A RESULT ON RICCI CURVATURE AND THE SECOND BETTI NUMBER

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ABSTRACT. We prove that the second Betti number of a compact Riemannian manifold vanishes under certain Ricci curved restriction.

1. INTRODUCTION

The studying of relation between curvature and topology is the central topic in Riemannian geometry. One of the strong tool is Bochner technique. It plays a very important role in understanding relation between curvature and Betti numbers. The first result in this field is Bochner's classical result (c.f. [5])

Theorem 1.1. *(Bochner 1946) Let M be a compact Riemannian manifold with Ricci curvature* $Ric_M > 0$*. Then the first Betti number* $b_1(M) = 0$*.*

Berger investigated that in what case the second Betti number vanishes. He proved the following (c.f. [1], also see [2] theorem 2.8)

Theorem 1.2. *(Berger) Let M be a compact Riemannian manifold of dimension* $n \geq 5$*. Suppose that n is odd and the sectional curvature satisfies that* $\frac{n-3}{4n-9} \leq K_M < 1$*. Then the second Betti number* $b_2(M) = 0$*.*

Consider a different curved condition, Micallef and Wang proved (c.f. [3], also see [2] theorem 2.7)

Theorem 1.3. *(Micallef-Wang) Let M be a compact Riemannian manifold of dimension* $n \geq 4$*. Suppose that n is even and M has positive isotropic curvature. Then the second Betti number* $b_2(M) = 0$*.*

Here positive isotropic curvature means, for any four othonormal vectors $e_1, e_2, e_3, e_4 \in T_pM$, the curvature tensor satisfies

$$R_{1313} + R_{1414} + R_{2323} + R_{2424} > 2|R_{1234}|.$$

Recall that the Rauch-Berger-Klingenberg's sphere theorem (c.f. [1]) states that a compact Riemannian manifold is homeomorphic to a sphere if the sectional curvatures lie in $(\frac{1}{4}, 1]$. A generalization of sphere theorem (dues to Micallef-Moore c.f. [4]) says that a compact simply connected Riemannian manifold with positive isotropic curvature is a homotopy sphere. Hence with the help of Poincare conjecture it is homeomorphic to a sphere. From the two theorems we know that the conditions in theorem 1.2 and 1.3 are very harsh.

In this note we shall use Ricci curvature to give a relaxedly sufficient condition for the second Betti number vanishing. Our main result is

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Theorem 1.4. *Let M be a compact Riemannian manifold. The dimension* $\dim(M) = 2m$ *or* $2m+1$. *Let $\bar{k}$ (resp. $\underline{k}$) be the maximal (resp. minimal) sectional curvature of M. If the Ricci curvature of M satisfies that*

$$Ric_M > \bar{k} + \frac{2m-2}{3}(\bar{k} - \underline{k}), \tag{1.1}$$

then the second Betti number $b_2(M) = 0$.

Particularly, if M is a compact Riemannian manifold with nonnegative sectional curvature, then the second Betti number vanishes provided

$$Ric_M > \frac{2m+1}{3}\bar{k}. \tag{1.2}$$

Note that there is no dimensional restriction in theorem 1.4.

Remark 1.5. 1) The condition (1.1) implies that the maximal sectional curvature $\bar{k} > 0$: If $\bar{k} \leq 0$, then

$$\bar{k} \geq Ric_M > \bar{k} + \frac{2m-2}{3}(\bar{k} - \underline{k}).$$

We get $\bar{k} < \underline{k}$. This is a contradiction.

2) Since $\bar{k} > 0$, of course (1.1) implies $Ric_M > 0$.

3) If the minimal sectional curvature $\underline{k} < 0$. Since $\bar{k} > 0$. If $\dim(M) = 2m+1$, from

$$2m\bar{k} \geq Ric_M > \bar{k} + \frac{2m-2}{3}(\bar{k} - \underline{k}),$$

one has

$$\bar{k} > \frac{2m-2}{4m-1}|\underline{k}|.$$

Similarly

$$\bar{k} > \frac{1}{2}|\underline{k}|$$

provided $\dim(M) = 2m$.

We use theorem 1.4 to test some simple examples.

Example 1.6. 1) The space form S^n, $\bar{k} = \underline{k} = 1$, $Ric = n-1 = \bar{k}$ for $n = 2$ and $Ric = n-1 > \bar{k}$ for $n \neq 2$, $b_2(S^2) = 1$ and $b_2(S^n) = 0$ for $n \neq 2$.

2) $S^2 \times S^2$ with product metric, $\bar{k} = 1, \underline{k} = 0$, $Ric = 1 < \bar{k} + \frac{2n-2}{3}(\bar{k} - \underline{k})$, $b_2(S^2 \times S^2) = 2$.

3) $S^m \times S^m, m > 4$ with product metric, $\bar{k} = 1, \underline{k} = 0$, $Ric = m-1 > \frac{2m+1}{3}\bar{k}$, $b_2 = 0$.

4) $\mathbb{CP}^n$ with Fubini-Study metric, $\bar{k} = 4, \underline{k} = 1$, $Ric = 2n+2 = \bar{k} + \frac{2n-2}{3}(\bar{k} - \underline{k})$, $b_2(\mathbb{CP}^n) = 1$.

From the examples we know that the inequality (1.1) is precise.

The proof of theorem 1.4 is also based on Bochner technique. But compare with Berger and Micallef-Wang's results, we consider a different side. This allows us get a uniform result (without dimensional restriction).

2. Proof of the theorem

2.1. **Bochner formula.**

Let M be a compact Riemannian manifold. Let

$$\Delta = d\delta + \delta d$$

be the Hodge-Laplacian, where d is the exterior differentiation and δ is the adjoint to d.

Let $\varphi \in \Omega^k(M)$ be a smooth k-form. Then we have the well-known Weitzenböck formula (c.f. [5])

$$\Delta\varphi = \sum_i \nabla^2_{v_i v_i}\varphi - \sum_{i,j} \omega^i \wedge i(v_j) R_{v_i v_j}\varphi, \tag{2.1}$$

here $\nabla^2_{XY} = \nabla_X\nabla_Y - \nabla_{\nabla_X Y}$ and $R_{XY} = -\nabla_X\nabla_Y + \nabla_Y\nabla_X + \nabla_{[X,Y]}$.

A k-form φ is called harmonic if $\Delta\varphi = 0$.

The famous *Hodge theorem* states that the de Rham cohomology $H^k_{dR}(M)$ is isomorphic to the space spanned by k-harmonic forms.

Let $\phi = \sum_{i,j} \phi_{ij}\omega^i \wedge \omega^j$ be a harmonic 2-form. By (2.1), under the normal frame we can get (c.f. [2] or [1])

$$\Delta\phi_{ij} = \sum_k (Ric_{ik}\varphi_{kj} + Ric_{jk}\varphi_{ik}) - 2\sum_{k,l} R_{ikjl}\varphi_{kl}, \tag{2.2}$$

where $R_{ijkl} =< R(v_i, v_j)v_k, v_l >$ is the curvature tensor and $Ric_{ij} = \sum_k < R(v_k, v_i)v_k, v_j >$ is the Ricci tensor.

So we have

$$\begin{aligned} \Delta|\varphi|^2 &= 2\sum_{i,j}\phi_{ij}\Delta\phi_{ij} + 2\sum_{i,j}\sum_k (v_k\phi_{ij})^2 \\ &\geq 2\sum_{i,j}\phi_{ij}\Delta\phi_{ij} \\ &\triangleq 2F(\varphi). \end{aligned}$$

Note that by (2.1) one has the global form of above formula

$$0 = - < \Delta\varphi, \varphi >= \sum_i |\nabla_{v_i}\varphi|^2 + < \sum_{i,j} \omega^i \wedge i(v_j)R_{v_i v_j}\varphi, \varphi > -\frac{1}{2}\Delta|\varphi|^2.$$

The $F(\varphi)$ is just the curvature term $< \sum_{i,j} \omega^i \wedge i(v_j)R_{v_i v_j}\varphi, \varphi >$.

2.2. **proof of theorem 1.4.** By Hodge theorem, we only need to show that every harmonic 2-form vanishes.

Case 1: Assume $\dim(M) = 2m$. For any $p \in M$, we can choose an othonormal basis $\{v_1, w_1, ..., v_m, w_m\}$ of T_pM such that $\varphi(p) = \sum_\alpha \lambda_\alpha v^*_\alpha \wedge w^*_\alpha$ (for instance c.f. [1] or [2]). Here $\{v^*_\alpha, w^*_\alpha\}$ is the dual basis. Then

$$F(\varphi) = \sum_{\alpha=1}^m \lambda^2_\alpha[Ric(v_\alpha, v_\alpha) + Ric(w_\alpha, w_\alpha)] - 2\sum_{\alpha,\beta=1}^m \lambda_\alpha\lambda_\beta R(v_\alpha, w_\alpha, v_\beta, w_\beta) \tag{2.3}$$

The term

$$
\begin{aligned}
-2\sum_{\alpha,\beta=1}^{m}\lambda_\alpha\lambda_\beta R(v_\alpha,w_\alpha,v_\beta,w_\beta) &= -2\sum_{\alpha\neq\beta}\lambda_\alpha\cdot\lambda_\beta\cdot R(v_\alpha,w_\alpha,v_\beta,w_\beta)-2\sum_{\alpha=1}^{m}\lambda_\alpha^2 R(v_\alpha,w_\alpha,v_\alpha,w_\alpha)\\
&\geq -\frac{4}{3}(\bar{k}-\underline{k})\sum_{\alpha\neq\beta}|\lambda_\alpha|\cdot|\lambda_\beta|-2\bar{k}\sum_{\alpha=1}^{m}\lambda_\alpha^2\\
&\geq -\frac{2}{3}(\bar{k}-\underline{k})\sum_{\alpha\neq\beta}(\lambda_\alpha^2+\lambda_\beta^2)-2\bar{k}|\varphi|^2\\
&= -\frac{2}{3}(\bar{k}-\underline{k})(2m-2)|\varphi|^2-2\bar{k}|\varphi|^2\\
&= -2[\bar{k}+\frac{2m-2}{3}(\bar{k}-\underline{k})]|\varphi|^2.
\end{aligned}
$$

The first " $\geq$ " follows from Berger's inequality (c.f. [1]): For any othonormal 4-frames $\{e_1,e_2,e_3,e_4\}$, one has

$$|R(e_1,e_2,e_3,e_4)|\leq\frac{2}{3}(\bar{k}-\underline{k}).$$

On the other hand, by the condition (1.1) we have

$$\sum_{\alpha=1}^{m}\lambda_\alpha^2[Ric(v_\alpha,v_\alpha)+Ric(w_\alpha,w_\alpha)]\geq 2[\bar{k}+\frac{2m-2}{3}(\bar{k}-\underline{k})]|\varphi|^2,$$

the equality holds if and only if $\varphi(p)=0$.

This leads to

$$F(\varphi)\geq 0$$

with equality if and only if $\varphi(p)=0$. Since

$$\int_M F(\varphi)\leq\frac{1}{4}\int_M\Delta|\varphi|^2=0,$$

we get

$$F(\varphi)\equiv 0.$$

Thus the harmonic 2-form $\varphi\equiv 0$.

Case 2: If $\dim(M)=2m+1$. For any $p\in M$, we also can choose an othonormal basis $\{u,v_1,w_1,...,v_m,w_m\}$ of T_pM such that $\varphi(p)=\sum_\alpha\lambda_\alpha v_\alpha^*\wedge w_\alpha^*$ (c.f. [1] or [2]). We also have

$$F(\varphi)=\sum_{\alpha=1}^{m}\lambda_\alpha^2[Ric(v_\alpha,v_\alpha)+Ric(w_\alpha,w_\alpha)]-2\sum_{\alpha,\beta=1}^{m}\lambda_\alpha\lambda_\beta R(v_\alpha,w_\alpha,v_\beta,w_\beta).$$

Thus the argument is same to the even dimensional case.

This completes the proof of the theorem.

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