ON THE INJECTIVITY RADIUS GROWTH OF COMPLETE NONCOMPACT RIEMANNNIAN MANIFOLDS

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Abstract. In this paper we introduce a global geometric invariant $\alpha(M)$ related to injectivity radius to complete non-compact Riemannian manifolds and prove: If $\alpha(M^n) > 1$, then M^n is isometric to $\mathbb{R}^n$ when Ricci curvature is non-negative, and is diffeomorphic to $\mathbb{R}^n$ for $n \neq 4$ and homeomorphic to $\mathbb{R}^4$ for $n = 4$ if without any curved assumption. .

1. introduction

The injectivity radius estimate plays an important role in the studying of global Riemannian geometry. For instance, see Klingenberg [8] and Cheeger [1]. But most work involves the injectivity radiuses of compact manifolds. Partial reason is that the injectivity radius of a compact manifold M

$$\text{injrad}(M) = \min\{\text{injrad}(p), p \in M\}$$

is always finite and positive. When the manifold is non-compact, we cannot say much about it.

In order to study complete non-compact Riemannian manifolds, we usually consider some objects involving infinity. Such as volume growth, Busemann function [9] (roughly speeking, a distance function from ∞) etc. In present paper we shall research the relationship between geometry and topology of complete non-compact Riemannian manifolds and the asymptotic properties of injectivity radiuses at infinity.

Let M be a complete Riemannian manifold. For a point $p \in M$, we denote the distance from p to x by $d(p, x)$. Recall that the injectivity radius of a point $p \in M$ is defined by

$$\text{injrad}(p) := \sup\{r|\exp_p : B(0, r) \to B(p, r) \text{ is a diffeomorphism}\},$$

where $B(0, r)$ and $B(p, r)$ denote the open ball of radius r and center at p in T_pM and M.

We define the *injectivity radius growth* by

$$\alpha(M) = \underline{\lim}_{r\to\infty} \frac{\text{injrad}(p, r)}{r}, \tag{1}$$

where $\text{injrad}(p, r) = \inf\{\text{injrad}(x)|x \in M, d(p, x) = r\}$. We can show that $\alpha(M)$ is well-defined, i.e., it is independent on the choice of $p \in M$ (see proposition 2.1).

Our first theorem can be stated as follows.

Theorem 1.1. *Let M^n be a complete non-compact Riemannian manifold with non-negative Ricci curvature. If $\alpha(M)$ defined by (1) satisfies $\alpha(M) > 1$, then M^n is isometric to $\mathbb{R}^n$.*

2010 *Mathematics Subject Classification.* Primary 53C20; Secondary 53C35.
Key words and phrases. Injectivity radius, complete non-compact manifold.
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Roughly speaking, theorem 1.1 says that if the injectivity radius at infinity is large enough, then a complete non-negative Ricci curved Riemannian manifold must be isometric to $\mathbb{R}^n$.

Without assumption of non-negative Ricci curvature in theorem 1.1, we have

Theorem 1.2. *Let M^n be a complete non-compact Riemannian manifold. If $\alpha(M)$ defined by (1) satisfying $\alpha(M) > 1$, then M^n is diffeomorphic to $\mathbb{R}^n$ for $n \neq 4$ and homeomorphic to $\mathbb{R}^4$ for $n = 4$.*

The proof of theorem 1.2 lies on some deep topological results. In fact we prove that if $\alpha(M) > 1$, then the manifold must be contractible and simple connected at infinity. We don't know whether one has a purely geometric method. We also don't know whether M^n is diffeomorphic to $\mathbb{R}^4$ for $n = 4$.

Remark 1.3. The $\alpha(M) > 1$ in theorem 1.1 and 1.2 is best possible. If $\alpha(M) \leq 1$, then we can construct counterexamples to theorem 1.1 and 1.2 (see section 5).

The rest of the paper is organized as follows: In section 2, we prove that $\alpha(M)$ is independent on the choice of point; We will give the proof of theorem 1.1 (resp. theorem 1.2) in section 3 (resp. section 4). The last section contains some examples and questions.

Acknowledgements The first author would like to thank his advisor professor Xiaochun Rong for his encouragement.

The second author would like to thank professor Xiaochun Rong for his many encouragement and help during his visits to Capital Normal University. A special thank goes to professor Kefeng Liu for his long-time encouragement. He also would like to thank professor Hongwei Xu for his many help.

2. ON THE INJECTIVITY RADIUS GROWTH

Let M be a complete non-compact Riemannian manifold. For a point $p \in M$, we write

$$\alpha(p) = \varliminf_{r\to\infty} \frac{\text{injrad}(p, r)}{r},$$

here $\text{injrad}(p, r) = \inf \{\text{injrad}(x) | x \in M, d(p, x) = r\}$.

Proposition 2.1. *The $\alpha(p)$ is independent on the choice of p. So we can write it as $\alpha(M)$.*

Proof. Let p, q be any two points of M. $d(p, q) = l$.

Case 1: $\alpha(p) = \infty$.

By the definition of $\alpha(p)$, for any $m > 0$, there exists $r_0 > 0$ such that for all $x \in M \setminus B(p, r_0)$, one has

$$\text{injrad}(x) \geq mr_1,$$

here $r_1 = d(p, x)$. Then

$$\frac{\text{injrad}(q, r_2)}{r_2} \geq \frac{mr_1}{r_2} \geq \frac{m(r_2 - l)}{r_2}$$

for all x such that $r_2 = d(q, x) \geq l + r_0$. Hence

$$\alpha(q) = \varliminf_{r\to\infty} \frac{\text{injrad}(q, r)}{r} \geq \lim_{r\to\infty} \frac{m(r - l)}{r} = m.$$

Since m is any positive number, we must have $\alpha(q) = \infty$.

Case 2: $\alpha(p) < \infty$. From case 1 one must have $\alpha(q) < \infty$

By the definition of $\alpha(q)$, for any $\epsilon > 0$, there exists $r_0 > 0$ such that for all $x \in M \setminus B(q, r_0)$, one has

$$\text{injrad}(x) \geq \text{injrad}(q, r_2) \geq (\alpha(q) - \epsilon) r_2,$$

where $r_2 = d(q, x)$. Hence for all x such that $d(p, x) = r_1 \geq l + r_0$, we have

$$\frac{\text{injrad}(p, r_1)}{r_1} \geq \frac{(\alpha(q) - \epsilon) r_2}{r_1} \geq \frac{(\alpha(q) - \epsilon)(r_1 - l)}{r_1}$$

when $\alpha(q) - \epsilon \geq 0$ and

$$\frac{\text{injrad}(p, r_1)}{r_1} \geq \frac{(\alpha(q) - \epsilon) r_2}{r_1} \geq \frac{(\alpha(q) - \epsilon)(r_1 + l)}{r_1}$$

when $\alpha(q) - \epsilon < 0$. Thus

$$\alpha(p) = \lim_{r \to \infty} \frac{\text{injrad}(p, r)}{r} \geq \alpha(q) - \epsilon.$$

Since ϵ is any positive real number, we get

$$\alpha(p) \geq \alpha(q).$$

Similarly we can get

$$\alpha(q) \geq \alpha(p).$$

So we have

$$\alpha(p) = \alpha(q).$$

□

Note that even the manifold is non-compact, $\alpha(M)$ may be equal to zero. The cylinder $S^1 \times \mathbb{R}$ is a simple example. Obviously the $\alpha(M)$ of a Cartan-Hadamard manifold is ∞.

3. A proof of Theorem 1.1

In order to prove theorem 1.1, we need two lemmas.

Lemma 3.1. *Let M^n be a complete non-compact Riemannian manifold with non-negative Ricci curvature. If $\alpha(M) > 1$, then M^n is isometric to $N \times \mathbb{R}^1$, where N is a complete non-negative Ricci curved manifold.*

Proof. Let p be a point of M. Let $\gamma_0(t)$ ($t \in [0, +\infty)$) be a ray starting at p. Let $\gamma(t)$ ($-\infty < t < +\infty$) be a geodesic through p such that $\gamma_0' = \gamma'$ at p. We claim that $\gamma(t)$ is a line.

Argue by contradiction. Assume that there exists $p_1,\ p_2 \in \gamma(t)$ such that p_2 is a cut point of p_1. Then there is another geodesic $\sigma(t)$ from p_1 to p_2.

Since $\alpha(M) > 1$, by the definition, for any $0 < \varepsilon < \frac{\alpha(M)-1}{2}$, there exists r_0 such that for all $r > r_0$, we have

$$\text{injrad}(p, r) \geq (\alpha(M) - \varepsilon) r > (1 + \varepsilon) r.^1$$

Hence for $r > \max\{\frac{\max\{d(p_1,p), d(p_2,p)\}}{\varepsilon}, r_0\}$, we can choose $q \in \gamma_0(t)$ such that

$$\text{injrad}(q) \geq \text{injrad}(p, r) > (1 + \varepsilon) r, \tag{2}$$

and

$$d(p, q) = r = \text{ the length of } \gamma_0(t) \text{ from } p \text{ to } q.$$

So we have

$$d(p_i, q) \leq d(p_i, p) + d(p, q) < \varepsilon r + r = (1 + \varepsilon) r, i = 1, 2. \tag{3}$$

[1]If $\alpha(M) = \infty$, we still have $\text{injrad}(p, r) > (1 + \varepsilon) r$ when $r > r_0$.

Therefore, we can conclude that from (2) and (3) that

$$p_1, p_2 \in B(q, (1+\varepsilon)r) \subset B(q, \mathrm{injrad}(q)). \tag{4}$$

Without losing generality, we assume that $p_1 = \gamma(t_1), p_2 = \gamma(t_2)$ and $t_1 < t_2$. Let $\gamma_1(t)$ be the curve from p_1 to q such that

$$\gamma_1(t)|_{[p_1,p_2]} = \sigma(t)$$

and

$$\gamma_1(t)|_{[p_2,q]} = \gamma(t)|_{[p_2,q]}.$$

Smoothing $\gamma_1(t)$ at p_2, we can obtain a smooth curve which the length is shorter than the length of $\gamma(t)|_{[p_1,q]}$. This is contradict to (4). Hence the claim is true.

Combining with the Cheeger-Gromoll splitting theorem [3], we complete the proof of the lemma. □

Lemma 3.2. *The N in lemma 3.1 is non-compact.*

Proof. If N is compact, then for any $q \in M = N \times \mathbb{R}^1$, one has $\mathrm{injrad}(q) \leq \mathrm{diam}(N)$. Hence $\alpha(M) = 0$. We get a contradiction. □

Proof of theorem 1.1: Since N is non-compact, M must contain another ray starting at p which is contained in N. Repeating the procedure of lemma 3.1, 3.2 and using Cheeger-Gromoll splitting theorem again, we have that M^n is isometric to $N' \times \mathbb{R}^2$, N' is non-compact. Step by step, we can conclude that M^n is isometric to $\mathbb{R}^n$.

4. A proof of Theorem 1.2

Lemma 4.1. *Let M be a complete non-compact Riemannian manifold. If $\alpha(M) > 1$, then for every compact set C (not need connected), we can find $q \in M$ such that $C \subset B(q, injrad(q))$.*

Proof. Let p be a point of M. Let $\gamma(t)$ be a ray starting at p. Similar to the proof of lemma 3.1. For any $0 < \varepsilon < \frac{\alpha(M)-1}{2}$, there exists r_0 such that for all $r > r_0$, we have

$$\mathrm{injrad}(p, r) \geq (\alpha(M) - \varepsilon)r > (1 + \varepsilon)r.$$

Let $s = \max\{d(p, x) | x \in C\}$. For $r > max\{\frac{s}{\varepsilon}, r_0\}$, we can choose $q \in \gamma(t)$ such that

$$\mathrm{injrad}(q) \geq \mathrm{injrad}(p, r) > (1 + \varepsilon)r,$$

and

$$d(p, q) = r = \text{ the length of } \gamma(t) \text{ from } p \text{ to } q.$$

So for any $x \in C$, one has

$$d(q, x) \leq d(x, p) + d(p, q) < \varepsilon r + r = (1 + \varepsilon)r,$$

Thus $C \subset B(q, \mathrm{injrad}(q))$. □

Corollary 4.2. *Let M be a complete non-compact Riemannian manifold. If $\alpha(M) > 1$, then M is contractible.*

Proof. We only need to showed that the homotopy group $\pi_i(M)$ is trivial for $i \geq 0$. Let $f : (S^i, s_0) \to (M, p)$ be an element of $\pi_i(M, p)$. By lemma 4.1, we know that $f(S^i)$ is contained in some $B(q, \mathrm{injrad}(q))$. Hence it is contractible in M. □

A topological space T is said to be *1-connected at infinity* [10]: If for each compact set C of T, there is a compact set D of T with $C \subset D \subset T$, such that $T \setminus D$ is 1-connected.

Corollary 4.3. *Let $M^n(n \geq 3)$ be a complete non-compact Riemannian manifold. If $\alpha(M) > 1$, then M is 1-connected at infinity.*

Proof. Let C be any compact set of M. By lemma 4.1, we can choose a compact ball $B(q, r)$ such that $B(q, r)$ is diffeomorphic to Euclid unit ball and $C \subset B(q, r)$. Since M is 1-connected, we know that $M \setminus B(q, r)$ is 1-connected. Hence M is 1-connected at infinity. □

To prove theorem 1.2, we need the following deep theorem.

Theorem 4.4. *Let $M^n(n \geq 3)$ be a contractible open smooth manifold and 1-connected at infinity. Then M^n is diffeomorphic to $\mathbb{R}^n$ for $n \neq 4$ and homeomorphic to $\mathbb{R}^4$ for $n = 4$.*

The case $n \geq 5$ is due to Stallings [10]. For $n = 3$, it is a consequence of Perelman's solution to Poincare conjecture and a theorem of Edwards [5] (see the theorem 1 and the third paragraph of [5]). The case $n = 4$ is due to Freedman [6] (see corollary 1.2 of [6]). Since Donaldson [4] found a smooth 4-manifold which is homeomorphic to $\mathbb{R}^4$ but not diffeomorphic to $\mathbb{R}^4$, we cannot get that M^n is diffeomorphic to $\mathbb{R}^4$ for $n = 4$.

Proof of theorem 1.2: For the dimension ≥ 3, it is a consequence of corollary 4.2, corollary 4.3 and theorem 4.4. It follows from the Riemann mapping theorem as dimension = 2.

5. EXAMPLES AND DISCUSSIONS

Now we give examples to show that the $\alpha(M)$ in theorem 1.1 and 1.2 is best possible.

Example 5.1. Let

$$x^2 + y^2 - (z\tan(\theta))^2 = 0, z \geq 0$$

be the cone in $\mathbb{R}^3$. Smoothing the original point $p = (0, 0, 0)$, we get a complete noncompact surface with non-negative Gauss curvature. Clearly it is not isometric to $\mathbb{R}^2$.

It is straightforward to compute that

$$\alpha(M) = \begin{cases} \sin(\pi \sin\theta), & \text{if } 0 < \theta < \frac{\pi}{6}; \\ 1, & \text{if } \frac{\pi}{6} \leq \theta < \frac{\pi}{2}. \end{cases}$$

Example 5.2. We glue the following two surfaces

$$x^2 + y^2 - (z\tan(\theta))^2 = 0, z \geq \epsilon > 0$$

and

$$x^2 + y^2 - (z\tan(\theta))^2 = 0, z \leq -\epsilon < 0$$

along their edges. It is a non-simple connected surface.

One also can easy to check that

$$\alpha(M) = \begin{cases} \sin(\pi \sin\theta), & \text{if } 0 < \theta < \frac{\pi}{6}; \\ 1, & \text{if } \frac{\pi}{6} \leq \theta < \frac{\pi}{2}. \end{cases}$$

Finally we propose two interesting questions.

Question 1 For a complete non-compact manifold, can we prove that every geodesic is a line as long as $\alpha(M) > 1$?

Question 2 Determining the minimal $\alpha_0 \in (0, 1]$ such that for any complete non-compact Riemannian manifold with non-negative Ricci curvature , if $\alpha(M) > \alpha_0$, then M^n is diffeomorphic to $\mathbb{R}^n$.

Let us compare with the following two classical theorems:

1) Cheng's maximal diameter theorem [2]: A complete Riemannian manifold with $Ric_M \geq n-1$ and $diam = \pi$ must be isometric to $S^n(1)$.

2) Grove-Shiohama's generalized sphere theorem [7]: A closed Riemannian manifold with $sec_M \geq 1$ and $diam > \frac{\pi}{2}$ is homeomorphic to a sphere.

Roughly speaking, theorem 1.1 is a non-compact analogue of Cheng's theorem. Question 2 is to seek a non-compact analogue of Grove-Shiohama theorem.

We hope that the $\alpha(M)$ gives more contributions to the research of complete non-compact Riemannian manifolds.

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