A Critical Exponent of Fujita Type for a Nonlinear Reaction-Diffusion System on Riemannian Manifold

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Abstract

In this paper, we study the global existence and nonexistence of positive solutions to the following nonlinear reaction-diffusion system

$$\begin{cases} u_t - \Delta u = W(x)v^p + S(x) & \text{in } \mathbb{M}^n \times (0,\infty), \\ v_t - \Delta v = F(x)u^d + G(x) & \text{in } \mathbb{M}^n \times (0,\infty), \\ u(x,0) = u_0(x) & \text{in } \mathbb{M}^n, \\ v(x,0) = v_0(x) & \text{in } \mathbb{M}^n, \end{cases}$$

where $\mathbb{M}^n$ $(n \geq 3)$ is a non-compact complete Riemannian manifold, Δ is the Laplace-Beltrami operator, and $S(x), G(x)$ are non-negative L^1_{loc} functions. We assume that both $u_0(x)$ and $v_0(x)$ are non-negative , smooth and bounded functions, constants $p\,, d > 1$. When $p = d$, there is an exponent p^* which is critical in the following sense. when $p \in (1, p^*]$, the above problem has no global positive solution for any non-negative constants $S(x), G(x)$ not identically zero; when $p \in [p^*, \infty)$, the problem has a global positive solution for some $S(x), G(x) > 0$ and $u_0(x), v_0(x) \geq 0$.

Key words and phrases: critical exponent; reaction-diffusion system; Riemannian manifold.

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1 Introduction

In this paper, we study the global existence and nonexistence of positive solutions to the following nonlinear reaction-diffusion system

$$\begin{cases} u_t - \Delta u = W(x)v^p + S(x) & \text{in } \mathbb{M}^n \times (0,\infty), \\ v_t - \Delta v = F(x)u^d + G(x) & \text{in } \mathbb{M}^n \times (0,\infty), \\ u(x,0) = u_0(x) & \text{in } \mathbb{M}^n, \\ v(x,0) = v_0(x) & \text{in } \mathbb{M}^n, \end{cases} \tag{1.1}$$

where $\mathbb{M}^n$ $(n \geq 3)$ is a non-compact complete Riemannian manifold, Δ is the Laplace-Beltrami operator, and $S(x), G(x)$ are non-negative L^1_{loc} functions. We assume that both $u_0(x)$ and $v_0(x)$ are non-negative , smooth and bounded functions, constants $p\,, d > 1$. When $p = d$, there is an exponent p^* which is critical in the following sense. when $p \in (1, p^*]$, the above problem has no global positive solution for any non-negative constants $S(x), G(x)$ not identically zero; when $p \in [p^*, \infty)$, the problem has a global positive solution for some $S(x), G(x) > 0$ and $u_0(x), v_0(x) \geq 0$.

As we all know, when the manifold $\mathbb{M}^n$ is Euclidean space $\mathbb{R}^n$, (1.1) provide a simple example of a parabolic system (see[1]). They can be used as a model to describe heat propagation in a two-component combustible mixture. System (1.1) and its elliptic counterpart arise in such diverse fields as chemistry, biology and physics (see[2]). In 1966, Fujita (see[3]) proved the following results for the problem

$$\begin{cases} u_t - \Delta u = u^p & \text{in } \mathbb{R}^n \times (0,\infty), \\ u(x,0) = u_0(x) & \text{in } \mathbb{R}^n, \end{cases} \tag{1.2}$$

(a) When $p \in (1, 1 + \frac{2}{n})$, and $u_0 > 0$, problem (1.2) possesses no global positive solution;
(b) When $p \in (1 + \frac{2}{n}, \infty)$ and u_0 is smaller than a small Gaussian, then (1.2) has global positive solutions.

Later Hayakawa (see[4]) showed the value $p = 1 + \frac{2}{n}$ belongs to the blow-up case when $n = 1, 2$, and the case in higher dimensions was established in [5, 6]. We call $p = 1 + \frac{2}{n}$ the critical exponent of the semi-linear heat equation (1.2). It plays an important role in the large-time behavior of the solutions to the semi-linear heat equation (1.2).

In 1991, Escobedo and Herrero generalized Fujita result to the homogeneous coupled systems (see[7]). In the past couple of years, there are a number of extensions of Fujita results in many directions(see [8]-[14]). There are only a few results when we investigate existence and non-existence of positive solutions to the parabolic system (1.1). It looks more imperative to fill this gap when we take into account the tremendous literature about the heat kernel of a complete Riemanian manifold (see[11]-[14]).

In recent years, many authors have undertaken the research on semi-linear elliptic operators on manifolds, including the well-known Yamabe problem (see[15, 16]). The study of Ricci flows also leads to semi-linear parabolic problems (see[17]). Not much literature has been done for their reaction-diffusion system on Riemannian manifold, so we need some new techniques to study the global existence and nonexistence of solutions to the reaction-diffusion system (1.1). The method we are using is based on some new inequalities involving the heat kernels. Qi S. Zhang has undertaken the research on semi-linear parabolic operators on Riemannian manifold, and obtains a lot of important results in the study of the global existence and blow-up of the following semi-linear parabolic Cauchy problem (see[11]-[14]):

$$\begin{cases} Hu \overset{\triangle}{=} H_0 + u^p = \Delta u - Ru - u_t + u^p = 0 & \text{in } \mathbb{M}^n \times (0,\infty), \\ u(x,0) = u_0(x) \geq 0 & \text{in } \mathbb{M}^n, \end{cases} \tag{1.3}$$

where $\mathbb{M}^n$ $(n \geq 3)$ is a non-compact complete Riemannian manifold. Δ is the Laplace-Beltrami operator and $R = R(x)$ is a bounded function. The method he uses is rather technical, and the main tools are fixed point theorems and many estimates, As an expansion, we take similar approaches to study the reaction-diffusion system and obtain several meaningful results.

Throughout the paper, for a fixed $x_0 \in \mathbb{M}^n$, we make the following assumptions (see[11, 12]):

(i) There are positive constants k, q and C, such that

$$|B(x,2r)| \leq C2^q|B(x,r)|, \quad r > 0; \quad \text{Ricci} \geq -k;$$

(ii) $G(x,y,t)$ is the fundamental solution of the linear operator $\triangle - \frac{\partial}{\partial t}$, and satisfies

$$\frac{C}{|B(x,t^{\frac{1}{2}})|} e^{-b\frac{d(x,y)^2}{t}} \geq G(x,y,t) \geq 0, \text{ in } \mathbb{M}^n \times (0,\infty)$$

and when $t - s \geq d(x,y)^2$, $G(x,y,t-s)$ satisfies

$$G(x,y,t-s) \geq \min\left\{ \frac{C}{|B(x,(t-s)^{\frac{1}{2}})|}, \frac{C}{|B(y,(t-s)^{\frac{1}{2}})|} \right\};$$

(iii) $\frac{\partial \log\ g^{\frac{1}{2}}}{\partial r} \leq \frac{C}{r}$, when $r = d(x,x_0)$ is smooth; here $g^{\frac{1}{2}}$ is the volume density of the manifold;

(iiii)There are positive constants $\alpha > 2$ and $m > -2$, such that

$C^{-1}r^\alpha \leq |B_r(x_0)| \leq Cr^\alpha$, when r is large and for all $x \in \mathbb{M}^n$;

$W(x),\ F(x)$ are non-negative L^∞_{loc} functions. and for large $r = d(x,x_0),\ C^{-1}r^m \le W(x),\ F(x) \le Cr^m$.

Since the above assumptions are satisfied, the following lemmas hold:

Lemma 1.(see[11]) *There exists positive constants C and R_0, for $R \ge R_0$ and $\frac{1}{p}+\frac{1}{q}=1$, such that*

$$\int_{B_R(x_0)} W^{-\frac{q}{p}}(x)dx \le C\ln R + CR^{-\frac{qm}{p}+\alpha},$$

$$\int_{B_R(x_0)} F^{-\frac{q}{p}}(x)dx \le C\ln R + CR^{-\frac{qm}{p}+\alpha}.$$

Lemma 2.(see[12]) *There exists a $C_0 > 0$, depending only on n, α and $\delta > 0$, such that*

$$\sup_x \int_{\mathbb{M}^n} \frac{1}{d(x,y)^{\alpha-2}[1+d(y,x_0)^{2+\delta}]}dy \le C_0.$$

Lemma 3. (see[12])*There exists a $C_1 > 0$, depending only on n, α and $\delta > 0$, such that*

$$\int_{\mathbb{M}^n} \frac{1}{d(x,y)^{\alpha-2}[1+d(y,x_0)^{\alpha+\delta}]}dy \le \frac{C_1}{1+d(x,x_0)^{\alpha-2}}.$$

Lemma 4. (see[12])*Let $\Gamma(x,y)$ be the Green' function for the Laplacian, then there exists a $C_2 > 0$, such that*

$$\int_{\mathbb{M}^n} \Gamma(x,y)\frac{1}{1+d(y,x_0)^{\alpha+\delta}}dy \le \frac{C_2}{1+d(x,x_0)^{\alpha-2}}.$$

Lemma 5.(see[12]) *Given $\delta > 0$, there exists a constant $C_3 > 0$, such that*

$$h(x,t) \triangleq \int_{\mathbb{M}^n} \frac{G(x,y,t)}{1+d(y,x_0)^{\alpha+\delta}}dy \le \frac{C_3}{1+d(x,x_0)^{\alpha}}.$$

Definition 1. $(u(x,t),v(x,t)) \in L^\infty_{Loc}(\mathbb{M}^n\times[0,\infty),\mathbb{R}^2), (x,t)\in\mathbb{M}^n\times(0,\infty)$ *is called a solution of* (1.1)*, if*

$$u(x,t) = \int_{\mathbb{M}^n} G(x,y,t)u_0(y)dy + \int_0^t\int_{\mathbb{M}^n} G(x,y,t-s)\left[W(y)v^p(y,s)+S(y)\right]dyds, \tag{1.4}$$

$$v(x,t) = \int_{\mathbb{M}^n} G(x,y,t)v_0(x)dy + \int_0^t\int_{\mathbb{M}^n} G(x,y,t-s)\left[F(y)u^d(y,s)+G(y)\right]dyds. \tag{1.5}$$

Definition 2. (see[18]) *On a complete Riemannian manifold, one defines the Green' function $\Gamma(x,y) \triangleq \int_0^\infty G(x,y,s)ds$, if the integral on the right hand side converges.*

One checks readily that $\Gamma(x,y) > 0, \Delta\Gamma = -\delta_x(y)$,

$$\int_0^t G(x,y,t-s)ds = \int_0^t G(x,y,\omega)d\omega \le \int_0^\infty G(x,y,\omega)d\omega = \Gamma(x,y). \tag{1.6}$$

Our results are as follows:

Theorem 1.1 . *For some* $x_0 \in \mathbb{M}^n$*, when* $p, d \in (\frac{\alpha+m}{\alpha-2}, \infty)$*, (1.1) has a global positive solution whenever* $0 < u_0(x), v_0(x), S(x), G(x) < \frac{\varepsilon}{1+d(x,x_0)^{\alpha+\delta}}$ *for some* $\delta > 0$ *and some sufficiently small* $\varepsilon > 0$*.*

Theorem 1.2 . *When* $d = p \in (1, \frac{\alpha+m}{\alpha-2})$ *and* $u_0(x), v_0(x), S(x), G(x) \geq 0$*, then (1.1) possesses no global positive solution unless* $S(x) \equiv 0, G(x) \equiv 0$*.*

Theorem 1.3 . *When* $d = p = \frac{\alpha+m}{\alpha-2}$ *and* $u_0(x), v_0(x), S(x), G(x) \geq 0$*, then (1.1) possesses no global positive solution unless* $S(x) \equiv 0, G(x) \equiv 0$*.*

Remark 1.1 . *By Theorem 1.1, Theorem 1.2 and Theorem 1.3, it is easy to see that* $\frac{\alpha+m}{\alpha-2}$ *is the critical exponent of the nonlinear reaction-diffusion system (1.1), when* $p = d$*.*

Theorems 1.1, 1.2 and 1.3 are proved in the sections 2, 3 and 4, respectively.

2 Global existence of solutions

Proof of Theorem 1.1. For $(u(x,t), v(x,t)) \in L^\infty_{Loc}(\mathbb{M}^n \times [0,\infty), \mathbb{R}^2)$, define the integral operator (Γ_1, Γ_2):

$$\Gamma_1 u(x,t) = \int_{\mathbb{M}^n} G(x,y,t) u_0(y) dy + \int_0^t \int_{\mathbb{M}^n} G(x,y,t-s) \left[W(y) v^p(y,s) + S(y)\right] dyds, \tag{2.1}$$

$$\Gamma_2 v(x,t) = \int_{\mathbb{M}^n} G(x,y,t) v_0(x) dy + \int_0^t \int_{\mathbb{M}^n} G(x,y,t-s) \left[F(y) u^d(y,s) + G(y)\right] dyds. \tag{2.2}$$

For $N \in (0,1)$, the set $\mathcal{H}_N$ is defined by

$$\mathcal{H}_N = \left\{ (u(x,t), v(x,t)) \in C(\mathbb{M}^n \times (0,\infty), \mathbb{R}^2) \mid 0 \leq u(x,t),\ v(x,t) \leq \frac{N}{1+d(x,x_0)^{\alpha-2}} \right\}. \tag{2.3}$$

Next, for the operator (Γ_1, Γ_2), we show that there exists a fixed point.

For $\varepsilon > 0$ and $\delta > 0$ to be chosen later, we select $u_0(x),\ S(x)$ satisfying

$$0 < u_0(x), S(x) < \frac{\varepsilon}{1+d(x,x_0)^{\alpha+\delta}}. \tag{2.4}$$

By Lemma 4, Lemma 5 and (1.6), we obtain

$$\begin{aligned} \int_0^t \int_{\mathbb{M}^n} G(x,y,t-s) S(y) dyds &= \int_{\mathbb{M}^n} \int_0^t G(x,y,t-s) ds S(y) dy \\ &\leq \int_{\mathbb{M}^n} \Gamma(x,y) \frac{\varepsilon}{1+d(y,x_0)^{\alpha+\delta}} dy \leq \frac{\varepsilon C_2}{1+d(x,x_0)^{\alpha-2}} \end{aligned} \tag{2.5}$$

and

$$\int_{\mathbb{M}^n} G(x,y,t)u_0(y)dy \le \varepsilon \int_{\mathbb{M}^n} \frac{G(x,y,t)}{1+d(y,x_0)^{\alpha+\delta}}dy \le \frac{\varepsilon C_3}{1+d(x,x_0)^{\alpha}} \le \frac{\varepsilon C_3 C_4}{1+d(x,x_0)^{\alpha-2}}. \tag{2.6}$$

By assumption(iiii) and (2.3), it is easy to obtain that

$$\int_0^t \int_{\mathbb{M}^n} G(x,y,t-s)W(y)v^p(y,s)dyds \le CN^p \int_0^t \int_{\mathbb{M}^n} G(x,y,t-s)\frac{d(y,x_0)^m}{[1+d(x,x_0)^{\alpha-2}]^p}dyds. \tag{2.7}$$

Since $p > \frac{\alpha+m}{\alpha-2}$, we can find $C_5 > 0$, and $\delta > 0$, such that

$$\frac{d(y,x_0)^m}{[1+d(x,x_0)^{\alpha-2}]^p} \le \frac{C_5}{1+d(x,x_0)^{\alpha+\delta}}. \tag{2.8}$$

Substituting (2.8) in the right-hand side of (2.7) and by Lemma 4, we obtain

$$\begin{aligned} \textstyle\int_0^t \int_{\mathbb{M}^n} G(x,y,t-s)W(y)v^p(y,s)dyds &\le CN^pC_5 \int_{\mathbb{M}^n} \Gamma(x,y)\frac{1}{1+d(y,x_0)^{\alpha+\delta}}dy \\ &\le \frac{CC_2C_5N^p}{1+d(x,x_0)^{\alpha-2}}. \end{aligned} \tag{2.9}$$

Merging (2.1), (2.5), (2.6) and (2.9), it follows that

$$\Gamma_1 u(x,t) \le \frac{\varepsilon C_3 C_4}{1+d(x,x_0)^{\alpha-2}} + \frac{\varepsilon C_2}{1+d(x,x_0)^{\alpha-2}} + \frac{CC_2C_5N^p}{1+d(x,x_0)^{\alpha-2}}. \tag{2.10}$$

Noticing that $p > 1$, we have

$$\Gamma_1 u(x,t) \le \frac{N}{1+d(x,x_0)^{\alpha-2}}, \tag{2.11}$$

when ε and N are sufficiently small. For $\Gamma_2 v(x,t)$, we have similar discussions,

$$\Gamma_2 v(x,t) \le \frac{N}{1+d(x,x_0)^{\alpha-2}}. \tag{2.12}$$

This shows that $(\Gamma_1,\Gamma_2)\mathcal{H}_N \subset \mathcal{H}_N$.

To obtain the global existence of positive solutions to (1.1), it is checked that (Γ_1,Γ_2) is continuous. Let $u_i(x,t)(i=1,2) \in \mathcal{H}_N$, then

$$\begin{aligned} &\int_{M^n} G(y,\omega,s)v_0(\omega)d\omega + \int_0^t \int_{M^n} G(y,\omega,s-z)\left[F(\omega)u_i^d(\omega,z)+G(\omega)\right]d\omega dz \\ \le\ & \frac{C_3C_4\varepsilon}{1+d(y,x_0)^{\alpha-2}} + \frac{CC_2C_5N^d}{1+d(y,x_0)^{\alpha-2}} + \frac{\varepsilon C_2}{1+d(y,x_0)^{\alpha-2}} \\ \le\ & \frac{N}{1+d(y,x_0)^{\alpha-2}}. \end{aligned} \tag{2.13}$$

when ε and N are sufficiently small.

Notice that

$$\mid u_1^p(\omega,z) - u_2^p(\omega,z) \mid \le p\max\{u_1^{p-1}(\omega,z), u_2^{p-1}(\omega,z)\} \mid u_1(\omega,z) - u_2(\omega,z) \mid . \tag{2.14}$$

By (2.14), we have

$$
\begin{aligned}
|\Gamma_1 u_1(x,t) - \Gamma_1 u_2(x,t)| &= \left| \int_0^t \int_{\mathbb{M}^n} G(x,y,t-s) \left\{ W(y) \left[v_1^p(y,s) - v_2^p(y,s) \right] \right\} dyds \right| \\
&\le p \int_0^t \int_{\mathbb{M}^n} G(x,y,t-s) W(y) \left[\frac{N}{1+d(y,x_0)^{\alpha-2}} \right]^{p-1} \times \\
&\quad \left| \int_0^s \int_{\mathbb{M}^n} G(y,\omega,s-z) F(\omega) \left(u_1^d(\omega,z) - u_2^d(\omega,z) \right) d\omega dz \right| dyds.
\end{aligned}
\tag{2.15}
$$

Denoting $||\cdot|| = \max_{x\in M^n,\, t>0} |\cdot|$, we have

$$
\begin{aligned}
& \left| \int_0^s \int_{\mathbb{M}^n} G(y,\omega,s-z) F(\omega) \left(u_1^d(\omega,z) - u_2^d(\omega,z) \right) d\omega dz \right| \\
\le & \int_0^s \int_{M^n} G(y,\omega,s-z) F(\omega) \mid u_1^d(\omega,z) - u_2^d(\omega,z) \mid d\omega dz \\
\le & \; d \mid\mid u_1 - u_2 \mid\mid \int_0^s \int_{M^n} (G(y,\omega,s-z) F(\omega) \max\{u_1^{d-1}(\omega,z), u_2^{d-1}(\omega,z)\} d\omega dz \\
\le & \; d \mid\mid u_1 - u_2 \mid\mid \int_0^s \int_{M^n} (G(y,\omega,s-z) F(\omega) (\frac{N}{1+d(\omega,x_0)^{\alpha-2}})^{d-1} d\omega dz \\
\le & \; dCN^{d-1} \mid\mid u_1 - u_2 \mid\mid \int_{M^n} \int_0^s G(y,\omega,s-z) dz \frac{d(\omega,x_0)^m}{[1+d(\omega,x_0)]^{(\alpha-2)(d-1)}} d\omega \\
\le & \; dCN^{d-1} \mid\mid u_1 - u_2 \mid\mid \int_{M^n} \Gamma(y,\omega) \frac{d(\omega,x_0)^m}{[1+d(\omega,x_0)]^{(\alpha-2)(d-1)}} d\omega.
\end{aligned}
\tag{2.16}
$$

Since $d > \frac{\alpha+m}{\alpha-2}$, we can select a constant $\delta > 0$, such that $(\alpha-2)(d-1) - m \ge 2 + \delta$. Hence there is a constant $C_6 > 0$, such that

$$
\begin{aligned}
& \left| \int_0^s \int_{\mathbb{M}^n} G(y,\omega,s-z) F(\omega) \left(u_1^d(\omega,z) - u_2^d(\omega,z) \right) d\omega dz \right| \\
\le & \; dCC_6 N^{d-1} \mid\mid u_1 - u_2 \mid\mid \int_{M^n} \Gamma(y,\omega) \frac{1}{[1+d(\omega,x_0)]^{2+\delta}} d\omega.
\end{aligned}
\tag{2.17}
$$

By [18], there is a nonnegative constant $\lambda > 0$ such that $\Gamma(x,y) \sim \frac{1}{d(x,y)^{\alpha-2}}$, when $d(x,y) \ge \lambda$, we have

$$
\begin{aligned}
& \left| \int_0^s \int_{\mathbb{M}^n} G(y,\omega,s-z) F(\omega) \left(u_1^d(\omega,z) - u_2^d(\omega,z) \right) d\omega dz \right| \\
\le & \; dCC_6 N^{d-1} \mid\mid u_1 - u_2 \mid\mid \int_{M^n} \frac{1}{d(y,\omega)^{\alpha-2}} \frac{1}{[1+d(\omega,x_0)]^{2+\delta}} d\omega \\
\le & \; dCC_0 C_6 N^{d-1} \mid\mid u_1 - u_2 \mid\mid .
\end{aligned}
\tag{2.18}
$$

Combining (2.15) and (2.18), and by Lemma 2, we obtain

$$
\begin{aligned}
& |\Gamma_1 u_1(x,t) - \Gamma_1 u_2(x,t)| \\
\leq\ & pdCC_0C_6N^{d-1}N^{p-1} \parallel u_1 - u_2 \parallel \int_0^t \int_{\mathbb{M}^n} G(x,y,t-s)W(y) \left[\frac{1}{1+d(y,x_0)^{\alpha-2}}\right]^{p-1} dyds \\
\leq\ & pdC^2C_0C_6N^{d-1}N^{p-1} \parallel u_1 - u_2 \parallel \int_{\mathbb{M}^n} \Gamma(x,y) \frac{d(y,x_0)^m}{[1+d(y,x_0)^{\alpha-2}]^{p-1}} dy \\
\leq\ & pdC^2C_0C_6C_7N^{d-1}N^{p-1} \parallel u_1 - u_2 \parallel \int_{\mathbb{M}^n} \Gamma(x,y) \frac{1}{1+d(y,x_0)^{2+\delta}} dy \\
\leq\ & pdC^2C_0C_6C_7N^{d-1}N^{p-1} \parallel u_1 - u_2 \parallel \sup_x \int_{\mathbb{M}^n} \frac{1}{[d(x,y)^{\alpha-2}][1+d(y,x_0)^{2+\delta}]} \\
\leq\ & pdC^2C_0^2C_6C_7N^{d+p-2} \parallel u_1 - u_2 \parallel .
\end{aligned}
\tag{2.19}
$$

If N is small enough so that $pdC^2C_0^2C_6C_7N^{d+p-2} < 1$, so Γ_1 is contractive in $\mathcal{H}_N$. For Γ_2, we have similar discussions. Hence, (1.1) has a global positive solution.

The proof of Theorem 1.1 is completed. □

3 Global non-existence of solutions

Proof of Theorem 2.2. From now on, C is always a constant that may change from line to line. Throughout the section, we let $\varphi, \eta \in C^\infty[0,\infty)$ be two functions satisfying

$$
\begin{cases}
\varphi(r) \in [0,1], & \text{if } r \in [0,\infty), \\
\varphi(r) = 1, & \text{if } r \in [0,\frac{1}{2}], \\
\varphi(r) = 0, & \text{if } r \in [1,\infty]; \\
\eta(t) \in [0,1], & \text{if } t \in [0,\infty), \\
\eta(t) = 1, & \text{if } t \in [0,\frac{1}{4}], \\
\eta(t) = 0, & \text{if } t \in [1,\infty]; \\
-C \leq \varphi(r)' \leq 0; \quad |\varphi(r)''| \leq C; \quad -C \leq \eta(t)' \leq 0.
\end{cases}
\tag{3.1}
$$

For $R > 0$, we define $Q_R = B_R(x_0) \times [0, R^2]$. We also need a cut-off function

$$
\psi_R = \varphi_R[d(x,x_0)]\eta_R(t), \tag{3.2}
$$

where $\varphi_R(r) = \varphi(\frac{r}{R})$ and $\eta_R(t) = \eta(\frac{t}{R^2})$. Clearly,

$$\frac{\partial \varphi_R}{\partial r} \in [-\frac{C}{R}, 0]; \quad \frac{\partial^2 \varphi_R}{\partial r^2} \in [-\frac{C}{R^2}, \frac{C}{R^2}]; \quad \frac{\partial \eta_R}{\partial t} \in [-\frac{C}{R^2}, 0]. \tag{3.3}$$

We use the method of contradiction. Suppose that $(u(x,t), v(x,t))$ is a global positive solution of (1.1). since $p = d$, For $R > 0$, we set

$$I_R \triangleq \int_{Q_R} W(x) v^p(x,t) \psi_R^q(x,t) dxdt \tag{3.4}$$

and

$$J_R \triangleq \int_{Q_R} F(x) u^p(x,t) \psi_R^q(x,t) dxdt, \tag{3.5}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Since $(u(x,t), v(x,t))$ is a solution of (1.1), we have

$$I_R = \int_{Q_R} [u_t(x,t) - \Delta u(x,t) - S(x)] \psi_R^q(x,t) dxdt \tag{3.6}$$

and

$$J_R = \int_{Q_R} [v_t(x,t) - \Delta v(x,t) - G(x)] \psi_R^q(x,t) dxdt. \tag{3.7}$$

Since non-negative constants $S(x), G(x)$ are not identically zero, notice that when $(x,t) \in Q_{\frac{R}{2}}$, $\psi(x,t) = 1$. there exists a $C_0 > 0$, such that

$$\int_{Q_R} S(x) \psi_R^q(x,t) dxdt, \int_{Q_R} G(x) \psi_R^q(x,t) dxdt \geq C_0 R^2 \tag{3.8}$$

Note that $\psi_R(x,t) \geq 0$, (3.6) and (3.7) yield

$$I_R + C_0 R^2 \leq \int_{Q_R} u_t(x,t) \psi_R^q(x,t) dxdt - \int_{Q_R} \Delta u(x,t) \psi_R^q(x,t) dxdt \tag{3.9}$$

and

$$J_R + C_0 R^2 \leq \int_{Q_R} v_t(x,t) \psi_R^q(x,t) dxdt - \int_{Q_R} \Delta v(x,t) \psi_R^q(x,t) dxdt. \tag{3.10}$$

By the Stokes formula and note that $\psi_R = 0$ on $\partial B_R(x_0)$, we have

$$\begin{aligned} I_R + C_0 R^2 &\leq \int_{Q_R} u_t(x,t) \psi_R^q(x,t) dxdt - \int_0^{R^2} \int_{\partial B_R(x_0)} \frac{\partial u(x,t)}{\partial n} \psi_R^q(x,t) dS_x dt + \int_{Q_R} \nabla u(x,t) \cdot \nabla \psi_R^q(x,t) dxdt \\ &\leq \int_{Q_R} u_t(x,t) \psi_R^q(x,t) dxdt + \int_{Q_R} \nabla u(x,t) \cdot \nabla \psi_R^q(x,t) dxdt \end{aligned} \tag{3.11}$$

and

$$\begin{aligned} J_R + C_0 R^2 &\leq \int_{Q_R} v_t(x,t) \psi_R^q(x,t) dxdt - \int_0^{R^2} \int_{\partial B_R(x_0)} \frac{\partial v(x,t)}{\partial n} \psi_R^q(x,t) dS_x dt + \int_{Q_R} \nabla v(x,t) \cdot \nabla \psi_R^q(x,t) dxdt \\ &\leq \int_{Q_R} v_t(x,t) \psi_R^q(x,t) dxdt + \int_{Q_R} \nabla v(x,t) \cdot \nabla \psi_R^q(x,t) dxdt, \end{aligned} \tag{3.12}$$

which imply, via integration by parts,

$$\begin{aligned} I_R + C_0 R^2 \quad \le \quad & \int_{B_R(x_0)} u(x,R^2)\psi_R^q(x,R^2)dx - \int_{B_R(x_0)} u(x,0)\psi_R^q(x,0)dx - \\ & q\int_{Q_R} u(x,t)\varphi_R^q(x)\eta_R^{q-1}(t)\eta_R'(t)dxdt + \int_0^{R^2}\int_{\partial B_R(x_0)} u(x,t)\frac{\partial\varphi_R^q}{\partial n}\eta_R^q(t)dS_x dt - \\ & \int_{Q_R} u(x,t)\Delta\varphi_R^q(x)\eta_R^q(t)dxdt \end{aligned} \tag{3.13}$$

and

$$\begin{aligned} J_R + C_0 R^2 \quad \le \quad & \int_{B_R(x_0)} v(x,R^2)\psi_R^q(x,R^2)dx - \int_{B_R(x_0)} v(x,0)\psi_R^q(x,0)dx - \\ & q\int_{Q_R} v(x,t)\varphi_R^q(x)\eta_R^{q-1}(t)\eta_R'(t)dxdt + \int_0^{R^2}\int_{\partial B_R(x_0)} v(x,t)\frac{\partial\varphi_R^q}{\partial n}\eta_R^q(t)dS_x dt - \\ & \int_{Q_R} v(x,t)\Delta\varphi_R^q(x)\eta_R^q(t)dxdt. \end{aligned} \tag{3.14}$$

We observe that $\psi_R^q(x,R^2)=0$; $u(x,0),v(x,0)\ge 0$ and $\frac{\partial\varphi_R^q}{\partial n} = q\varphi_R^{q-1}\varphi_R'(\frac{\partial r}{\partial n}) \le 0$ on $\partial B_R(x_0)$, so we obtain

$$I_R + C_0R^2 \quad \le \quad -q\int_{Q_R} u(x,t)\varphi_R^q(x)\eta_R^{q-1}(t)\eta_R'(t)dxdt - \int_{Q_R} u(x,t)\Delta\varphi_R^q(x)\eta_R^q(t)dxdt \tag{3.15}$$

and

$$J_R + C_0R^2 \quad \le \quad -q\int_{Q_R} v(x,t)\varphi_R^q(x)\eta_R^{q-1}(t)\eta_R'(t)dxdt - \int_{Q_R} v(x,t)\Delta\varphi_R^q(x)\eta_R^q(t)dxdt. \tag{3.16}$$

Since $\Delta\varphi_R^q(x) = q\varphi_R^{q-1}(x)\Delta\varphi_R(x) + q(q-1)\varphi_R^{q-2}(x)|\nabla\varphi_R(x)|^2$, (3.15) and (3.16) yield

$$I_R + C_0R^2 \quad \le \quad -q\int_{Q_R} u(x,t)\varphi_R^q(x)\eta_R^{q-1}(t)\eta_R'(t)dxdt - q\int_{Q_R} u(x,t)\varphi_R^{q-1}(x)\Delta\varphi_R(x)\eta_R^q(t)dxdt \tag{3.17}$$

and

$$J_R + C_0R^2 \quad \le \quad -q\int_{Q_R} v(x,t)\varphi_R^q(x)\eta_R^{q-1}(t)\eta_R'(t)dxdt - q\int_{Q_R} v(x,t)\varphi_R^{q-1}(x)\Delta\varphi_R(x)\eta_R^q(t)dxdt. \tag{3.18}$$

Recalling the supports of $\varphi_R(x)$ and $\eta_R(t)$, that is,

$$\begin{cases} \eta_R(t) = 1, \eta_R'(t) = 0, & \text{if } \ t \in [0, \dfrac{R^2}{4}], \\ \varphi_R(x) = 1, \Delta\varphi_R(x) = 0, & \text{if } \ r \in [0, \dfrac{R}{2}], \end{cases} \tag{3.19}$$

we can reduce (3.17) and (3.18) to

$$
\begin{aligned}
I_R + C_0R^2 \quad \leq \quad & -q\int_{\frac{R^2}{4}}^{R^2}\int_{B_R(x_0)} u(x,t)\varphi_R^q(x)\eta_R^{q-1}(t)\eta_R'(t)dxdt- \\
& q\int_0^{R^2}\int_{B_R(x_0)\backslash B_{\frac{R}{2}}(x_0)} u(x,t)\varphi_R^{q-1}(x)\Delta\varphi_R(x)\eta_R^q(t)dxdt
\end{aligned} \tag{3.20}
$$

and

$$
\begin{aligned}
J_R + C_0R^2 \quad \leq \quad & -q\int_{\frac{R^2}{4}}^{R^2}\int_{B_R(x_0)} v(x,t)\varphi_R^q(x)\eta_R^{q-1}(t)\eta_R'(t)dxdt- \\
& q\int_0^{R^2}\int_{B_R(x_0)\backslash B_{\frac{R}{2}}(x_0)} v(x,t)\varphi_R^{q-1}(x)\Delta\varphi_R(x)\eta_R^q(t)dxdt.
\end{aligned} \tag{3.21}
$$

Since φ_R is radial, we have

$$
\Delta\varphi_R = \varphi_R'' + \left[\frac{n-1}{r} + \frac{\partial\log g^{\frac{1}{2}}}{\partial r}\right]\varphi_R'. \tag{3.22}
$$

Taking R sufficiently large, by assumption (iii), that is, $\frac{\partial\log g^{\frac{1}{2}}}{\partial r} \leq \frac{C}{r}$, we obtain

$$
\Delta\varphi_R \geq -\frac{C}{R^2}, \tag{3.23}
$$

when $x \in B_R(x_0)\backslash B_{\frac{R}{2}}(x_0)$. Merging (3.20), (3.21), (3.23) and (3.3), we know

$$
\begin{aligned}
I_R + C_0R^2 \quad \leq \quad & \frac{Cq}{R^2}\int_{\frac{R^2}{4}}^{R^2}\int_{B_R(x_0)} u(x,t)\varphi_R^q(x)\eta_R^{q-1}(t)dxdt+ \\
& \frac{Cq}{R^2}\int_0^{R^2}\int_{B_R(x_0)\backslash B_{\frac{R}{2}}(x_0)} u(x,t)\varphi_R^{q-1}(x)\eta_R^q(t)dxdt
\end{aligned} \tag{3.24}
$$

and

$$
\begin{aligned}
J_R + C_0R^2 \quad \leq \quad & \frac{Cq}{R^2}\int_{\frac{R^2}{4}}^{R^2}\int_{B_R(x_0)} v(x,t)\varphi_R^q(x)\eta_R^{q-1}(t)dxdt+ \\
& \frac{Cq}{R^2}\int_0^{R^2}\int_{B_R(x_0)\backslash B_{\frac{R}{2}}(x_0)} v(x,t)\varphi_R^{q-1}(x)\eta_R^q(t)dxdt.
\end{aligned} \tag{3.25}
$$

Therefore, as $\varphi_R,\ \eta_R \leq 1$,

$$
\begin{aligned}
I_R + C_0R^2 \quad \leq \quad & \frac{Cq}{R^2}\int_{\frac{R^2}{4}}^{R^2}\int_{B_R(x_0)} u(x,t)\psi_R^{q-1}(x,t)dxdt+ \\
& \frac{Cq}{R^2}\int_0^{R^2}\int_{B_R(x_0)\backslash B_{\frac{R}{2}}(x_0)} u(x,t)\psi_R^{q-1}(x,t)dxdt \\
= \quad & \frac{Cq}{R^2}\int_{\frac{R^2}{4}}^{R^2}\int_{B_R(x_0)} F^{\frac{1}{p}}(x)u(x,t)\psi_R^{q-1}(x,t)F^{-\frac{1}{p}}(x)dxdt+ \\
& \frac{Cq}{R^2}\int_0^{R^2}\int_{B_R(x_0)\backslash B_{\frac{R}{2}}(x_0)} F^{\frac{1}{p}}(x)u(x,t)\psi_R^{q-1}(x,t)F^{-\frac{1}{p}}(x)dxdt
\end{aligned} \tag{3.26}
$$

and

$$
\begin{aligned}
J_R + C_0 R^2 &\leq \frac{Cq}{R^2}\int_{\frac{R^2}{4}}^{R^2}\int_{B_R(x_0)} v(x,t)\psi_R^{q-1}(x,t)dxdt + \\
&\quad \frac{Cq}{R^2}\int_0^{R^2}\int_{B_R(x_0)\setminus B_{\frac{R}{2}}(x_0)} v(x,t)\psi_R^{q-1}(x,t)dxdt \\
&= \frac{Cq}{R^2}\int_{\frac{R^2}{4}}^{R^2}\int_{B_R(x_0)} W^{\frac{1}{p}}(x)v(x,t)\psi_R^{q-1}(x,t)W^{-\frac{1}{p}}(x)dxdt + \\
&\quad \frac{Cq}{R^2}\int_0^{R^2}\int_{B_R(x_0)\setminus B_{\frac{R}{2}}(x_0)} W^{\frac{1}{p}}(x)v(x,t)\psi_R^{q-1}(x,t)W^{-\frac{1}{p}}(x)dxdt.
\end{aligned}
\tag{3.27}
$$

By the *Hö lder* inequality and notice $\frac{1}{p}+\frac{1}{q}=1$, we have

$$
\begin{aligned}
I_R + C_0 R^2 &\leq \frac{Cq}{R^2}\left[\int_{\frac{R^2}{4}}^{R^2}\int_{B_R(x_0)} F(x)u^p(x,t)\psi_R^q(x,t)dxdt\right]^{\frac{1}{p}} \times \left[\int_{\frac{R^2}{4}}^{R^2}\int_{B_R(x_0)} F^{-\frac{q}{p}}(x)dxdt\right]^{\frac{1}{q}} + \\
&\quad \frac{Cq}{R^2}\left[\int_0^{R^2}\int_{B_R(x_0)\setminus B_{\frac{R}{2}}(x_0)} F(x)u^p(x,t)\psi_R^q(x,t)dxdt\right]^{\frac{1}{p}} \times \left[\int_0^{R^2}\int_{B_R(x_0)\setminus B_{\frac{R}{2}}(x_0)} F^{-\frac{q}{p}}(x)dxdt\right]^{\frac{1}{q}} \\
&\leq \frac{Cq}{R^2}\left[J_R\right]^{\frac{1}{p}} \times \left[\int_{\frac{R^2}{4}}^{R^2}\int_{B_R(x_0)} F^{-\frac{q}{p}}(x)dxdt\right]^{\frac{1}{q}} + \\
&\quad \frac{Cq}{R^2}\left[J_R\right]^{\frac{1}{p}} \times \left[\int_0^{R^2}\int_{B_R(x_0)\setminus B_{\frac{R}{2}}(x_0)} F^{-\frac{q}{p}}(x)dxdt\right]^{\frac{1}{q}}
\end{aligned}
\tag{3.28}
$$

and

$$
\begin{aligned}
J_R + C_0 R^2 &\leq \frac{Cq}{R^2}\left[\int_{\frac{R^2}{4}}^{R^2}\int_{B_R(x_0)} W(x)v^p(x,t)\psi_R^q(x,t)dxdt\right]^{\frac{1}{p}} \times \left[\int_{\frac{R^2}{4}}^{R^2}\int_{B_R(x_0)} W^{-\frac{q}{p}}(x)dxdt\right]^{\frac{1}{q}} + \\
&\quad \frac{Cq}{R^2}\left[\int_0^{R^2}\int_{B_R(x_0)\setminus B_{\frac{R}{2}}(x_0)} W(x)v^p(x,t)\psi_R^q(x,t)dxdt\right]^{\frac{1}{p}} \times \left[\int_0^{R^2}\int_{B_R(x_0)\setminus B_{\frac{R}{2}}(x_0)} W^{-\frac{q}{p}}(x)dxdt\right]^{\frac{1}{q}} \\
&\leq \frac{Cq}{R^2}\left[I_R\right]^{\frac{1}{p}} \times \left[\int_{\frac{R^2}{4}}^{R^2}\int_{B_R(x_0)} W^{-\frac{q}{p}}(x)dxdt\right]^{\frac{1}{q}} + \\
&\quad \frac{Cq}{R^2}\left[I_R\right]^{\frac{1}{p}} \times \left[\int_0^{R^2}\int_{B_R(x_0)\setminus B_{\frac{R}{2}}(x_0)} W^{-\frac{q}{p}}(x)dxdt\right]^{\frac{1}{q}}.
\end{aligned}
\tag{3.29}
$$

From Lemma 1, we obtain

$$
\begin{aligned}
\left[\int_{\frac{R^2}{4}}^{R^2}\int_{B_R(x_0)} F^{-\frac{q}{p}}(x)dxdt\right]^{\frac{1}{q}} &\leq \left\{\int_{\frac{R^2}{4}}^{R^2}\left[C\ln R + CR^{-\frac{qm}{p}+\alpha}\right]dt\right\}^{\frac{1}{q}} \\
&\leq CR^{\frac{2}{q}}\ln R + CR^{-\frac{m}{p}+\frac{2+\alpha}{q}}
\end{aligned}
\tag{3.30}
$$

and

$$
\begin{aligned}
\left[\int_{\frac{R^2}{4}}^{R^2}\int_{B_R(x_0)} W^{-\frac{q}{p}}(x)dxdt\right]^{\frac{1}{q}} &\leq \left\{\int_{\frac{R^2}{4}}^{R^2}\left[C\ln R + CR^{-\frac{qm}{p}+\alpha}\right]dt\right\}^{\frac{1}{q}} \\
&\leq CR^{\frac{2}{q}}\ln R + CR^{-\frac{m}{p}+\frac{2+\alpha}{q}}.
\end{aligned}
\tag{3.31}
$$

Hence,

$$
\begin{aligned}
I_R + C_0R^2 &\leq \frac{Cq}{R^2}\left[J_R\right]^{\frac{1}{p}} \times \left[CR^{\frac{2}{q}}\ln R + CR^{-\frac{m}{p}+\frac{2+\alpha}{q}}\right] + \frac{Cq}{R^2}\left[J_R\right]^{\frac{1}{p}} \times \left[CR^{-\frac{m}{p}+\frac{2+\alpha}{q}}\right] \\
&\leq \frac{Cq}{R^2}\left[J_R\right]^{\frac{1}{p}} \times \left[CR^{\frac{2}{q}}\ln R + CR^{-\frac{m}{p}+\frac{2+\alpha}{q}}\right] \\
&\leq Cq\left[J_R\right]^{\frac{1}{p}} \times \left[CR^{\frac{2}{q}-2}\ln R + CR^{k}\right]
\end{aligned}
\tag{3.32}
$$

and

$$
\begin{aligned}
J_R + C_0R^2 &\leq \frac{Cq}{R^2}\left[I_R\right]^{\frac{1}{p}} \times \left[CR^{\frac{2}{q}}\ln R + CR^{-\frac{m}{p}+\frac{2+\alpha}{q}}\right] + \frac{Cq}{R^2}\left[I_R\right]^{\frac{1}{p}} \times \left[CR^{-\frac{m}{p}+\frac{2+\alpha}{q}}\right] \\
&\leq \frac{Cq}{R^2}\left[I_R\right]^{\frac{1}{p}} \times \left[CR^{\frac{2}{q}}\ln R + CR^{-\frac{m}{p}+\frac{2+\alpha}{q}}\right] \\
&\leq Cq\left[I_R\right]^{\frac{1}{p}} \times \left[CR^{\frac{2}{q}-2}\ln R + CR^{k}\right],
\end{aligned}
\tag{3.33}
$$

where $k \triangleq -\frac{m}{p} + \frac{2+\alpha}{q} - 2$.
If $J_R \leq 1$, then $Cq\left[J_R\right]^{\frac{1}{p}} R^{\frac{2}{q}-2}\ln R \leq \frac{C_0R^2}{2}$ for large R;
If $J_R > 1$, then $\left[J_R\right]^{\frac{1}{p}} \leq J_R$, and hence, $Cq\left[J_R\right]^{\frac{1}{p}} R^{\frac{2}{q}-2}\ln R \leq CqJ_RR^{\frac{2}{q}-2}\ln R \leq \frac{1}{2}I_R$ for large R .
In either case, we can find suitable positive constants C_1 and C_2, such that

$$
I_R + C_1R^2 \leq C_2\left[J_R\right]^{\frac{1}{p}} R^k. \tag{3.34}
$$

Similarly, we we can find suitable positive constants C_3 and C_4, such that

$$
J_R + C_3R^2 \leq C_4\left[I_R\right]^{\frac{1}{p}} R^k. \tag{3.35}
$$

Substitute (3.35) into the right-hand of (3.34), we obtain

$$
\begin{aligned}
I_R + C_1R^2 &\leq C_2\left[C_4\left[I_R\right]^{\frac{1}{p}} R^k - C_3R^2\right]^{\frac{1}{p}} R^k \\
&\leq C_2C_4\left[I_R\right]^{\frac{1}{p^2}} R^{k(1+\frac{1}{p})}.
\end{aligned}
\tag{3.36}
$$

Hence,

$$C_1 R^2 \quad \leq \quad C_2 C_4 \left[I_R\right]^{\frac{1}{p^2}} R^{k(1+\frac{1}{p})} \tag{3.37}$$

and

$$I_R \quad \leq \quad C_2 C_4 \left[I_R\right]^{\frac{1}{p^2}} R^{k(1+\frac{1}{p})}. \tag{3.38}$$

We can reduce (3.37) to

$$I_R \geq \left\{\tfrac{C_1}{C_2 C_4}\right\}^{p^2} R^{(2-h)p^2}, \tag{3.39}$$

where $h \triangleq k(1+\frac{1}{p})$.

By substituting (3.39) in the left-hand side of (3.38) and simplifying, we obtain

$$I_R \geq \frac{C_1^{(p^2)^2}}{(C_2 C_4)^{(p^2)^2+p^2}} R^{[2(p^2)^2 - h((p^2)^2+p^2)]}. \tag{3.40}$$

For any integer $j > 1$, iterations give

$$I_R \geq \frac{C_1^{(p^2)^j}}{(C_2 C_4)^{p^2+(p^2)^2+\cdots+(p^2)^j}} R^{\{2(p^2)^j - h[p^2+(p^2)^2+\cdots+(p^2)^j]\}}. \tag{3.41}$$

Next we observe that

$$\begin{aligned} 2(p^2)^j - h[p^2+(p^2)^2+\cdots+(p^2)^j] \quad &= \quad 2(p^2)^j - h\left\{\frac{(p^2)^{j+1}-1}{p^2-1}\right\} + h \\ &= \quad (p^2)^j\left\{2 - \frac{hp^2}{p^2-1}\right\} + \frac{h}{p^2-1} + h. \end{aligned} \tag{3.42}$$

Therefore, (3.41) and (3.42) show that there is a positive constant C_5, such that

$$I_R \geq C_5^{(p^2)^j} R^{\left[(p^2)^j\left\{2-\frac{hp^2}{p^2-1}\right\}\right]} R^{\left[\frac{h}{p^2-1}+h\right]}. \tag{3.43}$$

Since $p \in (1, \frac{\alpha+m}{\alpha-2})$, by direct calculation, we know that

$$\begin{aligned} 2 - \tfrac{hp^2}{p^2-1} \quad &= \quad 2 - k(1+\frac{1}{p})\frac{p^2}{p^2-1} \\ &= \quad 2 - k\frac{p}{p-1} \\ &= \quad 2 - \left[-\frac{m}{p} + \frac{2+\alpha}{q} - 2\right]\frac{p}{p-1} \\ &= \quad \frac{(m+\alpha)-(\alpha-2)p}{p-1} > 0. \end{aligned} \tag{3.44}$$

Therefore, if R is so large that $C_6 \triangleq C_5 R^{\left\{2-\frac{hp^2}{p^2-1}\right\}} > 1$, then (3.43) implies

$$I_R \geq C_6^{(p^2)^j} R^{\left[\frac{h}{p^2-1}+h\right]}. \tag{3.45}$$

Let $j \longrightarrow \infty$, we have

$$I_R = \int_{Q_R} W(x) v^p(x,t) \psi_R^q(x,t) dxdt = \infty, \tag{3.46}$$

which means that $v(x,t)$ has to blow-up when $t \leq R^2$. This is a contradiction.

Thus, the proof of Theorem 1.2 is completed. $\square$

4 Critical exponent of Fujita type

Proof of Theorem 1.3. Now $d = p = \frac{\alpha+m}{\alpha-2}$. In this section, C is always a constant that may change from line to line. Obviously all the arguments remain valid if we shift the parabolic cube $Q_R = B_R(x_0) \times [0, R^2]$ to $Q_R = B_R(x_0) \times [R^2, 2R^2]$ and shift $\eta_R(t) = \eta(\frac{t}{R^2})$ to $\eta_R(t) = \eta(\frac{t-R^2}{R^2})$. To save symbols, the latter is still called $Q_R, \eta_R(t)$. In this part,

$$I_R \triangleq \int_{R^2}^{2R^2} \int_{B_R(x_0)} W(x) v^p(x,t) \psi_R^q(x,t) dxdt \tag{4.1}$$

and

$$J_R \triangleq \int_{R^2}^{2R^2} \int_{B_R(x_0)} F(x) u^p(x,t) \psi_R^q(x,t) dxdt, \tag{4.2}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Just like (3.38), for large R, we now have

$$I_R \quad \leq \quad C_2 C_4 \left[I_R\right]^{\frac{1}{p^2}} R^{k(1+\frac{1}{p})}, \tag{4.3}$$

where $k \triangleq -\frac{m}{p} + \frac{2+\alpha}{q} - 2$.

It follows that

$$I_R^{1-\frac{1}{p^2}} \quad \leq \quad C_2 C_4 R^{k(1+\frac{1}{p})}. \tag{4.4}$$

Since $k\left(1-\frac{1}{p^2}\right)^{-1}\left(1+\frac{1}{p}\right) = 2$, there exists a $C > 0$, such that

$$\int_{R^2}^{\frac{5R^2}{4}} \int_{B_{\frac{R}{2}}(x_0)} W(x) v^p(x,t) dxdt \leq I_R \leq CR^2 \tag{4.5}$$

for all large $R > 0$.

From (4.5) the mean-value theorem shows

$$\inf_{R^2 \leq t \leq \frac{5R^2}{4}} \int_{B_{\frac{R}{2}}(x_0)} W(x) v^p(x,t) dxdt \leq C. \tag{4.6}$$

Hence, there exists a sequence R_j and $t_j \in [R_j^2, 2R_j^2]$, such that $\lim_{j\to\infty} R_j = \infty$, and

$$\int_{B_{R_j}(x_0)} W(x) v^p(x, t_j) dxdt \leq C. \tag{4.7}$$

Because $G(x)$ is not identically zero, we can find a compactly supported $G(x)_0$ being positive somewhere and $0 \leq G(x)_0 \leq G(x)$. Since $v(x,t)$ is a global solution of (1.1), we have

$$v(x,t) \geq \int_0^t \int_{\mathbb{M}^n} G(x,y,t-s)G(y)dyds \geq \int_0^t \int_{\mathbb{M}^n} G(x,y,t-s)G_0(y)dyds \overset{\triangle}{=} L(x,t). \tag{4.8}$$

From (4.7), we have

$$\int_{B_{R_j}(x_0)} W(x)L^p(x,t_j)dx \leq C. \tag{4.9}$$

By a change of the time variable, (4.8) yields

$$L(x,t) = \int_0^t \int_{\mathbb{M}^n} G(x,y,t-s)G_0(y)dyds = \int_{\mathbb{M}^n} \int_0^t G(x,y,t-s)dsG_0(y)dy = \int_{\mathbb{M}^n} \int_0^t G(x,y,s)dsG_0(y)dy. \tag{4.10}$$

Hence, we have the monotone convergence

$$\lim_{t\to\infty} L(x,t) = \int_{\mathbb{M}^n} \Gamma(x,y)G_0(y)dy \overset{\triangle}{=} L_\infty(x). \tag{4.11}$$

Combining (4.9) and (4.11), we have

$$\int_{B_R(x_0)} W(x)L^p_\infty(x)dx \leq \lim_{t\to\infty} \sup \int_{B_{R_j}(x_0)} W(x)L^p(x,t_j)dx \leq C \tag{4.12}$$

for any large $R > 0$.

By [18], $\Gamma(x,y) \sim \frac{1}{d(x,y)^{\alpha-2}}$ for large $d(x,y)$; it is easy to see that $L_\infty(x) \geq \frac{C}{d(x,x_0)^{\alpha-2}}$ when $r = d(x,x_0)$ is large. Using the assumption (iiii): $C^{-1}r^m \leq W(x) \leq Cr^m$, we can find an $R_0 > 0$, such that

$$\int_{B_R(x_0)\backslash B_{R_0}(x_0)} \frac{d(x,x_0)^m}{d(x,x_0)^{(\alpha-2)p}} \leq C \int_{B_R(x_0)} W(x)L^p_\infty(x)dx \leq C. \tag{4.13}$$

Recalling that $p = \frac{\alpha+m}{\alpha-2}$, we obtain

$$\int_{B_R(x_0)\backslash B_{R_0}(x_0)} \frac{1}{d(x,x_0)^\alpha}dx \leq C. \tag{4.14}$$

By the assumption (iiii): $|B_R(x_0)| \geq CR^\alpha$, (4.14) leads to a contradiction since the left-hand side of (4.14) goes to ∞ when $R \longrightarrow \infty$.

Thus, the proof of Theorem 1.3 is completed. $\square$

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