

MODULAR FORMS AND GENERALIZED ANOMALY CANCELLATION FORMULAS

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ABSTRACT. In this paper, we generalize the anomaly cancellation formulas in [1, 12, 7] to the cases that an auxiliary bundle W as well as a complex line bundle ξ are involved with no conditions on the first Pontryagin forms being assumed.

INTRODUCTION

In [1], gravitational anomaly cancellation formulas are derived from direct computations. In particular, in dimension 12, the Alvarez-Gaumé and Witten “miraculous cancellation” formula can be written as

$$(0.1) \quad \{\widehat{L}(TX, \nabla^{TX})\}^{(12)} = \{8\widehat{A}(TX, \nabla^{TX})\text{ch}(T_{\mathbf{C}}X, \nabla^{T_{\mathbf{C}}X})\}^{(12)} - 32\{\widehat{A}(TX, \nabla^{TX})\}^{(12)},$$

where X is a 12 dimensional Riemannian manifold, ∇^{TX} is the associated Levi-Civita connection, $T_{\mathbf{C}}X$ is the complexification of TX (with the induced Hermitian connection $\nabla^{T_{\mathbf{C}}X}$) and $\widehat{L}(TX, \nabla^{TX})$, $\widehat{A}(TX, \nabla^{TX})$ are the Hirzebruch characteristic forms (see (1.1)).

In [12], Liu generalizes (0.1) to general $8m + 4$ dimension by developing modular invariance properties of characteristic forms. Actually, in [12], Liu obtains a more general cancellation formula by including an auxiliary bundle W . More precisely, assume X to be $8m + 4$ dimensional and W be a rank $2l$ Euclidean vector bundle over X with a Euclidean connection ∇^W and curvature $R^W = \nabla^{W,2}$, if $p_1(TX, \nabla^{TX}) = p_1(W, \nabla^W)$, then the following identity holds,

$$(0.2) \quad \left\{ \widehat{A}(TX, \nabla^{TX}) \det^{1/2} \left(2 \cosh \left(\frac{\sqrt{-1}}{4\pi} R^W \right) \right) \right\}^{(8m+4)} \\ = \sum_{r=0}^m 2^{l+2m+1-6r} \left\{ \widehat{A}(TX, \nabla^{TX}) \text{ch}(b_r(T_{\mathbf{C}}X, W_{\mathbf{C}}, \mathbf{C}^2)) \right\}^{(8m+4)},$$

where $b_r(T_{\mathbf{C}}X, W_{\mathbf{C}}, \mathbf{C}^2)$'s are virtual complex vector bundles with connections over X canonically determined by (TX, ∇^{TX}) and (W, ∇^W) . In dimension 12, by direct computation, (0.2) becomes,

$$(0.3) \quad \left\{ \widehat{A}(TX, \nabla^{TX}) \det^{1/2} \left(2 \cosh \left(\frac{\sqrt{-1}}{4\pi} R^W \right) \right) \right\}^{(12)} \\ = 2^{l-3} \left\{ \widehat{A}(TX, \nabla^{TX}) \text{ch}(W_{\mathbf{C}}, \nabla^{W_{\mathbf{C}}}) \right\}^{(12)} - 2^{l-2}(l-4) \left\{ \widehat{A}(TX, \nabla^{TX}) \right\}^{(12)}.$$

When $(TX, \nabla^{TX}) = (W, \nabla^W)$, (0.2) gives,

$$(0.4) \quad \frac{1}{8} \left\{ \widehat{L}(TX, \nabla^{TX}) \right\}^{(8m+4)} \\ = \sum_{r=0}^m 2^{6m-6r} \left\{ \widehat{A}(TX, \nabla^{TX}) \text{ch}(b_r(T_{\mathbf{C}}X, T_{\mathbf{C}}X, \mathbf{C}^2)) \right\}^{(8m+4)}.$$

As an application ([11]), by the Atiyah-Hirzebruch divisibility [3], (0.4) implies the Ochanine divisibility [15], which asserts that the signature of an $8k + 4$ -dimensional smooth closed spin manifold is divisible by 16.

To study higher dimensional Rokhlin congruence, Han and Zhang ([6, 7]) extend the ‘‘miraculous cancellation’’ formulas of Alvarez-Gaumé, Witten and Liu to a twisted version where an extra complex line bundle (or equivalently a rank 2 real oriented vector bundle) is involved. More precisely, if ξ is a rank 2 real oriented Euclidean vector bundle equipped with a Euclidean connection ∇^ξ and $c = e(\xi, \nabla^\xi)$ is the associated Euler form, when $p_1(TX, \nabla^{TX}) = p_1(W, \nabla^W)$, the following identity holds,

$$(0.5) \quad \left\{ \frac{\widehat{A}(TX, \nabla^{TX}) \det^{1/2} \left(2 \cosh \left(\frac{\sqrt{-1}}{4\pi} R^W \right) \right)}{\cosh^2 \left(\frac{c}{2} \right)} \right\}^{(8m+4)} \\ = \sum_{r=0}^m 2^{l+2m+1-6r} \left\{ \widehat{A}(TX, \nabla^{TX}) \text{ch}(b_r(T_{\mathbf{C}}X, W_{\mathbf{C}}, \xi_{\mathbf{C}})) \cosh \left(\frac{c}{2} \right) \right\}^{(8m+4)},$$

where $b_r(T_{\mathbf{C}}X, W_{\mathbf{C}}, \xi_{\mathbf{C}})$'s are virtual complex vector bundles with connections over X canonically determined by (TX, ∇^{TX}) , (W, ∇^W) and (ξ, ∇^ξ) . Obviously, when ξ is trivial and $c = 0$, (0.5) reduces to (0.2).

When $\dim X = 12$ and $(TX, \nabla^{TX}) = (W, \nabla^W)$, (0.5) gives,

$$(0.6) \quad \left\{ \frac{\widehat{L}(TX, \nabla^{TX})}{\cosh^2 \left(\frac{c}{2} \right)} \right\}^{(12)} \\ = \left\{ \left[8\widehat{A}(TX, \nabla^{TX}) \text{ch}(W_{\mathbf{C}}, \nabla^{W_{\mathbf{C}}}) - 32\widehat{A}(TX, \nabla^{TX}) \right. \right. \\ \left. \left. - 24\widehat{A}(TX, \nabla^{TX})(e^c + e^{-c} - 2) \right] \cosh \left(\frac{c}{2} \right) \right\}^{(12)},$$

which extends the Alvarez-Gaumé and Witten “miraculous cancellation” formula (0.1) in dimension 12.

Note that (0.2) and (0.5) only hold under the condition $p_1(TX, \nabla^{TX}) = p_1(W, \nabla^W)$. In this paper, we study what if we remove this condition. We find that the difference between the left hand sides and the right hand sides in (0.2) and (0.5) can actually be written in the form

$$(p_1(TX, \nabla^{TX}) - p_1(W, \nabla^W)) \cdot \mathcal{R},$$

where \mathcal{R} is some characteristic forms canonically determined by (TX, ∇^{TX}) , (W, ∇^W) and (ξ, ∇^ξ) . For example, we find that in dimension 12, the following identity holds (for simplicity, we drop the connections),

$$\begin{aligned} (0.7) \quad & \left\{ \widehat{A}(TX) \det^{1/2} \left(2 \cosh \left(\frac{\sqrt{-1}}{4\pi} R^W \right) \right) \right\}^{(12)} \\ & - 2^{l-3} \left\{ \widehat{A}(TX) \text{ch}(W_{\mathbf{C}}) \right\}^{(12)} + 2^{l-2}(l-4) \left\{ \widehat{A}(TX) \right\}^{(12)} \\ = & (p_1(TX) - p_1(W)) \\ & \cdot \left\{ \frac{e^{\frac{1}{24}(p_1(TX) - p_1(W))} - 1}{p_1(TX) - p_1(W)} \right. \\ & \left. \cdot \left[\widehat{A}(TX) \left(2^{l-3} \text{ch}(W_{\mathbf{C}}) - 2^{l-2}(l-4) \right) - \widehat{A}(TX) \det^{1/2} \left(2 \cosh \left(\frac{\sqrt{-1}}{4\pi} R^W \right) \right) \right] \right\}^{(8)}. \end{aligned}$$

We will give similar general results for $8m+4$ and $8m$ dimensions in Theorem 1.1 and discuss various special cases in Corollaries 1.2-1.5. We obtain our generalized anomaly cancellation formulas still by developing modular invariance of characteristic forms.

To obtain our cancellation formulas, we were also inspired by the Green-Schwarz mechanism. In [5], Green and Schwarz discovered that the anomaly in type I string theory with the gauge group $SO(32)$ cancels because of an extra “classical” contribution from a 2-form field. One key step is that when the gauge group is 496 dimensional, the anomaly I_{12} can be written as (c.f. [16])

$$(0.8) \quad I_{12} = (p_1(Z) - p_1(F))I_8.$$

Our cancellation formulas in Theorem 1.1 and its corollaries are of same pattern. We hope they can find applications in physics.

1. RESULTS

The purpose of this section is to state our main results. We first recall the definitions of some characteristic forms to be used in Section 1.1 and then present our generalized anomaly cancellation formulas in Section 1.2.

1.1. Some characteristic forms. Let X be a $4k$ dimensional Riemannian manifold. Let ∇^{TX} be the associated Levi-Civita connection and $R^{TX} = \nabla^{TX, 2}$ be the curvature of ∇^{TX} . Let $\widehat{A}(TX, \nabla^{TX})$ and $\widehat{L}(TX, \nabla^{TX})$ be the Hirzebruch characteristic forms defined respectively by (cf. [18]):

$$(1.1) \quad \begin{aligned} \widehat{A}(TX, \nabla^{TX}) &= \det^{1/2} \left(\frac{\frac{\sqrt{-1}}{4\pi} R^{TX}}{\sinh \left(\frac{\sqrt{-1}}{4\pi} R^{TX} \right)} \right), \\ \widehat{L}(TX, \nabla^{TX}) &= \det^{1/2} \left(\frac{\frac{\sqrt{-1}}{2\pi} R^{TX}}{\tanh \left(\frac{\sqrt{-1}}{4\pi} R^{TX} \right)} \right). \end{aligned}$$

Let F, G be two Hermitian vector bundles over X carrying Hermitian connections ∇^F, ∇^G respectively. Let $R^F = \nabla^{F, 2}$ (resp. $R^G = \nabla^{G, 2}$) be the curvature of ∇^F (resp. ∇^G). If we set the formal difference $H = F - G$, then H carries an induced Hermitian connection ∇^H in an obvious sense. We define the associated Chern character form as (cf. [18])

$$(1.2) \quad \text{ch}(H, \nabla^H) = \text{tr} \left[\exp \left(\frac{\sqrt{-1}}{2\pi} R^F \right) \right] - \text{tr} \left[\exp \left(\frac{\sqrt{-1}}{2\pi} R^G \right) \right].$$

For any complex number t , let

$$\Lambda_t(F) = \mathbf{C}|_X + tF + t^2\Lambda^2(F) + \cdots, \quad S_t(F) = \mathbf{C}|_X + tF + t^2S^2(F) + \cdots$$

denote respectively the total exterior and symmetric powers of F , which live in $K(X)[[t]]$. The following relations between these two operations ([2], Chap. 3) hold,

$$(1.3) \quad S_t(F) = \frac{1}{\Lambda_{-t}(F)}, \quad \Lambda_t(F - G) = \frac{\Lambda_t(F)}{\Lambda_t(G)}.$$

The connections ∇^F, ∇^G naturally induce connections on $\Lambda_t F, S_t F$ etc. Moreover, if $\{\omega_i\}, \{\omega_j'\}$ are formal Chern roots for Hermitian vector bundles F, G respectively, then [[8], Chap. 1]

$$(1.4) \quad \text{ch} \left(\Lambda_t(F), \nabla^{\Lambda_t(F)} \right) = \prod_i (1 + e^{\omega_i t}).$$

We have the following formulas for Chern character forms,

$$(1.5) \quad \text{ch} \left(S_t(F), \nabla^{S_t(F)} \right) = \frac{1}{\text{ch} \left(\Lambda_{-t}(F), \nabla^{\Lambda_{-t}(F)} \right)} = \frac{1}{\prod_i (1 - e^{\omega_i t})},$$

$$(1.6) \quad \text{ch} \left(\Lambda_t(F - G), \nabla^{\Lambda_t(F - G)} \right) = \frac{\text{ch} \left(\Lambda_t(F), \nabla^{\Lambda_t(F)} \right)}{\text{ch} \left(\Lambda_t(G), \nabla^{\Lambda_t(G)} \right)} = \frac{\prod_i (1 + e^{\omega_i t})}{\prod_j (1 + e^{\omega_j' t})}.$$

1.2. Statement of results. We make the same assumptions and use the same notations as in Section 1.1.

Let W be a rank $2l$ real Euclidean vector bundle over X carrying a Euclidean connection ∇^W . Let $R^W = \nabla^{W,2}$ be the curvature of ∇^W . If W is spin, let $\Delta(W) = S^+(W) \oplus S^-(W)$ be the spinor bundle of W with the induced connection $\nabla^{\Delta(W)}$. It's not hard to see that

$$\text{ch}(\Delta(W), \nabla^{\Delta(W)}) = \det^{1/2} \left(2 \cosh \left(\frac{\sqrt{-1}}{4\pi} R^W \right) \right).$$

Let $p_1(TX, \nabla^X)$ and $p_1(W, \nabla^W)$ be the first Pontryagin forms of (TX, ∇^{TX}) and (W, ∇^W) respectively.

Let ξ be a rank two real oriented Euclidean vector bundle over X carrying a Euclidean connection ∇^ξ . Let $c = e(\xi, \nabla^\xi)$ be the Euler form canonically associated to ∇^ξ .

For simplicity, from now on, when there is no ambiguity, we will write characteristic forms without specifying the connections.

In the following, we will define some virtual bundles with connections and some differential forms on X associated to (TX, ∇^{TX}) , (W, ∇^W) and (ξ, ∇^ξ) .

If E is a vector bundle (real or complex) over X , set $\tilde{E} = E - \dim E$ in $KO(X)$ or $K(X)$.

If E is a real Euclidean vector bundle over X carrying a Euclidean connection ∇^E , then its complexification $E_{\mathbf{C}} = E \otimes \mathbf{C}$ is a complex vector bundle over X carrying a canonically induced Hermitian metric from that of E , as well as a Hermitian connection $\nabla^{E_{\mathbf{C}}}$ induced from ∇^E .

If ω is a differential form, denote the degree j -component of ω by $\omega^{(j)}$.

Let $q = e^{2\pi\sqrt{-1}\tau}$ with $\tau \in \mathbf{H}$, the upper half complex plane. Let $T_{\mathbf{C}}X$ be the complexification of TX .

Set

$$(1.7) \quad \begin{aligned} \Theta_2(T_{\mathbf{C}}X, W_{\mathbf{C}}, \xi_{\mathbf{C}}) &= \bigotimes_{u=1}^{\infty} S_{q^u}(\widetilde{T_{\mathbf{C}}X}) \otimes \bigotimes_{v=1}^{\infty} \Lambda_{-q^{v-\frac{1}{2}}}(\widetilde{W_{\mathbf{C}}} - 2\widetilde{\xi_{\mathbf{C}}}) \\ &\quad \otimes \bigotimes_{r=1}^{\infty} \Lambda_{q^{r-\frac{1}{2}}}(\widetilde{\xi_{\mathbf{C}}}) \otimes \bigotimes_{s=1}^{\infty} \Lambda_{q^s}(\widetilde{\xi_{\mathbf{C}}}), \end{aligned}$$

which is an element in $K(X)[[q^{\frac{1}{2}}]]$.

Clearly, $\Theta_2(T_{\mathbf{C}}X, W_{\mathbf{C}}, \xi_{\mathbf{C}})$ admits formal Fourier expansion in $q^{1/2}$ as

$$(1.8) \quad \Theta_2(T_{\mathbf{C}}X, W_{\mathbf{C}}, \xi_{\mathbf{C}}) = B_0(T_{\mathbf{C}}X, W_{\mathbf{C}}, \xi_{\mathbf{C}}) + B_1(T_{\mathbf{C}}X, W_{\mathbf{C}}, \xi_{\mathbf{C}})q^{1/2} + \cdots,$$

where the B_j 's are elements in the semi-group formally generated by complex vector bundles over X . Moreover, they carry canonically induced connections denoted by ∇^{B_j} and let ∇^{Θ_2} be the induced connections with $q^{1/2}$ -coefficients on Θ_2 from the ∇^{B_j} .

Consider the q -series:

$$(1.9) \quad \delta_1(\tau) = \frac{1}{4} + 6 \sum_{n=1}^{\infty} \sum_{\substack{d|n \\ d \text{ odd}}} dq^n = \frac{1}{4} + 6q + 6q^2 + \dots,$$

$$(1.10) \quad \varepsilon_1(\tau) = \frac{1}{16} + \sum_{n=1}^{\infty} \sum_{d|n} (-1)^d d^3 q^n = \frac{1}{16} - q + 7q^2 + \dots,$$

$$(1.11) \quad \delta_2(\tau) = -\frac{1}{8} - 3 \sum_{n=1}^{\infty} \sum_{\substack{d|n \\ d \text{ odd}}} dq^{n/2} = -\frac{1}{8} - 3q^{1/2} - 3q - \dots,$$

$$(1.12) \quad \varepsilon_2(\tau) = \sum_{n=1}^{\infty} \sum_{\substack{d|n \\ n/d \text{ odd}}} d^3 q^{n/2} = q^{1/2} + 8q + \dots,$$

and the Eisenstein series

$$E_2(\tau) = 1 - 24 \sum_{n=1}^{\infty} \left(\sum_{d|n} d \right) q^n = 1 - 24q - 72q^2 - 96q^3 - \dots.$$

Remark 1.1. δ_1 and ε_1 will only be used later in the proof of our results. We list them here for completeness.

Now we define the virtual bundles with connections and the differential forms on X associated to (TX, ∇^{TX}) , (W, ∇^W) and (ξ, ∇^ξ) . This will be done in two cases respectively.

Case 1: $\dim X = 8m + 4$.

Define virtual complex vector bundles $b_r(T_{\mathbf{C}}X, W_{\mathbf{C}}, \xi_{\mathbf{C}})$ on X , $0 \leq r \leq m$, via the equality

$$(1.13) \quad \Theta_2(T_{\mathbf{C}}X, W_{\mathbf{C}}, \xi_{\mathbf{C}}) \equiv \sum_{r=0}^m b_r (8\delta_2)^{2m+1-2r} \varepsilon_2^r \pmod{q^{\frac{m+1}{2}} \cdot K(X)[[q^{\frac{1}{2}}]]}.$$

It's not hard to see that each b_r , $0 \leq r \leq m$, is a canonical integral linear combination of $B_j(T_{\mathbf{C}}X, W_{\mathbf{C}}, \xi_{\mathbf{C}})$, $0 \leq j \leq r$. These b_r 's carry canonically induced metrics and connections. It's easy to calculate that

$$(1.14) \quad b_0 = -\mathbf{C}, \quad b_1 = W_{\mathbf{C}} - 3\xi_{\mathbf{C}} + \mathbf{C}^{48m-2l+30}.$$

Define degree $8m$ differential forms $\beta_r(\nabla^{TX}, \nabla^W, \nabla^\xi)$ on X , $0 \leq r \leq m$, via the equality

$$(1.15) \quad \left\{ \frac{e^{\frac{1}{24}E_2(\tau)(p_1(TX)-p_1(W))} - 1}{p_1(TX) - p_1(W)} \widehat{A}(TX) \cosh\left(\frac{c}{2}\right) \text{ch}(\Theta_2(T_{\mathbf{C}}X, W_{\mathbf{C}}, \xi_{\mathbf{C}})) \right\}^{(8m)} \\ \equiv \sum_{r=0}^m \beta_r (8\delta_2)^{2m+1-2r} \varepsilon_2^r \pmod{q^{\frac{m+1}{2}}} \cdot \Omega^{8m}(X)[[q^{\frac{1}{2}}]].$$

It's not hard to see that each β_r , $0 \leq r \leq m$, is a canonical linear combination of degree $8m$ forms of the type $c(p_1(TX)-p_1(W))^a \widehat{A}(TX) \cosh\left(\frac{c}{2}\right) \text{ch}(B_j)$, $0 \leq j \leq r$. It's easy to calculate that

$$(1.16) \quad \beta_0 = - \left\{ \frac{e^{\frac{1}{24}(p_1(TX)-p_1(W))} - 1}{p_1(TX) - p_1(W)} \widehat{A}(TX) \cosh\left(\frac{c}{2}\right) \right\}^{(8m)}, \\ \beta_1 = \left\{ \frac{e^{\frac{1}{24}(p_1(TX)-p_1(W))} - 1}{p_1(TX) - p_1(W)} \widehat{A}(TX) \cosh\left(\frac{c}{2}\right) (\text{ch}(W_{\mathbf{C}} - 3\xi_{\mathbf{C}}) + 48m - 2l + 30) \right\}^{(8m)}.$$

We would like to point out that although

$$\beta_0 = \left\{ \frac{e^{\frac{1}{24}(p_1(TX)-p_1(W))} - 1}{p_1(TX) - p_1(W)} \widehat{A}(TX) \cosh\left(\frac{c}{2}\right) \text{ch}(b_0) \right\}^{(8m)}$$

and

$$\beta_1 = \left\{ \frac{e^{\frac{1}{24}(p_1(TX)-p_1(W))} - 1}{p_1(TX) - p_1(W)} \widehat{A}(TX) \cosh\left(\frac{c}{2}\right) \text{ch}(b_1) \right\}^{(8m)},$$

generally,

$$\beta_r \neq \left\{ \frac{e^{\frac{1}{24}(p_1(TX)-p_1(W))} - 1}{p_1(TX) - p_1(W)} \widehat{A}(TX) \cosh\left(\frac{c}{2}\right) \text{ch}(b_r) \right\}^{(8m)}, \quad r > 2.$$

Case 2: $\dim X = 8m$.

Define virtual complex vector bundles $z_r(T_{\mathbf{C}}X, W_{\mathbf{C}}, \xi_{\mathbf{C}})$ on X , $0 \leq r \leq m$, via the equality

$$(1.17) \quad \Theta_2(T_{\mathbf{C}}X, W_{\mathbf{C}}, \xi_{\mathbf{C}}) \equiv \sum_{r=0}^m z_r (8\delta_2)^{2m-2r} \varepsilon_2^r \pmod{q^{\frac{m+1}{2}}} \cdot K(M)[[q^{\frac{1}{2}}]].$$

Similarly each $z_r(T_{\mathbf{C}}X, W_{\mathbf{C}}, \xi_{\mathbf{C}})$, $0 \leq r \leq m$, is a canonical integral linear combination of $B_j(T_{\mathbf{C}}X, W_{\mathbf{C}}, \xi_{\mathbf{C}})$, $0 \leq j \leq r$. These z_r 's carry canonically induced metrics and connections. It's easy to calculate that

$$(1.18) \quad z_0 = \mathbf{C}, \quad z_1 = -W_{\mathbf{C}} + 3\xi_{\mathbf{C}} - \mathbf{C}^{48m-2l+6}.$$

Define degree $8m-4$ differential forms $\zeta_r(\nabla^{TX}, \nabla^W, \nabla^\xi)$ on X , $0 \leq r \leq m$, via the equality

$$(1.19) \quad \left\{ \frac{e^{\frac{1}{24}E_2(\tau)(p_1(TX)-p_1(W))} - 1}{p_1(TX) - p_1(W)} \widehat{A}(TX) \cosh\left(\frac{c}{2}\right) \text{ch}(\Theta_2(T_{\mathbb{C}}X, W_{\mathbb{C}}, \xi_{\mathbb{C}})) \right\}^{(8m-4)} \\ \equiv \sum_{r=0}^m \zeta_r (8\delta_2)^{2m-2r} \varepsilon_2^r \pmod{q^{\frac{m+1}{2}}} \cdot \Omega^{8m-4}(X)[[q^{\frac{1}{2}}]].$$

It's not hard to see that each ζ_r , $0 \leq r \leq m$, is a canonical linear combination of degree $8m-4$ forms of the type $c(p_1(TX)-p_1(W))^a \widehat{A}(TX) \cosh\left(\frac{c}{2}\right) \text{ch}(B_j)$, $0 \leq j \leq r$. It's easy to calculate that

$$(1.20) \quad \zeta_0 = \left\{ \frac{e^{\frac{1}{24}(p_1(TX)-p_1(W))} - 1}{p_1(TX) - p_1(W)} \widehat{A}(TX) \cosh\left(\frac{c}{2}\right) \right\}^{(8m-4)}, \\ \zeta_1 = - \left\{ \frac{e^{\frac{1}{24}(p_1(TX)-p_1(W))} - 1}{p_1(TX) - p_1(W)} \widehat{A}(TX) \cosh\left(\frac{c}{2}\right) (\text{ch}(W_{\mathbb{C}} - 3\xi_{\mathbb{C}}) + 48m - 2l + 6) \right\}^{(8m-4)}.$$

Similar to β_r , generally,

$$\zeta_r \neq \left\{ \frac{e^{\frac{1}{24}(p_1(TX)-p_1(W))} - 1}{p_1(TX) - p_1(W)} \widehat{A}(TX) \cosh\left(\frac{c}{2}\right) \text{ch}(z_r) \right\}^{(8m-4)}, \quad r > 2.$$

We can now state our main theorem as follows.

Theorem 1.1. (1) *When $\dim X = 8m + 4$, one has*

$$(1.21) \quad \left\{ \frac{\widehat{A}(TX) \det^{1/2}\left(2 \cosh\left(\frac{\sqrt{-1}}{4\pi} RW\right)\right)}{\cosh^2\left(\frac{c}{2}\right)} \right\}^{(8m+4)} \\ - \sum_{r=0}^m 2^{l+2m+1-6r} \left\{ \widehat{A}(TX) \text{ch}(b_r(T_{\mathbb{C}}X, W_{\mathbb{C}}, \xi_{\mathbb{C}})) \cosh\left(\frac{c}{2}\right) \right\}^{(8m+4)} \\ = (p_1(TX) - p_1(W)) \cdot \mathfrak{B}(\nabla^{TX}, \nabla^W, \nabla^\xi),$$

where

$$\begin{aligned}
 (1.22) \quad & \mathfrak{B}(\nabla^{TX}, \nabla^W, \nabla^\xi) \\
 &= \sum_{r=0}^m 2^{l+2m+1-6r} \beta_r(\nabla^{TX}, \nabla^W, \nabla^\xi) \\
 & \quad - \left\{ \frac{e^{\frac{1}{24}(p_1(TX)-p_1(W))} - 1}{p_1(TX) - p_1(W)} \cdot \frac{\widehat{A}(TX) \det^{1/2} \left(2 \cosh \left(\frac{\sqrt{-1}}{4\pi} RW \right) \right)}{\cosh^2 \left(\frac{c}{2} \right)} \right\}^{(8m)}.
 \end{aligned}$$

(2) When $\dim X = 8m$, one has

$$\begin{aligned}
 (1.23) \quad & \left\{ \frac{\widehat{A}(TX) \det^{1/2} \left(2 \cosh \left(\frac{\sqrt{-1}}{4\pi} RW \right) \right)}{\cosh^2 \left(\frac{c}{2} \right)} \right\}^{(8m)} \\
 & - \sum_{r=0}^m 2^{l+2m-6r} \left\{ \widehat{A}(TX) \text{ch}(z_r(T_{\mathbf{C}}X, W_{\mathbf{C}}, \xi_{\mathbf{C}})) \cosh \left(\frac{c}{2} \right) \right\}^{(8m)} \\
 & = (p_1(TX) - p_1(W)) \cdot \mathfrak{Z}(\nabla^{TX}, \nabla^W, \nabla^\xi),
 \end{aligned}$$

where

$$\begin{aligned}
 (1.24) \quad & \mathfrak{Z}(\nabla^{TX}, \nabla^W, \nabla^\xi) \\
 &= \sum_{r=0}^m 2^{l+2m-6r} \zeta_r(\nabla^{TX}, \nabla^W, \nabla^\xi) \\
 & \quad - \left\{ \frac{e^{\frac{1}{24}(p_1(TX)-p_1(W))} - 1}{p_1(TX) - p_1(W)} \cdot \frac{\widehat{A}(TX) \det^{1/2} \left(2 \cosh \left(\frac{\sqrt{-1}}{4\pi} RW \right) \right)}{\cosh^2 \left(\frac{c}{2} \right)} \right\}^{(8m-4)}.
 \end{aligned}$$

We immediately obtain that

Corollary 1.2 (Han-Zhang, [7]) *If $p_1(TX, \nabla^{TX}) = p_1(W, \nabla^W)$, then (1) when $\dim X = 8m + 4$, the following identity holds,*

$$\begin{aligned}
 (1.25) \quad & \left\{ \frac{\widehat{A}(TX) \det^{1/2} \left(2 \cosh \left(\frac{\sqrt{-1}}{4\pi} RW \right) \right)}{\cosh^2 \left(\frac{c}{2} \right)} \right\}^{(8m+4)} \\
 & = \sum_{r=0}^m 2^{l+2m+1-6r} \left\{ \widehat{A}(TX) \text{ch}(b_r(T_{\mathbf{C}}X, W_{\mathbf{C}}, \xi_{\mathbf{C}})) \cosh \left(\frac{c}{2} \right) \right\}^{(8m+4)};
 \end{aligned}$$

(2) when $\dim X = 8m$, the following identity holds,

$$(1.26) \quad \left\{ \frac{\widehat{A}(TX) \det^{1/2} \left(2 \cosh \left(\frac{\sqrt{-1}}{4\pi} R^W \right) \right)}{\cosh^2 \left(\frac{c}{2} \right)} \right\}^{(8m)}$$

$$= \sum_{r=0}^m 2^{l+2m-6r} \left\{ \widehat{A}(TX) \text{ch}(z_r(T_{\mathbf{C}}X, W_{\mathbf{C}}, \xi_{\mathbf{C}})) \cosh \left(\frac{c}{2} \right) \right\}^{(8m)}.$$

In Corollary 1.2, when $\dim X = 8m + 4$ and $(W, \nabla^W) = (TX, \nabla^{TX})$, one has

$$(1.27) \quad \frac{1}{8} \left\{ \frac{\widehat{L}(TX)}{\cosh^2 \left(\frac{c}{2} \right)} \right\}^{(8m+4)}$$

$$= \sum_{r=0}^m 2^{6m-6r} \left\{ \widehat{A}(TX) \text{ch}(b_r(T_{\mathbf{C}}X, T_{\mathbf{C}}X, \xi_{\mathbf{C}})) \cosh \left(\frac{c}{2} \right) \right\}^{(8m+4)}.$$

This formula is used in [7] to study higher dimensional Rokhlin type congruences.

When $(\xi, \nabla^\xi) = (\mathbf{R}^2, d)$, from Theorem 1.1, we obtain that

Corollary 1.3 (1) When $\dim X = 8m + 4$, one has

$$(1.28) \quad \left\{ \widehat{A}(TX) \det^{1/2} \left(2 \cosh \left(\frac{\sqrt{-1}}{4\pi} R^W \right) \right) \right\}^{(8m+4)}$$

$$- \sum_{r=0}^m 2^{l+2m+1-6r} \left\{ \widehat{A}(TX) \text{ch}(b_r(T_{\mathbf{C}}X, W_{\mathbf{C}}, \mathbf{C}^2)) \right\}^{(8m+4)}$$

$$= (p_1(TX) - p_1(W)) \cdot \mathfrak{B}(\nabla^{TX}, \nabla^W, d),$$

where

$$(1.29) \quad \mathfrak{B}(\nabla^{TX}, \nabla^W, d)$$

$$= \sum_{r=0}^m 2^{l+2m+1-6r} \beta_r(\nabla^{TX}, \nabla^W, d)$$

$$- \left\{ \frac{e^{\frac{1}{24}(p_1(TX) - p_1(W))} - 1}{p_1(TX) - p_1(W)} \cdot \widehat{A}(TX) \det^{1/2} \left(2 \cosh \left(\frac{\sqrt{-1}}{4\pi} R^W \right) \right) \right\}^{(8m)}.$$

(2) When $\dim X = 8m$, one has

$$\begin{aligned}
 (1.30) \quad & \left\{ \widehat{A}(TX) \det^{1/2} \left(2 \cosh \left(\frac{\sqrt{-1}}{4\pi} R^W \right) \right) \right\}^{(8m)} \\
 & - \sum_{r=0}^m 2^{l+2m-6r} \left\{ \widehat{A}(TX) \text{ch}(z_r(T_{\mathbf{C}}X, W_{\mathbf{C}}, \mathbf{C}^2)) \right\}^{(8m)} \\
 & = (p_1(TX) - p_1(W)) \cdot \mathfrak{Z}(\nabla^{TX}, \nabla^W, d),
 \end{aligned}$$

where

$$\begin{aligned}
 (1.31) \quad & \mathfrak{Z}(\nabla^{TX}, \nabla^W, d) \\
 & = \sum_{r=0}^m 2^{l+2m-6r} \zeta_r(\nabla^{TX}, \nabla^W, d) \\
 & - \left\{ \frac{e^{\frac{1}{24}(p_1(TX) - p_1(W))} - 1}{p_1(TX) - p_1(W)} \cdot \widehat{A}(TX) \det^{1/2} \left(2 \cosh \left(\frac{\sqrt{-1}}{4\pi} R^W \right) \right) \right\}^{(8m-4)}.
 \end{aligned}$$

It's interesting to notice that the above anomaly cancellation formulas also imply some integrality results. From Corollary 1.3, we can see that if X is an $8m + 4$ dimensional closed spin manifold and W is a $2l$ dimensional spin vector bundle over X , then when $l \geq 4m - 1$,

$$(1.32) \quad \int_X (p_1(TX) - p_1(W)) \cdot \mathfrak{B}(\nabla^{TX}, \nabla^W, d)$$

is an integer. Moreover, if X is string, then

$$(1.33) \quad \int_X p_1(W) \cdot \mathfrak{B}(\nabla^{TX}, \nabla^W, d)$$

is an integer. Similarly, we can see that if X is an $8m$ dimensional closed spin manifold and W is a $2l$ dimensional spin vector bundle over X , then when $l \geq 4m$,

$$(1.34) \quad \int_X (p_1(TX) - p_1(W)) \cdot \mathfrak{Z}(\nabla^{TX}, \nabla^W, d)$$

is an integer. Moreover, if X is string, then

$$(1.35) \quad \int_X p_1(W) \cdot \mathfrak{Z}(\nabla^{TX}, \nabla^W, d)$$

is an integer.

From Corollary 1.3, we immediately obtain that

Corollary 1.4 (Liu, [12]) *If $p_1(TX, \nabla^{TX}) = p_1(W, \nabla^W)$, then*
 (1) *When $\dim X = 8m + 4$, the following identity holds,*

$$\begin{aligned}
(1.36) \quad & \left\{ \widehat{A}(TX) \det^{1/2} \left(2 \cosh \left(\frac{\sqrt{-1}}{4\pi} R^W \right) \right) \right\}^{(8m+4)} \\
& = \sum_{r=0}^m 2^{l+2m+1-6r} \left\{ \widehat{A}(TX) \text{ch}(b_r(T_{\mathbf{C}}X, W_{\mathbf{C}}, \mathbf{C}^2)) \right\}^{(8m+4)};
\end{aligned}$$

(2) When $\dim X = 8m$, one has

$$\begin{aligned}
(1.37) \quad & \left\{ \widehat{A}(TX) \det^{1/2} \left(2 \cosh \left(\frac{\sqrt{-1}}{4\pi} R^W \right) \right) \right\}^{(8m)} \\
& = \sum_{r=0}^m 2^{l+2m-6r} \left\{ \widehat{A}(TX) \text{ch}(z_r(T_{\mathbf{C}}X, W_{\mathbf{C}}, \mathbf{C}^2)) \right\}^{(8m)}.
\end{aligned}$$

In Corollary 1.4, when $\dim X = 8m + 4$ and $(W, \nabla^W) = (TX, \nabla^{TX})$, one has ([12], [11])

$$\begin{aligned}
(1.38) \quad & \frac{1}{8} \left\{ \widehat{L}(TX) \right\}^{(8m+4)} \\
& = \sum_{r=0}^m 2^{6m-6r} \left\{ \widehat{A}(TX) \text{ch}(b_r(T_{\mathbf{C}}X, T_{\mathbf{C}}X, \mathbf{C}^2)) \right\}^{(8m+4)}.
\end{aligned}$$

This formula implies the Ochanine divisibility [15], which asserts that the signature of an $8k + 4$ -dimensional smooth closed spin manifold is divisible by 16.

As examples, we give the explicit formulas when the dimension of X is 4, 8 and 12. Using (1.14), (1.16), (1.18) and (1.20), by direct computations, we have,

Corollary 1.5 (1) when $\dim X = 4$, the following identities hold,

$$\begin{aligned}
(1.39) \quad & \left\{ \frac{\widehat{A}(TX) \det^{1/2} \left(2 \cosh \left(\frac{\sqrt{-1}}{4\pi} R^W \right) \right)}{\cosh^2 \left(\frac{c}{2} \right)} \right\}^{(4)} + 2^{l+1} \left\{ \widehat{A}(TX) \cosh \left(\frac{c}{2} \right) \right\}^{(4)} \\
& = -2^{l-3} (p_1(TX) - p_1(W)),
\end{aligned}$$

$$\begin{aligned}
(1.40) \quad & \left\{ \widehat{A}(TX) \det^{1/2} \left(2 \cosh \left(\frac{\sqrt{-1}}{4\pi} R^W \right) \right) \right\}^{(4)} + 2^{l+1} \left\{ \widehat{A}(TX) \right\}^{(4)} \\
& = -2^{l-3} (p_1(TX) - p_1(W));
\end{aligned}$$

(2) when $\dim X = 8$, the following identities hold,

$$\begin{aligned}
 (1.41) \quad & \left\{ \frac{\widehat{A}(TX) \det^{1/2} \left(2 \cosh \left(\frac{\sqrt{-1}}{4\pi} R^W \right) \right)}{\cosh^2 \left(\frac{c}{2} \right)} \right\}^{(8)} \\
 & - \left\{ \left[-2^{l-4} \widehat{A}(TX) \operatorname{ch}(W_{\mathbf{C}}) + 2^{l-3} (l+8) \widehat{A}(TX) + 3 \cdot 2^{l-4} \widehat{A}(TX) (e^c + e^{-c} - 2) \right] \cosh \left(\frac{c}{2} \right) \right\}^{(8)} \\
 = & (p_1(TX) - p_1(W)) \\
 & \cdot \left\{ \frac{e^{\frac{1}{24}(p_1(TX) - p_1(W))} - 1}{p_1(TX) - p_1(W)} \right. \\
 & \cdot \left[\widehat{A}(TX) \cosh \left(\frac{c}{2} \right) \left(-2^{l-4} \operatorname{ch}(W_{\mathbf{C}}) + 2^{l-3} (l+8) + 3 \cdot 2^{l-4} (e^c + e^{-c} - 2) \right) \right. \\
 & \left. \left. - \frac{\widehat{A}(TX) \det^{1/2} \left(2 \cosh \left(\frac{\sqrt{-1}}{4\pi} R^W \right) \right)}{\cosh^2 \left(\frac{c}{2} \right)} \right] \right\}^{(4)},
 \end{aligned}$$

$$\begin{aligned}
 (1.42) \quad & \left\{ \widehat{A}(TX) \det^{1/2} \left(2 \cosh \left(\frac{\sqrt{-1}}{4\pi} R^W \right) \right) \right\}^{(8)} \\
 & + 2^{l-4} \left\{ \widehat{A}(TX) \operatorname{ch}(W_{\mathbf{C}}) \right\}^{(8)} - 2^{l-3} (l+8) \left\{ \widehat{A}(TX) \right\}^{(8)} \\
 = & (p_1(TX) - p_1(W)) \\
 & \cdot \left\{ \frac{e^{\frac{1}{24}(p_1(TX) - p_1(W))} - 1}{p_1(TX) - p_1(W)} \right. \\
 & \cdot \left[\widehat{A}(TX) \left(-2^{l-4} \operatorname{ch}(W_{\mathbf{C}}) + 2^{l-3} (l+8) \right) - \widehat{A}(TX) \det^{1/2} \left(2 \cosh \left(\frac{\sqrt{-1}}{4\pi} R^W \right) \right) \right] \right\}^{(4)};
 \end{aligned}$$

(3) when $\dim X = 12$, the following identities hold,

(1.43)

$$\begin{aligned}
& \left\{ \frac{\widehat{A}(TX) \det^{1/2} \left(2 \cosh \left(\frac{\sqrt{-1}}{4\pi} R^W \right) \right)}{\cosh^2 \left(\frac{c}{2} \right)} \right\}^{(12)} \\
& - \left\{ \left[2^{l-3} \widehat{A}(TX) \operatorname{ch}(W_{\mathbf{C}}) - 2^{l-2} (l-4) \widehat{A}(TX) - 3 \cdot 2^{l-3} \widehat{A}(TX) (e^c + e^{-c} - 2) \right] \cosh \left(\frac{c}{2} \right) \right\}^{(12)} \\
& = (p_1(TX) - p_1(W)) \\
& \cdot \left\{ \frac{e^{\frac{1}{24}(p_1(TX) - p_1(W))} - 1}{p_1(TX) - p_1(W)} \right. \\
& \quad \cdot \left[\widehat{A}(TX) \cosh \left(\frac{c}{2} \right) \left(2^{l-3} \operatorname{ch}(W_{\mathbf{C}}) - 2^{l-2} (l-4) - 3 \cdot 2^{l-3} (e^c + e^{-c} - 2) \right) \right. \\
& \quad \left. \left. - \frac{\widehat{A}(TX) \det^{1/2} \left(2 \cosh \left(\frac{\sqrt{-1}}{4\pi} R^W \right) \right)}{\cosh^2 \left(\frac{c}{2} \right)} \right] \right\}^{(8)},
\end{aligned}$$

(1.44)

$$\begin{aligned}
& \left\{ \widehat{A}(TX) \det^{1/2} \left(2 \cosh \left(\frac{\sqrt{-1}}{4\pi} R^W \right) \right) \right\}^{(12)} \\
& - 2^{l-3} \left\{ \widehat{A}(TX) \operatorname{ch}(W_{\mathbf{C}}) \right\}^{(12)} + 2^{l-2} (l-4) \left\{ \widehat{A}(TX) \right\}^{(12)} \\
& = (p_1(TX) - p_1(W)) \\
& \cdot \left\{ \frac{e^{\frac{1}{24}(p_1(TX) - p_1(W))} - 1}{p_1(TX) - p_1(W)} \right. \\
& \quad \cdot \left[\widehat{A}(TX) \left(2^{l-3} \operatorname{ch}(W_{\mathbf{C}}) - 2^{l-2} (l-4) \right) - \widehat{A}(TX) \det^{1/2} \left(2 \cosh \left(\frac{\sqrt{-1}}{4\pi} R^W \right) \right) \right] \right\}^{(8)}.
\end{aligned}$$

Remark 1.2. It's not hard to see that (1.41)-(1.44) are respectively equivalent to the following identities,

(1.45)

$$\begin{aligned}
& \left\{ e^{\frac{1}{24}(p_1(TX) - p_1(W))} \right. \\
& \quad \cdot \left[\frac{\widehat{A}(TX) \det^{1/2} \left(2 \cosh \left(\frac{\sqrt{-1}}{4\pi} R^W \right) \right)}{\cosh^2 \left(\frac{c}{2} \right)} \right. \\
& \quad \left. \left. - \widehat{A}(TX) \cosh \left(\frac{c}{2} \right) \left(-2^{l-4} \operatorname{ch}(W_{\mathbf{C}}) + 2^{l-3} (l+8) + 3 \cdot 2^{l-4} (e^c + e^{-c} - 2) \right) \right] \right\}^{(8)} = 0,
\end{aligned}$$

$$(1.46) \quad \left\{ e^{\frac{1}{24}(p_1(TX) - p_1(W))} \cdot \left[\widehat{A}(TX) \det^{1/2} \left(2 \cosh \left(\frac{\sqrt{-1}}{4\pi} R^W \right) \right) - \widehat{A}(TX) \left(-2^{l-4} \text{ch}(W_{\mathbf{C}}) + 2^{l-3}(l+8) \right) \right] \right\}^{(8)} = 0;$$

$$(1.47) \quad \left\{ e^{\frac{1}{24}(p_1(TX) - p_1(W))} \cdot \left[-\frac{\widehat{A}(TX) \det^{1/2} \left(2 \cosh \left(\frac{\sqrt{-1}}{4\pi} R^W \right) \right)}{\cosh^2 \left(\frac{c}{2} \right)} - \widehat{A}(TX) \cosh \left(\frac{c}{2} \right) \left(2^{l-3} \text{ch}(W_{\mathbf{C}}) - 2^{l-2}(l-4) - 3 \cdot 2^{l-3}(e^c + e^{-c} - 2) \right) \right] \right\}^{(12)} = 0,$$

$$(1.48) \quad \left\{ e^{\frac{1}{24}(p_1(TX) - p_1(W))} \cdot \left[\widehat{A}(TX) \det^{1/2} \left(2 \cosh \left(\frac{\sqrt{-1}}{4\pi} R^W \right) \right) - \widehat{A}(TX) \left(2^{l-3} \text{ch}(W_{\mathbf{C}}) - 2^{l-2}(l-4) \right) \right] \right\}^{(12)} = 0.$$

These formulas are simply in a form of the multiplication of $e^{\frac{1}{24}(p_1(TX) - p_1(W))}$ to the original anomaly cancellation formulas in dimension 8 and 12 holding under the condition $p_1(X) = p_1(W)$.

However as pointed out on page 7 and page 8 about the patterns of β_r and ζ_r for $r \geq 2$, we know that the higher (> 12) dimensional anomaly cancellation formulas are not as simple as the above lower anomaly cancellation formulas, i.e. they are not simply in a form of the multiplication of $e^{\frac{1}{24}(p_1(TX) - p_1(W))}$ to the original anomaly cancellation formulas holding under the condition $p_1(X) = p_1(W)$.

2. PROOFS

In this section, we give the proof of Theorem 1.1. To prepare for the proof in Section 2.2, we will first recall some basic knowledge about the Jacobi theta functions, modular forms and Eisenstein series in Section 2.1.

2.1. Preliminaries. Let

$$SL_2(\mathbf{Z}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| a, b, c, d \in \mathbf{Z}, ad - bc = 1 \right\}$$

as usual be the modular group. Let

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

be the two generators of $SL_2(\mathbf{Z})$. Their actions on \mathbf{H} are given by

$$S : \tau \rightarrow -\frac{1}{\tau}, \quad T : \tau \rightarrow \tau + 1.$$

The four Jacobi theta functions are defined as follows (cf. [4]):

$$\theta(v, \tau) = 2q^{1/8} \sin(\pi v) \prod_{j=1}^{\infty} \left[(1 - q^j)(1 - e^{2\pi\sqrt{-1}v} q^j)(1 - e^{-2\pi\sqrt{-1}v} q^j) \right],$$

$$\theta_1(v, \tau) = 2q^{1/8} \cos(\pi v) \prod_{j=1}^{\infty} \left[(1 - q^j)(1 + e^{2\pi\sqrt{-1}v} q^j)(1 + e^{-2\pi\sqrt{-1}v} q^j) \right],$$

$$\theta_2(v, \tau) = \prod_{j=1}^{\infty} \left[(1 - q^j)(1 - e^{2\pi\sqrt{-1}v} q^{j-1/2})(1 - e^{-2\pi\sqrt{-1}v} q^{j-1/2}) \right],$$

$$\theta_3(v, \tau) = \prod_{j=1}^{\infty} \left[(1 - q^j)(1 + e^{2\pi\sqrt{-1}v} q^{j-1/2})(1 + e^{-2\pi\sqrt{-1}v} q^{j-1/2}) \right].$$

They are all holomorphic functions for $(v, \tau) \in \mathbf{C} \times \mathbf{H}$, where \mathbf{C} is the complex plane and \mathbf{H} is the upper half plane.

If we act theta-functions by S and T , the theta functions obey the following transformation laws (cf. [4]),

(2.1)

$$\theta(v, \tau+1) = e^{\frac{\pi\sqrt{-1}}{4}} \theta(v, \tau), \quad \theta(v, -1/\tau) = \frac{1}{\sqrt{-1}} \left(\frac{\tau}{\sqrt{-1}} \right)^{1/2} e^{\pi\sqrt{-1}\tau v^2} \theta(\tau v, \tau);$$

(2.2)

$$\theta_1(v, \tau+1) = e^{\frac{\pi\sqrt{-1}}{4}} \theta_1(v, \tau), \quad \theta_1(v, -1/\tau) = \left(\frac{\tau}{\sqrt{-1}} \right)^{1/2} e^{\pi\sqrt{-1}\tau v^2} \theta_2(\tau v, \tau);$$

(2.3)

$$\theta_2(v, \tau+1) = \theta_3(v, \tau), \quad \theta_2(v, -1/\tau) = \left(\frac{\tau}{\sqrt{-1}} \right)^{1/2} e^{\pi\sqrt{-1}\tau v^2} \theta_1(\tau v, \tau);$$

(2.4)

$$\theta_3(v, \tau+1) = \theta_2(v, \tau), \quad \theta_3(v, -1/\tau) = \left(\frac{\tau}{\sqrt{-1}} \right)^{1/2} e^{\pi\sqrt{-1}\tau v^2} \theta_3(\tau v, \tau).$$

Definition 2.1. Let Γ be a subgroup of $SL_2(\mathbf{Z})$. A modular form over Γ is a holomorphic function $f(\tau)$ on $\mathbf{H} \cup \{\infty\}$ such that for any

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma,$$

the following property holds

$$f(g\tau) := f\left(\frac{a\tau + b}{c\tau + d}\right) = \chi(g)(c\tau + d)^l f(\tau),$$

where $\chi : \Gamma \rightarrow \mathbf{C}^*$ is a character of Γ and l is called the weight of f .

Let

$$(2.5) \quad E_{2k} = 1 - \frac{4k}{B_{2k}} \sum_{n=1}^{\infty} \left(\sum_{d|n} d^{2k-1} \right) q^n$$

be the Eisenstein series, where B_{2k} is the $2k$ -th Bernoulli number.

When $k > 1$, E_{2k} is a modular form of weight $2k$ over $SL_2(\mathbf{Z})$. However, unlike other Eisenstein theories, $E_2(\tau)$ is not a modular form over $SL(2, \mathbf{Z})$, instead $E_2(\tau)$ is a quasimodular form over $SL(2, \mathbf{Z})$, satisfying:

$$(2.6) \quad E_2 \left(\frac{a\tau + b}{c\tau + d} \right) = (c\tau + d)^2 E_2(\tau) - \frac{6\sqrt{-1}c(c\tau + d)}{\pi}.$$

In particular, we have

$$(2.7) \quad E_2(\tau + 1) = E_2(\tau),$$

$$(2.8) \quad E_2 \left(-\frac{1}{\tau} \right) = \tau^2 E_2(\tau) - \frac{6\sqrt{-1}\tau}{\pi}.$$

For the precise definition of quasimodular forms, see [10].

In the following, let's review some level 2 modular forms.

Let

$$\Gamma_0(2) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbf{Z}) \mid c \equiv 0 \pmod{2} \right\},$$

$$\Gamma^0(2) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbf{Z}) \mid b \equiv 0 \pmod{2} \right\}$$

be the two modular subgroups of $SL_2(\mathbf{Z})$. It is known that the generators of $\Gamma_0(2)$ are T, ST^2ST and the generators of $\Gamma^0(2)$ are STS, T^2STS . (cf. [4]).

Writing simply $\theta_j = \theta_j(0, \tau)$, $1 \leq j \leq 3$, we have (cf. [9] and [14]),

$$\delta_1(\tau) = \frac{1}{8}(\theta_2^4 + \theta_3^4), \quad \varepsilon_1(\tau) = \frac{1}{16}\theta_2^4\theta_3^4,$$

$$\delta_2(\tau) = -\frac{1}{8}(\theta_1^4 + \theta_3^4), \quad \varepsilon_2(\tau) = \frac{1}{16}\theta_1^4\theta_3^4.$$

If Γ is a modular subgroup, let $M_{\mathbf{R}}(\Gamma)$ denote the ring of modular forms over Γ with real Fourier coefficients.

Lemma 2.1 (cf. [12]). *One has that $\delta_1(\tau)$ (resp. $\varepsilon_1(\tau)$) is a modular form of weight 2 (resp. 4) over $\Gamma_0(2)$, $\delta_2(\tau)$ (resp. $\varepsilon_2(\tau)$) is a modular form of weight 2 (resp. 4) over $\Gamma^0(2)$ and moreover $M_{\mathbf{R}}(\Gamma^0(2)) = \mathbf{R}[\delta_2(\tau), \varepsilon_2(\tau)]$. Moreover, we have transformation laws*

$$(2.9) \quad \delta_2 \left(-\frac{1}{\tau} \right) = \tau^2 \delta_1(\tau), \quad \varepsilon_2 \left(-\frac{1}{\tau} \right) = \tau^4 \varepsilon_1(\tau).$$

2.2. **Proof of Theorem 1.1.** Set

$$(2.10) \quad \Theta_1(T_{\mathbf{C}}X, W_{\mathbf{C}}, \xi_{\mathbf{C}}) = \bigotimes_{u=1}^{\infty} S_{q^u}(\widetilde{T_{\mathbf{C}}X}) \otimes \bigotimes_{v=1}^{\infty} \Lambda_{q^v}(\widetilde{W_{\mathbf{C}}} - 2\widetilde{\xi_{\mathbf{C}}}) \\ \otimes \bigotimes_{r=1}^{\infty} \Lambda_{q^{r-1/2}}(\widetilde{\xi_{\mathbf{C}}}) \otimes \bigotimes_{s=1}^{\infty} \Lambda_{-q^{s-1/2}}(\widetilde{\xi_{\mathbf{C}}}).$$

(2.11) $\Theta_1(T_{\mathbf{C}}X, W_{\mathbf{C}}, \xi_{\mathbf{C}})$ admits formal Fourier expansion in $q^{1/2}$ as

$$\Theta_1(T_{\mathbf{C}}X, W_{\mathbf{C}}, \xi_{\mathbf{C}}) = A_0(T_{\mathbf{C}}X, W_{\mathbf{C}}, \xi_{\mathbf{C}}) + A_1(T_{\mathbf{C}}X, W_{\mathbf{C}}, \xi_{\mathbf{C}})q^{1/2} + \cdots,$$

where the A_j 's are elements in the semi-group formally generated by complex vector bundles over X . Moreover, they carry canonically induced connections denoted by ∇^{A_j} , and let ∇^{Θ_1} be the induced connections with $q^{1/2}$ -coefficients on Θ_1 from the ∇^{A_j} .

To prove part 1 of Theorem 1.1 ($8m + 4$ -dimensional case), set

$$(2.12) \quad P_1(\tau) := \left\{ e^{\frac{1}{24}E_2(\tau)(p_1(TX) - p_1(W))} \cdot \frac{\widehat{A}(TX) \det^{1/2} \left(2 \cosh \left(\frac{\sqrt{-1}}{4\pi} RW \right) \right)}{\cosh^2 \left(\frac{c}{2} \right)} \operatorname{ch} (\Theta_1(T_{\mathbf{C}}X, W_{\mathbf{C}}, \xi_{\mathbf{C}})) \right\}^{(8m+4)},$$

$$(2.13) \quad P_2(\tau) := \left\{ \widehat{A}(TX) \cosh \left(\frac{c}{2} \right) \operatorname{ch} (\Theta_2(T_{\mathbf{C}}X, W_{\mathbf{C}}, \xi_{\mathbf{C}})) \right\}^{(8m+4)}$$

and

$$(2.14) \quad \Xi_2(\tau) := \left\{ \frac{e^{\frac{1}{24}E_2(\tau)(p_1(TX) - p_1(W))} - 1}{p_1(TX) - p_1(W)} \cdot \widehat{A}(TX) \cosh \left(\frac{c}{2} \right) \operatorname{ch} (\Theta_2(T_{\mathbf{C}}X, W_{\mathbf{C}}, \xi_{\mathbf{C}})) \right\}^{(8m)}.$$

We have

Proposition 2.1. $P_1(\tau)$ is a modular form of weight $4m + 2$ over $\Gamma_0(2)$ while $P_2(\tau) + (p_1(TX) - p_1(W))\Xi_2(\tau)$ is a modular form of weight $4m + 2$ over $\Gamma^0(2)$. Moreover, the following identity holds,

$$(2.15) \quad P_1 \left(-\frac{1}{\tau} \right) = 2^l \tau^{4m+2} (P_2(\tau) + (p_1(TX) - p_1(W))\Xi_2(\tau)).$$

Proof. Let $\{\pm 2\pi\sqrt{-1}y_k\}$ (resp. $\{\pm 2\pi\sqrt{-1}x_j\}$) be the formal Chern roots for $(W_{\mathbf{C}}, \nabla^{W_{\mathbf{C}}})$ (resp. $(TM_{\mathbf{C}}, \nabla^{TM_{\mathbf{C}}})$). Let $c = 2\pi\sqrt{-1}u$.

By the Chern root algorithm, we have

$$(2.16) \quad P_1(\tau) = 2^l \left\{ e^{\frac{1}{24}E_2(\tau)(p_1(TX)-p_1(W))} \left(\prod_{j=1}^{4m+2} \left(x_j \frac{\theta'(0, \tau)}{\theta(x_j, \tau)} \right) \right) \left(\prod_{k=1}^l \frac{\theta_1(y_k, \tau)}{\theta_1(0, \tau)} \right) \cdot \frac{\theta_1^2(0, \tau) \theta_3(u, \tau) \theta_2(u, \tau)}{\theta_1^2(u, \tau) \theta_3(0, \tau) \theta_2(0, \tau)} \right\}^{(8m+4)},$$

and

$$(2.17) \quad \begin{aligned} & P_2(\tau) + (p_1(TX) - p_1(W))\Xi_2(\tau) \\ &= \left\{ e^{\frac{1}{24}E_2(\tau)(p_1(TX)-p_1(W))} \widehat{A}(TX) \cosh\left(\frac{c}{2}\right) \text{ch}(\Theta_2(T_{\mathbf{C}}X, W_{\mathbf{C}}, \xi_{\mathbf{C}})) \right\}^{(8m+4)} \\ &= \left\{ e^{\frac{1}{24}E_2(\tau)(p_1(TX)-p_1(W))} \left(\prod_{j=1}^{4m+2} \left(x_j \frac{\theta'(0, \tau)}{\theta(x_j, \tau)} \right) \right) \left(\prod_{j=1}^l \frac{\theta_2(y_j, \tau)}{\theta_2(0, \tau)} \right) \cdot \frac{\theta_2^2(0, \tau) \theta_3(u, \tau) \theta_1(u, \tau)}{\theta_2^2(u, \tau) \theta_3(0, \tau) \theta_1(0, \tau)} \right\}^{(8m+4)}. \end{aligned}$$

Then we can apply the transformation laws (2.1)-(2.4) for theta functions as well as the transformation laws (2.7)-(2.8) to (2.16) and (2.17) to get the desired results. \square

We can now proceed to prove part 1 of Theorem 1.1 as follows.

Combining Lemma 2.1 and Proposition 2.1, we can write

$$(2.18) \quad \begin{aligned} & P_2(\tau) + (p_1(TX) - p_1(W))\Xi_2(\tau) \\ &= h_0(8\delta_2)^{2m+1} + h_1(8\delta_2)^{2m-1}\varepsilon_2 + \cdots + h_m(8\delta_2)\varepsilon_2^m, \end{aligned}$$

where $h_r \in \Omega^{8m+4}(X)$, $0 \leq r \leq m$.

By the definitions of $b_r(T_{\mathbf{C}}X, W_{\mathbf{C}}, \xi_{\mathbf{C}})$ and $\beta_r(\nabla^{TX}, \nabla^W, \nabla^\xi)$, it's easy to see that for $0 \leq r \leq m$,

$$(2.19) \quad h_r = \left\{ \widehat{A}(TX) \cosh\left(\frac{c}{2}\right) \text{ch}(b_r(T_{\mathbf{C}}X, W_{\mathbf{C}}, \xi_{\mathbf{C}})) \right\}^{(8m+4)} + (p_1(TX) - p_1(W))\beta_r(\nabla^{TX}, \nabla^W, \nabla^\xi).$$

Therefore (simply denote $b_r(T_{\mathbf{C}}X, W_{\mathbf{C}}, \xi_{\mathbf{C}})$ and $\beta_r(\nabla^{TX}, \nabla^W, \nabla^\xi)$ by b_r and β_r),

$$(2.20) \quad \begin{aligned} & P_2(\tau) + (p_1(TX) - p_1(W))\Xi_2(\tau) \\ &= \sum_{r=0}^m \left(\left\{ \widehat{A}(TX) \cosh\left(\frac{c}{2}\right) \text{ch}(b_r) \right\}^{(8m+4)} + (p_1(TX) - p_1(W))\beta_r \right) (8\delta_2)^{2m+1-r} \varepsilon_2^r. \end{aligned}$$

By (2.9) and (2.15), we have

$$\begin{aligned}
(2.21) \quad & P_1(\tau) \\
&= \frac{2^l}{\tau^{4m+2}} \left[P_2 \left(-\frac{1}{\tau} \right) + (p_1(TX) - p_1(W)) \Xi_2 \left(-\frac{1}{\tau} \right) \right] \\
&= \frac{2^l}{\tau^{4m+2}} \left[\left(\left\{ \widehat{A}(TX) \cosh \left(\frac{c}{2} \right) \text{ch}(b_0) \right\}^{(8m+4)} + (p_1(TX) - p_1(W)) \beta_0 \right) \left(8\delta_2 \left(-\frac{1}{\tau} \right) \right)^{2m+1} \right. \\
&\quad + \dots \\
&\quad \left. + \left(\left\{ \widehat{A}(TX) \cosh \left(\frac{c}{2} \right) \text{ch}(b_m) \right\}^{(8m+4)} + (p_1(TX) - p_1(W)) \beta_m \right) \right. \\
&\quad \left. \cdot 8\delta_2 \left(-\frac{1}{\tau} \right) \left(\varepsilon_2 \left(-\frac{1}{\tau} \right) \right)^m \right] \\
&= 2^l \left[\left(\left\{ \widehat{A}(TX) \cosh \left(\frac{c}{2} \right) \text{ch}(b_0) \right\}^{(8m+4)} + (p_1(TX) - p_1(W)) \beta_0 \right) (8\delta_1)^{2m+1} \right. \\
&\quad + \dots \\
&\quad \left. + \left(\left\{ \widehat{A}(TX) \cosh \left(\frac{c}{2} \right) \text{ch}(b_m) \right\}^{(8m+4)} + (p_1(TX) - p_1(W)) \beta_m \right) (8\delta_1) \varepsilon_1^m \right].
\end{aligned}$$

Comparing the constant terms of both sides of (2.21), one has

$$\begin{aligned}
(2.22) \quad & e^{\frac{1}{24}(p_1(TX) - p_1(W))} \frac{\widehat{A}(TX) \det^{1/2} \left(2 \cosh \left(\frac{\sqrt{-1}}{4\pi} RW \right) \right)}{\cosh^2 \left(\frac{c}{2} \right)} \\
&= \sum_{r=0}^m 2^{l+2m+1-6r} \left(\left\{ \widehat{A}(TX) \cosh \left(\frac{c}{2} \right) \text{ch}(b_r) \right\}^{(8m+4)} + (p_1(TX) - p_1(W)) \beta_r \right).
\end{aligned}$$

So we have

$$\begin{aligned}
(2.23) \quad & \left\{ \frac{\widehat{A}(TX) \det^{1/2} \left(2 \cosh \left(\frac{\sqrt{-1}}{4\pi} RW \right) \right)}{\cosh^2 \left(\frac{c}{2} \right)} \right\}^{(8m+4)} \\
&\quad - \sum_{r=0}^m 2^{l+2m+1-6r} \left\{ \widehat{A}(TX) \text{ch}(b_r(T_{\mathbb{C}}X, W_{\mathbb{C}}, \xi_{\mathbb{C}})) \cosh \left(\frac{c}{2} \right) \right\}^{(8m+4)} \\
&= (p_1(TX) - p_1(W)) \cdot \mathfrak{B}(\nabla^{TX}, \nabla^W, \nabla^\xi),
\end{aligned}$$

where

$$\begin{aligned} & \mathfrak{B}(\nabla^{TX}, \nabla^W, \nabla^\xi) \\ &= \sum_{r=0}^m 2^{l+2m+1-6r} \beta_r(\nabla^{TX}, \nabla^W, \nabla^\xi) \\ & \quad - \left\{ \frac{e^{\frac{1}{24}(p_1(TX)-p_1(W))} - 1}{p_1(TX) - p_1(W)} \cdot \frac{\widehat{A}(TX) \det^{1/2} \left(2 \cosh \left(\frac{\sqrt{-1}}{4\pi} R^W \right) \right)}{\cosh^2 \left(\frac{c}{2} \right)} \right\}^{(8m)}. \end{aligned}$$

To prove part 2 of Theorem 1.1 ($8m$ -dimensional case), set

$$(2.24) \quad Q_1(\tau) := \left\{ e^{\frac{1}{24} E_2(\tau)(p_1(TX)-p_1(W))} \cdot \frac{\widehat{A}(TX) \det^{1/2} \left(2 \cosh \left(\frac{\sqrt{-1}}{4\pi} R^W \right) \right)}{\cosh^2 \left(\frac{c}{2} \right)} \text{ch} \left(\Theta_1(T_{\mathbf{C}}X, W_{\mathbf{C}}, \xi_{\mathbf{C}}) \right) \right\}^{(8m)},$$

$$(2.25) \quad Q_2(\tau) := \left\{ \widehat{A}(TX) \cosh \left(\frac{c}{2} \right) \text{ch} \left(\Theta_2(T_{\mathbf{C}}X, W_{\mathbf{C}}, \xi_{\mathbf{C}}) \right) \right\}^{(8m)}$$

and

$$(2.26) \quad \begin{aligned} \Pi_2(\tau) := & \left\{ \frac{e^{\frac{1}{24} E_2(\tau)(p_1(TX)-p_1(W))} - 1}{p_1(TX) - p_1(W)} \right. \\ & \left. \cdot \widehat{A}(TX) \cosh \left(\frac{c}{2} \right) \text{ch} \left(\Theta_2(T_{\mathbf{C}}X, W_{\mathbf{C}}, \xi_{\mathbf{C}}) \right) \right\}^{(8m-4)}. \end{aligned}$$

Similar to the $8m + 4$ dimensional case, one has

Proposition 2.2. $Q_1(\tau)$ is a modular form of weight $4m$ over $\Gamma_0(2)$ while $Q_2(\tau) + (p_1(TX) - p_1(W))\Pi_2(\tau)$ is a modular form of weight $4m$ over $\Gamma^0(2)$. Moreover, the following identity holds,

$$(2.27) \quad Q_1 \left(-\frac{1}{\tau} \right) = 2^l \tau^{4m} (Q_2(\tau) + (p_1(TX) - p_1(W))\Pi_2(\tau)).$$

Then one can prove part 2 of Theorem 1.1 by adopting similar idea as in the above proof of part 1 of Theorem 1.1.

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REFERENCES

- [1] L. Alvarez-Gaumé and E. Witten, Gravitational anomalies, *Nucl. Physics*, B234, 269-330, (1983).
- [2] M. F. Atiyah, *K – theory*. Benjamin, New York, 1967.
- [3] M. F. Atiyah and F. Hirzebruch, Riemann-Roch for differentiable manifolds. *Bull. Amer. Math. Soc.* 65(1959), 276-281.
- [4] K. Chandrasekharan, *Elliptic Functions*. Springer-Verlag, 1985.
- [5] M. B. Green, J. H. Schwarz, Anomaly Cancellations in Supersymmetric D=10 Gauge Theory and Superstring Theory, *Physics Letters* B149 (1984) pp. 11722.
- [6] F. Han and W. Zhang, Spin^c-manifold and elliptic genera. *C. R. Acad. Sci. Paris, Série I.* 336 (2003), 1011-1014.
- [7] F. Han and W. Zhang, Modular invariance, characteristic numbers and η invariants. *Journal of Differential Geometry* 67 (2004), 257-288.
- [8] F. Hirzebruch, *Topological Methods in Algebraic Geometry*. Springer-Verlag, 1966.
- [9] F. Hirzebruch, T. Berger and R. Jung, *Manifolds and Modular Forms*. Aspects of Mathematics, vol. E20, Vieweg, Braunschweig 1992.
- [10] M. Kaneko and D. Zagier, A generalized Jacobi theta function and quasimodular forms. in *The moduli space of curves*, Ed. R. Dijkgraaf, C. Faber and G. van der Geer, Boston: Birkhäuser, (1995), 165-172.
- [11] P. S. Landweber, Elliptic cohomology and modular forms. in *Elliptic Curves and Modular Forms in Algebraic Topology*, p. 55-68. Ed. P. S. Landweber. Lecture Notes in Mathematics Vol. 1326, Springer-Verlag (1988).
- [12] K. Liu, Modular invariance and characteristic numbers, *Comm. Math. Phys.*, 174, 29-42 (1995).
- [13] K. Liu, On elliptic genera and theta-functions, *Topology* 35 (1996), 617-640.
- [14] K. Liu, Modular forms and topology. *Moonshine, the Monster, and related topics (South Hadley, MA, 1994)*, 237-262, *Contemp. Math.*, 193, Amer. Math. Soc., Providence, RI, 1996.
- [15] S. Ochanine, Signature modulo 16, invariants de Kervaire généralisés et nombre caractéristiques dans la K -théorie réelle. *Mémoire Soc. Math. France*, Tom. 109 (1987), 1-141.
- [16] J. H. Schwarz, Anomaly cancellation: a retrospective from a modern perspective, *Arxiv*: hep-th/0107059.
- [17] E. Witten, The index of the Dirac operator in loop space, in P.S. Landweber, ed., *Elliptic Curves and Modular Forms in Algebraic Topology* (Proceedings, Princeton 1986), *Lecture Notes in Math.*, 1326, pp. 161-181, Springer, 1988, MR 0970288, Zbl 0679.58045.
- [18] W. Zhang, *Lectures on Chern-Weil Theory and Witten Deformations*. Nankai Tracts in Mathematics Vol. 4, World Scientific, Singapore, 2001.

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