PINCHING THEOREMS FOR SELF-SHRINKERS OF HIGHER CODIMENSIONS

SHUNJUAN CAO, HONGWEI XU, AND ENTAO ZHAO

Abstract. In this paper, we investigate the pinching phenomena of the tracefree second fundamental form of complete self-shrinkers of higher codimension. Firstly, assuming the mean curvature is nonzero everywhere and the self-shrinker is of polynomial volume growth, we prove that if the tracefree second fundamental form $\mathring{A}$ satisfies $||\mathring{A}||_n < C(n)$ for a positive constant $C(n)$ depending only on the dimension n of the self-shrinker, then it is isometric to the sphere $\mathbb{S}^n(\sqrt{2n})$. Secondly, we show if the mean curvature vector H of the self-shrinker satisfies $\sup|H| < \sqrt{\frac{n}{2}}$ and $\mathring{A}$ satisfies $||\mathring{A}||_n < D(n, \sup|H|)$ for a positive constant $D(n, \sup|H|)$ depending n and $\sup|H|$, then it is isometric to the Euclidean space $\mathbb{R}^n$. We also obtain some rigidity theorems for self-shrinkers satisfying pointwise curvature pinching conditions on $|\mathring{A}|^2$.

1. Introduction

The mean curvature flow is a one-parameter family of smooth immersions $F : M \times [0, T) \to \mathbb{R}^{n+p}$ satisfying

$$\begin{cases} \frac{\partial}{\partial t}F(x,t) = H(x,t), \\ F(x,0) = F_0(x), \end{cases} \tag{1.1}$$

where $H(x,t)$ is the mean curvature vector of $F_t(M)$ that is defined to be the trace of the second fundamental form A, $F_t(x) = F(x,t)$ and F_0 is some given immersion.

It is useful to investigate an important class of solutions to the mean curvature flow (1.1), called self-shrinkers. An immersion $F : M^n \to \mathbb{R}^{n+p}$ is called a self-shrinker if it satisfies

$$H(x) = -\frac{1}{2}F(x)^{\perp} \tag{1.2}$$

for all $x \in M$. Here $(\ )^{\perp}$ denotes the normal part of a vector field on $\mathbb{R}^{n+p}$. The self-shrinker is a time slice of a self-similar solution of the mean curvature flow, which shrinks as time increases. It is well-known that self-shrinkers play an important role in the study of mean curvature flow for they describe the singularity models of the mean curvature flow and they arise as tangent flows of mean curvature flow at singularities, see [11, 16, 18, 28], etc. If M is a curve in $\mathbb{R}^2$, all solutions of (1.2) have been classified by Abresch-Langer [1]. In higher codimension the theorem of Abresch-Langer applies as well since a self-shrinking curve in $\mathbb{R}^N$ lies in a flat linear two-space $\mathbb{R}^2 \subset \mathbb{R}^N$. In the higher dimension case, it was proved in [16] that a closed hypersurface in $\mathbb{R}^{n+1}$ satisfying (1.2) with positive mean curvature is $\mathbb{S}^n(\sqrt{2n})$. Later this was extended in [17] to complete noncompact hypersurfaces in $\mathbb{R}^{n+1}$ with nonnegative mean curvature, bounded second fundamental form and

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polynomial volume growth. Recently, Colding-Minicozzi [11] showed that Huisken's classification theorem still holds without boundedness of the second fundamental form. Moreover, they showed that the only embedded entropy stable self-shrinkers with polynomial volume growth in $\mathbb{R}^{n+1}$ are hyperplanes, n-spheres, and cylinders.

In higher dimension and higher codimension case, the classification of self-shrinkers is much more complicated. Smoczyk [27] made an extension of the classification theorems in [16, 17] for the self-shrinkers with nowhere vanishing mean curvature and parallel normalized mean curvature vector. Recently, a gap theorem of the squared norm of the second fundamental form for self-shrinkers with polynomial volume growth was obtained by Cao-Li [5], which generalized the gap theorem in [21] to arbitrary codimension.

Theorem 1.1 ([5]). *Let $F : M^n \to \mathbb{R}^{n+p}$ be an n-dimensional complete self-shrinker with polynomial volume growth. If M satisfies $|A|^2 \leqslant \frac{1}{2}$, then M is isometric to one of $\mathbb{S}^k(\sqrt{2k}) \times \mathbb{R}^{n-k} \subset \mathbb{R}^{n+1}$, $0 \leqslant k \leqslant n$.*

More recently, Ding-Xin [14] proved a gap theorem for complete self-shrinkers under an integral curvature pinching condition.

Theorem 1.2 ([14]). *Let $F : M^n \to \mathbb{R}^{n+p}$ $(n \geqslant 3)$ be a complete self-shrinker of the mean curvature flow. If M satisfies $||A||_n < \sqrt{\frac{4}{3nS}}$, then it is isometric to $\mathbb{R}^n$. Here S is a positive constant which appears in the Sobolev inequality on submanifolds in the Euclidean space.*

Without the assumption of polynomial volume growth, Cheng and Peng [9] proved the following rigidity theorem.

Theorem 1.3 ([9]). *Let $F : M^n \to \mathbb{R}^{n+p}$ be an n-dimensional complete self-shrinker. If M satisfies $\sup_M |A|^2 < \frac{1}{2}$, then M is isometric to $\mathbb{R}^n$.*

Some other classification and rigidity theorems for self-shrinkers satisfying certain curvature conditions have been proved in [6, 9, 10, 14, 23, 24], etc.

In this paper, we study the gap phenomena for the tracefree second fundamental form of self-shrinkers. Let $\mathring{A}$ denote the tracefree second fundamental form, which is defined by $\mathring{A} = A - \frac{1}{n} g \otimes H$ with g denoting the induced metric on M. Motivated by Theorem 1.2 and integral pinching theorems in [29], we prove the following

Theorem 1.4. *Let $F : M^n \to \mathbb{R}^{n+p}$ $(n \geqslant 3)$ be a complete self-shrinker of the mean curvature flow with polynomial volume growth. Suppose the mean curvature is nowhere vanishing. If M satisfies $||\mathring{A}||_n < C(n)$, where $C(n)$ is an explicit given positive constant depending only on n, then it is isometric to $\mathbb{S}^n(\sqrt{2n})$.*

Without the assumption of polynomial volume growth, we also prove an integral curvature pinching theorem under the condition that the mean curvature is suitably bounded.

Theorem 1.5. *Let $F : M^n \to \mathbb{R}^{n+p}$ $(n \geqslant 3)$ be a complete self-shrinker of the mean curvature flow. Suppose the mean curvature satisfies $\sup_M |H| < \sqrt{\frac{n}{2}}$. If M satisfies $||\mathring{A}||_n < D(n, \sup_M |H|)$, where $D(n, \sup_M |H|)$ is an explicit given positive constant depending only on n and $\sup_M |H|$, then it is isometric to $\mathbb{R}^n$.*

Comparing with Theorems 1.1 and 1.3, we prove the following

Theorem 1.6. *Let $F : M^n \to \mathbb{R}^{n+p}$ $(n \geqslant 2)$ be a complete self-shrinker of the mean curvature flow. Suppose the mean curvature is nowhere vanishing.*

(1) If M has polynomial volume growth and satisfies $|\mathring{A}|^2 \leqslant \frac{1}{4}$, then it is isometric to one of $\mathbb{S}^k(\sqrt{2k}) \times \mathbb{R}^{n-k}$, $[\frac{n}{2}] \leqslant k \leqslant n$.

(2) If M satisfies $\sup_M |\mathring{A}|^2 < \frac{1}{4}$*, then it is isometric to one of* $\mathbb{S}^k(\sqrt{2k}) \times \mathbb{R}^{n-k}$, $[\frac{n}{2}] < k \leqslant n$.

Note that $\mathbb{S}^k(\sqrt{2k}) \times \mathbb{R}^{n-k}$ satisfies $|\mathring{A}|^2 = \frac{1}{2}(1-\frac{k}{n}) \leqslant \frac{n-1}{2n} < \frac{1}{2}$ for $k > 0$. So it is natural to conjecture that the best pinching constant is $\frac{1}{2}$. We prove the following

Theorem 1.7. *Let $F : M^n \to \mathbb{R}^{n+p}$ $(n \geqslant 2)$ be a complete embedded self-shrinker of the mean curvature flow. Suppose the mean curvature is nowhere vanishing and the normal bundle is flat.*

(1) If M has polynomial volume growth and satisfies $|\mathring{A}|^2 < \frac{1}{2}$, then it is isometric to one of $\mathbb{S}^k(\sqrt{2k}) \times \mathbb{R}^{n-k}$, $1 \leqslant k \leqslant n$.

(2) If M satisfies $\sup_M |\mathring{A}|^2 < \frac{1}{2}$, then it is isometric to one of $\mathbb{S}^k(\sqrt{2k}) \times \mathbb{R}^{n-k}$, $1 \leqslant k \leqslant n$.

We also prove some rigidity results for closed self-shrinkers under pointwise curvature pinching conditions.

Theorem 1.8. *Let $F : M^n \to \mathbb{R}^{n+p}$ $(n \geqslant 2)$ be a closed self-shrinker of the mean curvature flow. Suppose the mean curvature is nowhere vanishing and the normalized mean curvature vector is parallel in the normal bundle.*

(1) If M satisfies

$$|\mathring{A}|^2 \leqslant \frac{1}{3},$$

then M is one of the following:

(i) $\mathbb{S}^n(\sqrt{2n}) \subset \mathbb{R}^{n+1} \subset \mathbb{R}^{n+p}$, $p \geqslant 1$;

(ii) $\mathbb{S}^2(2\sqrt{3}) \to \mathbb{S}^4(2) \subset \mathbb{R}^5 \subset \mathbb{R}^{2+p}$, $p \geqslant 3$, where $\mathbb{S}^2(2\sqrt{3}) \to \mathbb{S}^4(2)$ is the Veronese surface.

(2) If $n = 2$ and M satisfies

$$\frac{1}{3} \leqslant |\mathring{A}|^2 \leqslant \frac{5}{12},$$

then M is one of the following:

(i) $\mathbb{S}^2(2\sqrt{3}) \to \mathbb{S}^4(2) \subset \mathbb{S}^{1+p}(2) \subset \mathbb{R}^{2+p}$, $p \geqslant 3$, where $\mathbb{S}^2(2\sqrt{3}) \to \mathbb{S}^4(2)$ is the Veronese surface;

(ii) $\mathbb{S}^2(2\sqrt{6}) \to \mathbb{S}^6(2) \subset \mathbb{S}^{1+p}(2) \subset \mathbb{R}^{2+p}$, $p \geqslant 5$, where $\mathbb{S}^2(2\sqrt{6}) \to \mathbb{S}^6(2)$ is the standard immersion, see [19].

(3) If $p = 2$ and M satisfies

$$|\mathring{A}|^2 \leqslant \frac{1}{2},$$

then M is one of $\mathbb{S}^k(\sqrt{2k}) \times \mathbb{S}^{n-k}(\sqrt{2(n-k)}) \subset \mathbb{S}^{n+1}(\sqrt{2n}) \subset \mathbb{R}^{n+2}$, $1 \leqslant k \leqslant n$.

(4) If $p = 2$, there is a positive constant $\delta(n)$ depending only on n such that if M satisfies

$$\frac{1}{2} \leqslant |\mathring{A}|^2 \leqslant \frac{1}{2} + \delta(n),$$

then M is one of $\mathbb{S}^k(\sqrt{2k}) \times \mathbb{S}^{n-k}(\sqrt{2(r-k)}) \subset \mathbb{S}^{n+1}(\sqrt{2n}) \subset \mathbb{R}^{n+2}$, $1 \leqslant k \leqslant n-1$.

The paper is organized as follows. In Section 2, we recall some basic equations in submanifold geometry and introduce an elliptic operator for self-shrinkers. In Section 3, we study the gap theorem of self-shrinkers with nowhere vanishing mean curvature under integral or pointwise curvature pinching conditions. In Section 4, we consider the integral pinching theorem for $\mathring{A}$ under the assumption that the mean curvature is suitably bounded. In Section 5, we discuss the gap phenomena of $\mathring{A}$ for closed self-shrinkers with parallel normalized mean curvature vector under pointwise pinching conditions.

2. Preliminaries

Throughout this paper, let M^n be an n-dimensional complete smooth manifold isometrically immersed into an $(n+p)$-dimensional Euclidean space $\mathbb{R}^{n+p}$. Denote by g the induced metric on M. We shall make use of the following convention on the range of indices:

$$1 \leqslant A, B, C, \cdots \leqslant n+p;\ 1 \leqslant i, j, k, \cdots \leqslant n;\ n+1 \leqslant \alpha, \beta, \gamma, \cdots \leqslant n+p.$$

Choose a local field of orthonormal frame $\{e_A\}$ in $\mathbb{R}^{n+p}$ such that, restricted to M, e_i's are tangent to M. Let $\{\omega_A\}$ and $\{\omega_{AB}\}$ be the dual frame field and connection 1-forms of $\mathbb{R}^{n+p}$, respectively. Restricting these forms to M, we have

$$\omega_{\alpha i} = \sum_j h^\alpha_{ij}\omega_j,\ \ h^\alpha_{ij} = h^\alpha_{ji},$$

$$A = \sum_{\alpha,i,j} h^\alpha_{ij}\omega_i \otimes \omega_j \otimes e_\alpha = \sum_{i,j} h_{ij}\omega_i \otimes \omega_j,$$

$$H = \sum_{\alpha,i} h^\alpha_{ii} e_\alpha = \sum_\alpha H^\alpha e_\alpha,$$

$$R_{ijkl} = \sum_\alpha (h^\alpha_{ik}h^\alpha_{jl} - h^\alpha_{il}h^\alpha_{jk}),$$

$$R_{\alpha\beta kl} = \sum_i (h^\alpha_{ik}h^\beta_{il} - h^\alpha_{il}h^\beta_{ik}),$$

where $A, H, R_{ijkl}, R_{\alpha\beta kl}$ are the second fundamental form, the mean curvature vector, the Riemannian curvature tensor, the normal curvature tensor of M, respectively. The tracefree second fundamental form is defined by $\mathring{A} = A - \frac{1}{n} g \otimes H$. Let Δ be the Laplacian of M.

Denoting the first and second covariant derivatives of h^α_{ij} by h^α_{ijk} and h^α_{ijkl} respectively, we have

$$\sum_k h^\alpha_{ijk}\omega_k = dh^\alpha_{ij} - \sum_k h^\alpha_{ik}\omega_{kj} - \sum_k h^\alpha_{kj}\omega_{ki} - \sum_\beta h^\beta_{ij}\omega_{\beta\alpha}, \tag{2.1}$$

$$\sum_l h^\alpha_{ijkl}\omega_l = dh^\alpha_{ijk} - \sum_l h^\alpha_{ijl}\omega_{lk} - \sum_l h^\alpha_{ilk}\omega_{lj} - \sum_l h^\alpha_{ljk}\omega_{li} - \sum_\beta h^\beta_{ijk}\omega_{\beta\alpha}.$$

Then we have

$$h^\alpha_{ijk} = h^\alpha_{ikj},$$

$$h^\alpha_{ijkl} - h^\alpha_{ijlk} = \sum_m h^\alpha_{im}R_{mjkl} + \sum_m h^\alpha_{mj}R_{mikl} - \sum_\beta h^\beta_{ij}R_{\alpha\beta kl}.$$

Hence

$$\begin{aligned}\Delta h^\alpha_{ij} =& \sum_k h^\alpha_{ijkk} \\ =& \sum_k h^\alpha_{kkij} + \sum_k \Big(\sum_m h^\alpha_{km}R_{mijk} + \sum_m h^\alpha_{mi}R_{mkjk} - \sum_\beta h^\beta_{ki}R_{\alpha\beta jk}\Big).\end{aligned} \tag{2.2}$$

From the self-shrinker equation (1.2) we obtain

$$\nabla_i H^\alpha = \frac{1}{2}\sum_k \langle F, e_k\rangle\, h^\alpha_{ik}, \tag{2.3}$$

and

$$\nabla_j\nabla_i H^\alpha = \frac{1}{2}h^\alpha_{ij} - \sum_k \langle H, h_{jk}\rangle h^\alpha_{ik} + \frac{1}{2}\sum_k \langle F, e_k\rangle h^\alpha_{kij}. \tag{2.4}$$

Let div and $d\mu$ be the divergence and volume form on M, respectively. Colding-Minicozzi [11] introduced a linear operator

$$\mathcal{L} = \Delta - \frac{1}{2}\langle F, \nabla(\cdot)\rangle = e^{\frac{|F|^2}{4}}\mathrm{div}(e^{-\frac{|F|^2}{4}}\nabla(\cdot))$$

for Euclidean submanifolds. Here F is considered as a vector in $\mathbb{R}^{n+p}$. They showed that $\mathcal{L}$ is self-adjoint with respect to the measure $e^{-\frac{|F|^2}{4}}d\mu$.

3. Self-shrinkers with nowhere vanishing mean curvature

Suppose the mean curvature of the self-shrinker M is nowhere vanishing. We choose $e_{n+1} = \frac{H}{|H|}$. The second fundamental form can be written as $A = \sum\limits_{\alpha} h^\alpha e_\alpha$, where h^α, $n+1 \leqslant \alpha \leqslant n+p$, are symmetric 2-tensors. By the choice of e_{n+1}, we see that $\mathrm{tr}h^{n+1} = |H|$ and $\mathrm{tr}h^\alpha = 0$ for $\alpha \geqslant n+2$. The tracefree second fundamental form may be rewritten as $\mathring{A} = \sum\limits_{\alpha} \mathring{h}^\alpha e_\alpha$, where $\mathring{h}^{n+1} = h^{n+1} - \frac{|H|}{n}\mathrm{Id}$ and $\mathring{h}^\alpha = h^\alpha$ for $\alpha \geqslant n+2$. We set $A_H = h^{n+1}e_{n+1}$, $A_I = \sum\limits_{\alpha\neq n+1} h^\alpha e_\alpha$, $\mathring{A}_H = \mathring{h}^{n+1}e_{n+1}$ and $\mathring{A}_I = \sum\limits_{\alpha\neq n+2} \mathring{h}^\alpha e_\alpha$. Then we have

$$|A_I|^2 = \sum_{\alpha\neq n+1} |h^\alpha|^2 = |A|^2 - |A_H|^2,$$
$$|\mathring{A}_I|^2 = \sum_{\alpha\neq n+1} |\mathring{h}^\alpha|^2 = |\mathring{A}|^2 - |\mathring{A}_H|^2.$$

Note that $|\mathring{A}_H|^2 = |A_H|^2 - \frac{|H|^2}{n}$ and $|\mathring{A}_I|^2 = |A_I|^2$. Since e_{n+1} is chosen globally, $|A_H|^2$, $|\mathring{A}_H|^2$ and $|A_I|^2$ are defined globally and independent of the choice of $\{e_i\}$.

To prove Theorem 1.4, we need to give an estimate of $\mathcal{L}|A_I|^2$. Combining (2.2) and (2.4), we obtain

(3.1)
$$\begin{aligned}\sum_{\alpha\neq n+1}\sum_{i,j} h^\alpha_{ij}\Delta h^\alpha_{ij} =& \sum_{\alpha\neq n+1}\sum_{i,j} h^\alpha_{ij}\nabla_j\nabla_i H^\alpha\\ &+ \sum_{\alpha\neq n+1}\sum_{i,j,k} h^\alpha_{ij}\Big(\sum_m h^\alpha_{km}R_{mijk} + \sum_m h^\alpha_{mi}R_{mkjk} - \sum_\beta h^\beta_{ki}R_{\alpha\beta jk}\Big)\\ =&\frac{1}{2}|A_I|^2 - |H|\sum_{\alpha\neq n+1}\sum_{i,j,k} h^{n+1}_{jk}h^\alpha_{ik}h^\alpha_{ij} + \frac{1}{4}\langle F, \nabla|A_I|^2\rangle\\ &+ \sum_{\alpha\neq n+1}\sum_{i,j,k} h^\alpha_{ij}\Big(\sum_m h^\alpha_{km}R_{mijk} + \sum_m h^\alpha_{mi}R_{mkjk} - \sum_\beta h^\beta_{ki}R_{\alpha\beta jk}\Big).\end{aligned}$$

From (3.1) and the definition of $\mathcal{L}$, we get

(3.2)
$$\begin{aligned}\mathcal{L}|A_I|^2 =&\Delta|A_I|^2 - \frac{1}{2}\langle F, \nabla|A_I|^2\rangle\\ =&2\sum_{i,j,\alpha\neq n+1} h^\alpha_{ij}\Delta h^\alpha_{ij} + 2|\nabla A_I|^2 - \frac{1}{2}\langle F, \nabla|A_I|^2\rangle\\ =&2|\nabla A_I|^2 + |A_I|^2 - 2|H|\sum_{\alpha\neq n+1}\mathrm{tr}[A^{n+1}(A^\alpha)^2]\\ &+ 2\sum_{\alpha\neq n+1}\sum_{i,j,k} h^\alpha_{ij}\Big(\sum_m h^\alpha_{km}R_{mijk} + \sum_m h^\alpha_{mi}R_{mkjk} - \sum_\beta h^\beta_{ki}R_{\alpha\beta jk}\Big).\end{aligned}$$

Set $A^\alpha=(h^\alpha_{ij})_{n\times n}$ and define $N(B)=\mathrm{tr}(B^tB)$ for a matrix B. Then by direct computation, we have

$$\begin{aligned}\sum_{\alpha\neq n+1}\sum_{i,j,k,m}h^\alpha_{ij}h^\alpha_{km}R_{mijk}=&\sum_{\alpha\neq n+1}\mathrm{tr}(A^{n+1}A^\alpha)^2-\sum_{\alpha\neq n+1}[\mathrm{tr}(A^{n+1}A^\alpha)]^2\\&+\sum_{\alpha,\beta\neq n+1}\mathrm{tr}(A^\alpha A^\beta)^2-\sum_{\alpha,\beta\neq n+1}[\mathrm{tr}(A^\alpha A^\beta)]^2,\end{aligned}\tag{3.3}$$

$$\begin{aligned}\sum_{\alpha\neq n+1}\sum_{i,j,k,m}h^\alpha_{ij}h^\alpha_{mi}R_{mkjk}=&|H|\sum_{\alpha\neq n+1}\mathrm{tr}[A^{n+1}(A^\alpha)^2]\\&-\sum_{\alpha\neq n+1}\mathrm{tr}[(A^{n+1})^2(A^\alpha)^2]-\sum_{\alpha,\beta\neq n+1}\mathrm{tr}(A^\alpha A^\beta A^\beta A^\alpha),\end{aligned}\tag{3.4}$$

$$\begin{aligned}\sum_{\alpha\neq n+1,\beta}\sum_{i,j,k}h^\alpha_{ij}h^\beta_{ki}R_{\beta\alpha jk}=&\sum_{\alpha,\beta\neq n+1}\mathrm{tr}(A^\alpha A^\beta)^2-\sum_{\alpha,\beta\neq n+1}\mathrm{tr}(A^\alpha A^\beta A^\beta A^\alpha)\\&+\sum_{\alpha\neq n+1}\mathrm{tr}(A^\alpha A^{n+1})^2-\sum_{\alpha\neq n+1}\mathrm{tr}(A^\alpha A^{n+1}A^{n+1}A^\alpha).\end{aligned}\tag{3.5}$$

Combining (3.2)-(3.5), we get

$$\begin{aligned}\mathcal{L}|A_I|^2=&2|\nabla A_I|^2+|A_I|^2-2\sum_{\alpha\neq n+1}[\mathrm{tr}(A^{n+1}A^\alpha)]^2\\&-2\sum_{\alpha,\beta\neq n+1}N(A^\alpha A^\beta-A^\beta A^\alpha)-2\sum_{\alpha,\beta\neq n+1}[\mathrm{tr}(A^\alpha A^\beta)]^2\\&+2\sum_{\alpha\neq n+1}\mathrm{tr}(A^{n+1}A^\alpha)^2-2\sum_{\alpha\neq n+1}\mathrm{tr}[(A^{n+1})^2(A^\alpha)^2]\\&+2\sum_{\alpha\neq n+1}\mathrm{tr}(A^\alpha A^{n+1})^2-2\sum_{\alpha\neq n+1}\mathrm{tr}(A^\alpha A^{n+1}A^{n+1}A^\alpha).\end{aligned}\tag{3.6}$$

By Theorem 1 in [22], we have

$$-2\sum_{\alpha,\beta\neq n+1}N(A^\alpha A^\beta-A^\beta A^\alpha)-2\sum_{\alpha,\beta\neq n+1}[\mathrm{tr}(A^\alpha A^\beta)]^2\geqslant-3|A_I|^4.\tag{3.7}$$

We also have

$$\begin{aligned}&-2\sum_{\alpha\neq n+1}[\mathrm{tr}(A^{n+1}A^\alpha)]^2+2\sum_{\alpha\neq n+1}\mathrm{tr}(A^{n+1}A^\alpha)^2-2\sum_{\alpha\neq n+1}\mathrm{tr}[(A^{n+1})^2(A^\alpha)^2]\\=&-2\sum_{\alpha\neq n+1}[\mathrm{tr}(\mathring{A}^{n+1}A^\alpha)]^2+2\sum_{\alpha\neq n+1}\mathrm{tr}(\mathring{A}^{n+1}A^\alpha)^2-2\sum_{\alpha\neq n+1}\mathrm{tr}[(\mathring{A}^{n+1})^2(A^\alpha)^2]\\\geqslant&-2|\mathring{A}_H|^2|A_I|^2.\end{aligned}\tag{3.8}$$

Here $\mathring{A}^{n+1}=(\mathring{h}^{n+1}_{ij})_{n\times n}$ and for the inequality we have used Lemma 3.2 in [8].

We choose $\{e_i\}$ such that $h_{ij}^{n+1}=\lambda_i\delta_{ij}$. Then $\mathring{h}_{ij}^{n+1}=\mathring{\lambda}_i\delta_{ij}$, where $\mathring{\lambda}_i=\lambda_i-\frac{|H|}{n}$. We have the following estimate.

$$
\begin{aligned}
&2\sum_{\alpha\neq n+1}\mathrm{tr}(A^\alpha A^{n+1})^2-2\sum_{\alpha\neq n+1}\mathrm{tr}(A^\alpha A^{n+1}A^{n+1}A^\alpha)\\
=&2\sum_{\alpha\neq n+1}\sum_{i,j,k,l}h_{ij}^\alpha h_{jk}^{n+1}h_{kl}^\alpha h_{li}^{n+1}-2\sum_{\alpha\neq n+1}\sum_{i,j,k,l}h_{ij}^\alpha h_{jk}^{n+1}h_{kl}^{n+1}h_{li}^\alpha\\
=&2\sum_{\alpha\neq n+1}\sum_{i,j}\lambda_i\lambda_j(h_{ij}^\alpha)^2-2\sum_{\alpha\neq n+1}\sum_{i,j}\lambda_i^2(h_{ij}^\alpha)^2\\
=&-\sum_{\alpha\neq n+1}\sum_{i\neq j}\big(\lambda_i-\lambda_j\big)^2(h_{ij}^\alpha)^2\\
=&-\sum_{\alpha\neq n+1}\sum_{i\neq j}\big(\mathring{\lambda}_i-\mathring{\lambda}_j\big)^2(h_{ij}^\alpha)^2\\
\geqslant&-2\Big(\sum_i\mathring{\lambda}_i^2\Big)\Big(\sum_{\alpha\neq n+1}\sum_{i\neq j}(h_{ij}^\alpha)^2\Big)\\
=&-2\Big(\sum_i\mathring{\lambda}_i^2\Big)\Big(|\mathring{A}_I|^2-\sum_{\alpha\neq n+1}\sum_i(\mathring{h}_{ii}^\alpha)^2\Big)\\
\geqslant&-2|\mathring{A}_H|^2|A_I|^2. \qquad (3.9)
\end{aligned}
$$

From (3.6)-(3.9), we obtain

$$\mathcal{L}|A_I|^2\geqslant 2|\nabla A_I|^2+|A_I|^2(1-4|\mathring{A}_H|^2-3|A_I|^2). \tag{3.10}$$

We need the following Sobolev inequality for submanifolds in the Euclidean space, which is a consequence of the classical Sobolev inequality due to Michael-Simon [26].

Lemma 3.1 ([29]). *Let M^n $(n\geqslant 3)$ be a complete submanifold in the Euclidean space $\mathbb{R}^{n+p}$. Let f be a nonnegative C^1 function with compact support. Then we have*

$$||\nabla f||_2^2\geqslant\frac{(n-2)^2}{4(n-1)^2(1+t)}\left[\frac{1}{D^2(n)}||f||^2_{\frac{2n}{n-2}}-\left(1+\frac{1}{t}\right)\frac{1}{n^2}|||H|f||_2^2\right],$$

where $D(n)=2^n(1+n)^{\frac{n+1}{n}}(n-1)^{-1}\sigma_n^{-\frac{1}{n}}$ and σ_n denotes the volume of the unit ball in $\mathbb{R}^n$.

Now we give the proof of Theorem 1.4.

Proof. From (3.10), we have

$$\mathcal{L}|A_I|^2\geqslant 2|\nabla A_I|^2+|A_I|^2(1-4|\mathring{A}|^2). \tag{3.11}$$

Set $f_\varepsilon=[|A_I|^2+n(p-1)\varepsilon^2]^{\frac{1}{2}}$ for a constant $\varepsilon>0$. Since A_I is a Codazzi tensor, we have

$$|\nabla A_I|^2\geqslant\frac{n+2}{n}|\nabla f_\varepsilon|^2.$$

From (3.11) we have

$$\mathcal{L}f_\varepsilon^2\geqslant\frac{2(n+2)}{n}|\nabla f_\varepsilon|^2+|A_I|^2(1-4|\mathring{A}|^2). \tag{3.12}$$

For a fixed point $x_0\in M$ and every $r>0$, define a smooth cut-off function ϕ_r by

$$\phi_r(x)=\begin{cases}1 & x\in B_r(x_0),\\ \phi_r(x)\in[0,1]\text{ and }|\nabla\phi_r|\leqslant\frac{2}{r} & x\in B_{2r}(x_0)\setminus B_r(x_0),\\ 0 & x\in M\setminus B_{2r}(x_0).\end{cases}$$

Multiplying both sides of (3.12) by $\phi_r^2 f_\varepsilon^{n-2}$ and integrating by parts with respect to the measure $e^{-\frac{|F|^2}{4}}d\mu$ on M give

$$
\begin{aligned}
0 \geqslant & \int_M \frac{2(n+2)}{n}|\nabla f_\varepsilon|^2\phi_r^2 f_\varepsilon^{n-2}e^{-\frac{|F|^2}{4}}d\mu + \int_M |A_I|^2\phi_r^2 f_\varepsilon^{n-2}e^{-\frac{|F|^2}{4}}d\mu \\
& - 4\int_M |A_I|^2|\mathring{A}|^2\phi_r^2 f_\varepsilon^{n-2}e^{-\frac{|F|^2}{4}}d\mu - \int_M \mathcal{L}f_\varepsilon^2\phi_r^2 f_\varepsilon^{n-2}e^{-\frac{|F|^2}{4}}d\mu \\
= & \int_M \frac{2(n+2)}{n}|\nabla f_\varepsilon|^2\phi_r^2 f_\varepsilon^{n-2}e^{-\frac{|F|^2}{4}}d\mu + \int_M |A_I|^2\phi_r^2 f_\varepsilon^{n-2}e^{-\frac{|F|^2}{4}}d\mu \\
& - 4\int_M |A_I|^2|\mathring{A}|^2\phi_r^2 f_\varepsilon^{n-2}e^{-\frac{|F|^2}{4}}d\mu + \int_M \langle\nabla f_\varepsilon^2, \nabla(\phi_r^2 f_\varepsilon^{n-2})\rangle e^{-\frac{|F|^2}{4}}d\mu \\
= & \frac{2(n^2-n+2)}{n}\int_M |\nabla f_\varepsilon|^2\phi_r^2 f_\varepsilon^{n-2}e^{-\frac{|F|^2}{4}}d\mu + \int_M |A_I|^2\phi_r^2 f_\varepsilon^{n-2}e^{-\frac{|F|^2}{4}}d\mu \\
& - 4\int_M |A_I|^2|\mathring{A}|^2\phi_r^2 f_\varepsilon^{n-2}e^{-\frac{|F|^2}{4}}d\mu + 4\int_M \phi_r f_\varepsilon^{n-1}\langle\nabla f_\varepsilon, \nabla\phi_r\rangle e^{-\frac{|F|^2}{4}}d\mu \\
\geqslant & \left(\frac{2(n^2-n+2)}{n} - \frac{(4-\sigma)\rho}{2}\right)\int_M |\nabla f_\varepsilon|^2\phi_r^2 f_\varepsilon^{n-2}e^{-\frac{|F|^2}{4}}d\mu \\
& + \int_M |A_I|^2\phi_r^2 f_\varepsilon^{n-2}e^{-\frac{|F|^2}{4}}d\mu - 4\int_M |A_I|^2|\mathring{A}|^2\phi_r^2 f_\varepsilon^{n-2}e^{-\frac{|F|^2}{4}}d\mu \\
& + \sigma\int_M \phi_r f_\varepsilon^{n-1}\langle\nabla f_\varepsilon, \nabla\phi_r\rangle e^{-\frac{|F|^2}{4}}d\mu - \frac{4-\sigma}{2\rho}\int_M f_\varepsilon^n|\nabla\phi_r|^2 e^{-\frac{|F|^2}{4}}d\mu.
\end{aligned}
\tag{3.13}
$$

Here $\sigma, \rho \in \mathbb{R}^+$.

By a direct computation, we have

$$
|\nabla(\phi_r f_\varepsilon^{\frac{n}{2}})|^2 = f_\varepsilon^n|\nabla\phi_r|^2 + n\phi_r f_\varepsilon^{n-1}\langle\nabla\phi_r, \nabla f_\varepsilon\rangle + \frac{n^2}{4}\phi_r^2 f_\varepsilon^{n-2}|\nabla f_\varepsilon|^2. \tag{3.14}
$$

Pick $\sigma, \rho > 0$ such that $\frac{2(n^2-n+2)}{n} - \frac{(4-\sigma)\rho}{2} = \frac{n\sigma}{4}$. Then we have

$$
\begin{aligned}
0 \geqslant & \frac{n\sigma}{4}\left(\frac{4}{n^2}\int_M |\nabla(\phi_r f_\varepsilon^{\frac{n}{2}})|^2 e^{-\frac{|F|^2}{4}}d\mu - \frac{4}{n}\int_M \phi_r f_\varepsilon^{n-1}\langle\nabla f_\varepsilon, \nabla\phi_r\rangle e^{-\frac{|F|^2}{4}}d\mu \right.\\
& \left. - \frac{4}{n^2}\int_M f_\varepsilon^n|\nabla\phi_r|^2 e^{-\frac{|F|^2}{4}}\right) \\
& + \int_M |A_I|^2\phi_r^2 f_\varepsilon^{n-2}e^{-\frac{|F|^2}{4}}d\mu - 4\int_M |A_I|^2|\mathring{A}|^2\phi_r^2 f_\varepsilon^{n-2}e^{-\frac{|F|^2}{4}}d\mu \\
& + \sigma\int_M \phi_r f_\varepsilon^{n-1}\langle\nabla f_\varepsilon, \nabla\phi_r\rangle e^{-\frac{|F|^2}{4}}d\mu - \frac{4-\sigma}{2\rho}\int_M f_\varepsilon^n|\nabla\phi_r|^2 e^{-\frac{|F|^2}{4}}d\mu \\
= & \frac{\sigma}{n}\int_M |\nabla(f_\varepsilon^{\frac{n}{2}}\phi_r)|^2 e^{-\frac{|F|^2}{4}}d\mu - \left(\frac{4-\sigma}{2\rho} + \frac{\sigma}{n}\right)\int_M f_\varepsilon^n|\nabla\phi_r|^2 e^{-\frac{|F|^2}{4}}d\mu \\
& + \int_M |A_I|^2\phi_r^2 f_\varepsilon^{n-2}e^{-\frac{|F|^2}{4}}d\mu - 4\int_M |A_I|^2|\mathring{A}|^2\phi_r^2 f_\varepsilon^{n-2}e^{-\frac{|F|^2}{4}}d\mu.
\end{aligned}
\tag{3.15}
$$

Similar as in [14], we have

(3.16)

$$\begin{aligned}\int_M |\nabla(\phi_r f_\varepsilon^{\frac{n}{2}} e^{-\frac{|F|^2}{8}})|^2 d\mu =& \int_M |\nabla(f_\varepsilon^{\frac{n}{2}}\phi_r)|^2 e^{-\frac{|F|^2}{4}} d\mu - \frac{1}{8}\int_M \phi_r^2 f_\varepsilon^n |F^N|^2 e^{-\frac{|F|^2}{4}} d\mu \\ &+ \frac{n}{4}\int_M \phi_r^2 f_\varepsilon^n e^{-\frac{|F|^2}{4}} d\mu - \frac{1}{16}\int_M \phi_r^2 f_\varepsilon^n |F^T|^2 e^{-\frac{|F|^2}{4}} d\mu \\ \leqslant& \int_M |\nabla(f_\varepsilon^{\frac{n}{2}}\phi_r)|^2 e^{-\frac{|F|^2}{4}} d\mu - \frac{1}{2}\int_M \phi_r^2 f_\varepsilon^n |H|^2 e^{-\frac{|F|^2}{4}} d\mu \\ &+ \frac{n}{4}\int_M \phi_r^2 f_\varepsilon^n e^{-\frac{|F|^2}{4}} d\mu.\end{aligned}$$

Then we have

$$\begin{aligned}0 \geqslant& \frac{\sigma}{n}\int_M |\nabla(\phi_r f_\varepsilon^{\frac{n}{2}} e^{-\frac{|F|^2}{8}})|^2 d\mu + \frac{\sigma}{2n}\int_M \phi_r^2 f_\varepsilon^n |H|^2 e^{-\frac{|F|^2}{4}} d\mu \\ &- \frac{\sigma}{4}\int_M \phi_r^2 f_\varepsilon^n e^{-\frac{|F|^2}{4}} d\mu - \left(\frac{4-\sigma}{2\rho} + \frac{\sigma}{n}\right)\int_M f_\varepsilon^n |\nabla\phi_r|^2 e^{-\frac{|F|^2}{4}} d\mu \\ &+ \int_M |A_I|^2 \phi_r^2 f_\varepsilon^{n-2} e^{-\frac{|F|^2}{4}} d\mu - 4\int_M |A_I|^2 |\mathring{A}|^2 \phi_r^2 f_\varepsilon^{n-2} e^{-\frac{|F|^2}{4}} d\mu \\ \geqslant& \frac{(n-2)^2\sigma}{4n(n-1)^2 D^2(n)(1+t)} \left|\left|\phi_r^2 f_\varepsilon^n e^{-\frac{|F|^2}{4}}\right|\right|_{\frac{n}{n-2}} \\ &+ \left(\frac{\sigma}{2n} - \frac{(n-2)^2\sigma}{4n^3(n-1)^2 t}\right)\int_M \phi_r^2 f_\varepsilon^n |H|^2 e^{-\frac{|F|^2}{4}} d\mu \\ &- \frac{\sigma}{4}\int_M \phi_r^2 f_\varepsilon^n e^{-\frac{|F|^2}{4}} d\mu - \left(\frac{4-\sigma}{2\rho} + \frac{\sigma}{n}\right)\int_M f_\varepsilon^n |\nabla\phi_r|^2 e^{-\frac{|F|^2}{4}} d\mu \\ &+ \int_M |A_I|^2 \phi_r^2 f_\varepsilon^{n-2} e^{-\frac{|F|^2}{4}} d\mu - 4\int_M |A_I|^2 |\mathring{A}|^2 \phi_r^2 f_\varepsilon^{n-2} e^{-\frac{|F|^2}{4}} d\mu.\end{aligned} \tag{3.17}$$

Letting $\varepsilon \to 0$, we obtain

$$\begin{aligned}0 \geqslant& \frac{(n-2)^2\sigma}{4n(n-1)^2 D^2(n)(1+t)} \left|\left|\phi_r^2 |A_I|^n e^{-\frac{|F|^2}{4}}\right|\right|_{\frac{n}{n-2}} \\ &+ \left(\frac{\sigma}{2n} - \frac{(n-2)^2\sigma}{4n^3(n-1)^2 t}\right)\int_M \phi_r^2 |A_I|^n |H|^2 e^{-\frac{|F|^2}{4}} d\mu \\ &- \frac{\sigma}{4}\int_M \phi_r^2 |A_I|^n e^{-\frac{|F|^2}{4}} d\mu - \left(\frac{4-\sigma}{2\rho} + \frac{\sigma}{n}\right)\int_M |A_I|^n |\nabla\phi_r|^2 e^{-\frac{|F|^2}{4}} d\mu \\ &+ \int_M |A_I|^n \phi_r^2 e^{-\frac{|F|^2}{4}} d\mu - 4\int_M |A_I|^n |\mathring{A}|^2 \phi_r^2 e^{-\frac{|F|^2}{4}} d\mu.\end{aligned} \tag{3.18}$$

Now we set $\sigma = 4 - \theta$ for small $\theta > 0$, $\rho = \rho_\theta = \frac{2}{\theta}\left(\frac{2(n^2-n+2)}{n} - \frac{n(4-\theta)}{4}\right)$ and $t = \frac{(n-2)^2}{2n^2(n-1)^2}$. Then we have

(3.19)

$$\begin{aligned}0 \geqslant& \frac{(n-2)^2(4-\theta)}{2D^2(n)[2n^2(n-1)^2 + (n-2)^2]} \left|\left|\phi_r^2 |A_I|^n e^{-\frac{|F|^2}{4}}\right|\right|_{\frac{n}{n-2}} \\ &- \left(\frac{\theta}{2\rho_\theta} + \frac{4-\theta}{n}\right)\int_M |A_I|^n |\nabla\phi_r|^2 e^{-\frac{|F|^2}{4}} d\mu - 4\int_M |A_I|^n |\mathring{A}|^2 \phi_r^2 e^{-\frac{|F|^2}{4}} d\mu.\end{aligned}$$

By the Cauchy inequality, we have

$$\int_M |A_I|^n |\mathring{A}|^2 \phi_r^2 e^{-\frac{|F|^2}{4}} d\mu \leqslant \left|\left|\phi_r^2 |A_I|^n e^{-\frac{|F|^2}{4}}\right|\right|_{\frac{n}{n-2}} \cdot \left|\left||\mathring{A}|^2\right|\right|_{\frac{n}{2}}. \tag{3.20}$$

Hence

$$
\begin{aligned}
0 \geqslant & \left(\frac{(n-2)^2(4-\theta)}{2D^2(n)[2n^2(n-1)^2+(n-2)^2]} - 4\|\mathring{A}\|_n^2\right)\left\|\phi_r^2|A_I|^n e^{-\frac{|F|^2}{4}}\right\|_{\frac{n}{n-2}} \\
& - \left(\frac{\theta}{2\rho_\theta} + \frac{4-\theta}{n}\right)\int_M |A_I|^n|\nabla\phi_r|^2 e^{-\frac{|F|^2}{4}}d\mu.
\end{aligned} \tag{3.21}
$$

Since $|\nabla\phi_r| \leqslant \frac{2}{r}$ and $\int_M |A_I|^n d\mu \leqslant \int_M |\mathring{A}|^n d\mu < \infty$, we have

$$\lim_{r\to\infty}\int_M |A_I|^n|\nabla\phi_r|^2 e^{-\frac{|F|^2}{4}}d\mu = 0.$$

Hence

$$0 \geqslant \left(\frac{(n-2)^2(4-\theta)}{2D^2(n)[2n^2(n-1)^2+(n-2)^2]} - 4\|\mathring{A}\|_n^2\right)\lim_{r\to\infty}\left\|\phi_r^2|A_I|^n e^{-\frac{|F|^2}{4}}\right\|_{\frac{n}{n-2}}.$$

Now we let $\theta \to 0$. If $\|\mathring{A}\|_n < C(n)$, where $C^2(n) = \frac{(n-2)^2}{2D^2(n)[2n^2(n-1)^2+(n-2)^2]}$, then $\lim_{r\to\infty}\left\||A_I|^n e^{-\frac{|F|^2}{4}}\right\|_{\frac{n}{n-2}} = 0$, which implies that $A_I = 0$ and $\nabla A_I = 0$. From (2.1), for $\alpha \neq n+1$, we have

$$0 = \sum_{i,k} h^{\alpha}_{iik}\omega_k = -|H|\omega_{n+1\alpha}.$$

Since H is nowhere vanishing, we obtain $\omega_{n+1\alpha} = 0$. Also, since $\nabla^{\perp}e_{n+1} = \sum_\alpha \omega_{n+1\alpha}\otimes\omega_\alpha = 0$, e_{n+1} is parallel in the normal bundle. The first normal space $N_1(x)$ at $x \in M$, which is defined to be the orthogonal complement of $\{\xi \in N_xM | W_\xi = 0\}$ in N_xM with W_ξ denoting the shape operator with respect to ξ, is just $\{\lambda e_{n+1} | \lambda \in \mathbb{R}\}$, hence it is invariant under parallel translation in the normal bundle. By the codimension reduction theorem in [15], M in fact lies in an $(n+1)$-dimensional affine subspace of $\mathbb{R}^{n+p}$. Then from [11, 16, 17] we see that M is isometric to $\mathbb{S}^n(\sqrt{2n})$, $\mathbb{R}^n$, the product $\mathbb{S}^k(\sqrt{2k})\times\mathbb{R}^{n-k}$ with $1 \leqslant k \leqslant n-1$, or the product $\Gamma\times\mathbb{R}^{n-1}$, where Γ denotes one of the Abresch-Langer curves. Since we assume that the mean curvature is nonzero and $\|\mathring{A}\|_n < \infty$, the latter three cases are excluded. This completes the proof of the theorem. □

By using equation (3.11), we give the proof of Theorem 1.6.

Proof. To prove (1), since M has polynomial volume growth, by (3.11) and the assumption $|\mathring{A}|^2 \leqslant \frac{1}{4}$, we have $\nabla A_I = 0$ and $|A_I| = $ constant. By (3.10), we get

$$0 \geqslant |A_I|^2(1-4|\mathring{A}|^2) + |A_I|^4 \geqslant 0.$$

This implies $|A_I| = 0$. So, as in the proof of Theorem 1.4, M is isometric to $\mathbb{S}^k(\sqrt{2k})\times\mathbb{R}^{n-k}$ with $1 \leqslant k \leqslant n$ or the product $\Gamma\times\mathbb{R}^{n-1}$ with Γ denoting one of the Abresch-Langer curves. For the first case, by a direct computation, we have $|\mathring{A}|^2 = \frac{1}{2}\left(1-\frac{k}{n}\right)$. Since $|\mathring{A}|^2 \leqslant \frac{1}{4}$, $k \geqslant \left[\frac{n}{2}\right]$. For the second case, by the arguments in Proposition 3.4.1 and Appendix E in [25], except the circle $\mathbb{S}^1(\sqrt{2}) \subset \mathbb{R}^2$, the curvature κ of $\Gamma \subset \mathbb{R}^2$ satisfies $\kappa_{\max} > \frac{\sqrt{2}}{2}$. So either $\Gamma = \mathbb{S}^1(\sqrt{2})$, or there is point in $\Gamma\times\mathbb{R}^{n-1} \subset \mathbb{R}^{n+1}$ such that only one of the principal curvatures at this point is nonzero and it is grater than $\frac{\sqrt{2}}{2}$. Hence either $n = 2$ and $M = \mathbb{S}^1(\sqrt{2})\times\mathbb{R}$, or $\sup|\mathring{A}|^2 > \frac{1}{2}\left(1-\frac{1}{n}\right) \geqslant \frac{1}{4}$, and the latter sub-case is excluded. This completes the proof of (1).

We now use a generalized maximum principle for the $\mathcal{L}$ operator on self-shrinkers that was proved in [9] to prove (2). By an inequality in [7, 30], the sectional curvature K_M of M satisfies

$$K_M \geqslant \frac{1}{2}\bigg(\frac{|H|^2}{n-1} - |A|^2\bigg) = \frac{1}{2}\bigg(\frac{|H|^2}{n(n-1)} - |\mathring{A}|^2\bigg) > -\frac{1}{8}.$$

We also have $|A_I|^2 \leqslant |\mathring{A}|^2 < \frac{1}{4}$. Hence the generalized maximum principle for the $\mathcal{L}$ operator on self-shrinkers applies. So, there exists a sequence of points $\{x_k\}_{k=1}^{\infty}$ in M such that

$$\lim_{k\to\infty} |A_I|^2(x_k) = \sup_M |A_I|^2, \ \limsup_{k\to\infty} \mathcal{L}|A_I|^2(x_k) \leqslant 0.$$

Then by (3.11), we have

$$\begin{aligned} 0 \geqslant& \limsup_{k\to\infty} \mathcal{L}|A_I|^2(x_k) \\ \geqslant& \lim_{k\to\infty} |A_I|^2(x_k)(1 - 4\lim_{k\to\infty} |\mathring{A}|^2(x_k)) \\ \geqslant& \sup_M |A_I|^2(1 - 4\sup_M |\mathring{A}|^2). \end{aligned}$$

Under the assumption $\sup_M |\mathring{A}|^2 < \frac{1}{4}$, we obtain $|A_I|^2 = 0$. Hence as in the proof of (1), M is isometric to one of $\mathbb{S}^k(\sqrt{2k}) \times \mathbb{R}^{n-k}$, $[\frac{n}{2}] < k \leqslant n$. □

We use similar argument to prove Theorem 1.7.

Proof. Since the normal bundle if flat, we have $A^\alpha A^\beta = A^\beta A^\alpha$ for all α, β. To estimate the right hand side of (3.6), we have

$$\begin{aligned} &-2\sum_{\alpha,\beta\neq n+1} N(A^\alpha A^\beta - A^\beta A^\alpha) - 2\sum_{\alpha,\beta\neq n+1} [\mathrm{tr}(A^\alpha A^\beta)]^2 \\ =&-2\sum_{\alpha,\beta\neq n+1} [\mathrm{tr}(A^\alpha A^\beta)]^2 \\ \geqslant&-2|A_I|^4, \end{aligned}$$

$$\begin{aligned} &-2\sum_{\alpha\neq n+1} [\mathrm{tr}(A^{n+1}A^\alpha)]^2 + 2\sum_{\alpha\neq n+1} \mathrm{tr}(A^{n+1}A^\alpha)^2 - 2\sum_{\alpha\neq n+1} \mathrm{tr}[(A^{n+1})^2(A^\alpha)^2] \\ =&-2\sum_{\alpha\neq n+1} [\mathrm{tr}(\mathring{A}^{n+1}A^\alpha)]^2 \\ \geqslant&-2|\mathring{A}_H|^2|A_I|^2, \end{aligned}$$

$$2\sum_{\alpha\neq n+1} \mathrm{tr}(A^\alpha A^{n+1})^2 - 2\sum_{\alpha\neq n+1} \mathrm{tr}(A^\alpha A^{n+1}A^{n+1}A^\alpha) = 0.$$

Then by (3.6), we obtain

$$\begin{aligned} \mathcal{L}|A_I|^2 \geqslant& 2|\nabla A_I|^2 + |A_I|^2 - 2|A_I|^4 - 2|\mathring{A}_H|^2|A_I|^2 \\ =& 2|\nabla A_I|^2 + |A_I|^2(1 - 2|\mathring{A}|^2). \end{aligned}$$

If M has polynomial volume growth and $|\mathring{A}|^2 < \frac{1}{2}$, then $\nabla A_I = 0$ and $|A_I| = 0$. By a similar argument as in the proof of Theorem 1.6, M is isometric to $\mathbb{S}^k(\sqrt{2k}) \times \mathbb{R}^{n-k}$ with $1 \leqslant k \leqslant n$ or the product $\Gamma \times \mathbb{R}^{n-1}$ with Γ denoting one of the Abresch-Langer curves. Since we have assumed that M is embedded, it is isometric to one of $\mathbb{S}^k(\sqrt{2k}) \times \mathbb{R}^{n-k}$, $1 \leqslant k \leqslant n$.

If $\sup_M |\mathring{A}|^2 < \frac{1}{2}$, we have $K_M > -\frac{1}{4}$. Also, $|\mathring{A}|^2$ is bounded. In a similar way, we can prove that M is isometric to one of $\mathbb{S}^k(\sqrt{2k}) \times \mathbb{R}^{n-k}$, $1 \leqslant k \leqslant n$. □

4. Self-shrinkers with bounded mean curvature

Combining (2.2) and (2.4), we obtain

$$
\begin{aligned}
\sum_{i,j,\alpha} h_{ij}^{\alpha}\Delta h_{ij}^{\alpha} =& \sum_{i,j,\alpha} h_{ij}^{\alpha}\sum_k \nabla_j\nabla_i H^{\alpha}\\
&+\sum_{i,j,k,\alpha} h_{ij}^{\alpha}\Big(\sum_m h_{km}^{\alpha}R_{mijk}+\sum_m h_{mi}^{\alpha}R_{mkjk}-\sum_\beta h_{ki}^{\beta}R_{\alpha\beta jk}\Big)\\
=&\frac{1}{2}|A|^2-\sum_{i,j,k,\alpha,\beta} H^{\beta}h_{jk}^{\beta}h_{ik}^{\alpha}h_{ij}^{\alpha}+\frac{1}{4}\langle F,\nabla|A|^2\rangle\\
&+\sum_{i,j,k,\alpha} h_{ij}^{\alpha}\Big(\sum_m h_{km}^{\alpha}R_{mijk}+\sum_m h_{mi}^{\alpha}R_{mkjk}-\sum_\beta h_{ki}^{\beta}R_{\alpha\beta jk}\Big).
\end{aligned}
\tag{4.1}
$$

Then we have

(4.2)

$$
\begin{aligned}
\mathcal{L}|A|^2 =&\Delta|A|^2-\frac{1}{2}\langle F,\nabla|A|^2\rangle\\
=&2\sum_{i,j,\alpha} h_{ij}^{\alpha}\Delta h_{ij}^{\alpha}+2|\nabla A|^2-\frac{1}{2}\langle F,\nabla|A|^2\rangle\\
=&2|\nabla A|^2+|A|^2-2\sum_{i,j,k,\alpha,\beta} H^{\beta}h_{jk}^{\beta}h_{ik}^{\alpha}h_{ij}^{\alpha}\\
&+2\sum_{i,j,k,\alpha} h_{ij}^{\alpha}\Big(\sum_m h_{km}^{\alpha}R_{mijk}+\sum_m h_{mi}^{\alpha}R_{mkjk}-\sum_\beta h_{ki}^{\beta}R_{\alpha\beta jk}\Big)\\
=&2|\nabla A|^2+|A|^2-2\sum_{\alpha,\beta}\Big(\sum_{i,j} h_{ij}^{\alpha}h_{ij}^{\beta}\Big)^2-2\sum_{i,j,\alpha,\beta}\Big(\sum_p (h_{pj}^{\alpha}h_{pj}^{\beta}-h_{ip}^{\alpha}h_{ip}^{\beta})\Big)^2.
\end{aligned}
$$

On the other hand, from (2.4) we have

$$
\Delta|H|^2=2|\nabla H|^2+|H|^2-2\sum_{i,j}\Big(\sum_\alpha H^{\alpha}h_{ij}^{\alpha}\Big)^2+\frac{1}{2}\langle F,\nabla|H|^2\rangle,
$$

where $H^{\alpha}=\sum_i h_{ii}^{\alpha}$. It follows that

$$
\mathcal{L}|H|^2=2|\nabla H|^2+|H|^2-2\sum_{i,j}\Big(\sum_\alpha H^{\alpha}h_{ij}^{\alpha}\Big)^2.
\tag{4.3}
$$

Combining (4.2) and (4.3), we have

$$
\begin{aligned}
\mathcal{L}|\mathring{A}|^2 =&2|\nabla\mathring{A}|^2+|\mathring{A}|^2-2\sum_{\alpha,\beta}\Big(\sum_{i,j} h_{ij}^{\alpha}h_{ij}^{\beta}\Big)^2\\
&-2\sum_{i,j,\alpha,\beta}\Big(\sum_p (h_{ip}^{\alpha}h_{pj}^{\beta}-h_{jp}^{\alpha}h_{pi}^{\beta})\Big)^2+\frac{2}{n}\sum_{i,j}\Big(\sum_\alpha H^{\alpha}h_{ij}^{\alpha}\Big)^2.
\end{aligned}
\tag{4.4}
$$

At the point where the mean curvature is zero, we have

$$
\begin{aligned}
&-2\sum_{\alpha,\beta}\Big(\sum_{i,j} h_{ij}^{\alpha}h_{ij}^{\beta}\Big)^2-2\sum_{i,j,\alpha,\beta}\Big(\sum_p (h_{ip}^{\alpha}h_{pj}^{\beta}-h_{jp}^{\alpha}h_{pi}^{\beta})\Big)^2+\frac{2}{n}\sum_{i,j}\Big(\sum_\alpha H^{\alpha}h_{ij}^{\alpha}\Big)^2.\\
=&-2\sum_{\alpha,\beta}N(\mathring{A}^{\alpha}\mathring{A}^{\beta}-\mathring{A}^{\beta}\mathring{A}^{\alpha})-2\sum_{\alpha,\beta}[\mathrm{tr}(\mathring{A}^{\alpha}\mathring{A}^{\beta})]^2,
\end{aligned}
$$

where $\mathring{A}^{\alpha}=(\mathring{h}_{ij}^{\alpha})_{n\times n}$. By Theorem 1 in [22] this is not less than $-3|\mathring{A}|^4$. Hence we have

$$
\mathcal{L}|\mathring{A}|^2\geqslant 2|\nabla\mathring{A}|^2+|\mathring{A}|^2-3|\mathring{A}|^4.
\tag{4.5}
$$

At the point where the mean curvature is nonzero, we choose $e_{n+1} = \frac{H}{|H|}$ and define A_H, $\mathring{A}_H$, A_I and $\mathring{A}_I$ in the same way as in Section 3. Then we have

$$\begin{aligned}\sum_{\alpha,\beta}\Big(\sum_{i,j} h^\alpha_{ij}h^\beta_{ij}\Big)^2 =&|\mathring{A}_H|^4 + \frac{2}{n}|H|^2|\mathring{A}_H|^2 + \frac{1}{n^2}|H|^4\\ &+ 2\sum_{\alpha\neq n+1}\Big(\sum_{i,j}\mathring{h}^{n+1}_{ij}\mathring{h}^\alpha_{ij}\Big)^2 + \sum_{\alpha,\beta\neq n+1}\Big(\sum_{i,j}\mathring{h}^\alpha_{ij}\mathring{h}^\beta_{ij}\Big)^2,\end{aligned}\tag{4.6}$$

$$\begin{aligned}\sum_{i,j,\alpha,\beta}\Big(\sum_p(h^\alpha_{ip}h^\beta_{pj} - h^\alpha_{jp}h^\beta_{pi})\Big)^2 =&2\sum_{\alpha\neq n+1}\sum_{i,j}\Big(\sum_p(h^{n+1}_{ip}\mathring{h}^\alpha_{pj} - h^{n+1}_{jp}\mathring{h}^\alpha_{pi})\Big)^2\\ &+\sum_{\alpha,\beta\neq n+1}\sum_{i,j}\Big(\sum_p(\mathring{h}^\alpha_{pj}\mathring{h}^\beta_{pj} - \mathring{h}^\alpha_{ip}\mathring{h}^\beta_{ip})\Big)^2,\end{aligned}\tag{4.7}$$

$$\sum_{i,j}\Big(\sum_\alpha H^\alpha h^\alpha_{ij}\Big)^2 = |H|^2|\mathring{A}_H|^2 + \frac{1}{n}|H|^4.\tag{4.8}$$

From (4.6), (4.7) and (4.8) we obtain the following

$$\begin{aligned}&2\sum_{\alpha,\beta}\Big(\sum_{i,j} h^\alpha_{ij}h^\beta_{ij}\Big)^2 + 2\sum_{i,j,\alpha,\beta}\Big(\sum_p(h^\alpha_{ip}h^\beta_{pj} - h^\alpha_{jp}h^\beta_{pi})\Big)^2 - \frac{2}{n}\sum_{i,j}\Big(\sum_\alpha H^\alpha h^\alpha_{ij}\Big)^2\\ =&2|\mathring{A}_H|^4 + \frac{2}{n}|H|^2|\mathring{A}_H|^2\\ &+4\sum_{\alpha\neq n+1}\Big(\sum_{i,j}\mathring{h}^{n+1}_{ij}\mathring{h}^\alpha_{ij}\Big)^2 + 4\sum_{\alpha\neq n+1}\sum_{i,j}\Big(\sum_p(h^{n+1}_{ip}\mathring{h}^\alpha_{pj} - h^{n+1}_{jp}\mathring{h}^\alpha_{pi})\Big)^2\\ &+2\sum_{\alpha,\beta\neq n+1}\sum_{i,j}\Big(\sum_{i,j}\mathring{h}^\alpha_{ij}\mathring{h}^\beta_{ij}\Big)^2 + 2\sum_{\alpha,\beta\neq n+1}\sum_{i,j}\Big(\sum_p(\mathring{h}^\alpha_{pj}\mathring{h}^\beta_{pj} - \mathring{h}^\alpha_{ip}\mathring{h}^\beta_{ip})\Big)^2.\end{aligned}\tag{4.9}$$

We choose $\{e_i\}$ such that $h^{n+1}_{ij} = \lambda_i\delta_{ij}$. Then $\mathring{h}^{n+1}_{ij} = \mathring{\lambda}_i\delta_{ij}$, where $\mathring{\lambda}_i = \lambda_i - \frac{|H|}{n}$. We have the following estimates.

$$\begin{aligned}4\sum_{\alpha\neq n+1}\Big(\sum_{i,j}\mathring{h}^{n+1}_{ij}\mathring{h}^\alpha_{ij}\Big)^2 =&4\sum_{\alpha\neq n+1}\Big(\sum_i\mathring{\lambda}_i\mathring{h}^\alpha_{ii}\Big)^2\\ \leqslant&4\Big(\sum_i\mathring{\lambda}_i^2\Big)\Big(\sum_{\alpha\neq n+1}\sum_i(\mathring{h}^\alpha_{ii})^2\Big)\\ =&4|\mathring{A}_H|^2\sum_{\alpha\neq n+1}\sum_i(\mathring{h}^\alpha_{ii})^2,\end{aligned}\tag{4.10}$$

$$
\begin{aligned}
&4\sum_{\alpha\neq n+1}\sum_{i,j}\Big(\sum_p(h^{n+1}_{ip}\mathring{h}^{\alpha}_{pj}-h^{n+1}_{jp}\mathring{h}^{\alpha}_{pi})\Big)^2\\
=&4\sum_{\alpha\neq n+1}\sum_{i\neq j}(\lambda_i-\lambda_j)^2(\mathring{h}^{\alpha}_{ij})^2\\
=&4\sum_{\alpha\neq n+1}\sum_{i\neq j}(\mathring{\lambda}_i-\mathring{\lambda}_j)^2(\mathring{h}^{\alpha}_{ij})^2\\
\leqslant&8\sum_{\alpha\neq n+1}\sum_{i\neq j}(\mathring{\lambda}_i^2+\mathring{\lambda}_j^2)(\mathring{h}^{\alpha}_{ij})^2\\
\leqslant&8|\mathring{A}_H|^2\sum_{\alpha\neq n+1}\sum_{i\neq j}(\mathring{h}^{\alpha}_{ij})^2\\
=&8|\mathring{A}_H|^2\Big(|\mathring{A}_I|^2-\sum_{\alpha\neq n+1}\sum_i(\mathring{h}^{\alpha}_{ii})^2\Big).\\
\leqslant&8|\mathring{A}_H|^2|\mathring{A}_I|^2.
\end{aligned}
\tag{4.11}
$$

By Theorem 1 in [22], we have

$$
\begin{aligned}
&-2\sum_{\alpha,\beta\neq n+1}\Big(\sum_{i,j}\mathring{h}^{\alpha}_{ij}\mathring{h}^{\beta}_{ij}\Big)^2-2\sum_{\alpha,\beta\neq n+1}\sum_{i,j}\Big(\sum_p(\mathring{h}^{\alpha}_{pj}\mathring{h}^{\beta}_{pj}-\mathring{h}^{\alpha}_{ip}\mathring{h}^{\beta}_{ip})\Big)^2.\\
=&-2\sum_{\alpha,\beta\neq n+1}[\mathrm{tr}(\mathring{A}^{\alpha}\mathring{A}^{\beta})]^2-2\sum_{\alpha,\beta\neq n+1}N(\mathring{A}^{\alpha}\mathring{A}^{\beta}-\mathring{A}^{\beta}\mathring{A}^{\alpha})\\
\geqslant&-3|\mathring{A}_I|^4.
\end{aligned}
\tag{4.12}
$$

Putting (4.4), (4.9), (4.10), (4.11) and (4.12) together, we obtain

$$
\begin{aligned}
\mathcal{L}|\mathring{A}|^2\geqslant&2|\nabla\mathring{A}|^2+|\mathring{A}|^2-2|\mathring{A}_H|^4-\frac{2}{n}|H|^2|\mathring{A}_H|^2-8|\mathring{A}_H|^2|\mathring{A}_I|^2-3|\mathring{A}_I|^4\\
\geqslant&2|\nabla\mathring{A}|^2+|\mathring{A}|^2-4|\mathring{A}|^4-\frac{2}{n}|H|^2|\mathring{A}_H|^2.
\end{aligned}
\tag{4.13}
$$

From (4.5) and (4.13), we always have the following estimate.

$$
\mathcal{L}|\mathring{A}|^2\geqslant 2|\nabla\mathring{A}|^2+|\mathring{A}|^2-4|\mathring{A}|^4-\frac{2}{n}|H|^2|\mathring{A}|^2.
\tag{4.14}
$$

Now we give the proof of Theorem 1.5.

Proof. Set $f_\varepsilon=[|\mathring{A}|^2+np\varepsilon^2]^{\frac{1}{2}}$ for a constant $\varepsilon>0$. Then we have

$$
|\nabla\mathring{A}|^2\geqslant|\nabla f_\varepsilon|^2.
$$

For the proof one can see [31]. From (4.14) we have

$$
\mathcal{L}f_\varepsilon^2\geqslant 2|\nabla f_\varepsilon|^2+|\mathring{A}|^2-4|\mathring{A}|^4-\frac{2}{n}|H|^2|\mathring{A}|^2.
\tag{4.15}
$$

Multiplying both sides of (4.15) by $\phi_r^2 f_\varepsilon^{n-2}$ and integrating by parts with respect to the measure $e^{-\frac{|F|^2}{4}}d\mu$ on M give

$$\begin{aligned}
0 \geqslant & 2\int_M |\nabla f_\varepsilon|^2\phi_r^2 f_\varepsilon^{n-2}e^{-\frac{|F|^2}{4}}d\mu + \int_M |\mathring{A}|^2\phi_r^2 f_\varepsilon^{n-2}e^{-\frac{|F|^2}{4}}d\mu\\
&-4\int_M |\mathring{A}|^4\phi_r^2 f_\varepsilon^{n-2}e^{-\frac{|F|^2}{4}}d\mu - \frac{2}{n}\int_M \phi_r^2 f_\varepsilon^{n}|H|^2e^{-\frac{|F|^2}{4}}d\mu\\
&-\int_M \phi_r^2 f_\varepsilon^{n-2}\mathcal{L}(f_\varepsilon^2)e^{-\frac{|F|^2}{4}}d\mu\\
=& 2\int_M |\nabla f_\varepsilon|^2\phi_r^2 f_\varepsilon^{n-2}e^{-\frac{|F|^2}{4}}d\mu + \int_M |\mathring{A}|^2\phi_r^2 f_\varepsilon^{n-2}e^{-\frac{|F|^2}{4}}d\mu\\
&-4\int_M |\mathring{A}|^4\phi_r^2 f_\varepsilon^{n-2}e^{-\frac{|F|^2}{4}}d\mu - \frac{2}{n}\int_M \phi_r^2 f_\varepsilon^{n}|H|^2e^{-\frac{|F|^2}{4}}d\mu\\
&+\int_M \langle\nabla f_\varepsilon^2, \nabla(\phi_r^2 f_\varepsilon^{n-2})\rangle e^{-\frac{|F|^2}{4}}d\mu\\
=& 2(n-1)\int_M |\nabla f_\varepsilon|^2\phi_r^2 f_\varepsilon^{n-2}e^{-\frac{|F|^2}{4}}d\mu + \int_M |\mathring{A}|^2\phi_r^2 f_\varepsilon^{n-2}e^{-\frac{|F|^2}{4}}d\mu\\
&-4\int_M |\mathring{A}|^4\phi_r^2 f_\varepsilon^{n-2}e^{-\frac{|F|^2}{4}}d\mu - \frac{2}{n}\int_M \phi_r^2 f_\varepsilon^{n}|H|^2e^{-\frac{|F|^2}{4}}d\mu\\
&+4\int_M \phi_r f_\varepsilon^{n-1}\langle\nabla f_\varepsilon, \nabla\phi_r\rangle e^{-\frac{|F|^2}{4}}d\mu\\
\geqslant & \left(2(n-1)-\frac{\sigma\rho}{2}\right)\int_M |\nabla f_\varepsilon|^2\phi_r^2 f_\varepsilon^{n-2}e^{-\frac{|F|^2}{4}}d\mu + \int_M |\mathring{A}|^2\phi_r^2 f_\varepsilon^{n-2}e^{-\frac{|F|^2}{4}}d\mu\\
&-4\int_M |\mathring{A}|^4\phi_r^2 f_\varepsilon^{n-2}e^{-\frac{|F|^2}{4}}d\mu - \frac{2}{n}\int_M \phi_r^2 f_\varepsilon^{n}|H|^2e^{-\frac{|F|^2}{4}}d\mu\\
&+(4+\sigma)\int_M \phi_r f_\varepsilon^{n-1}\langle\nabla f_\varepsilon, \nabla\phi_r\rangle e^{-\frac{|F|^2}{4}}d\mu - \frac{\sigma}{2\rho}\int_M f_\varepsilon^n|\nabla\phi_r|^2e^{-\frac{|F|^2}{4}}d\mu.
\end{aligned}$$

Here $\sigma, \rho \in \mathbb{R}^+$.

As (3.16), we have

$$|\nabla(\phi_r f_\varepsilon^{\frac{n}{2}})|^2 = f_\varepsilon^n|\nabla\phi_r|^2 + n\phi_r f_\varepsilon^{n-1}\langle\nabla\phi_r, \nabla f_\varepsilon\rangle + \frac{n^2}{4}\phi_r^2 f_\varepsilon^{n-2}|\nabla f_\varepsilon|^2.$$

Pick $\sigma, \rho > 0$ such that $2(n-1) - \frac{(4-\sigma)\rho}{2} = \frac{n\sigma}{4}$. Then we get

$$\begin{aligned}
0 \geqslant & \frac{n\sigma}{4}\left(\frac{4}{n^2}\int_M |\nabla(\phi_r f_\varepsilon^{\frac{n}{2}})|^2e^{-\frac{|F|^2}{4}}d\mu - \frac{4}{n}\int_M \phi_r f_\varepsilon^{n-1}\langle\nabla f_\varepsilon, \nabla\phi_r\rangle e^{-\frac{|F|^2}{4}}d\mu\right.\\
&\left.-\frac{4}{n^2}\int_M f_\varepsilon^n|\nabla\phi_r|^2e^{-\frac{|F|^2}{4}}\right) + \int_M |\mathring{A}|^2\phi_r^2 f_\varepsilon^{n-2}e^{-\frac{|F|^2}{4}}d\mu\\
&-4\int_M |\mathring{A}|^4\phi_r^2 f_\varepsilon^{n-2}e^{-\frac{|F|^2}{4}}d\mu - \frac{2}{n}\int_M \phi_r^2 f_\varepsilon^{n}|H|^2e^{-\frac{|F|^2}{4}}d\mu\\
&+\sigma\int_M \phi_r f_\varepsilon^{n-1}\langle\nabla f_\varepsilon, \nabla\phi_r\rangle e^{-\frac{|F|^2}{4}}d\mu - \frac{4-\sigma}{2\rho}\int_M f_\varepsilon^n|\nabla\phi_r|^2e^{-\frac{|F|^2}{4}}d\mu.\\
=& \frac{\sigma}{n}\int_M |\nabla(f_\varepsilon^{\frac{n}{2}}\phi_r)|^2e^{-\frac{|F|^2}{4}}d\mu - \left(\frac{4-\sigma}{2\rho}+\frac{\sigma}{n}\right)\int_M f_\varepsilon^n|\nabla\phi_r|^2e^{-\frac{|F|^2}{4}}d\mu\\
&+\int_M |\mathring{A}|^2\phi_r^2 f_\varepsilon^{n-2}e^{-\frac{|F|^2}{4}}d\mu - 4\int_M |\mathring{A}|^4\phi_r^2 f_\varepsilon^{n-2}e^{-\frac{|F|^2}{4}}d\mu - \frac{2}{n}\int_M \phi_r^2 f_\varepsilon^{n}|H|^2e^{-\frac{|F|^2}{4}}d\mu.
\end{aligned}$$

From [14], we have

$$\begin{aligned}\int_M |\nabla(\phi_r f_\varepsilon^{\frac{n}{2}} e^{-\frac{|F|^2}{8}})|^2 d\mu =& \int_M |\nabla(f_\varepsilon^{\frac{n}{2}}\phi_r)|^2 e^{-\frac{|F|^2}{4}} d\mu - \frac{1}{8}\int_M \phi_r^2 f_\varepsilon^n |F^N|^2 e^{-\frac{|F|^2}{4}} d\mu \\ &+ \frac{n}{4}\int_M \phi_r^2 f_\varepsilon^n e^{-\frac{|F|^2}{4}} d\mu - \frac{1}{16}\int_M \phi_r^2 f_\varepsilon^n |F^T|^2 e^{-\frac{|F|^2}{4}} d\mu \\ \leqslant& \int_M |\nabla(f_\varepsilon^{\frac{n}{2}}\phi_r)|^2 e^{-\frac{|F|^2}{4}} d\mu - \frac{1}{2}\int_M \phi_r^2 f_\varepsilon^n |H|^2 e^{-\frac{|F|^2}{4}} d\mu \\ &+ \frac{n}{4}\int_M \phi_r^2 f_\varepsilon^n e^{-\frac{|F|^2}{4}} d\mu.\end{aligned}$$

Then we have

$$\begin{aligned}0 \geqslant& \frac{\sigma}{n}\int_M |\nabla(\phi_r f_\varepsilon^{\frac{n}{2}} e^{-\frac{|F|^2}{8}})|^2 d\mu + \frac{\sigma}{2n}\int_M \phi_r^2 f_\varepsilon^n |H|^2 e^{-\frac{|F|^2}{4}} d\mu \\ &- \frac{\sigma}{4}\int_M \phi_r^2 f_\varepsilon^n e^{-\frac{|F|^2}{4}} d\mu - \left(\frac{4-\sigma}{2\rho} + \frac{\sigma}{n}\right)\int_M f_\varepsilon^n |\nabla\phi_r|^2 e^{-\frac{|F|^2}{4}} d\mu \\ &+ \int_M |\mathring{A}|^2 \phi_r^2 f_\varepsilon^{n-2} e^{-\frac{|F|^2}{4}} d\mu - 4\int_M |\mathring{A}|^4 \phi_r^2 f_\varepsilon^{n-2} e^{-\frac{|F|^2}{4}} d\mu \\ &- \frac{2}{n}\int_M \phi_r^2 f_\varepsilon^n |H|^2 e^{-\frac{|F|^2}{4}} d\mu \\ \geqslant& \frac{(n-2)^2\sigma}{4n(n-1)^2 D^2(n)(1+t)} ||\phi_r^2 f_\varepsilon^n e^{-\frac{|F|^2}{4}}||_{\frac{n}{n-2}} \\ &+ \left(\frac{\sigma}{2n} - \frac{(n-2)^2\sigma}{4n^3(n-1)^2 t} - \frac{2}{n}\right)\int_M \phi_r^2 f_\varepsilon^n |H|^2 e^{-\frac{|F|^2}{4}} d\mu \\ &- \frac{\sigma}{4}\int_M \phi_r^2 f_\varepsilon^n e^{-\frac{|F|^2}{4}} d\mu - \left(\frac{4-\sigma}{2\rho} + \frac{\sigma}{n}\right)\int_M f_\varepsilon^n |\nabla\phi_r|^2 e^{-\frac{|F|^2}{4}} d\mu \\ &+ \int_M |\mathring{A}|^2 \phi_r^2 f_\varepsilon^{n-2} e^{-\frac{|F|^2}{4}} d\mu - 4\int_M |\mathring{A}|^4 \phi_r^2 f_\varepsilon^{n-2} e^{-\frac{|F|^2}{4}} d\mu.\end{aligned}$$

Letting $\varepsilon \to 0$, we obtain

$$\begin{aligned}0 \geqslant& \frac{(n-2)^2\sigma}{4n(n-1)^2 D^2(n)(1+t)} ||\phi_r^2 |\mathring{A}|^n e^{-\frac{|F|^2}{4}}||_{\frac{n}{n-2}} \\ &+ \left(\frac{\sigma}{2n} - \frac{(n-2)^2\sigma}{4n^3(n-1)^2 t} - \frac{2}{n}\right)\int_M \phi_r^2 |\mathring{A}|^n |H|^2 e^{-\frac{|F|^2}{4}} d\mu \\ &+ \left(1 - \frac{\sigma}{4}\right)\int_M \phi_r^2 |\mathring{A}|^n e^{-\frac{|F|^2}{4}} d\mu \\ &- \left(\frac{4-\sigma}{2\rho} + \frac{\sigma}{n}\right)\int_M |\mathring{A}|^n |\nabla\phi_r|^2 e^{-\frac{|F|^2}{4}} d\mu - 4\int_M \phi_r^2 |\mathring{A}|^{n+2} e^{-\frac{|F|^2}{4}} d\mu.\end{aligned}$$

Since $H_0 := \sup_M |H| < \sqrt{\frac{n}{2}}$, for any $\eta \in (0, \frac{1}{H_0^2})$, from the inequality above we get

$$\begin{aligned}0 \geqslant& \frac{(n-2)^2\sigma}{4n(n-1)^2 D^2(n)(1+t)} ||\phi_r^2 |\mathring{A}|^n e^{-\frac{|F|^2}{4}}||_{\frac{n}{n-2}} \\ &\left(\frac{\sigma}{2n} - \frac{(n-2)^2\sigma}{4n^3(n-1)^2 t} - \frac{2}{n} + \eta\right)\int_M \phi_r^2 |\mathring{A}|^n |H|^2 e^{-\frac{|F|^2}{4}} d\mu \\ &+ \left(1 - \frac{\sigma}{4} - \eta H_0^2\right)\int_M \phi_r^2 |\mathring{A}|^n e^{-\frac{|F|^2}{4}} d\mu \\ &- \left(\frac{4-\sigma}{2\rho} + \frac{\sigma}{n}\right)\int_M |\mathring{A}|^n |\nabla\phi_r|^2 e^{-\frac{|F|^2}{4}} d\mu - 4\int_M \phi_r^2 |\mathring{A}|^{n+2} e^{-\frac{|F|^2}{4}} d\mu.\end{aligned}$$

Now we let $1-\frac{\sigma}{4}-\eta H_0^2=0$ and $\frac{\sigma}{2n}-\frac{(n-2)^2\sigma}{4n^3(n-1)^2t}-\frac{2}{n}+\eta=0$. Then we have

$$0\geqslant\frac{(n-2)^2\sigma}{4n(n-1)^2D^2(n)}\cdot\frac{(1-\eta H_0^2)\eta(1-\frac{2}{n}H_0^2)}{\eta(1-\frac{2}{n}H_0^2)+c(n)(1-\eta H_0^2)}\Big|\Big|\phi_r^2|\mathring{A}|^n e^{-\frac{|F|^2}{4}}\Big|\Big|_{\frac{n}{n-2}}$$
$$-4\int_M|\mathring{A}|^{n+2}\phi_r^2e^{-\frac{|F|^2}{4}}d\mu-\left(\frac{4-\sigma}{2\rho}+\frac{\sigma}{n}\right)\int_M|\mathring{A}|^n|\nabla\phi_r|^2e^{-\frac{|F|^2}{4}}d\mu.$$

Here $c(n)=\frac{(n-2)^2}{n(n-1)^2}$. By the Cauchy inequality, we have

$$\int_M|\mathring{A}|^{n+2}\phi_r^2e^{-\frac{|F|^2}{4}}d\mu\leqslant\Big|\Big|\phi_r^2|\mathring{A}|^ne^{-\frac{|F|^2}{4}}\Big|\Big|_{\frac{n}{n-2}}\cdot\Big|\Big||\mathring{A}|^2\Big|\Big|_{\frac{n}{2}}.$$

Hence

$$0\geqslant\left(\frac{(n-2)^2}{n(n-1)^2D^2(n)}\cdot\frac{(1-\eta H_0^2)\eta(1-\frac{2}{n}H_0^2)}{\eta(1-\frac{2}{n}H_0^2)+c(n)(1-\eta H_0^2)}-4||\mathring{A}||_n^2\right)\Big|\Big|\phi_r^2|\mathring{A}|^ne^{-\frac{|F|^2}{4}}\Big|\Big|_{\frac{n}{n-2}}$$
$$-\left(\frac{4-\sigma}{2\rho}+\frac{\sigma}{n}\right)\int_M|\mathring{A}|^n|\nabla\phi_r|^2e^{-\frac{|F|^2}{4}}d\mu.$$

The above inequality holds for any $\eta\in(0,\frac{1}{H_0^2})$. Let $\eta=\eta_0=\eta_0(H_0)=\frac{1}{H_0^2}\cdot\frac{\sqrt{d}}{1+\sqrt{d}}$ with $d=d(H_0)=\frac{c(n)}{\frac{1}{H_0^2}-\frac{2}{n}}$. Then we get

$$0\geqslant\left(4D(n,H_0)^2-4||\mathring{A}||_n^2\right)||\phi_r^2|\mathring{A}|^ne^{-\frac{|F|^2}{4}}||_{\frac{n}{n-2}}-\left(\frac{4-\sigma}{2\rho}+\frac{\sigma}{n}\right)\int_M|\mathring{A}|^n|\nabla\phi_r|^2e^{-\frac{|F|^2}{4}}d\mu,$$

where

$$D(n,H_0)^2=\frac{(n-2)^2}{4n(n-1)^2D^2(n)}\cdot\frac{(1-\eta_0H_0^2)\eta_0(1-\frac{2}{n}H_0^2)}{\eta_0(1-\frac{2}{n}H_0^2)+c(n)(1-\eta_0H_0^2)}.$$

By direct computation, $\frac{4-\sigma}{2\rho}+\frac{\sigma}{n}$ has an upper bounded $E(n)$ that depends only on n. Since $|\nabla\phi_r|\leqslant\frac{2}{r}$ and $\int_M|\mathring{A}|^nd\mu<\infty$, we have

$$\lim_{r\to\infty}\int_M|\mathring{A}|^n|\nabla\phi_r|^2e^{-\frac{|F|^2}{4}}d\mu=0.$$

If $||\mathring{A}||_n<D(n,H_0)$, then we get

$$0\geqslant\left(D(n,H_0)^2-||\mathring{A}||_n^2\right)\lim_{r\to\infty}\Big|\Big|\phi_r^2|\mathring{A}|^ne^{-\frac{|F|^2}{4}}\Big|\Big|_{\frac{n}{n-2}}\geqslant0.$$

Hence $\Big|\Big||\mathring{A}|^ne^{-\frac{|F|^2}{4}}|\Big|\Big|=0$, which implies that $\mathring{A}=0$ and M is isometric to $\mathbb{R}^n$ or $\mathbb{S}^n(\sqrt{2n})$. Since we have assumed that $\sup_M|H|<\sqrt{\frac{n}{2}}$, the second case is excluded. This completes the proof of the theorem. □

5. Self-shrinkers with parallel normalized mean curvature vector

In this section, we assume that the mean curvature of the self-shrinker M is nowhere vanishing and the normalized mean curvature vector is parallel in the normal bundle. To prove our results, we need the following theorem proved by Smoczyk [27].

Theorem 5.1 ([27]). *Let $F:M^n\to\mathbb{R}^{n+p}$ be a closed self-shrinker of the mean curvature flow. Then M is a minimal submanifold of the sphere $\mathbb{S}^{n+p-1}(\sqrt{2n})$ if and only if $H\neq0$ and $\frac{H}{|H|}$ is parallel in the normal bundle.*

We will use Theorem 5.1 to prove Theorem 1.8.

Proof. (1) From the assumption and Theorem 5.1, we see that M is a minimal submanifold of the sphere $\mathbb{S}^{n+p-1}(\sqrt{2n}) \subset \mathbb{R}^{n+p}$. Let $\tilde{A}$ and $\tilde{H}$ denote the second fundamental form and mean curvature vector of M in $\mathbb{S}^{n+p-1}(\sqrt{2n})$. Then by the Gauss equation, we have

$$\frac{n-1}{2} + |\tilde{H}|^2 - |\tilde{A}|^2 = |H|^2 - |A|^2.$$

For H and $\tilde{H}$, we have the relation $|H|^2 = |\tilde{H}|^2 + \frac{n}{2}$. Substituting this to the equality above we get

$$|\mathring{\tilde{A}}|^2 = |\mathring{A}|^2. \tag{5.1}$$

Here $\mathring{\tilde{A}}$ is the tracefree second fundamental form of M in $\mathbb{S}^{n+p-1}(\sqrt{2n})$. Since M is minimal in $\mathbb{S}^{n+p-1}(\sqrt{2n})$, we get $|\tilde{A}|^2 = |\mathring{\tilde{A}}|^2$. Hence the pinching condition assumption is equivalent to

$$|\tilde{A}|^2 \leqslant \frac{1}{3}.$$

Then from the intrinsic rigidity theorem in [22], we see that either M is the totally geodesic sphere $\mathbb{S}^n(\sqrt{2n})$ in $\mathbb{S}^{n+p-1}(\sqrt{2n})$, or $n = 2$ and M is the Veronese surface $\mathbb{S}^2(2\sqrt{3}) \to \mathbb{S}^4(2) \subset \mathbb{R}^5 \subset \mathbb{R}^{2+p}$.

(2) From the assumption and Theorem 5.1, we see that M is a minimal submanifold of the sphere $\mathbb{S}^{1+p}(2) \subset \mathbb{R}^{2+p}$. Hence the second fundamental form $\tilde{A}$ of M in $\mathbb{S}^{1+p}(2)$ satisfies

$$\frac{1}{3} \leqslant |\tilde{A}|^2 \leqslant \frac{5}{12}.$$

Then by Theorem C in [19], either $|\tilde{A}|^2 = \frac{1}{3}$ and $M = \mathbb{S}^2(2\sqrt{3}) \to \mathbb{S}^4(2) \subset \mathbb{S}^{1+p}(2) \subset \mathbb{R}^{2+p}$, or $|\tilde{A}|^2 = \frac{5}{12}$ and $M = \mathbb{S}^2(2\sqrt{6}) \to \mathbb{S}^6(2) \subset \mathbb{S}^{1+p}(2) \subset \mathbb{R}^{2+p}$. Here $\mathbb{S}^2(2\sqrt{6}) \to \mathbb{S}^6(2)$ is a canonical immersion, see [19].

(3) From the assumption and Theorem 5.1, we see that M is a minimal hypersurface of the sphere $\mathbb{S}^{n+1}(\sqrt{2n})$ with second fundamental form $\tilde{A}$ satisfying $|\tilde{A}|^2 \leqslant \frac{1}{2}$. Then by the rigidity theorem in [20], either M is totally geodesic, or M is one of the Clifford hypersurfaces $\mathbb{S}^k(\sqrt{2k}) \times \mathbb{S}^{n-k}(\sqrt{2(n-k)})$ in $\mathbb{S}^{n+1}(\sqrt{2n}) \subset \mathbb{R}^{n+2}$.

(4) From the assumption and Theorem 5.1, M is a minimal submanifold of the sphere $\mathbb{S}^{n+1}(\sqrt{2n})$ with second fundamental form $\tilde{A}$. Then by the gap theorem in [12], there is a positive constant $\delta(n)$ depending only on n such that if $\frac{1}{2} \leqslant |\tilde{A}|^2 \leqslant \frac{1}{2} + \delta(n)$, then M is one of the Clifford hypersurfaces $\mathbb{S}^k(\sqrt{2k}) \times \mathbb{S}^{n-k}(\sqrt{2(n-k)})$ in $\mathbb{S}^{n+1}(\sqrt{2n}) \subset \mathbb{R}^{n+2}$, $1 \leqslant k \leqslant n-1$. □

At the end of this section, we present the following theorem for 2-dimensional self-shrinkers.

Theorem 5.2. *Let $F : M^2 \to \mathbb{R}^{2+p}$ ($p \geqslant 2$) be a 2-dimensional closed self-shrinker of the mean curvature flow. Suppose the mean curvature is nowhere vanishing and the normalized mean curvature vector is parallel in the normal bundle.*

(i) If the genus of M is zero and the group of isometries of the induced metric on M contains a non-trivial 1-parameter subgroup, and if for some integer $s \geqslant 1$, the Gaussian curvature K satisfies

$$\frac{1}{2(s+1)(s+2)} \leqslant K \leqslant \frac{1}{2s(s+1)},$$

then $K = \frac{1}{2(s+1)(s+2)}$ or $K = \frac{1}{2s(s+1)}$.

(ii) If $p = 2$ and the genus of M is zero, then M is $\mathbb{S}^2(2) \subset \mathbb{S}^3(2) \subset \mathbb{R}^4$.

(iii) If $p = 2$, M is embedded and the genus of M is one, then M is $\mathbb{S}^1(\sqrt{2}) \times \mathbb{S}^1(\sqrt{2}) \subset \mathbb{S}^3(2) \subset \mathbb{R}^4$.

Proof. The proof is just a combination of Theorem 5.1 and the rigidity theorems for minimal surfaces in spheres in [2, 3, 4]. □

It is an interesting question that if the conditions that the mean curvature is nowhere vanishing and the normalized mean curvature vector is parallel in the theorems proved in this section could be removed.

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