k+1}(A,S,M)$ by:

$$\begin{array}{rcl} d_{0,0} & := & am, \\ d_{k,0}(a\otimes a_1\otimes\cdots\otimes a_k\otimes m) & := & aa_1\otimes a_2\otimes\cdots\otimes a_k\otimes m, \\ d_{k,j}(a\otimes a_1\otimes\cdots\otimes a_k\otimes m) & := & a\otimes a_1\otimes\cdots\otimes a_ja_{j+1}\otimes\cdots\otimes a_k\otimes m,\ 1\leq j\leq k-1, \\ d_{k,k}(a\otimes a_1\otimes\cdots\otimes a_k\otimes m) & := & a\otimes a_1\otimes\cdots\otimes a_{k-1}\otimes a_km, \\ s_{-1} & := & e\otimes m, \\ s_k(a\otimes a_1\otimes\cdots\otimes a_k\otimes m) & := & e\otimes\widetilde{p}(a)\otimes a_1\otimes\cdots\otimes a_k\otimes m, 0\leq k. \end{array}$$

Define d_k: $B_k(A,S,M) \to B_{k-1}(A,S,M)$ by:

$$d_k := \sum_{t=0}^{k}(-1)^t d_{k,t}$$

.

Proposition 3.10. *The sequence $(B_k(A,S,M), d_k)_{k\geq -1}$ is a complex of left A-module. Moreover, we have this relations:*

$$\begin{array}{rcl} d_0s_{-1} & = & id_{B_{-1}(A,S,M)}, \\ d_{k+1}s_k + s_{k-1}d_k & = & id_{B_k(A,S,M)},\ 0\leq k. \end{array}$$

Thus, these maps $\{s_k\}_{k\geq 1}$ give a splitting of $B_*(A,S,M)$ in the category of S-modules. In particular, $(B_k(A,S,M), d_k)_{k\geq -1}$ is exact.

Proof. It is easy to see that

$$\begin{array}{rcl} d_{k,i}d_{k+1,j} & = & d_{k,j-1}d_{k+1,i},\ 0\leq i<j\leq k, \\ s_{k-1}d_{k,j} & = & d_{k+1,j+1}s_k,\ 1\leq j\leq k. \end{array}$$

Moreover for $k \geq 0$

$$\begin{array}{rcl} d_{0,0}s_{-1}(m) & = & d_{0,0}(e\otimes m) = m, \\ d_{k+1,0}s_k(a\otimes a_1\otimes\cdots\otimes a_k\otimes m) & = & \widetilde{p}(a)\otimes a_1\otimes\cdots\otimes a_k\otimes m, \\ d_{k+1,1}s_k(a\otimes a_1\otimes\cdots\otimes a_k\otimes m) & = & e\otimes\widetilde{p}(a)a_1\otimes\cdots\otimes a_k\otimes m, \\ s_{-1}d_{0,0}(a\otimes m) & = & s_{-1}(am), \\ s_{k-1}d_{k,0}(a\otimes a_1\otimes\cdots\otimes a_k\otimes m) & = & e\otimes aa_1\otimes a_2\otimes\cdots\otimes a_k\otimes m. \end{array}$$

Note that in the last identity we used the fact that $aa_1 \in I$ which implies that $\widetilde{p}(aa_1) = aa_1$. Taking into account that $\widetilde{p} := a - p(a)$ and $p(a) \in S$.

$$\begin{array}{rl} & (d_{k+1,0}s_k - d_{k+1,1}s_k + s_{k-1}d_{k,0})(a\otimes a_1\otimes\cdots\otimes a_k\otimes m) \\ = & a\otimes a_1\otimes\cdots\otimes a_k\otimes m - e\otimes p(a)a_1\otimes\cdots\otimes a_k\otimes m \\ - & e\otimes aa_1\otimes a_2\otimes\cdots\otimes a_k\otimes m + e\otimes p(a)a_1\otimes a_2\otimes\cdots\otimes a_k\otimes m + e\otimes aa_1\otimes a_2\otimes\cdots\otimes a_k\otimes m \\ = & a\otimes a_1\otimes\cdots\otimes a_k\otimes m. \end{array}$$

Thus $d_{k+1,0}s_k - d_{k+1,1}s_k + s_{k-1}d_{k,0} = id_{B_k(A,S,M)}$, $k \geq 0$. The above computations show that $(B_k(A,S,M), d_k)_{k\geq -1}$ is exact. □

Definition 3.11. Let J be the anti-automorphism of the q-Schur algebra $S_R(n,r)$, which is defined as

$$S_R(n,r) \to S_R(n,r) \qquad \psi^d_{\lambda\mu} \mapsto \psi^{d^{-1}}_{\mu\lambda}$$

It is clear that with definition of $S_R(n,r)$, $S_R^+(n,r)$ and $S_R^-(n,r) := J(S_R^+(n,r))$. This anti-automorphism J can be restricted as a ring homomorphism $S_R^+(n,r) \to S_R^-(n,r)$. Following the definition of [24], Appendix 7, we define a functor which is called *contravariant dual* $\mathcal{J}$:

$$\begin{array}{ccc} S_R(n,r)\text{-mod} & \to & S_R(n,r)\text{-mod} \\ V & \mapsto & V^{\circledast} \end{array}$$

where the dual $V^{\circledast} = Hom_R(V,R)$ is a left $S_R(n,r)$-mod if one defines $(\xi\theta)(v) = \theta(J(\xi)v)$, for $\theta \in Hom_R(V,R)$, $\xi \in S_R(n,r)$, $v \in V$.

Then we may consider the right exact functor

$$F = S_R(n,r) \otimes_{S_R^-(n,r)} \_ : \quad S_R^-(n,r)\text{-mod} \to S_R(n,r)\text{-mod},$$

and the left exact functor

$$G = Hom_{S_R^+(n,r)}(S_R(n,r), \_) : \quad S_R^+(n,r)\text{-mod} \to S_R(n,r)\text{-mod}$$

Lemma 3.12. *With the notation above, there is a $S_R(n,r)$-isomorphism*

$$F(V^{\circledast}) \cong (G(V))^{\circledast}$$

naturally in $V \in S_R^+(n,r)$-mod

Proof. In [24](section 7), the author proved a more general result for these algebras. □

Let $L_{n,r} := \bigoplus_{\lambda\in\Lambda} R\psi_\lambda$, with $\psi_\lambda := \psi^1_{\lambda\lambda}$. Then $L_{n,r}$ is a commutative R-subalgebra of $S_R^+(n,r)$, and $S_R^+(n,r) = L_{n,r} \oplus J_1$. For every $\lambda \in \Lambda(n,r)$ we have a free R-module module $R_\lambda := R\psi_\lambda$ of rank one over $L_{n,r}$. Note that ψ_λ acts on R_λ by identity, and ψ_μ, $\mu \neq \lambda$, acts by zero. We will denote in the same way the module over $S_R^+(n,r)$ obtained from R_λ by inflating along the natural projection of $S_R^+(n,r)$ on $L_{n,r}$, i.e., J_1 and $\psi_\mu(\mu \neq \lambda)$ act on R_λ as zero, and ψ_λ as identity.

Note that if R is a field the algebra $L_{n,r}$ is semi-simple, and so J_1 is the radical of $S_R^+(n,r)$. In this case $\{R_\lambda | \lambda \in \Lambda(n,r)\}$ is a complete set of pairwise non-isomorphic simple modules over $S_R^+(n,r)$.

For $\lambda \in \Lambda(n,r)$ we denote the bar resolution $B_*(S_R^+(n,r), L_{n,r}, R_\lambda)$ defined in 3.9 by $B_*^+(R_\lambda)$. Then

$$B_R^+(R_\lambda) := S_R^+(n,r) \otimes J_1 \otimes \cdots \otimes J_1 \otimes R_\lambda, \tag{3.5}$$

where all tensor products are over $L_{n,r}$ and there are k factors J_1.

Let M be a right $L_{n,r}$-module and N a left $L_{n,r}$-module. It follows from Corollary 9.3 in [20] that $M \otimes_{L_{n,r}} N \cong \bigoplus_{\lambda\in\Lambda(n,r)} M\psi_\lambda \otimes_{R\psi_\lambda} \psi_\lambda N$.

Therefore $B_k^+(R_\lambda)$ is the direct sum over all sequence $\mu^{(1)}, \ldots, \mu^{(k+1)} \in \Lambda(n,r)$ of the $S_R^+(n,r)$-modules

$$S_R^+(n,r)\psi_{\mu^{(1)}} \otimes \psi_{\mu^{(1)}} J_1 \psi_{\mu^{(2)}} \otimes \cdots \otimes \psi_{\mu^{(k)}} J_1 \psi_{\mu^{(k+1)}} \otimes \psi_{\mu^{(k+1)}} R_\lambda. \tag{3.6}$$

where all tensor products are over R. Since $\psi_{\mu^{(k+1)}}R_\lambda = 0$ unless $\psi^{(k+1)} = \lambda$, the summation is in fact over the sequences $\mu^{(1)}, \ldots, \mu^{(k)} \in \Lambda(n, r)$.

Proposition 3.13. *Let $\nu, \mu \in \Lambda$ and $n \geq 0$. Then $\psi_\nu J_n \psi_\mu = 0$ unless $\nu \rhd \mu$ (which means $\nu \unrhd \mu$ but $\nu \neq \mu$). If $\nu \rhd \mu$, then*

$$\{\psi^d_{\nu\mu} | d \in \Omega^{\succeq n}_{\nu\mu}\}$$

is an R-basis of the free R-module $\psi_\nu J_n \psi_\mu$.

Proof. From Remark 3.3 it follow that the set $J_n = \bigoplus_{\substack{d\in\Omega^{\succeq n}_{\theta\eta}\\ \theta,\eta\in\Lambda(n,r)}} R\psi^d_{\theta\eta}$. Then, we have $\psi_\nu J_n \psi_\mu = \psi^1_{\nu\nu} \cdot (\bigoplus_{\substack{d\in\Omega^{\succeq n}_{\theta\eta}\\ \theta,\eta\in\Lambda(n,r)}} R\psi^d_{\theta\eta}) \cdot \psi^1_{\mu\mu} = \bigoplus_{d\in\Omega^{\succeq n}_{\nu\mu}} R\psi^d_{\nu\mu}$. The first statement follows from the fact $\Omega^{\succeq n}_{\nu\mu} \subseteq \Omega^{\succeq}_{\nu\mu}$ and $d \in \Omega^{\succeq}_{\nu\mu}$ shows that $\nu \rhd \mu$.

□

Corollary 3.14. *Let N be the length of the maximal strictly decreasing sequence in $(\Lambda(n, r), \rhd)$. Then $B^+_k(R_\lambda) = 0$ for $k > N$.*

We conclude that the resolutions $B^+_*(R_\lambda)$ are finite, for all $\lambda \in \Lambda(n, r)$. We also have $B^+_0(R_\lambda) \cong S^+_R(n, r)\psi_\lambda$, and for $k \geq 1$

$$B^+_k(R_\lambda) = \bigoplus_{\mu^{(1)}\rhd\cdots\rhd\mu^{(k)}\rhd\lambda} S^+_R(n, r)\psi_{\mu^{(1)}} \otimes_R \psi_{\mu^{(1)}} J_1 \psi_{\mu^{(2)}} \otimes_R \cdots \otimes_R \psi_{\mu^{(k)}} J_1 \psi_\lambda,$$

In particular, $B^+_k(R_\lambda)$ is a projective $S^+_R(n, r)$-module for any $k \geq 0$. So we have the following result.

Proposition 3.15. *Let $\lambda \in \Lambda(n, r)$. Then $B^+_*(R_\lambda)$ is a projective resolution of the module R_λ over $S^+_R(n, r)$.*

Proof. The summand $S^+_R(n, r)\psi_{\mu^{(1)}} \otimes_R \psi_{\mu^{(1)}} J_1 \psi_{\mu^{(2)}} \otimes_R \cdots \otimes_R \psi_{\mu^{(k)}} J_1 \psi_\lambda$ of $B^+_*(R_\lambda)$ is a projective $S^+_R(n, r)$-module, since $S^+_R(n, r)\psi_{\mu^{(1)}}$ is projective and every $\psi_{\mu^{(i)}} J_1 \psi_{\mu^{(i+1)}}$ is isomorphic to a free R-module with finite rank. Then, we can say that $B^+_k(R_\lambda)$ is a projective $S^+_R(n, r)$-module and $B^+_*(R_\lambda)$ is a projective resolution of the module R_λ over $S^+_R(n, r)$.

□

4. Quantized enveloping algebras

We write $\mathscr{A}$ for the ring $\mathbb{Z}[q, q^{-1}]$ of integral Laurent polynomials in the indeterminate q. We write $\mathscr{B}$ for the localisation of $\mathscr{A}$ at the prime ideal $q\mathscr{A}$. Denote by $a \mapsto \bar{a}$ the automorphism of $\mathbb{Q}(q)$ interchangeing q and q^{-1}, and write $\overline{\mathscr{B}}$ for the image of $\mathscr{C}$ under the map. We write R for an arbitrary commutative $\mathscr{A}$-algebra. If $X_{\mathscr{A}}$ is a structure defined over $\mathscr{A}$, i.e. $X_{\mathscr{A}}$ is a module or algebra over $\mathscr{A}$.

In order to show the vanishing theorem in next section, we need some notations about quantum groups here. Recall the definition of the quantised enveloping algebra in the version given by Kashiwara [18]: let $\mathbb{A} = (\mathbb{A}_{i,j})_{i,j\in I}$ be a finite-type Cartan matrix. Fix positive integers $(d_i)_i \in I$ such that $d_i\mathbb{A}_{i,j} = d_j\mathbb{A}_{j,i}$ for all $i, j \in I$. Fix the following root datum:

(i) A perfect pairing $<,>: P^* \times P \to \mathbb{Z}$ of finitely generated free $\mathbb{Z}$-modules.

(ii) Linearly independent subsets $\{\alpha_i | i \in I\}$ of P and $\{\alpha_i^\vee | i \in I\}$ of P^+, satisfying $\mathbb{A}_{i,j} =< \alpha_i^\vee, \alpha_j >$ for all i and j.

Definition 4.1. Let U be the $\mathbb{Q}(q)$-algebra with generators e_i, f_i, q^h, $1 \leq i \leq n$, $h \in P^*$, and relations

$$\begin{aligned}
q^0 &= 1, \qquad q^h q^{h'} = q^{h+h'}, \\
q^h e_i &= q^{<h,\alpha_i>} e_i q^h, \\
q^h f_i &= q^{-<h,\alpha_i>} f_i q^h, \\
[e_i, f_j] &= \delta_{i,j} \frac{t_i - t_i^{-1}}{q_i - q_i^{-1}}, \\
\sum_{l=0}^{a} (-1)^l e_i^{(l)} e_j e_i^{(a-l)} &= \sum_{l=0}^{a} (-1)^l f_i^{(l)} f_j f_i^{(a-l)} = 0, \\
& \text{for } i \neq j, \ a = 1 - \mathbb{A}_{i,j}.
\end{aligned}$$

We have used the following abbreviations: $t_i = q^{d_i \alpha_i^\vee}$, $q_i = q^{d_i}$, $e_i^{(l)} = e_i^l / [l]_i^!$, $f_i^{(l)} = f_i^l / [l]_i^!$. (The subscript i signifies that the q in $[l]!$ is replaced by q_i.) If x is a unit in a $\mathbb{Q}(q)$-algebra, $c \in \mathbb{Z}$, $t \in \mathbb{N}$, put

$$\begin{bmatrix} x; c \\ t \end{bmatrix} = \prod_{s=1}^{t} \frac{x q^{(c-s+1)} - x^{-1} q^{-(c-s+1)}}{q^s - q^{-s}}.$$

Write $\begin{bmatrix} x \\ t \end{bmatrix}$ for $\begin{bmatrix} x; 0 \\ t \end{bmatrix}$.

P and P^* are called the lattices of *weight* and *coweights*, respectively; the α_i are the *simple roots*, the $\alpha_i^\vee$ are the *simple coroot*. The *dominance order* on P is defined by $\lambda \geq \mu$ if and only if $\lambda - \mu$ can be written as a sum of simple roots. A weight λ is *dominant* (resp. *antidominant*) if all $< \alpha_i^\vee, \lambda >$ are nonnegative (resp. nonpositive). The Weyl group W of the root system acts in the usual way on the weight and coweight lattices. We write w_0 for the longest element of W. Every weight λ is W-conjugate to a unique dominant (resp. antidominant) weight which we write as λ^+ (resp. λ^-.)

Take $J \subseteq I$. Put

$$\begin{aligned}
\mathscr{E} &= \{e_i^{(s)} | s \geq 0, i \in J\}. \\
\mathscr{F} &= \{f_i^{(s)} | s \geq 0, i \in J\}. \\
\mathscr{H} &= \{q^h, \begin{bmatrix} q^h \\ s \end{bmatrix}\}
\end{aligned}$$

Let $U_{\mathscr{A}}^-(J)$, $U_{\mathscr{A}}^+(K)$, and $U_{\mathscr{A}}(J,K)$ be the $\mathscr{A}$-subalgebras of U generated respectively by $\mathscr{F}_J$, $\mathscr{E}_K$, and $\mathscr{F}_J \cup \mathscr{H} \cup \mathscr{E}_K$. For the corresponding algebras over $\mathbb{Q}(q)$, We use the notation $U(J,K)$ omitted the subscript $\mathscr{A}$ (resp. $U_R(J,K)$) (resp. obtained by base changed to R, i.e., U is defined as $U_{\mathscr{A}} \otimes_{\mathscr{A}} R$.). Then, we use the following abbreviations: $U_{\mathscr{A}}^{\pm} = U_{\mathscr{A}}^{\pm}(I,I)$, $U_{\mathscr{A}}^0 = U_{\mathscr{A}}(\varnothing,\varnothing)$, $U_{\mathscr{A}}^{\flat} = U_{\mathscr{A}}(I,\varnothing)$, $U_{\mathscr{A}}^{\#} = U_{\mathscr{A}}(\varnothing, I)$, and define $U^{\pm}, U^0, U^{\flat}$ (resp. $U_R^{\pm}, U_R^0, U_R^{\flat}$) similarly with the ground ring $\mathbb{Q}(q)$ (resp. R).

Definition 4.2. A left $U_R(J,K)$-module V is *integrable* if it satisfies:

(i) V is the sum of its weight spaces.

(ii) For each $v \in V$, if $i \in J$ (resp. $i \in K$), we have

$$f_i^{(s)} v = 0 \qquad \text{(resp. } e_i^{(s)} = 0) \qquad \forall s \gg 0.$$

Integrability of right modules is defined analogously.

For $\lambda \in P$, R_λ denotes the integrable $U_R^\flat$ (resp. $U_R^\#$)-module with underlying R-module R, on which U_R^0 acts via λ and $f_i^{(s)}$ (resp. $e_i^{(s)}$) as 0 if $s > 0$. The *Verma module* of highest weight λ is the U_R-module $M_R(\lambda) = U_R \otimes_{U_R^\#} R_\lambda$. We write v_λ for its highest weight vector $1 \otimes 1$. This is the sum of weight spaces, but is not integrable for U_R. As a U_R^--module, $M_R(\lambda)$ is free of rank 1; in particular, it is free over R.

Write $U_R(J,K)$-Int for the category of all integrable $U_R(J,K)$-modules V. It is known as in Lusztig [19] that U-Int is semisimple with simple modules $\triangle(\lambda)$, $\lambda \in P^+$, where $\triangle(\lambda)$ is the unique maximal integrable quotient of $M(\lambda)$. We write the image in $\triangle(\lambda)$ of $v\lambda \in M(\lambda)$ by the same symbol. Put $\triangle_{\mathscr{A}}(\lambda) = U_{\mathscr{A}} v_\lambda$ and $\triangle_R(\lambda) = R \otimes_{\mathscr{A}} \triangle_\lambda$, the *Weyl module* of highest weight λ. This is free over R, see [29] (3.3), with character given by the Weyl's character formula. If R is a field, $\triangle_R(\lambda)$ has a unique simple quotient, which we write $L_R(\lambda)$. These modules form a complete set of simple integrable U_R-modules.

There is a $\mathbb{Q}(q)$-algebra anti-automorphism $u \mapsto u^\tau$ of U given by

$$e_i^\tau = f_i, \qquad f_i^\tau = e_i, \qquad (q^h)^\tau = q^h,$$

which interchanges U^- and U^+ and fixed U^0 pointwise. This maps $U_{\mathscr{A}}$ onto itself, hence inducing an anti-automorphism of each U_R. If V is a left (resp. right) U_R-module, we write V^τ for V made into a right (resp. left) U_R-module via τ. If $V \in U_R$-Mod, its *contravariant dual* V° is the linear dual $Hom_R(V,R)$, with its natural right U_R-action transferred to the left via τ. If V is integrable and finitely generated and projective over R, this is again integrable. Put $\nabla_R(\lambda) = \triangle_R(\lambda)^\circ$.

5. Woodcock's condition and Kempf's Vanishing theorem

In this section we explain a condition which appears in Kashiwara and Woodcock's work [17], [29], and explain how the result of Kempf's Vanishing theorem can be applied to prove that $S_R^+(n,r)$-module R_λ, $\lambda \in \Lambda^+(n,r)$, is acyclic for the induction functor $S_R^+(n,r)$-mod $\to S_R(n,r)$-mod. We start with a definition.

To circumvent the deficiencies of the coefficient ring, for R an arbitrary commutative ring with a unit, we build an explicit weight space decomposition in comodule category:

Definition 5.1. ([29]). Let P be a set. For $\lambda \in P$ let $R(\lambda)$ be a copy of the trivial R-coalgebra R, and put

$$C^0 = \coprod_{\lambda \in P} R(\lambda), \tag{5.1}$$

a coproduct of coalgebras. Let 1_λ be the element of C^0 whose μ-component is $\delta_{\lambda,\mu}$. Let (V,ρ) be a left C^0-comodule. The decomposition 5.1 gives rise to a functorial decomposition of V: for $\lambda \in P$ put

$${}^\lambda V := \{v \in V | \rho(v) = 1_\lambda \otimes v\}.$$

Then we have $V = \otimes_{\lambda \in P}{}^{\lambda}V$. We refer to the elements of P as weights and use the usual terminology of weight spaces. We use analogous notation for right weight spaces.

Suppose we are given a partical order $\leq$ on a poset P and a subset P^+ of P which is a *locally finite* poset under the restriction of the partial order on P, i.e., for each $\lambda \in P$ there are only finitely many $\mu \in P^+$ with $\mu \leq \lambda$. Let $(A(\Lambda), \mu_\Gamma^\Lambda)$ be a filtered system of R-coalgebras indexed by the finite ideals in P^+. Assume that each map $\mu_\Gamma^\Lambda : A(\Gamma) \to A(\Lambda)$ is injective. Assume also that we have a maps $A(\Lambda) \to C^0$ compatible with μ_Λ^Γ.

Put $C = \varinjlim_{\Lambda} A(\Lambda)$. Note that the canonical map $C \to C^0$ gives us a notion of weight spaces on C-comodule. Assume that for each finite ideal Λ in P^+ and each maximal $\lambda \in \Lambda$ we say that $(A(\Lambda), P^+, C)$ satisfy a *Woodcock condition*, if we have an isomorphism of bicomodules

$$A(\Lambda)/A(\Lambda \setminus \{\lambda\}) \cong \nabla(\lambda) \otimes \nabla'(\lambda).$$

where $\nabla(\lambda)$ and $\nabla'(\lambda)$ are, respectively, left and right $A(\lambda)$-comodules satisfying:
(i) ${}^{\lambda}\nabla(\lambda) \cong \nabla'(\lambda)^{\lambda} \cong k$.
(ii) For all $\lambda \in P$, ${}^{\mu}\nabla(\lambda) \neq 0$ or $\nabla'(\lambda)^{\mu} \neq 0$ implies $\mu \leq \lambda$.
(iii) $\nabla(\lambda)$ and $\nabla'(\lambda)$ are finitely generated and projective over k.

Note from (iii) and the local finiteness of P^+ that each $A(\Lambda)$ is finitely generated and projective over k. Thus C is flat over k. Moverover, let $S(\Lambda)$ be the dual algebra of $A(\Lambda)$. Since $A(\Lambda)$ is finitely generated and projective over k, we have $A(\Lambda)$-Comod $\cong S(\Lambda)$-Mod.

Remark 5.2. For $\mu \in P$ let $\varepsilon_\mu \in S(\Lambda)$ be the composition $A(\Lambda) \to C^0 \to k(\mu)$, where the last map is the natural projection. It can be alternatively describe as the map on $A(\Lambda)$ given by projection along the weight grading to ${}^{\mu}A(\Lambda)^{\mu}$ followed by the counit. It is easy to see that ε_μ are pairwise orthogonal idempotents whose sum is the identity element of $S(\Lambda)$. For any $V \in S(\Lambda)$-Mod we have ${}^{\mu}V = \varepsilon_\mu V$ and $V^{\mu} = V\varepsilon_\mu$.

The most valuable example of Woodcock condition appears in quantum algebra and its *coordinate algebra.*

Example 5.3. [17][29] Let U be a quantum algebra.Put

$$C := \{c \in U^* | Uc, cU \text{both integrable}\}.$$

Then, it is called as *coordinate algebra* of U. This has a $\mathbb{Q}(q)$-coalgebra structure where the comultiplication is defined by $\triangle\ (c)(u \otimes v) = c(vu)$ and the counit by $\varepsilon(c) = c(1)$, with $c \in C, \quad u, v \in U$. Note the twist in the definition of the comultiplication-cf. (**4.3**). By Kashiwara [[17], sSection 7] there is an embedding of U-bimodules:

$$\begin{aligned} \Phi_\lambda : \nabla(\lambda) \otimes \nabla(\lambda)' &\to C, \\ x \otimes y &\mapsto (u \mapsto < ux, y >) \end{aligned}$$

for each $\lambda \in P^+$, and we have $C \cong \bigoplus_{\lambda \in P^+} im\Phi_\lambda$. Put

$$\begin{aligned}
C_{\mathscr{A}} &= \{a \in C | a(U_{\mathscr{A}}) \subseteq \mathscr{A}\}, \\
C_{\mathscr{B}} &= \sum_{\lambda \in P^+} \Phi_\lambda(\nabla_{\mathscr{B}}(\lambda) \otimes \nabla_{\mathscr{B}}(\lambda)'), \\
C_{\overline{\mathscr{B}}} &= \sum_{\lambda \in P^+} \Phi_\lambda(\nabla_{\overline{\mathscr{B}}}(\lambda) \otimes \nabla_{\overline{\mathscr{B}}}(\lambda)'), \\
B(C) &= \bigcup_{\lambda \in P^+} \Phi_\lambda(B(\lambda)^\circ \otimes B(\lambda)^\circ).
\end{aligned}$$

and $(C_{\mathscr{C}})_{\mathscr{A},\mathscr{B},\overline{\mathscr{B}}}$ is a balanced triple in C. We have $B(C)$ as a crystal basis of the triple.

In [29], Woodcock showed that, if Λ is a finite ideal in P^+ and $V \in U$-int, let $O_\Lambda V$ be the largest submodule V' of V such that ${}^\lambda V' \neq 0$, $\lambda \in P^+$ implies $\lambda \in \Lambda$. Put $A(\Lambda) := O_\Lambda C$, we know that $(A(\Lambda), P^+, C)$ is a triple which satisfies woodcock's condition.

Definition 5.4. For $\mu \in W \cdot \lambda$ with $\lambda \in P^+$, let b_μ (resp. ν_μ) be the element of weight μ in the lower crystal (resp. global) basis of $\triangle\ (\lambda)$. If $w \in W$ with $m =< \alpha^\vee, w\lambda >\geq 0$, we have

$$\nu_{s_i w\lambda} \;=\; f_i^{(m)} \nu_{w\lambda}, \nu_{w\lambda} = e_i^{(m)} \nu_{s_i w\lambda}.$$

Put $\triangle^{\#}\ (\mu) := U^{\#} \cdot \nu_\mu$. There are the *Demazure modules* associated to μ. Define

$$B^{\#}(\mu) = \{b \in B(\lambda) | G(b) \in \triangle^{\#}\ (\mu)\}.$$

Remark 5.5. From Kashiwara [17] we have:
(i) $GB^{\#}(\mu)$ is an $\mathscr{A}$-basis of $\triangle^{\#}_{\mathscr{A}}\ (\mu)$,
(ii) $\triangle^{\#}_{\mathscr{A}}\ (\mu) = \sum_{k_1,\ldots,k_l \geq 0} \mathscr{A} f_{i_1}^{(k_1)} \cdots f_{i_l}^{(k_l)} \nu_\lambda$.
We infer the $\triangle^{\#}_R\ (\mu) \cong R \otimes_{\mathscr{A}} \triangle^{\#}_{\mathscr{A}}\ (\mu)$ and that $GB^{\#}(\mu)$ is a R-basis of $\triangle^{\#}_R\ (\mu)$.

Definition 5.6. Let W^λ denote the set of minimal length coset representation of the stabiliser of λ in W; let $\preceq$ be the partial order on the Weyl orbit $W\lambda$ induced by the opposite of the Bruhat order on W^λ. (So $\mu \preceq \nu$ implies $\mu \leq \nu$, but the converse is false in general.) For later use we follow van der Kallen [26] and extend $\preceq$ to a partial order on the whole of P, the *antipodal excellent order*, by $\mu \prec \nu$ iff $\mu^+ < \nu^+$ when μ and ν lie in different Weyl orbits.

We have (see Kashiwara [17], **3.2.2**) $\triangle^{\#}_R\ (\nu) \supseteq \triangle^{\#}_R\ (\mu)$ if and only if $\nu \preceq \mu$. Let $\overline{\triangle}^{\#}_R(\mu)$ be the quotiemt of $\triangle^{\#}_R\ (\mu)$ by the sum of all $\triangle^{\#}_R\ (\nu)$ with $\nu \prec \mu$. Put

$$\overline{B}^{\#}(\mu) = B^{\#}(\mu) \setminus \bigcup_{\mu \prec \nu} B^{\#}(\nu).$$

We see from 5.5 that $\overline{\triangle}^{\#}_R(\mu)$ and $\overline{\triangle}^{\#}_{\mathscr{A}}(\mu)$ have bases consisting of the images of the $G(b)$ for $b \in \overline{B}^{\#}(\mu)$.

Remark 5.7. [29] This is the integral version of Example 5.3 which appears in [17]. In that case, Kashiwara Put $\mathscr{A}' = \mathbb{Q}[q, q^{-1}]$ and proof above results about quantum algebras in triple $(C_{\mathscr{C}'})_{C' \in \{\mathscr{A}', \mathscr{B}, \overline{\mathscr{B}}\}}$, which is balanced triple in C. In fact, we can follow Kashiwara and check that the argument for balanced triple which carries through in the integral case. This progress is taken in Woodcock's paper [29] (**18**).

Then, put $C_R = R \otimes_{\mathscr{A}} C_{\mathscr{A}}$ a R-coalgebra with R-basis $GB(C)$. If Λ is a finite ideal in P^+, we put $A_R(\Lambda) = R \otimes_{\mathscr{A}} A_{\mathscr{A}}$, where $\mathscr{A}(\Lambda)$ is as defined in definition above. This is a R-subalgebra of C_R, and is spanned by the global basis elements it contains. Furthermore, if λ is maximal in Λ and $\Gamma = \Lambda \setminus \{\lambda\}$, we still have a short exact sequence in U-bimodule

$$0 \to A_R(\Gamma) \to A_R(\Lambda) \to \nabla_R(\lambda) \otimes \nabla_R(\lambda)' \to 0, \tag{5.2}$$

which maps those global basis elements in $A_R(\Lambda) \setminus A_R(\Gamma)$ bijectively onto the standard global basis of $\nabla(\lambda) \otimes \nabla(\lambda)'$.

Therefore, we find a triple $(A_R(\lambda), P^+, C_R)$ whose ground ring can be an arbitrary commutative ring with a ring morphism $\mathbb{Z}[q, q^{-1}] \to R$, which satisfy the woodcock condition defined above. In addition, the dual algebra of $A_R(\Lambda)$ we also state it as $S_R(\Lambda)$ which is the (generalised) Schur algebras. They are quantised analogues of Donkin's generlised Schur algebras[8], and have been defined in a slight different manner in Du and Scott [14] and Lusztig[[19], 29.2].

Example 5.8. As in [29] dualizing 5.2 gives a short exact sequence

$$0 \to \triangle_R(\lambda) \otimes \triangle_R(\lambda)^\tau \to S_R(\Lambda) \to S_R(\Gamma) \to 0. \tag{5.3}$$

Using 5.3 and induction on $|\Lambda|$, we see that the algebra map $U_R \to S_R$, $u \mapsto u \cdot 1$ is surjective. Write $S_R^{\mathfrak{b}}$ for the image of $U_R^{\mathfrak{b}}$ under this map. When we need to make the role of Λ explicit, we write $S_R^{\mathfrak{b}} = S_R^{\mathfrak{b}}(W\Lambda)$, with W is the weyl group defined in Section 1. We call these algebra *Borel Schur algebras*. There were first studied by Green [15] in the case of GL_n. In the classical setting they are considered in Woodcock [30].

We will show that the Borel Schur algebras still fit the framework of Woodcock's condition:

Put $\Xi \leq P$ as a finite ideal for the antipodal excellent order (see 5.6), $W\Xi \cap P^+$ is a finite ideal in P^+. We may thus suppose that it is our chosen ideal Λ. Let $F_\Xi : S_R^{\mathfrak{b}} - Mod \to S_R^{\mathfrak{b}} - Mod$ take V to its largest quotient with weight in Ξ, a right exact functor.

Put $S_R^{\mathfrak{b}}(\Xi) = F_\Xi S_R^{\mathfrak{b}}$. By the alternative version of [[26], 3.11], if μ is maximal in Ξ for the antipodal excellent order, then there is a short exact sequence of bimodules

$$0 \to \triangle_R^{\mathfrak{b}}(\mu) \otimes \bar{\triangle}_R^{\#}(\mu)^\tau \to S_R^{\mathfrak{b}}(\Xi) \to S_R^{\mathfrak{b}}(\Xi \setminus \{\mu\}) \to 0. \tag{5.4}$$

Induction on $|\Xi|$ now shows that $S_R^{\mathfrak{b}}$ is finitely generated free over R; hence the $A_R^{\mathfrak{b}}(\Xi) := (S_R^{\mathfrak{b}}(\Xi))^*$ forms a filtered system of R-coalgebras. We write $C_R^{\mathfrak{b}}$ for the colimit of this system, a free R-coalgebra. If we take P for both P and P^+, $\preceq$ for order $\leq$, and define the maps $A_R^{\mathfrak{b}} \to C^0$ using the idempotent ε_μ, the woodcock's condition is satisfied in triple $(A_R^{\mathfrak{b}}, P, C_R^{\mathfrak{b}})$. The roles of the $\triangle(\mu)$ and $\triangle'(\mu)$ are played by $\triangle_R^{\mathfrak{b}}(\mu)$ and $\bar{\triangle}_R^{\#}(\mu)'$, respectively.

The triple $(A(\Lambda), P^+, C)$ with Woodcock condition have some very interesting homological consequences [29]. One of them is called *Ext-reciprocity*. For $V \in S(\Lambda)$-Mod, $X \in R$-Mod, and $\mu \in P$, we have

$$Ext^i_{S(\Lambda)}(S(\Lambda)^\mu \otimes X, V) \cong Ext^i_R(X, {}^\mu V) \qquad \forall i \geq 0, \tag{5.5}$$

$$Ext^i_{S(\Lambda)}(V, A(\Lambda)^\mu \otimes X) \cong Ext^i_R({}^\mu V, X) \qquad \forall i \geq 0. \tag{5.6}$$

A q-analogue Kempf's vanishing theorem was established by some properties of the crystal basis proved by Kashiwara in order to obtain the refined Demazure character formula in [17][23]. Woodcock [29] showed that the ideal of using the properties of the cystal basisi to obtain the quantized Kempf's vanishing theorem also works from a Schur algebra point of view by 5.5:

Let us recall the *Kempf's vanishing theorem.* For all $X \in R$-Mod and $\mu \in P$, note that μ^+ as the unique dominant weight in orbit $W\mu$. We have

$$\begin{aligned} Ext^i_{S_R^{\mathfrak{b}}}(S_R, \nabla_R^{\mathfrak{b}}(\mu) \otimes X) &\cong Ext^i_R({}^{\mu}S_R, X) \cong Ext^i_R(\triangle_R(\mu^+)^{\tau}, X) \\ &\cong \begin{cases} \triangle_R(\mu^+) \otimes X & if\ i = 0 \\ 0 & if\ i > 0 \end{cases} \end{aligned}$$

Remark 5.9. The case $\mu = \mu^+$ of 5.7 is Kempf's vanishing theorem in [23], since $H^i(U_R/U_R^{\mathfrak{b}}) \cong Ext^i_{S_R^{\mathfrak{b}}(W\Lambda)}(S_R(\Lambda), \triangle^{\lambda})$ for all $i \geq 0$. If Λ is a finite ideal in P^+, and $V \in S_R^{\mathfrak{b}}(W\Lambda)$-Mod.

Moreover, in this situation, we have $\nabla_R^{\mathfrak{b}}(\mu) = R_{\mu}$. Thanks to the definition of Demazure modules in 5.4.

Theorem 5.10. *Let $\lambda \in \Lambda^+(n,r)$. The complex $S_R(n,r) \otimes_{S_R^+(n,r)} B_*^+(R_\lambda)$ is a projective resolution of $W_\lambda^R := S_R(n,r) \otimes_{S_R^+(n,r)} R_\lambda$ over $S_R(n,r)$.*

Proof. Fix $\lambda \in \Lambda^+(n,r)$ and denote the complex $S_R \otimes_{S_R^+} B_*^+(R_\lambda)$ by $\mathbb{X}(R)$. Then all R-module in $\mathbb{X}(R)$ are free R-modules, since $S_R \otimes_{S_R^+} B_k^+(R_\lambda)$

$$S_R \otimes_{S_R^+} B_k^+(R_\lambda) \cong \bigoplus_{\mu^{(1)} \rhd \cdots \rhd \mu^{(k)} \rhd \lambda} S_R(n,r)\psi_{\mu^{(1)}} \otimes_R \psi_{\mu^{(1)}} J_1 \psi_{\mu^{(2)}} \otimes_R \cdots \otimes_R \psi_{\mu^{(k)}} J_1 \psi_\lambda,$$

and $S_R(n,r)\psi_{\mu^{(1)}}$, $\psi_{\mu^{(i)}} J_1 \psi_{\mu^{(i+1)}}$ with $1 \leq i \leq k$, $\psi_{\mu^{(k)}} J_1 \psi_\lambda$ are all free R-module.

Moreover we know that $\mathbb{X}(\mathbb{Z}) \otimes_{\mathbb{Z}} R \cong \mathbb{X}(R)$. Now by the Universal Coefficient Theorem we have a short exact sequence

$$0 \to H_i(\mathbb{X}(\mathbb{Z})) \otimes R \to H_i(\mathbb{X}(R)) \to Tor_1^{\mathbb{Z}}(H_{i-1}(\mathbb{X}(\mathbb{Z})), R) \to 0.$$

Thus to show that the complex $\mathbb{X}(R)$ are acyclic it is enough to check that the complex $\mathbb{X}(\mathbb{Z})$ is acyclic.

Now we already know that $H_i(\mathbb{X}(\mathbb{Z}))$ is a finitely generated abelian group. Therefore

$$H_i(\mathbb{X}(\mathbb{Z})) \cong \mathbb{Z}^t \bigoplus_{p\ prime} \bigoplus_{s \geq 1} (\mathbb{Z}/p^s\mathbb{Z})^{t_{ps}},$$

where only finitely many of the integer t, t_{ps} are different from zero. For every prime p denote by $\bar{\mathbb{F}}_p$ the algebraic closure $\mathbb{F}_p$. Then we get

$$H_i(\mathbb{X}(\mathbb{Z})) \otimes_{\mathbb{Z}} \bar{\mathbb{F}}_p \cong \bar{\mathbb{F}}_p^{\sum_s \geq 1} t_{ps}$$

and also

$$H_k(\mathbb{X}(\mathbb{Z})) \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q}'.$$

Hence if we can show that $H_i(\mathbb{X}(\mathbb{Z})) \otimes_{\mathbb{Z}} \bar{\mathbb{F}}_p$ for all prime numbers p and $H_i(\mathbb{X}(\mathbb{Z})) \otimes_{\mathbb{Z}} \mathbb{Q} = 0$, this will imply that $H_i(\mathbb{X}(\mathbb{Z})) = 0$.

Let $\mathbb{K}$ be one of the field $\bar{\mathbb{F}}_p$, p prime, or $\mathbb{Q}$. Then, by the Universal Coefficient Theorem again, $H_i(\mathbb{X}(\mathbb{Z})) \otimes_{\mathbb{Z}} \mathbb{K}$ is a submodule of $H_i(\mathbb{X}(\mathbb{K}))$. Therefore it is enough to show that $H_i(\mathbb{X}(\mathbb{K})) = 0$.

As in Section 3.11, we note that the algebra $S_{\mathbb{K}}(n,r)$ has an anti-involution $\mathcal{J}: S_{\mathbb{K}}(n,r) \to S_{\mathbb{K}}(n,r)$ defined on the basis elements by $\mathcal{J}: \varphi^d_{\lambda\mu} \mapsto \varphi^{d^{-1}}_{\mu\lambda}$. The image of $S^+_{\mathbb{K}}(n,r)$ under $\mathcal{J}$ is the subalgebra $S^-_{\mathbb{K}}(n,r)$ of $S_{\mathbb{K}}(n,r)$. Now for each $S^+_{\mathbb{K}}(n,r)$-module M we define a structure of $\mathcal{J}(S^+_{\mathbb{K}}(n,r))$-module on $M^* := Hom_{\mathbb{K}}(M,\mathbb{K})$ by $(\xi\theta)(m) := \theta(\mathcal{J}(\xi)m)$, for $\theta \in M^*$, $\xi \in \mathcal{J}(S^+_{\mathbb{K}}(n,r))$. This induces a contravariant equivariant of categories $\mathcal{J}_*: S^+_{\mathbb{K}}(n,r)\text{-mod} \to S^-_{\mathbb{K}}(n,r)\text{-mod}$.

By Lemma 3.12, the following functors are exact,????

$$\mathcal{J}_* \circ Hom_{S^-_{\mathbb{K}}(n,r)}(S_{\mathbb{K}}, -) \circ \mathcal{J} : S^+_{\mathbb{K}}(n,r)\text{-mod} \to S_{\mathbb{K}}(n,r)\text{-mod}$$
$$S_{\mathbb{K}}(n,r) \otimes_{S^+_{\mathbb{K}}(n,r)} - : S^+_{\mathbb{K}}(n,r)\text{-mod} \to S_{\mathbb{K}}(n,r)\text{-mod}$$

By the Kempf's vanishing theorem we already know that the module R_λ is $Hom_{S^-_{\mathbb{K}}}(S_{\mathbb{K}}, -)$-acyclic, then, the complex

$$Hom_{S^-_{\mathbb{K}}}(S_{\mathbb{K}}, \mathcal{J}_*(B_*(\mathbb{K}_\lambda))).$$

is exact for $\lambda \in \Lambda^+(n,r)$. Applying $\mathcal{J}_*$ we get that the complex $\mathbb{X}(\mathbb{K})$ is exact. □

6. The Boltje-Maisch complex

Suppose that $n \geq r$, then we know that there is a partition $\delta := (1,\dots,1,0,\dots,0) \in \Lambda(n,r)$. Then there is a obvious isomorphism of algebras $\phi : \mathcal{H}_r \cong \psi_\delta S_R(n,r)\psi_\delta$. It is obvious that if M is an $S_R(n,r)$-module then the $\psi_\delta M$ is a $\psi_\delta S_R(n,r)\psi_\delta$-module. In fact the map is functorial, and the resulting functor $\mathfrak{S} : S_R(n,r)\text{-mod} \to R\mathcal{H}_r$-mod was named *Schur functor* in [16]. It is obvious that $\mathfrak{S}$ is exact.

In this section we show that for $\lambda \in \Lambda^+(n,r)$ the complex $\mathfrak{S}(B^+_*(R_\lambda) \otimes_{S^+_R} S_R)$ is isomorphic to the complex constructed in [23], which we called *Boltje-Maisch complex* here.

We start by summarizing the notations and conventions of [23]. They introduce, for any compositions $\lambda, \mu \in \Lambda(n,r)$, and R-submodule $Hom^\wedge_{\mathcal{H}}(M^\mu, M^\lambda)$ of $Hom_{\mathcal{H}}(M^\mu, M^\lambda)$.

Definition 6.1. Let $\lambda, \mu \in \Lambda(n,r)$ be composition of r. We say that $T \in \mathcal{T}^{rs}(\lambda,\mu)$ is *ascending* if, for every $i \in \mathbb{N}$, the i-th row of T contains only entries which are greater than or equal to i, i.e., we can identify this row-semistandard tableau T with a unique element $d \in \mathfrak{S}_r$ which satisfies $d \in \Omega^{\succeq}_{\lambda\mu}$ and $T^\lambda_\mu d = T$. Moreover, Boltje and Hartmann in [22] define

$$Hom^\wedge_{\mathcal{H}}(M^\mu, M^\lambda) := \bigoplus_{d\in\Omega^{\succeq}_{\lambda\mu}} R\psi^d_{\lambda\mu} \subseteq Hom_{\mathcal{H}}(M^\mu, M^\lambda). \tag{6.1}$$

As a free R-module, $\psi_\lambda S^+_R(n,r)\psi_\mu = \psi^1_{\lambda\lambda} \bigoplus_{\substack{d\in\Omega^{\succeq}_{\theta,\eta} \\ \theta,\eta\in\Lambda(n,r)}} R\psi^d_{\theta\eta}\psi^1_{\mu\mu} = \bigoplus_{d\in\Omega^{\succeq}_{\lambda\mu}} R\psi^d_{\lambda\mu} = Hom^\wedge_{\mathcal{H}}(M^\mu, M^\lambda)$. Moreover, as $S^+_R(n,r) = J_1 \oplus L_{n,r}$, we have $Hom^\wedge_{\mathcal{H}}(M^\mu, M^\lambda) = \psi_\lambda J_1 \psi_\mu$ if $\lambda \rhd \mu$.

In Section 3.1 of [5] there is a complex $\widetilde{C}^\lambda_*$ which is defined as follows, for some $\lambda \in \Lambda^+(n,r)$:

$\widetilde{C}^\lambda_{-1}$ is the co-Specht module that corresponds the partition λ, $\widetilde{C}^\lambda_0 = Hom_R(M^\lambda, R)$. For $k \geq 1$ the $\mathcal{H}$-module $\widetilde{C}^\lambda_k$ is defined as the direct sum over all sequence

$$\gamma = (\mu^{(1)} \rhd \cdots \rhd \mu^{(k)} \rhd \lambda) \tag{6.2}$$

as

$$\bigoplus_{\substack{\mu^{(1)},\dots,\mu^{(k)}\in\Lambda^+(n,r)\\ \mu^{(1)}\rhd\cdots\rhd\mu^{(k)}\rhd\lambda}} Hom_R(M^{\mu^{(1)}},R)\otimes_R Hom^{\wedge}_{\mathcal{H}}(M^{\mu^{(2)}},M^{\mu^{(1)}})\otimes_R\cdots\otimes_R Hom^{\wedge}_{\mathcal{H}}(M^{\lambda},M^{\mu^{(k)}}).$$

The differential d_k, $k\geq 1$, in $\widetilde{C}^{\lambda}_*$ is given by the formula

$$d_k(f_0\otimes f_1\otimes\cdots\otimes f_k)=\sum_{t=0}^{k-1}(-1)^t f_0\otimes\cdots\otimes f_t\circ f_{t+1}\otimes\cdots\otimes f_k. \tag{6.3}$$

and when $k=0$, we put

$$d^{\lambda}_0:\widetilde{C}^{\lambda}_0=Hom_R(M^{\lambda},R)\to Hom_R(S^{\lambda},R)=:\widetilde{C}^{\lambda}_{-1},\qquad \varepsilon\mapsto\varepsilon|_{S^{\lambda}}. \tag{6.4}$$

and finally obtain a chain complex with only finite no trivial terms:

$$\widetilde{C}^{\lambda}_*:\qquad 0\to\widetilde{C}^{\lambda}_{a(\lambda)}\xrightarrow{d^{\lambda}_{a(\lambda)}}\widetilde{C}^{\lambda}_{a(\lambda)-1}\xrightarrow{d^{\lambda}_{a(\lambda)-1}}\cdots\xrightarrow{d^{\lambda}_1}\widetilde{C}^{\lambda}_0\xrightarrow{d^{\lambda}_0}\widetilde{C}^{\lambda}_{-1}\to 0, \tag{6.5}$$

for every integer $n\geq 0$, we write Γ^{λ}_n for the set of chain γ as in 6.2 with $\lambda^{(0)}=\lambda$. Then, here, $a(\lambda)$ is defined as the length of the longest possible strictly ascending chain γ as in 6.2 with $\lambda^{(0)}=\lambda$. In other word, $a(\lambda)$ is the largest integer with $\Gamma^{\lambda}_n\neq\emptyset$.

Theorem 6.2. *For $\lambda\in\Lambda^+(n,r)$, the complex $\widetilde{C}^{\lambda}_*$ is isomorphic to $\mathfrak{S}(S_R(n,r)\otimes_{S^+_R(n,r)} B^+_*(R_\lambda))$.*

Proof. We only need to establish the isomorphism in the non-negative degrees. The isomorphism in the degree -1 will follow, since the complex $\widetilde{C}^{\lambda}_*$ is isomorphic to $\mathfrak{S}(S_R(n,r)\otimes_{S^+_R(n,r)} B^+_*(R_\lambda))$ is exact, and the complex $\widetilde{C}^{\lambda}_*$ is exact in degree 0 and -1 by Theorems **4.2** and **4.4** in [22].

We define the complex $\widehat{C}^{\lambda}_*$ in the same way as the complex $\widetilde{C}^{\lambda}_*$ with the only difference that the summands 6.3 are replaced by

$$Hom_{\mathcal{H}}(M^{\mu^{(1)}},\mathcal{H})\otimes_R Hom^{\wedge}_{\mathcal{H}}(M^{\mu^{(2)}},M^{\mu^{(1)}})\otimes_R\cdots\otimes_R Hom^{\wedge}_{\mathcal{H}}(M^{\lambda},M^{\mu^{(k)}}),$$

which equals to

$$\psi_\delta S_R(n,r)\psi_{\mu^{(1)}}\otimes_R\psi_{\mu^{(1)}}J_1\psi_{\mu^{(2)}}\otimes_R\cdots\otimes_R\psi_{\mu^{(k)}}J_1\psi_\lambda\cong\mathfrak{S}(S_R(n,r)\otimes_{S^+_R(n,r)} B^+_*(R_\lambda)).$$

Then it is straightforward that the isomorphism can induce an isomorphism between the complexes $\mathfrak{S}(S_R(n,r)\otimes_{S^+_R(n,r)}$ and $\widehat{C}^{\lambda}_*$ in non-negative degrees.

To show that the complex $\widetilde{C}^{\lambda}_*$ and $\widehat{C}^{\lambda}_*$ are isomorphic in non-negative degree, it is enough to find for every $\nu\in\Lambda(n,r)$, an isomorphism of $\mathcal{H}$-modules ϕ_ν : $Hom_{\mathcal{H}}(M^{\nu},\mathcal{H})\to Hom_R(M^{\nu},R)$. Note that the action of $\mathcal{H}$ on $Hom_{\mathcal{H}}(M^{\nu},\mathcal{H})$ and $Hom_R(M^{\mu},R)$ is given by the formula $(f\sigma)(m)=f(m\cdot\chi(\sigma))$, with the natural anti-automorphism of $\chi:\mathcal{H}\to\mathcal{H}$ and $f\in Hom_{\mathcal{H}}(M^{\nu},\mathcal{H})$ (respectively $f\in Hom_R(M^{\nu},R)$), $m\in M^{\nu}$, and $\sigma\in\mathcal{H}$.

Let $f\in Hom_{\mathcal{H}}(M^{\nu},\mathcal{H})$, $m\in M^{\nu}$. We define $\phi_\nu(f)(m)$ to be the coefficient of T_{id} (i.e., 1) in $f(m)\in\mathcal{H}$, with the fact $\{T_w|w\in\mathfrak{S}_r\}$ is an R-basis of $\mathcal{H}$. Now we check that ϕ_ν is a homomorphism of $\mathcal{H}$-modules. For

$$(\phi_\nu(f)\cdot T_w)(m) = \phi_\nu(f)(mT_{w^{-1}}) = \text{coefficient of } 1 \text{ in } f(mT_{w^{-1}})$$
$$\phi_\nu(fT_w)(m) = \text{coefficient of } 1 \text{ in } (f\cdot T_w)(m) = \text{coefficient of } 1 \text{ in } f(mT_{w^{-1}}),$$

we have shown that $\phi_\nu(f)\cdot T_w = \phi_\nu(fT_w)$.

Then we assume that $\phi_\nu(f) = 0$, which means the coefficient of 1 in $f(m)$ equals to 0, for any $m \in M^\nu$. Then, by induction on the length of elements of $\mathfrak{S}_r$, we assume that the coefficient of T_w in $f(m)$ equals to 0, for $\ell(w) \leq n$ and any $m \in M^\nu$. We use the formula which appears in Mathas's book [21]. Suppose that $w_0 \in \mathfrak{S}_r$ with $\ell(w_0) = n+1$. We can find $s = (i, i+1) \in \mathfrak{S}_r$ for some $1 \leq i \leq r-1$, such that this following formula satisfied:

$$T_{w_0}T_s = qT_{w_0s} + (q-1)T_{w_0}, \qquad \textit{and } n = \ell(ws) < \ell(w) = n+1.$$

Meanwhile, $f(m)T_s = f(mT_s)$ and suppose $f(m) = \sum\limits_{\ell(w)>n} \alpha_w T_w$ for some $\alpha \in R$. We find $f(mT_s) = \sum\limits_{\ell(w)>n} \alpha_w T_w \cdot T_s = \alpha_{w_0} qT_{w_0s} + \sum\limits_{\substack{\ell(w)\geq n \\ w\neq w_0s}} \alpha'_w T_w$, which implies that $q\alpha_{w_0} = 0$ and the homomorphism is injective.

Finally, as free R-modules, it is obvious that the rank of $Hom_{\mathcal{H}}(M^\nu, \mathcal{H})$ equals to $Hom_R(M^\nu, R)$, which means the homomorphism ϕ_ν is surjective.

So we have proved that $\widetilde{C}_*^\lambda$ and $\widehat{C}_*^\lambda$ are isomorphic.

$\square$

Remark 6.3. With help of the above theorem and Theorem 5.10, we have shown that the Boltje-Maisch complex $\widetilde{C}_*^\lambda$ is a exact for $\lambda \in \Lambda^+(n, r)$. From it, we arrive at the positive answer of the conjecture in [5].

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