ANOMALY CANCELLATION AND MODULARITY

FEI HAN, KEFENG LIU, AND WEIPING ZHANG

Abstract. It has been shown that the Alvarez-Gaumé-Witten miraculous anomaly cancellation formula in type IIB superstring theory and its various generalizations can be derived from modularity of certain characteristic forms. In this paper, we show that the Green-Schwarz formula and the Schwarz-Witten formula in type I superstring theory can also be derived from the modularity of those characteristic forms and thus unify the Alvarez-Gaumé-Witten formula, the Green-Schwarz formula as well as the Schwarz-Witten formula in the same framework. Various generalizations of these remarkable formulas are also established.

Introduction

Let $Z \to X \to B$ be a fiber bundle with fiber Z being 10 dimensional. Let TZ be the vertical tangent bundle equipped with a metric g^{TZ} and an associated Levi-Civita connection ∇^{TZ} (cf. [3, Proposition 10.2]). Let $R^{TZ} = (\nabla^{TZ})^2$ be the curvature of ∇^{TZ}. Let $T_{\mathbf{C}}Z$ be the complexification of TZ with the induced Hermitian connection $\nabla^{T_{\mathbf{C}}Z}$. Let $\widehat{A}(TZ, \nabla^{TZ})$, $L(TZ, \nabla^{TZ})$ and $\mathrm{ch}(T_{\mathbf{C}}Z, \nabla^{T_{\mathbf{C}}Z})$ be the Hirzebruch $\widehat{A}$-form, the Hirzebruch L-form and the Chern character form respectively (cf. [3] and [18]).

The Alvarez-Gaumé-Witten "miraculous anomaly cancellation formula" [1] in type IIB superstring theory asserts that

$$\{L(TZ, \nabla^{TZ})\}^{(12)} - 8\{\widehat{A}(TZ, \nabla^{TZ})\mathrm{ch}(T_{\mathbf{C}}Z, \nabla^{T_{\mathbf{C}}Z})\}^{(12)} + 16\{\widehat{A}(TZ, \nabla^{TZ})\}^{(12)} = 0, \tag{0.1}$$

which assures that the corresponding theory is anomaly-free.

On the other hand, Green and Schwarz ([5], see also [16]) discovered that the anomaly in type I superstring theory with gauge group $SO(32)$ cancels. They found that when the gauge group is $SO(32)$, the anomaly factorizes so that there is a Chern-Simons counterterm making the anomaly cancelled. More precisely, let F be a 32 dimensional Euclidean vector bundle over X with Euclidean connection ∇^F and $F_{\mathbf{C}}$ the complexfication of F with the induced Hermitian connection $\nabla^{F_{\mathbf{C}}}$, then the Green-Schwarz formula reads as follows,[1]

$$\begin{aligned} &\{\widehat{A}(TZ)\mathrm{ch}(\wedge^2 F_{\mathbf{C}})\}^{(12)} + \{\widehat{A}(TZ)\mathrm{ch}(T_{\mathbf{C}}Z)\}^{(12)} - 2\{\widehat{A}(TZ)\}^{(12)} \\ =&(p_1(TZ) - p_1(F)) \cdot \frac{1}{24}\left(\frac{-3p_1(TZ)^2 + 4p_2(TZ)}{8} - 2p_1(F)^2 + 4p_2(F) + \frac{1}{2}p_1(TZ)p_1(F)\right), \end{aligned} \tag{0.2}$$

where $p_i(TZ)$, $p_i(F), 1 \le i \le 2$, are the Pontryagin forms of $(TZ, \nabla^{TZ}), (F, \nabla^F)$ respectively.

The above formulas of Alvarez-Gaumé-Witten and Green-Schwarz have played crucial roles in the early development of superstring theory.

More recently, Schwarz and Witten [17] analyzed the anomaly in type I theory with additional spacetime-filling D-branes and anti-D-branes pairs and found a similar factorization. More precisely, let F_1 be an m dimensional Euclidean vector bundle over X equipped with a Euclidean connection ∇^{F_1} and F_2 be an n dimensional Euclidean vector bundle over X equipped with a Euclidean

[1] In what follows, we will write characteristic forms without specifying the connections when there is no confusion.

connection ∇^{F_2}, when $m = n + 32$, one has

(0.3)

$$\begin{aligned}
&\{\widehat{A}(TZ)\mathrm{ch}(\wedge^2 F_{1\mathbf{C}})\}^{(12)} + \{\widehat{A}(TZ)\mathrm{ch}(S^2 F_{2\mathbf{C}})\}^{(12)} - \{\widehat{A}(TZ)\mathrm{ch}(F_{1\mathbf{C}} \otimes F_{2\mathbf{C}})\}^{(12)} \\
&+ \{\widehat{A}(TZ)\mathrm{ch}(T_{\mathbf{C}}Z)\}^{(12)} - 2\{\widehat{A}(TZ)\}^{(12)} \\
&= (p_1(TZ) - p_1(F_1) + p_1(F_2)) \\
&\cdot \frac{1}{24}\left(\frac{-3p_1(TZ)^2 + 4p_2(TZ)}{8} - 2p_1(F_1)^2 + 4p_2(F_1) + 2p_1(F_2)^2 - 4p_2(F_2) + \frac{1}{2}p_1(TZ)(p_1(F_1) - p_1(F_2))\right).
\end{aligned}$$

Therefore, similarly, there is a Chern-Simons counterterm to make the anomaly cancelled.

In [12, 6], it is shown that the Alvarez-Gaumé-Witten "miraculous anomaly cancellation formula" can be derived from the modularity of certain characteristic forms. In fact, let V be a Euclidean vector bundle equipped with a Euclidean connection over X, one can construct two characteristic forms $P_1(TZ, V, \tau)$ and $P_2(TZ, V, \tau)$ such that when $p_1(TZ) = p_1(V)$, $P_1(TZ, V, \tau)$ and $P_2(TZ, V, \tau)$ are level 2 modular forms over $\Gamma_0(2)$ and $\Gamma^0(2)$ respectively. Moreover they are modularly related and form what we call a modular pair (see page 9 for more details). The Alvarez-Gaumé-Witten formula can then be deduced from this modular pair $(P_1(TZ, V, \tau), P_2(TZ, V, \tau))$ if one sets $V = TZ$. This construction is further generalized in [8] to the case where a complex line bundle is involved, in dealing with the Ochanine congruence [15] on spin^c manifolds.

In a recent article [7], in using the Eisenstein series $E_2(\tau)$, we constructed a pair of modularly related characteristic forms $(\mathcal{P}_1(TZ, V, \xi, \tau), \mathcal{P}_2(TZ, V, \xi, \tau))$ without assuming $p_1(TZ) = p_1(V)$. When $p_1(TZ) = p_1(V)$ and ξ is trivial, $(\mathcal{P}_1(TZ, V, \xi, \tau), \mathcal{P}_2(TZ, V, \xi, \tau))$ degenerates to $(P_1(TZ, V, \tau), P_2(TZ, V, \tau))$.

In the current paper, we will show that the formulas due to Green-Schwarz (0.3) and Schwarz-Witten (0.4) can also be deduced from the modularity of the pair $(\mathcal{P}_1(TZ, V, \xi, \tau), \mathcal{P}_2(TZ, V, \xi, \tau))$. Actually, we need only to make use of the modularity of $\mathcal{P}_2(TZ, V, \xi, \tau)$ by replacing V by a super vector bundle $F_1 - F_2$. Our method also generates many generalizations of the Green-Schwarz and Schwarz-Witten formulas. See Theorem 1.1 and its corollaries for more details.

It is quite amazing that all of the three anomaly cancellation formulas due to Alvarez-Gaumé-Witten, Green-Schwarz, as well as Schwarz-Witten, can be unified through a single modular pair $(\mathcal{P}_1(TZ, V, \xi, \tau), \mathcal{P}_2(TZ, V, \xi, \tau))$. It illustrates one of the deep implications of modularity in physics.

In the rest of this paper, we will first present the Green-Schwarz type factorization formulas in Section 1 and then show how to derive them from modularity in Section 2.

1. Green-Schwarz Type Factorization Formulas

The purpose of this section is to present various generalizations of the Green-Schwarz formula and the Schwarz-Witten formula.

Let $Z \to X \to B$ be a fiber bundle with fiber Z being 10 dimensional. Let TZ be the vertical tangent bundle equipped with a metric g^{TZ} and an associated Levi-Civita connection ∇^{TZ} (cf. [3, Proposition 10.2]). Let $R^{TZ} = (\nabla^{TZ})^2$ be the curvature of ∇^{TZ}. Let $T_{\mathbf{C}}Z$ be the complexification of TZ with the induced Hermitian connection $\nabla^{T_{\mathbf{C}}Z}$.

Let F_1 be an m dimensional Euclidean vector bundle over X equipped with a Euclidean connection ∇^{F_1} and F_2 be an n dimensional Euclidean vector bundle over X equipped with a Euclidean connection ∇^{F_2}.

Let ξ be a rank two real oriented Euclidean vector bundle over X carrying a Euclidean connection ∇^{ξ}. Let $c = e(\xi, \nabla^{\xi})$ be the Euler form canonically associated to ∇^{ξ}.

If E is a real (resp. complex) vector bundle over X, set $\widetilde{E} = E - \dim E \in KO(X)$ (resp. $K(X)$).

If ω is a differential form, denote the degree j-component of ω by $\omega^{(j)}$.

Theorem 1.1. *The following identity holds,*

$$
\begin{aligned}
&\{\widehat{A}(TZ)e^{\frac{c}{2}}\mathrm{ch}(\wedge^2 F_{1\mathbf{C}})\}^{(12)}+\{\widehat{A}(TZ)e^{\frac{c}{2}}\mathrm{ch}(S^2F_{2\mathbf{C}})\}^{(12)}-\{\widehat{A}(TZ)e^{\frac{c}{2}}\mathrm{ch}(F_{1\mathbf{C}}\otimes F_{2\mathbf{C}})\}^{(12)}\\
&+\{\widehat{A}(TZ)e^{\frac{c}{2}}\mathrm{ch}(T_{\mathbf{C}}Z)\}^{(12)}+\left(\frac{(m-n-32)(m-n-31)}{2}-2\right)\{\widehat{A}(TZ)e^{\frac{c}{2}}\}^{(12)}\\
&-(m-n-32)\{\widehat{A}(TZ)e^{\frac{c}{2}}\mathrm{ch}(F_{1\mathbf{C}}-F_{2\mathbf{C}})\}^{(12)}\\
&+5\{\widehat{A}(TZ)e^{\frac{c}{2}}\mathrm{ch}(\widetilde{\xi_{\mathbf{C}}}\otimes\widetilde{\xi_{\mathbf{C}}})\}^{(12)}+3\{\widehat{A}(TZ)e^{\frac{c}{2}}\mathrm{ch}((m-n-31-F_{1\mathbf{C}}+F_{2\mathbf{C}})\otimes\widetilde{\xi_{\mathbf{C}}})\}^{(12)}\\
=&(p_1(TZ)-p_1(F_1)+p_1(F_2))\\
&\cdot\left\{-\frac{e^{\frac{1}{24}(p_1(TZ)-p_1(F_1)+p_1(F_2))}-1}{p_1(TZ)-p_1(F_1)+p_1(F_2)}\widehat{A}(TZ)e^{\frac{c}{2}}\mathrm{ch}(\mathfrak{A})+e^{\frac{1}{24}(p_1(TZ)-p_1(F_1)+p_1(F_2))}\widehat{A}(TZ)e^{\frac{c}{2}}\right\}^{(8)},
\end{aligned}
\tag{1.1}
$$

where

$$
\begin{aligned}
\mathfrak{A}=&\wedge^2 F_{1\mathbf{C}}+S^2F_{2\mathbf{C}}-F_{1\mathbf{C}}\otimes F_{2\mathbf{C}}+T_{\mathbf{C}}Z+\frac{(m-n-32)(m-n-31)}{2}-2\\
&-(m-n-32)(F_{1\mathbf{C}}-F_{2\mathbf{C}})+5\widetilde{\xi_{\mathbf{C}}}\otimes\widetilde{\xi_{\mathbf{C}}}+3(m-n-31-F_{1\mathbf{C}}+F_{2\mathbf{C}})\otimes\widetilde{\xi_{\mathbf{C}}};
\end{aligned}
\tag{1.2}
$$

if ξ is trivial, the following identity holds,

$$
\begin{aligned}
&\{\widehat{A}(TZ)\mathrm{ch}(\wedge^2 F_{1\mathbf{C}})\}^{(12)}+\{\widehat{A}(TZ)\mathrm{ch}(S^2F_{2\mathbf{C}})\}^{(12)}-\{\widehat{A}(TZ)\mathrm{ch}(F_{1\mathbf{C}}\otimes F_{2\mathbf{C}})\}^{(12)}\\
&+\{\widehat{A}(TZ)\mathrm{ch}(T_{\mathbf{C}}Z)\}^{(12)}+\left(\frac{(m-n-32)(m-n-31)}{2}-2\right)\{\widehat{A}(TZ)\}^{(12)}\\
&-(m-n-32)\{\widehat{A}(TZ)\mathrm{ch}(F_{1\mathbf{C}}-F_{2\mathbf{C}})\}^{(12)}\\
=&(p_1(TZ)-p_1(F_1)+p_1(F_2))\\
&\cdot\left\{-\frac{e^{\frac{1}{24}(p_1(TZ)-p_1(F_1)+p_1(F_2))}-1}{p_1(TZ)-p_1(F_1)+p_1(F_2)}\widehat{A}(TZ)\mathrm{ch}(\mathfrak{B})+e^{\frac{1}{24}(p_1(TZ)-p_1(F_1)+p_1(F_2))}\widehat{A}(TZ)\right\}^{(8)},
\end{aligned}
\tag{1.3}
$$

where

$$
\begin{aligned}
\mathfrak{B}=&\wedge^2 F_{1\mathbf{C}}+S^2F_{2\mathbf{C}}-F_{1\mathbf{C}}\otimes F_{2\mathbf{C}}+T_{\mathbf{C}}Z+\frac{(m-n-32)(m-n-31)}{2}-2\\
&-(m-n-32)(F_{1\mathbf{C}}-F_{2\mathbf{C}}).
\end{aligned}
\tag{1.4}
$$

Putting $m=n+32$ in Theorem 1, we get

Corollary 1. *If* $\dim F_1-\dim F_2=32$, *the following identity holds,*

$$
\begin{aligned}
&\{\widehat{A}(TZ)e^{\frac{c}{2}}\mathrm{ch}(\wedge^2 F_{1\mathbf{C}})\}^{(12)}+\{\widehat{A}(TZ)e^{\frac{c}{2}}\mathrm{ch}(S^2F_{2\mathbf{C}})\}^{(12)}-\{\widehat{A}(TZ)e^{\frac{c}{2}}\mathrm{ch}(F_{1\mathbf{C}}\otimes F_{2\mathbf{C}})\}^{(12)}\\
&+\{\widehat{A}(TZ)e^{\frac{c}{2}}\mathrm{ch}(T_{\mathbf{C}}Z)\}^{(12)}-2\{\widehat{A}(TZ)e^{\frac{c}{2}}\}^{(12)}\\
&+5\{\widehat{A}(TZ)e^{\frac{c}{2}}\mathrm{ch}(\widetilde{\xi_{\mathbf{C}}}\otimes\widetilde{\xi_{\mathbf{C}}})\}^{(12)}+3\{\widehat{A}(TZ)e^{\frac{c}{2}}\mathrm{ch}((1-F_{1\mathbf{C}}+F_{2\mathbf{C}})\otimes\widetilde{\xi_{\mathbf{C}}})\}^{(12)}\\
=&(p_1(TZ)-p_1(F_1)+p_1(F_2))\\
&\cdot\left\{-\frac{e^{\frac{1}{24}(p_1(TZ)-p_1(F_1)+p_1(F_2))}-1}{p_1(TZ)-p_1(F_1)+p_1(F_2)}\widehat{A}(TZ)e^{\frac{c}{2}}\mathrm{ch}(\mathfrak{C})+e^{\frac{1}{24}(p_1(TZ)-p_1(F_1)+p_1(F_2))}\widehat{A}(TZ)e^{\frac{c}{2}}\right\}^{(8)},
\end{aligned}
\tag{1.5}
$$

where

$$
\begin{aligned}
\mathfrak{C}=&\wedge^2 F_{1\mathbf{C}}+S^2F_{2\mathbf{C}}-F_{1\mathbf{C}}\otimes F_{2\mathbf{C}}+T_{\mathbf{C}}Z-2\\
&+5\widetilde{\xi_{\mathbf{C}}}\otimes\widetilde{\xi_{\mathbf{C}}}+3(1-F_{1\mathbf{C}}+F_{2\mathbf{C}})\otimes\widetilde{\xi_{\mathbf{C}}};
\end{aligned}
\tag{1.6}
$$

if ξ is trivial, we obtain the Schwarz-Witten formula (0.4),

$$\begin{aligned}
&\{\widehat{A}(TZ)\mathrm{ch}(\wedge^2 F_{1\mathbf{C}})\}^{(12)}+\{\widehat{A}(TZ)\mathrm{ch}(S^2F_{2\mathbf{C}})\}^{(12)}-\{\widehat{A}(TZ)\mathrm{ch}(F_{1\mathbf{C}}\otimes F_{2\mathbf{C}})\}^{(12)}\\
&+\{\widehat{A}(TZ)\mathrm{ch}(T_{\mathbf{C}}Z)\}^{(12)}-2\{\widehat{A}(TZ)\}^{(12)}\\
=&(p_1(TZ)-p_1(F_1)+p_1(F_2))\\
&\cdot\left\{-\frac{e^{\frac{1}{24}(p_1(TZ)-p_1(F_1)+p_1(F_2))}-1}{p_1(TZ)-p_1(F_1)+p_1(F_2)}\widehat{A}(TZ)\mathrm{ch}(\mathfrak{D})+e^{\frac{1}{24}(p_1(TZ)-p_1(F_1)+p_1(F_2))}\widehat{A}(TZ)\right\}^{(8)},
\end{aligned}\tag{1.7}$$

where

$$\mathfrak{D}=\wedge^2 F_{1\mathbf{C}}+S^2F_{2\mathbf{C}}-F_{1\mathbf{C}}\otimes F_{2\mathbf{C}}+T_{\mathbf{C}}Z-2.\tag{1.8}$$

Remark 1.1. *It can be checked by direct computation that when $m=n+32$, one indeed has*

$$\begin{aligned}
&\frac{1}{24}\left(\frac{-3p_1(TZ)^2+4p_2(TZ)}{8}-2p_1(F_1)^2+4p_2(F_1)+2p_1(F_2)^2-4p_2(F_2)+\frac{1}{2}p_1(TZ)(p_1(F_1)-p_1(F_2)\right)\\
=&\left\{-\frac{e^{\frac{1}{24}(p_1(TZ)-p_1(F_1)+p_1(F_2))}-1}{p_1(TZ)-p_1(F_1)+p_1(F_2)}\widehat{A}(TZ)\mathrm{ch}(\mathfrak{D})+e^{\frac{1}{24}(p_1(TZ)-p_1(F_1)+p_1(F_2))}\widehat{A}(TZ)\right\}^{(8)}.
\end{aligned}\tag{1.9}$$

If we set $n=0$ in Theorem 1, we get

Corollary 2. *If $\dim F=m$, then the following identity holds,*

$$\begin{aligned}
&\{\widehat{A}(TZ)e^{\frac{c}{2}}\mathrm{ch}(\wedge^2 F_{\mathbf{C}})\}^{(12)}+\{\widehat{A}(TZ)e^{\frac{c}{2}}\mathrm{ch}(T_{\mathbf{C}}Z)\}^{(12)}+\left(\frac{(m-32)(m-31)}{2}-2\right)\{\widehat{A}(TZ)e^{\frac{c}{2}}\}^{(12)}\\
&-(m-32)\{\widehat{A}(TZ)e^{\frac{c}{2}}\mathrm{ch}(F_{\mathbf{C}})\}^{(12)}\\
&+5\{\widehat{A}(TZ)e^{\frac{c}{2}}\mathrm{ch}(\widetilde{\xi_{\mathbf{C}}}\otimes\widetilde{\xi_{\mathbf{C}}})\}^{(12)}+3\{\widehat{A}(TZ)e^{\frac{c}{2}}\mathrm{ch}((m-31-F_{\mathbf{C}})\otimes\widetilde{\xi_{\mathbf{C}}})\}^{(12)}\\
=&(p_1(TZ)-p_1(F))\\
&\cdot\left\{-\frac{e^{\frac{1}{24}(p_1(TZ)-p_1(F))}-1}{p_1(TZ)-p_1(F)}\widehat{A}(TZ)e^{\frac{c}{2}}\mathrm{ch}(\mathfrak{E})+e^{\frac{1}{24}(p_1(TZ)-p_1(F))}\widehat{A}(TZ)e^{\frac{c}{2}}\right\}^{(8)},
\end{aligned}\tag{1.10}$$

where

$$\begin{aligned}
\mathfrak{E}=&\wedge^2 F_{1\mathbf{C}}+T_{\mathbf{C}}Z+\frac{(m-32)(m-31)}{2}-2\\
&-(m-32)(F_{\mathbf{C}})+5\widetilde{\xi_{\mathbf{C}}}\otimes\widetilde{\xi_{\mathbf{C}}}+3(m-31-F_{\mathbf{C}})\otimes\widetilde{\xi_{\mathbf{C}}};
\end{aligned}\tag{1.11}$$

if ξ is trivial, the following identity holds,

$$\begin{aligned}
&\{\widehat{A}(TZ)\mathrm{ch}(\wedge^2 F_{\mathbf{C}})\}^{(12)}+\{\widehat{A}(TZ)\mathrm{ch}(T_{\mathbf{C}}Z\}^{(12)}+\left(\frac{(m-32)(m-31)}{2}-2\right)\{\widehat{A}(TZ)\}^{(12)}\\
&-(m-32)\{\widehat{A}(TZ)\mathrm{ch}(F_{\mathbf{C}})\}^{(12)}\\
=&(p_1(TZ)-p_1(F))\left\{-\frac{e^{\frac{1}{24}(p_1(TZ)-p_1(F))}-1}{p_1(TZ)-p_1(F)}\widehat{A}(TZ)\mathrm{ch}(\mathfrak{F})+e^{\frac{1}{24}(p_1(TZ)-p_1(F))}\widehat{A}(TZ)\right\}^{(8)},
\end{aligned}\tag{1.12}$$

where

$$\mathfrak{F}=\wedge^2 F_{\mathbf{C}}+T_{\mathbf{C}}Z+\frac{(m-32)(m-31)}{2}-2-(m-32)F_{\mathbf{C}}.\tag{1.13}$$

Putting $m = 32$ in the above corollary, we get

Corollary 3. *If* $\dim F = 32$*, the following identity holds,*

$$\begin{aligned}
&\{\widehat{A}(TZ)e^{\frac{c}{2}}\mathrm{ch}(\wedge^2 F_{\mathbf{C}})\}^{(12)} + \{\widehat{A}(TZ)e^{\frac{c}{2}}\mathrm{ch}(T_{\mathbf{C}}Z)\}^{(12)} - 2\{\widehat{A}(TZ)e^{\frac{c}{2}}\}^{(12)} \\
&+ 5\{\widehat{A}(TZ)e^{\frac{c}{2}}\mathrm{ch}(\widetilde{\xi_{\mathbf{C}}}\otimes\widetilde{\xi_{\mathbf{C}}})\}^{(12)} + 3\{\widehat{A}(TZ)e^{\frac{c}{2}}\mathrm{ch}((1-F_{\mathbf{C}})\otimes\widetilde{\xi_{\mathbf{C}}})\}^{(12)} \\
=&(p_1(TZ)-p_1(F)) \\
&\cdot\left\{-\frac{e^{\frac{1}{24}(p_1(TZ)-p_1(F))}-1}{p_1(TZ)-p_1(F)}\widehat{A}(TZ)e^{\frac{c}{2}}\mathrm{ch}(\mathfrak{G}) + e^{\frac{1}{24}(p_1(TZ)-p_1(F))}\widehat{A}(TZ)e^{\frac{c}{2}}\right\}^{(8)},
\end{aligned} \tag{1.14}$$

where

$$\mathfrak{G} = \wedge^2 F_{\mathbf{C}} + T_{\mathbf{C}}Z - 2 + 5\widetilde{\xi_{\mathbf{C}}}\otimes\widetilde{\xi_{\mathbf{C}}} + 3(1-F_{\mathbf{C}})\otimes\widetilde{\xi_{\mathbf{C}}}; \tag{1.15}$$

if ξ *is trivial, we obtain the Green-Schwarz formula (0.3),*

(1.16)

$$\begin{aligned}
&\{\widehat{A}(TZ)\mathrm{ch}(\wedge^2 F_{\mathbf{C}})\}^{(12)} + \{\widehat{A}(TZ)\mathrm{ch}(T_{\mathbf{C}}Z\}^{(12)} - 2\{\widehat{A}(TZ)\}^{(12)} \\
=&(p_1(TZ)-p_1(F))\left\{-\frac{e^{\frac{1}{24}(p_1(TZ)-p_1(F))}-1}{p_1(TZ)-p_1(F)}\widehat{A}(TZ)\mathrm{ch}(\wedge^2 F_{\mathbf{C}} + T_{\mathbf{C}}Z - 2) + e^{\frac{1}{24}(p_1(TZ)-p_1(F))}\widehat{A}(TZ)\right\}^{(8)}.
\end{aligned}$$

2. Derivation of the Green-Schwarz Type Factorizations from Modularity

In this section, we will derive the Green-Schwarz type factorization formulas presented in Section 1 via the modularity of $\mathcal{P}_2(TZ, F_1 - F_2, \xi, \tau)$.

2.1. **Preliminaries.** In this subsection, we recall some basic knowledge about the Jacobi theta functions, modular forms and Eisenstein series. Although we will not use all the things recalled here, we still put them in this subsection for completeness.

Let $SL_2(\mathbf{Z}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| a, b, c, d \in \mathbf{Z},\ ad - bc = 1 \right\}$ as usual be the modular group. Let $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ be the two generators of $SL_2(\mathbf{Z})$. Their actions on $\mathbf{H}$ are given by $S : \tau \to -\frac{1}{\tau}$, $\quad T : \tau \to \tau + 1$.

The four Jacobi theta functions are defined as follows (cf. [4]):

$$\theta(v,\tau) = 2q^{1/8}\sin(\pi v)\prod_{j=1}^{\infty}\left[(1-q^j)(1-e^{2\pi\sqrt{-1}v}q^j)(1-e^{-2\pi\sqrt{-1}v}q^j)\right],$$

$$\theta_1(v,\tau) = 2q^{1/8}\cos(\pi v)\prod_{j=1}^{\infty}\left[(1-q^j)(1+e^{2\pi\sqrt{-1}v}q^j)(1+e^{-2\pi\sqrt{-1}v}q^j)\right],$$

$$\theta_2(v,\tau) = \prod_{j=1}^{\infty}\left[(1-q^j)(1-e^{2\pi\sqrt{-1}v}q^{j-1/2})(1-e^{-2\pi\sqrt{-1}v}q^{j-1/2})\right],$$

$$\theta_3(v,\tau) = \prod_{j=1}^{\infty}\left[(1-q^j)(1+e^{2\pi\sqrt{-1}v}q^{j-1/2})(1+e^{-2\pi\sqrt{-1}v}q^{j-1/2})\right].$$

They are all holomorphic functions for $(v,\tau) \in \mathbf{C}\times\mathbf{H}$, where $\mathbf{C}$ is the complex plane and $\mathbf{H}$ is the upper half plane.

When acted by S and T, the theta functions obey the following transformation laws (cf. [4]),

$$\theta(v,\tau+1)=e^{\frac{\pi\sqrt{-1}}{4}}\theta(v,\tau),\quad \theta\left(v,-1/\tau\right)=\frac{1}{\sqrt{-1}}\left(\frac{\tau}{\sqrt{-1}}\right)^{1/2}e^{\pi\sqrt{-1}\tau v^2}\theta\left(\tau v,\tau\right)\ ; \tag{2.1}$$

$$\theta_1(v,\tau+1)=e^{\frac{\pi\sqrt{-1}}{4}}\theta_1(v,\tau),\quad \theta_1\left(v,-1/\tau\right)=\left(\frac{\tau}{\sqrt{-1}}\right)^{1/2}e^{\pi\sqrt{-1}\tau v^2}\theta_2(\tau v,\tau)\ ; \tag{2.2}$$

$$\theta_2(v,\tau+1)=\theta_3(v,\tau),\quad \theta_2\left(v,-1/\tau\right)=\left(\frac{\tau}{\sqrt{-1}}\right)^{1/2}e^{\pi\sqrt{-1}\tau v^2}\theta_1(\tau v,\tau)\ ; \tag{2.3}$$

$$\theta_3(v,\tau+1)=\theta_2(v,\tau),\quad \theta_3\left(v,-1/\tau\right)=\left(\frac{\tau}{\sqrt{-1}}\right)^{1/2}e^{\pi\sqrt{-1}\tau v^2}\theta_3(\tau v,\tau)\ . \tag{2.4}$$

Definition 2.1. *Let Γ be a subgroup of $SL_2(\mathbf{Z})$. A modular form over Γ is a holomorphic function $f(\tau)$ on $\mathbf{H}\cup\{\infty\}$ such that for any $g=\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in\Gamma$, the following property holds,*

$$f(g\tau):=f\left(\frac{a\tau+b}{c\tau+d}\right)=\chi(g)(c\tau+d)^l f(\tau),$$

where $\chi:\Gamma\to\mathbf{C}^*$ is a character of Γ and l is called the weight of f.

Let

$$E_{2k}=1-\frac{4k}{B_{2k}}\sum_{n=1}^{\infty}\left(\sum_{d|n}d^{2k-1}\right)q^n \tag{2.5}$$

be the Eisenstein series, where B_{2k} is the $2k$-th Bernoulli number.

When $k>1$, E_{2k} is a modular form of weight $2k$ over $SL_2(\mathbf{Z})$. However, unlike other Eisenstein series, $E_2(\tau)=1-24\sum_{n=1}^{\infty}\left(\sum_{d|n}d\right)q^n=1-24q-72q^2-96q^3-\cdots$ is not a modular form over $SL(2,\mathbf{Z})$, instead it is a quasimodular form over $SL(2,\mathbf{Z})$ satisfying

$$E_2\left(\frac{a\tau+b}{c\tau+d}\right)=(c\tau+d)^2E_2(\tau)-\frac{6\sqrt{-1}c(c\tau+d)}{\pi}. \tag{2.6}$$

In particular, we have

$$E_2(\tau+1)=E_2(\tau), \tag{2.7}$$

$$E_2\left(-\frac{1}{\tau}\right)=\tau^2E_2(\tau)-\frac{6\sqrt{-1}\tau}{\pi}. \tag{2.8}$$

For the precise definition of quasimodular forms, see [11].

Let $\Gamma_0(2)=\left\{\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in SL_2(\mathbf{Z})\middle| c\equiv 0 \pmod 2\right\}$, $\Gamma^0(2)=\left\{\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in SL_2(\mathbf{Z})\middle| b\equiv 0 \pmod 2\right\}$ be the two modular subgroups of $SL_2(\mathbf{Z})$. It is known that the generators of $\Gamma_0(2)$ are T, ST^2S and the generators of $\Gamma^0(2)$ are STS, T^2STS (cf. [4]).

Consider the q-series:

$$\delta_1(\tau)=\frac{1}{4}+6\sum_{n=1}^{\infty}\sum_{\substack{d|n\\ d\ odd}}dq^n=\frac{1}{4}+6q+6q^2+\cdots, \tag{2.9}$$

$$\varepsilon_1(\tau) = \frac{1}{16} + \sum_{n=1}^{\infty} \sum_{d|n} (-1)^d d^3 q^n = \frac{1}{16} - q + 7q^2 + \cdots , \tag{2.10}$$

$$\delta_2(\tau) = -\frac{1}{8} - 3 \sum_{n=1}^{\infty} \sum_{\substack{d|n \\ d \ odd}} d q^{n/2} = -\frac{1}{8} - 3q^{1/2} - 3q - \cdots , \tag{2.11}$$

$$\varepsilon_2(\tau) = \sum_{n=1}^{\infty} \sum_{\substack{d|n \\ n/d \ odd}} d^3 q^{n/2} = q^{1/2} + 8q + \cdots . \tag{2.12}$$

Simply writing $\theta_j = \theta_j(0, \tau),\ 1 \leq j \leq 3$, then we have (cf. [10] and [13]),

$$\delta_1(\tau) = \frac{1}{8}(\theta_2^4 + \theta_3^4), \quad \varepsilon_1(\tau) = \frac{1}{16}\theta_2^4\theta_3^4 ,$$
$$\delta_2(\tau) = -\frac{1}{8}(\theta_1^4 + \theta_3^4), \quad \varepsilon_2(\tau) = \frac{1}{16}\theta_1^4\theta_3^4 .$$

If Γ is a modular subgroup, let $M_{\mathbf{R}}(\Gamma)$ denote the ring of modular forms over Γ with real Fourier coefficients.

Lemma 2.1 (cf. [12])**.** *One has that $\delta_1(\tau)$ (resp. $\varepsilon_1(\tau)$) is a modular form of weight 2 (resp. 4) over $\Gamma_0(2)$, $\delta_2(\tau)$ (resp. $\varepsilon_2(\tau)$) is a modular form of weight 2 (resp. 4) over $\Gamma^0(2)$ and moreover $M_{\mathbf{R}}(\Gamma^0(2)) = \mathbf{R}[\delta_2(\tau), \varepsilon_2(\tau)]$. Moreover, we have transformation laws*

$$\delta_2\left(-\frac{1}{\tau}\right) = \tau^2 \delta_1(\tau), \qquad \varepsilon_2\left(-\frac{1}{\tau}\right) = \tau^4 \varepsilon_1(\tau). \tag{2.13}$$

2.2. **The modular form $\mathcal{P}_2(TZ, F_1 - F_2, \xi, \tau)$.** Let F (resp. G) be a Hermitian vector bundle over X equipped with a Hermitian connection ∇^F (resp. ∇^G). For any complex number t, let

$$\Lambda_t(F) = \mathbf{C}|_X + tF + t^2\Lambda^2(F) + \cdots , \quad S_t(F) = \mathbf{C}|_X + tF + t^2 S^2(F) + \cdots$$

denote respectively the total exterior and symmetric powers of F, which live in $K(X)[[t]]$. The following relations between these two operations hold (cf. [2, Chap. 3]),

$$S_t(F) = \frac{1}{\Lambda_{-t}(F)}, \quad \Lambda_t(F - G) = \frac{\Lambda_t(F)}{\Lambda_t(G)}. \tag{2.14}$$

The connections ∇^F, ∇^G naturally induce connections on $\Lambda_t(F), S_t(F)$ etc. Moreover, if $\{\omega_i\}$, $\{\omega_j{}'\}$ are formal Chern roots for the Hermitian vector bundles F, G respectively, then (cf. [9, Chap. 1]),

$$\mathrm{ch}\left(\Lambda_t(F), \nabla^{\Lambda_t(F)}\right) = \prod_i (1 + e^{\omega_i} t) \tag{2.15}$$

and we have the following formulas for Chern character forms,

$$\mathrm{ch}\left(S_t(F), \nabla^{S_t(F)}\right) = \frac{1}{\mathrm{ch}\left(\Lambda_{-t}(F), \nabla^{\Lambda_{-t}(F)}\right)} = \frac{1}{\prod_i (1 - e^{\omega_i} t)} , \tag{2.16}$$

$$\mathrm{ch}\left(\Lambda_t(F - G), \nabla^{\Lambda_t(F-G)}\right) = \frac{\mathrm{ch}\left(\Lambda_t(F), \nabla^{\Lambda_t(F)}\right)}{\mathrm{ch}\left(\Lambda_t(G), \nabla^{\Lambda_t(G)}\right)} = \frac{\prod_i (1 + e^{\omega_i} t)}{\prod_j (1 + e^{\omega_j{}'} t)} . \tag{2.17}$$

Let $q = e^{2\pi\sqrt{-1}\tau}$ with $\tau \in \mathbf{H}$, the upper half complex plane.

Set

$$\begin{aligned}\Theta_2(T_{\mathbf{C}}Z, F_{1\mathbf{C}} - F_{2\mathbf{C}}, \xi_{\mathbf{C}}) =& \bigotimes_{u=1}^{\infty} S_{q^u}(\widetilde{T_{\mathbf{C}}Z}) \otimes \bigotimes_{v=1}^{\infty} \Lambda_{-q^{v-\frac{1}{2}}}(\widetilde{F_{1\mathbf{C}}} - \widetilde{F_{2\mathbf{C}}} - 2\widetilde{\xi_{\mathbf{C}}}) \\ & \otimes \bigotimes_{r=1}^{\infty} \Lambda_{q^{r-\frac{1}{2}}}(\widetilde{\xi_{\mathbf{C}}}) \otimes \bigotimes_{s=1}^{\infty} \Lambda_{q^s}(\widetilde{\xi_{\mathbf{C}}}),\end{aligned} \tag{2.18}$$

which is an element in $K(X)[[q^{\frac{1}{2}}]]$.

Clearly, $\Theta_2(T_{\mathbf{C}}Z, F_{1\mathbf{C}} - F_{2\mathbf{C}}, \xi_{\mathbf{C}})$ admits a formal Fourier expansion in $q^{1/2}$ as

$$\Theta_2(T_{\mathbf{C}}Z, F_{1\mathbf{C}} - F_{2\mathbf{C}}, \xi_{\mathbf{C}}) = B_0 + B_1 q^{1/2} + B_2 q \cdots, \tag{2.19}$$

where the B_j's are elements in the semi-group formally generated by complex vector bundles over X. Moreover, they carry canonically induced connections denoted by ∇^{B_j} and let ∇^{Θ_2} be the induced connections with $q^{1/2}$-coefficients on Θ_2.

Set

$$\begin{aligned}&\mathcal{P}_2(TZ, F_1 - F_2, \xi, \tau) \\ &:= \left\{ e^{\frac{1}{24}E_2(\tau)(p_1(TZ) - p_1(F_1) + p_1(F_2))} \widehat{A}(TZ) \cosh\left(\frac{c}{2}\right) \mathrm{ch}\left(\Theta_2(T_{\mathbf{C}}Z, F_{1\mathbf{C}} - F_{2\mathbf{C}}, \xi_{\mathbf{C}})\right) \right\}^{(12)}.\end{aligned} \tag{2.20}$$

Proposition 2.1. *$\mathcal{P}_2(TZ, F_1 - F_2, \xi, \tau)$ is a modular form of weight 6 over $\Gamma^0(2)$.*

Proof. Let $\{\pm 2\pi\sqrt{-1}y_k\}$ (resp. $\{\pm 2\pi\sqrt{-1}z_k\}$, $\{\pm 2\pi\sqrt{-1}x_j\}$) be the formal Chern roots for $(F_{1\mathbf{C}}, \nabla^{F_{1\mathbf{C}}})$ (resp. $(F_{2\mathbf{C}}, \nabla^{F_{2\mathbf{C}}})$, $(TZ_{\mathbf{C}}, \nabla^{TZ_{\mathbf{C}}})$). Let $c = 2\pi\sqrt{-1}u$. By the Chern root algorithm, we have

$$\begin{aligned}&\mathcal{P}_2(TZ, F_1 - F_2, \xi, \tau) \\ &= \left\{ e^{\frac{1}{24}E_2(\tau)(p_1(TZ) - p_1(F_1) + p_1(F_2))} \widehat{A}(TZ) \cosh\left(\frac{c}{2}\right) \mathrm{ch}\left(\Theta_2(T_{\mathbf{C}}Z, F_{1\mathbf{C}} - F_{2\mathbf{C}}, \xi_{\mathbf{C}})\right) \right\}^{(12)} \\ &= \left\{ e^{\frac{1}{24}E_2(\tau)(p_1(TZ) - p_1(F_1) + p_1(F_2))} \left(\prod_{j=1}^{5} \left(x_j \frac{\theta'(0,\tau)}{\theta(x_j,\tau)} \right) \right) \left(\prod_{j=1}^{\left[\frac{m}{2}\right]} \frac{\theta_2(y_j,\tau)}{\theta_2(0,\tau)} \right) \right. \\ &\quad \left. \cdot \left(\prod_{k=1}^{\left[\frac{n}{2}\right]} \frac{\theta_2(0,\tau)}{\theta_2(z_k,\tau)} \right) \cdot \frac{\theta_2^2(0,\tau)}{\theta_2^2(u,\tau)} \frac{\theta_3(u,\tau)}{\theta_3(0,\tau)} \frac{\theta_1(u,\tau)}{\theta_1(0,\tau)} \right\}^{(12)}.\end{aligned} \tag{2.21}$$

Then we can apply the transformation laws (2.1)-(2.4) for theta functions as well as the transformation laws (2.7), (2.8) to (2.21) to get the desired results. □

In addition to the above modular form, we have also constructed in [7] the modular form

$$\begin{aligned}\mathcal{P}_1(TZ, V, \xi, \tau) :=& \left\{ e^{\frac{1}{24}E_2(\tau)(p_1(TZ) - p_1(V))} \right. \\ & \left. \cdot \frac{\widehat{A}(TZ) \det^{1/2}\left(2\cosh\left(\frac{\sqrt{-1}}{4\pi}R^V\right)\right)}{\cosh^2\left(\frac{c}{2}\right)} \mathrm{ch}\left(\Theta_1(T_{\mathbf{C}}Z, V_{\mathbf{C}}, \xi_{\mathbf{C}})\right) \right\}^{(12)},\end{aligned} \tag{2.22}$$

where

$$\begin{aligned}\Theta_1(T_{\mathbf{C}}Z, V_{\mathbf{C}}, \xi_{\mathbf{C}}) =& \bigotimes_{u=1}^{\infty} S_{q^u}(\widetilde{T_{\mathbf{C}}Z}) \otimes \bigotimes_{v=1}^{\infty} \Lambda_{q^v}(\widetilde{V_{\mathbf{C}}} - 2\widetilde{\xi_{\mathbf{C}}}) \\ & \otimes \bigotimes_{r=1}^{\infty} \Lambda_{q^{r-1/2}}(\widetilde{\xi_{\mathbf{C}}}) \otimes \bigotimes_{s=1}^{\infty} \Lambda_{-q^{s-1/2}}(\widetilde{\xi_{\mathbf{C}}}).\end{aligned} \tag{2.23}$$

We showed in [7] that $\mathcal{P}_1(TZ,V,\xi,\tau)$ is a modular form of weight 6 over $\Gamma_0(2)$ while $\mathcal{P}_2(TZ,V,\xi,\tau)$ is a modular form of weight 6 over $\Gamma^0(2)$ and moreover they are modularly related in the sense that

$$\mathcal{P}_1\left(TZ,V,\xi,-\frac{1}{\tau}\right)=2^{[\frac{\dim V}{2}]}\tau^6\mathcal{P}_2(TZ,V,\xi,\tau).$$

We call such a pair of modular forms a modular pair (see [7] for the cases of general dimensions).

One can use this modular pair $(\mathcal{P}_1(TZ,V,\xi,\tau),\mathcal{P}_2(TZ,V,\xi,\tau))$ to derive the Alvarez-Gaumé-Witten miraculous anomaly cancellation formula by setting $V=TZ$, $\xi=\mathbf{C}$ and obtain its various generalizations (see [12, 13, 8, 6, 7, 14] for details).

In the following subsection, we will use the modularity of $\mathcal{P}_2(TZ,F_1-F_2,\xi,\tau)$ to derive the Green-Schwarz type factorization formulas. It's amazing to see that all these anomaly cancellations due to Alvarez-Gaumé-Witten, Green-Schwarz as well as Schwarz-Witten can be derived from the modular pair $(\mathcal{P}_1(TZ,V,\xi,\tau),\mathcal{P}_2(TZ,V,\xi,\tau))$.

2.3. **Derivation of Green-Schwarz type factorizations from modularity.** From Proposition 2.1, we see that $\mathcal{P}_2(TZ,F_1-F_2,\xi,\tau)$ is a modular form of weight 6 over $\Gamma^0(2)$. Therefore, by Lemma 2.1, there exist $h_1,h_2\in\Omega^{12}(X)$ such that

$$\mathcal{P}_2(TZ,F_1-F_2,\xi,\tau)=h_0(8\delta_2)^3+h_1(8\delta_2)\varepsilon_2. \tag{2.24}$$

Therefore

$$\begin{aligned}&\left\{e^{\frac{1}{24}(1-24q+\cdots)(p_1(TZ)-p_1(F_1)+p_1(F_2))}\widehat{A}(TZ)\cosh\left(\frac{c}{2}\right)\mathrm{ch}\left(B_0+B_1q^{\frac{1}{2}}+B_2q+\cdots\right)\right\}^{(12)}\\ =&h_0(8\delta_2)^3+h_1(8\delta_2)\varepsilon_2\\ =&h_0(-1-24q^{\frac{1}{2}}-24q-\cdots)^3+h_1(-1-24q^{\frac{1}{2}}-24q-\cdots)(q^{\frac{1}{2}}+8q+\cdots).\end{aligned} \tag{2.25}$$

Comparing the coefficients of 1, $q^{\frac{1}{2}}$ and q in both sides of (2.25), we have

$$\left\{e^{\frac{1}{24}(p_1(TZ)-p_1(F_1)+p_1(F_2))}\widehat{A}(TZ)\cosh\left(\frac{c}{2}\right)\mathrm{ch}(B_0)\right\}^{(12)}=-h_0, \tag{2.26}$$

$$\left\{e^{\frac{1}{24}(p_1(TZ)-p_1(F_1)+p_1(F_2))}\widehat{A}(TZ)\cosh\left(\frac{c}{2}\right)\mathrm{ch}(B_1)\right\}^{(12)}=-h_1-72h_0, \tag{2.27}$$

$$\begin{aligned}&\left\{e^{\frac{1}{24}(p_1(TZ)-p_1(F_1)+p_1(F_2))}(-(p_1(TZ)+p_1(F_1)-p_1(F_2)))\widehat{A}(TZ)\cosh\left(\frac{c}{2}\right)\mathrm{ch}(B_0)\right.\\ &\left.+e^{\frac{1}{24}(p_1(TZ)-p_1(F_1)+p_1(F_2))}\widehat{A}(TZ)\cosh\left(\frac{c}{2}\right)\mathrm{ch}(B_2)\right\}^{(12)}\\ =&-32h_1-1800h_0.\end{aligned} \tag{2.28}$$

By (2.26)-(2.28), we see that

$$\begin{aligned}&\left\{e^{\frac{1}{24}(p_1(TZ)-p_1(F_1)+p_1(F_2))}(-(p_1(TZ)+p_1(F_1)-p_1(F_2)))\widehat{A}(TZ)\cosh\left(\frac{c}{2}\right)\mathrm{ch}(B_0)\right.\\ &\left.+e^{\frac{1}{24}(p_1(TZ)-p_1(F_1)+p_1(F_2))}\widehat{A}(TZ)\cosh\left(\frac{c}{2}\right)\mathrm{ch}(B_2)\right\}^{(12)}\\ =&\left\{e^{\frac{1}{24}(p_1(TZ)-p_1(F_1)+p_1(F_2))}\widehat{A}(TZ)\cosh\left(\frac{c}{2}\right)\mathrm{ch}(32B_1-504B_0)\right\}^{(12)}.\end{aligned} \tag{2.29}$$

In the following, let's expand $\Theta_2(T_{\mathbf{C}}Z, F_{1\mathbf{C}} - F_{2\mathbf{C}}, \xi_{\mathbf{C}})$ to find B_0, B_1, B_2. In fact, we have

(2.30)

$$
\begin{aligned}
&\Theta_2(T_{\mathbf{C}}Z, F_{1\mathbf{C}} - F_{2\mathbf{C}}, \xi_{\mathbf{C}}) \\
=&\bigotimes_{u=1}^{\infty} S_{q^u}(\widetilde{T_{\mathbf{C}}Z}) \otimes \bigotimes_{v=1}^{\infty} \Lambda_{-q^{v-\frac{1}{2}}}(\widetilde{F_{1\mathbf{C}}} - \widetilde{F_{2\mathbf{C}}} - 2\widetilde{\xi_{\mathbf{C}}}) \\
&\otimes \bigotimes_{r=1}^{\infty} \Lambda_{q^{r-\frac{1}{2}}}(\widetilde{\xi_{\mathbf{C}}}) \otimes \bigotimes_{s=1}^{\infty} \Lambda_{q^s}(\widetilde{\xi_{\mathbf{C}}}) \\
=&\bigotimes_{u=1}^{\infty} S_{q^u}(\widetilde{T_{\mathbf{C}}Z}) \\
&\otimes \bigotimes_{v=1}^{\infty} \Lambda_{-q^{v-\frac{1}{2}}}(\widetilde{F_{1\mathbf{C}}}) \\
&\otimes \frac{1}{\bigotimes_{v=1}^{\infty} \Lambda_{-q^{v-\frac{1}{2}}}(\widetilde{F_{2\mathbf{C}}})} \\
&\otimes \frac{1}{\bigotimes_{v=1}^{\infty} (\Lambda_{-q^{v-\frac{1}{2}}}(\widetilde{\xi_{\mathbf{C}}}))^2} \\
&\otimes \bigotimes_{r=1}^{\infty} \Lambda_{q^{r-\frac{1}{2}}}(\widetilde{\xi_{\mathbf{C}}}) \\
&\otimes \bigotimes_{s=1}^{\infty} \Lambda_{q^s}(\widetilde{\xi_{\mathbf{C}}}) \\
=&(1 + (T_{\mathbf{C}}Z - 10)q + O(q^2)) \\
&\otimes \left(1 + (m - F_{1\mathbf{C}})q^{\frac{1}{2}} + \left(\wedge^2 F_{1\mathbf{C}} - mF_{1\mathbf{C}} + \frac{m(m+1)}{2}\right) q + O(q^{\frac{3}{2}})\right) \\
&\otimes \left(1 + (F_{2\mathbf{C}} - n)q^{\frac{1}{2}} + \left(S^2 F_{2\mathbf{C}} - nF_{2\mathbf{C}} + \frac{n(n-1)}{2}\right) q + O(q^{\frac{3}{2}})\right) \\
&\otimes \left(1 + 2\widetilde{\xi_{\mathbf{C}}}q^{\frac{1}{2}} + \left(3\widetilde{\xi_{\mathbf{C}}} \otimes \widetilde{\xi_{\mathbf{C}}} + 4\widetilde{\xi_{\mathbf{C}}}\right) q + O(q^{\frac{3}{2}})\right) \\
&\otimes \left(1 + \widetilde{\xi_{\mathbf{C}}}q^{\frac{1}{2}} - 2\widetilde{\xi_{\mathbf{C}}}q + O(q^{\frac{3}{2}})\right) \\
&\otimes (1 + \widetilde{\xi_{\mathbf{C}}}q + O(q^2)) \\
=&1 + (m - F_{1\mathbf{C}} + F_{2\mathbf{C}} - n + 3\widetilde{\xi_{\mathbf{C}}})q^{\frac{1}{2}} \\
&+ \Bigg(\wedge^2 F_{1\mathbf{C}} + S^2 F_{2\mathbf{C}} - F_{1\mathbf{C}} \otimes F_{2\mathbf{C}} + T_{\mathbf{C}}Z + \frac{(m-n)^2 + (m-n)}{2} - 10 - (m-n)(F_{1\mathbf{C}} - F_{2\mathbf{C}}) \\
&\qquad +5\widetilde{\xi_{\mathbf{C}}} \otimes \widetilde{\xi_{\mathbf{C}}} + 3(m - F_{1\mathbf{C}} + F_{2\mathbf{C}} - n + 1) \otimes \widetilde{\xi_{\mathbf{C}}}\Bigg) q + O(q^{\frac{3}{2}}).
\end{aligned}
$$

Therefore, we have

(2.31)

$$\begin{aligned}
B_0 =&1,\\
B_1 =&m-F_{1\mathbf{C}}+F_{2\mathbf{C}}-n+3\widetilde{\xi_{\mathbf{C}}},\\
B_2 =&\wedge^2 F_{1\mathbf{C}}+S^2F_{2\mathbf{C}}-F_{1\mathbf{C}}\otimes F_{2\mathbf{C}}+T_{\mathbf{C}}Z+\frac{(m-n)^2+(m-n)}{2}-10-(m-n)(F_{1\mathbf{C}}-F_{2\mathbf{C}})\\
&+5\widetilde{\xi_{\mathbf{C}}}\otimes\widetilde{\xi_{\mathbf{C}}}+3(m-F_{1\mathbf{C}}+F_{2\mathbf{C}}-n+1)\otimes\widetilde{\xi_{\mathbf{C}}}
\end{aligned}$$

From (2.29), we see that

$$\begin{aligned}
&\left\{\widehat{A}(TZ)\cosh\left(\frac{c}{2}\right)\mathrm{ch}(B_2-32B_1+504B_0)\right\}^{(12)}\\
=&(p_1(TZ)-p_1(F_1)+p_1(F_2))\\
&\cdot\left\{-\frac{e^{\frac{1}{24}(p_1(TZ)-p_1(F_1)+p_1(F_2))}-1}{p_1(TZ)-p_1(F_1)+p_1(F_2)}\widehat{A}(TZ)e^{\frac{c}{2}}\mathrm{ch}(B_2-32B_1+504B_0)\right.\\
&\left.+e^{\frac{1}{24}(p_1(TZ)-p_1(F_1)+p_1(F_2))}\widehat{A}(TZ)e^{\frac{c}{2}}\mathrm{ch}(B_0)\right\}^{(8)}.
\end{aligned}\tag{2.32}$$

However, from (2.31), we have

$$\begin{aligned}
&B_2-32B_1+504B_0\\
=&\wedge^2 F_{1\mathbf{C}}+S^2F_{2\mathbf{C}}-F_{1\mathbf{C}}\otimes F_{2\mathbf{C}}+T_{\mathbf{C}}Z+\frac{(m-n-32)(m-n-31)}{2}-2\\
&-(m-n-32)(F_{1\mathbf{C}}-F_{2\mathbf{C}})+5\widetilde{\xi_{\mathbf{C}}}\otimes\widetilde{\xi_{\mathbf{C}}}+3(m-n-31-F_{1\mathbf{C}}+F_{2\mathbf{C}})\otimes\widetilde{\xi_{\mathbf{C}}}.
\end{aligned}\tag{2.33}$$

Theorem 1.1 follows from (2.32) and (2.33).

Acknowledgements. The first author is partially supported by a start-up grant from National University of Singapore. The second author is partially supported by NSF. The third author is partially supported by MOE and NNSFC.

References

[1] L. Alvarez-Gaumé and E. Witten, Gravitational anomalies, *Nucl. Physics*, B234, 269-330, (1983).

[2] M. F. Atiyah, *$K-theory$*. Benjamin, New York, 1967.

[3] N. Berline, E. Getzler and M. Vergne, *Heat Kernels and Dirac Operators*. Springer, 2002.

[4] K. Chandrasekharan, *Elliptic Functions*. Springer-Verlag, 1985.

[5] M. B. Green, J. H. Schwarz, Anomaly cancellations in supersymmetric d=10 gauge theory and superstring theory, *Physics Letters* B149 (1984) pp. 11722.

[6] F. Han and K. Liu, Gravitational anomaly cancellation and modular invariance, *Arxiv:1007.5295*.

[7] F. Han, K. Liu and W. Zhang, Modular forms and generalized anomaly cancellation formulas, *J. Geom. Phys.*, 62 (2012), pp. 1038-1053.

[8] F. Han and W. Zhang, Modular invariance, characteristic numbers and η invariants. *J. Diff. Geom.* 67 (2004), 257-288.

[9] F. Hirzebruch, *Topological Methods in Algebraic Geometry*. Springer-Verlag, 1966.

[10] F. Hirzebruch, T. Berger and R. Jung, *Manifolds and Modular Forms*. Aspects of Mathematics, vol. E20, Vieweg, Braunschweig 1992.

[11] M. Kaneko and D. Zagier, A generalized Jacobi theta function and quasimodular forms. in *The moduli space of curves,* Ed. R. Dijkgraaf, C. Faber and G. van der Geer, Boston: Birkhäuser, (1995), 165-172.

[12] K. Liu, Modular invariance and characteristic numbers, *Comm. Math. Phys*, 174, 29-42 (1995).

[13] K. Liu, Modular forms and topology. *Moonshine, the Monster, and related topics (South Hadley, MA, 1994), 237–262, Contemp. Math.*, 193, Amer. Math. Soc., Providence, RI, 1996.

[14] K. Liu and Y. Wang, A note on modular forms and generalized anomaly cancellation formulas, *Arxiv:1109.5794*.
[15] S. Ochanine, Signature modulo 16, invariants de Kervaire généralisés et nombre caractéristiques dans la K-théorie reelle. *Mémoire Soc. Math. France*, Tom. 109 (1987), 1-141.
[16] J. H. Schwarz, Anomaly cancellation: a retrospective from a modern perspective, *Arxiv*: hep-th/0107059.
[17] J. H. Schwarz and E. Witten, Anomaly analysis and brane-antibrane system, *JHEP*, 0103, 032 (2001).
[18] W. Zhang, *Lectures on Chern-Weil Theory and Witten Deformations*. Nankai Tracts in Mathematics Vol. 4, World Scientific, Singapore, 2001.

Fei Han, Department of Mathematics, National University of Singapore, Block S17, 10 Lower Kent Ridge Road, Singapore 119076 (mathanf@nus.edu.sg)

Kefeng Liu, Department of Mathematics, University of California at Los Angeles, Los Angeles, CA 90095, USA (liu@math.ucla.edu) and Center of Mathematical Sciences, Zhejiang University, 310027, P.R. China

Weiping Zhang, Chern Institute of Mathematics & LPMC, Nankai University, Tianjin 300071, P.R. China. (weiping@nankai.edu.cn)