A new geometric flow with rotational invariance

De-Xing Kong and Qiang Ru

Department of Mathematics, Zhejiang University

Hangzhou 310027, China

Abstract

In this paper we introduce a new geometric flow with rotational invariance and prove that, under this kind of flow, an arbitrary smooth closed contractible hypersurface in the Euclidean space $\mathbb{R}^{n+1}$ $(n \geq 1)$ converges to $\mathbb{S}^n$ in the C^∞-topology as t goes to the infinity. This result covers the well-known theorem of Gage and Hamilton in [4] for the curvature flow of plane curves and the famous result of Huisken in [5] on the flow by mean curvature of convex surfaces, respectively.

Key words and phrases: System of hyperbolic-parabolic equations, rotational invariance, global smooth solution, time-asymptotic behavior, hypersurface.

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1 Introduction

Since the last quarter of twentieth century, using partial differential equations to formulate and solve geometric problems has become a trend and a dominating force. A new area called geometric analysis was born. When looking back at the history of geometric analysis, one could see numerous success stories of utilizing differential equations to tackle important problems in geometry, topology and physics. Typical and important examples would include Yau's solution to the Calabi conjecture using the complex Monge-Ampere equation (see Yau [16]), Schoen's solution of the Yamabe conjecture (see Schoen [13]), Schoen-Yau's proof of the positive mass conjecture (see Schoen-Yau [14]), Donaldson's work on 4-dimensional smooth manifolds using the Yang-Mills equation (see Donaldson [3]), and recently, Perelman's solution to the century-old Poincaré conjecture using Hamilton's beautiful theory on the Ricci flow, which is just a nonlinear version of the classical heat equation (see [10]-[12]). However, despite all these success, the equations studied and utilized in geometry so far are almost exclusively of elliptic or parabolic type. With few exceptions, hyperbolic equations have not yet found their way into the study of geometric or topological problems. More recently, Kong et al introduced the hyperbolic geometric flow which is a fresh start of an attempt to introduce hyperbolic partial differential equations into the realm of geometry (see [6] or [7]). The kind of flow is a very natural tool to understand the wave character of metrics, the wave phenomenon of curvatures, the evolution of manifolds and their structures (see [2], [8]-[9]).

In this paper, we introduce a new geometric flow with rotational invariance. This flow is described by, formally a system of parabolic partial differential equations, essentially a coupled system of hyperbolic-parabolic partial differential equations with rotational invariance. More precisely, let $\mathscr{S}_t$ be a family of hypersurfaces in the $(n+1)$-dimensional Euclidean space $\mathbb{R}^{n+1}$ with coordinates $(x_1, \cdots, x_{n+1})$, without loss of generality, we may assume that the family of hypersurfaces $\mathscr{S}_t$ is given by

$$x = x(t, \theta_1, \cdots, \theta_n), \tag{1.1}$$

where $x = (x_1, \cdots, x_{n+1})^T$ is a vector-valued smooth function of t and $\theta = (\theta_1, \cdots, \theta_n)$, the new flow considered here is given by the following evolution equation

$$\frac{\partial x}{\partial t} + \sum_{i=1}^{n} \frac{\partial (f_i(|x|)x)}{\partial \theta_i} = \frac{x}{|x|}\Delta|x|, \tag{1.2}$$

where $f_i(\nu)$ $(i = 1, \cdots, n)$ are n given smooth functions, $\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial \theta_i^2}$ is the Laplacian operator, and $| \bullet |$ stands for the norm of the vector $\bullet$ in $\mathbb{R}^{n+1}$. It is easy to verify that the equation (1.2) possesses the rotational invariance which plays an important role in the present paper.

We are interested in the deformation of a smooth closed contractible hypersurface $x = x_0(\theta_1, \cdots, \theta_n)$ under the flow (1.2), that is, we consider how the hypersurface x_0 is smoothly deformed, say, embedded into a smooth family of hypersurfaces depending on a time parameter. This can be reduced to solve the Cauchy problem for (1.2) with the initial data

$$t = 0: \quad x = x_0(\theta_1, \cdots, \theta_n). \tag{1.3}$$

Obviously, in the present situation, $x_0 = x_0(\theta_1, \cdots, \theta_n)$ is a vector-valued periodic function, say, defined on $[0, 1]^n$. In Section 2, we shall prove

Theorem 1.1 *If $f \in C^1$, $x_0 \in L^\infty$and $|x_0(\theta_1, \cdots, \theta_n)| > 0$, then the Cauchy problem (1.2), (1.3) admits a unique global smooth solution on $[0, \infty) \times \mathbb{R}^n$.*

In particular, the following theorem will be proved in Section 3.

Theorem 1.2 *Suppose that $f_i(\nu)$ are all constants, i.e., $f_i(\nu) \equiv c_i$ $(i = 1, \cdots, n)$, suppose furthermore that $x_0 = x_0(\theta_1, \cdots, \theta_n)$ is a smooth vector-valued periodic function with the period $[0, 1]^n$, and satisfies*

$$|x_0(\theta_1, \cdots, \theta_n)| > 0, \quad \forall\, (\theta_1, \cdots, \theta_n) \in [0, 1]^n. \tag{1.4}$$

Then the Cauchy problem (1.2)-(1.3) has a global smooth solution $x = x(t, \theta_1, \cdots, \theta_n)$, and the solution satisfies

$$|x(t, \theta_1, \cdots, \theta_n)| \longrightarrow \bar{r}_0 \triangleq \int_{[0,1]^n} |x_0(\theta_1, \cdots, \theta_n)| d\theta_1 \cdots d\theta_n \qquad \text{as} \quad t \nearrow \infty. \tag{1.5}$$

Moreover, if the hypersurfaces undergo suitable homotheties, then the normalized hypersurfaces converge to a sphere in the C^∞-topology as t goes to the infinity.

Remark 1.1 *The geometric meaning of the result in Theorem 1.2 can be shown in the following figure:*

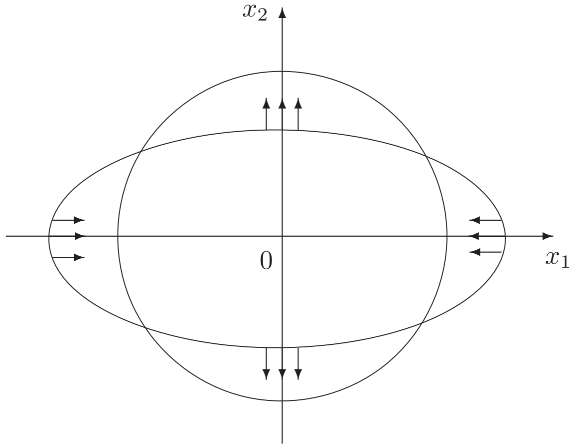

Figure 1: Deformation of an ellipse

Remark 1.2 *In the case $n = 1$, Theorem 1.1 covers the well-known theorem of Gage and Hamilton in [4] for the curvature flow of plane curves; while for the case of general n, Theorem 1.1 covers the famous result of Huisken in [5] on the flow by mean curvature of convex surfaces into spheres. In particular, we would like to point out that, in Theorem 1.1, we do NOT require the assumption that the hypersurface is convex.*

The paper is organized as follows. In Section 2 we prove the global existence and uniqueness of smooth solutions for the Cauchy problem (1.2), (1.3); Sections 3 is devoted to the proof of Theorem 1.2; In Section 4, we state conclusions obtained in the present paper and give some open problems. In Appendix, we investigate the time-asymptotic behavior of global smooth solutions for the equation (1.2).

2 Global existence and uniqueness of smooth solutions

This section is devoted to the global existence and uniqueness of smooth solution of the following equation

$$\frac{\partial x}{\partial t} + \sum_{i=1}^{m} \frac{\partial (f_i(|x|)x)}{\partial \theta_i} = \frac{x}{|x|}\Delta |x|, \tag{2.1}$$

where $x = (x_1, \cdots, x_n)^T$ is the unknown vector-valued function, $f(\nu) = (f_1(\nu), \cdots, f_m(\nu))^T$ is a given smooth vector-valed function, $\Delta = \sum_{i=1}^{m} \frac{\partial^2}{\partial \theta_i^2}$ is the Laplacian operator, and $|\bullet|$ stands for the norm of the vector $\bullet$ in $\mathbb{R}^n$.

Let

$$x = rP, \quad r = |x|, \quad P = (p_1, \cdots, p_n) \in \mathbb{S}^{n-1}. \tag{2.2}$$

Then it is easy to verify that the equation (2.1) can be rewritten as

$$\frac{\partial r}{\partial t} + \sum_{i=1}^{m} \frac{\partial (f_i(r)r)}{\partial \theta_i} = \Delta r \tag{2.1a}$$

and

$$\frac{\partial P}{\partial t} + \sum_{i=1}^{m} \frac{\partial P}{\partial \theta_i} f_i(r) = 0 \tag{2.1b}$$

for smooth solutions.

We now consider the Cauchy problem for the equation (2.1), equivalently, the system (2.1a)-(2.1b) with initial data

$$t = 0: \quad r = r_0(\theta), \quad P = P_0(\theta), \tag{2.3}$$

where $r_0(\theta)$ is a given scalar function of θ, and $P_0(\theta)$ is a given vector-valued function of θ. In what follows, we first investigate the local existence of smooth solution of the above Cauchy problem.

As the standard way, let $K(t, \theta)$ be the fundamental solution associated with the operator $\dfrac{\partial}{\partial t} - \triangle$. That is to say,

$$K(t, \theta) = (4\pi t)^{-\frac{n}{2}} \exp\left\{-\frac{|\theta|^2}{4t}\right\}. \tag{2.4}$$

Then the solution $r = r(t, \theta)$ of the Cauchy problem

$$\begin{cases} \dfrac{\partial r}{\partial t} + \displaystyle\sum_{i=1}^{m} \frac{\partial (f_i(r)r)}{\partial \theta_i} = \Delta r, \\ \\ t = 0: \quad r = r_0(\theta) \end{cases} \tag{2.5}$$

has the following integral representation

$$r(t, \theta) = K(t, \theta) * r_0(\theta) + \sum_{j=1}^{m} \int_0^t K_{\theta_j}(t - s, \theta) * (f_j(r(s, \theta))r(s, \theta))ds, \tag{2.6}$$

where $*$ denotes the convolution with the space variables. We have

Lemma 2.1 *Assume that*

$$f \in C^1, \quad r_0 \in L^\infty, \tag{2.7}$$

then there exists a positive constant T such that the Cauchy problem (2.5) admits a unique smooth solution $= r(t, \theta)$ on the strip

$$\Pi_T = \{(t, \theta) \,|\, t \in [0, T], \ \theta \in \mathbb{R}^m\}, \tag{2.8}$$

where

$$T = \min\left\{ \left(\frac{M\pi^{\frac{1}{2}}}{2H}\right)^2, \left(\frac{\pi^{\frac{1}{2}}}{4Hm}\right)^2 \right\}, \tag{2.9}$$

in which

$$M = \|r_0(\theta)\|_{L^\infty}, \quad H = \max_{i,j=1,\cdots,m} \left\{ \sup_{|r|\le(m+1)M} |g_i(r(t,\theta))|, \quad \sup_{|r|\le(m+1)M} \left|\frac{\partial}{\partial\theta_j} g_i(r(t,\theta))\right| \right\}, \tag{2.10}$$

here $g_j = rf_j(r)$ $(j = 1, \cdots, m)$.

Proof. Set

$$G_T = \{r : [0,T]\times\mathbb{R}^m \to L^\infty(\mathbb{R}^m) \,|\, \|r(t,\cdot)\|_{L^\infty} \le (m+1)M \quad \text{for} \;\; t\in[0,T]\} \tag{2.11}$$

and let $\mathcal{T}$ be the following integral operator

$$\mathcal{T}r(t,\theta) = K(t,\theta) * r_0(\theta) + \sum_{j=1}^m \int_0^t K_{\theta_j}(t-s,\theta) * (f_j(r(s,\theta))r(s,\theta))ds. \tag{2.12}$$

The solution $r = r(t,\theta)$ can be obtained as the L^∞-limit of the sequence $\{r^k\}$ defined by

$$r^0(t,\theta) = K(t,\theta) * r_0(\theta), \quad r^{k+1} = \mathcal{T}r^k \quad (n = 0, 1\cdots). \tag{2.13}$$

To prove the above statement, we first claim that, for any $t \in [0,T]$, it holds that

$$\|r^k(t,\theta)\|_{L^\infty} \le (m+1)M, \quad \forall\, k \in \{0,1,2,\cdots\}. \tag{2.14}$$

In what follows, we prove (2.14) by the method of induction.

When $k = 0$, we have

$$\|r^0(t,\theta)\|_{L^\infty} = \|K(t,\theta) * r_0(\theta)\|_{L^\infty}. \tag{2.15}$$

By Young's inequality, we obtain

$$\|r^0(t,\theta)\|_{L^\infty} \le \|K(t,\theta)\|_{L^1}\|r_0(\theta)\|_{L^\infty} = \|r_0(\theta)\|_{L^\infty} = M \le (m+1)M. \tag{2.16}$$

Now we assume that $\|r^k(t,\theta)\|_{L^\infty} \le (m+1)M$ $(k \in \mathbb{N})$ holds. We next prove

$$\|r^{k+1}(t,\theta)\|_{L^\infty} \le (m+1)M. \tag{2.17}$$

In fact,

$$\begin{aligned} \|r^{k+1}(t,\theta)\|_{L^\infty} &= \|\mathcal{T}r^k(t,\theta)\|_{L^\infty} \\ &\le \|K(t,x) * r_0(\theta)\|_{L^\infty} + \sum_{j=1}^m \int_0^t \|K_{x_j}(t-s,\theta) * (f_j(r^k(s,\theta))r^k(s,\theta))\|_{L^\infty} ds \\ &\le M + \sum_{j=1}^m \int_0^t \|K_{\theta_j}(t-s,\theta)\|_{L^1} \|(f_j(r^k(s,\theta))r^k(s,\theta))\|_{L^\infty} ds. \end{aligned} \tag{2.18}$$

Notice that

$$\begin{aligned}
\textstyle\int_{\mathbb{R}^m} |K_{\theta_j}(t-s,\theta)|d\theta &= \int_{\mathbb{R}^m} [4\pi(t-s)]^{-\frac{m}{2}} \frac{\theta_j}{2(t-s)} \exp\left\{-\frac{|\theta|^2}{4(t-s)}\right\} d\theta \\
&= [4\pi(t-s)]^{-\frac{m}{2}} \int_{\mathbb{R}^{m-1}} \left\{\exp\left\{-\frac{\theta_1^2}{4(t-s)}\right\} + \cdots + \exp\left\{-\frac{\theta_{j-1}^2}{4(t-s)}\right\} + \right. \\
&\quad \left. \exp\left\{-\frac{\theta_{j+1}^2}{4(t-s)}\right\} + \cdots + \exp\left\{-\frac{\theta_m^2}{4(t-s)}\right\}\right\} d\theta_1 \cdots d\theta_{j-1} d\theta_{j+1} \cdots d\theta_m \times \\
&\quad \int_{\mathbb{R}} \frac{\theta_j}{2(t-s)} \exp\left\{-\frac{\theta_j^2}{4(t-s)}\right\} d\theta_j \\
&= [4\pi(t-s)]^{-\frac{m}{2}} \left\{[4(t-s)]^{\frac{1}{2}} \int_{\mathbb{R}} \exp\left\{-\frac{\theta_1^2}{4(t-s)}\right\} d\left(-\frac{\theta_1}{[4(t-s)]^{\frac{1}{2}}}\right)\right\}^{m-1} \times \\
&\quad \int_{\mathbb{R}} \frac{\theta_j}{2(t-s)} \exp\left\{-\frac{\theta_j^2}{4(t-s)}\right\} d\theta_j \\
&= [4\pi(t-s)]^{-\frac{1}{2}} \int_{\mathbb{R}} \frac{\theta_j}{2(t-s)} \exp\left\{-\frac{\theta_j^2}{4(t-s)}\right\} d\theta_j \\
&= [4\pi(t-s)]^{-\frac{1}{2}} \left[-2\int_0^{\infty} \exp\left\{-\frac{\theta_j^2}{4(t-s)}\right\} d\left(-\frac{\theta_j^2}{4(t-s)}\right)\right] \\
&= [\pi(t-s)]^{-\frac{1}{2}}.
\end{aligned} \tag{2.19}$$

It follows from (2.18) that

$$\begin{aligned}
\|r^{k+1}(t,\theta)\|_{L^\infty} &\le M + \sum_{j=1}^m \int_0^t \|K_{\theta_j}(t-s,\theta)\|_{L^1} \|(f_j(r^k(s,\theta))r^k(s,\theta))\|_{L^\infty} ds \\
&\le M + \sum_{j=1}^m \pi^{-\frac{1}{2}} \int_0^t (t-s)^{-\frac{1}{2}} \|(f_j(r^k(s,\theta))r^k(s,\theta))\|_{L^\infty} ds \\
&\le M + m\pi^{-\frac{1}{2}} H \int_0^t (t-s)^{-\frac{1}{2}} ds \\
&= M + 2m\pi^{-\frac{1}{2}} H t^{\frac{1}{2}} \le M + 2m\pi^{-\frac{1}{2}} H T^{\frac{1}{2}} \le (m+1)M.
\end{aligned} \tag{2.20}$$

This is the desired estimate (2.14). Thus, the proof of (2.14) is completed.

In what follows, we prove that $\{r^k(t,\theta)\}$ is uniformly convergent in the strip $(0,T] \times \mathbb{R}^m$. To do so, it suffices to show that

$$\sum_{k=1}^{\infty} \left[r^{k+1}(t,\theta) - r^k(t,\theta)\right]$$

is uniformly convergent in the strip $(0,T] \times \mathbb{R}^m$.

In fact, it holds that

$$
\begin{aligned}
\|r^{k+1}-r^k\|_{L^\infty} &\le \sum_{j=1}^m \int_0^t \|K_{\theta_j}(t-s,\theta) * \left[(f_j(r^k(s,\theta))r^k(s,\theta)) - (f_j(r^{k-1}(s,\theta))r^{k-1}(s,\theta))\right]\|_{L^\infty} ds \\
&\le \sum_{j=1}^m \int_0^t \|K_{\theta_j}(t-s,\theta)\|_{L^1} \| \left[(f_j(r^k(s,\theta))r^k(s,\theta)) - (f_j(r^{k-1}(s,\theta))r^{k-1}(s,\theta))\right]\|_{L^\infty} ds \\
&\le \sum_{j=1}^m \int_0^t \|K_{\theta_j}(t-s,\theta)\|_{L^1} \|g_j(r^k(s,\theta)) - g_j(r^{k-1}(s,\theta))\|_{L^\infty} ds \\
&\le \sum_{j=1}^m \int_0^t \|K_{\theta_j}(t-s,\theta)\|_{L^1} \|\nabla g_j(\beta_k)\|_{L^\infty} \|r^k(s,\theta) - r^{k-1}(s,\theta)\|_{L^\infty} ds \\
&\le mH \max\left\{|r^k(s,\theta) - r^{k-1}(s,\theta)|\right\} \int_0^t \|K_{\theta_j}(t-s,\theta)\|_{L^1} ds \\
&\le 2\pi^{-\frac{1}{2}} mHt^{\frac{1}{2}} \max\left\{|r^k(s,\theta) - r^{k-1}(s,\theta)|\right\} \\
&\le 2\pi^{-\frac{1}{2}} mHT^{\frac{1}{2}} \max\left\{|r^k(s,\theta) - r^{k-1}(s,\theta)|\right\} \\
&\le \left(2\pi^{-\frac{1}{2}} mHT^{\frac{1}{2}}\right)^2 \max\left\{|r^{k-1}(s,\theta) - r^{k-2}(s,\theta)|\right\} \\
&\le \cdots \\
&\le \left(2\pi^{-\frac{1}{2}} mHT^{\frac{1}{2}}\right)^k \max\left\{|r^1(s,\theta) - r^0(s,\theta)|\right\},
\end{aligned}
\tag{2.21}
$$

where

$$\beta_k \in \left[\min\{r^k(s,x), r^{k-1}(s,x)\}, \max\{r^k(s,x), r^{k-1}(s,x)\}\right].$$

Noting

$$\|r^1(s,\theta) - r^0(s,\theta)\|_{L^\infty} \le 2\pi^{-\frac{1}{2}} mHT^{\frac{1}{2}}, \tag{2.22}$$

we obtain from (2.21) that

$$\|r^{k+1}-r^k\|_{L^\infty} \le \left(2\pi^{-\frac{1}{2}} mHT^{\frac{1}{2}}\right)^{k+1}. \tag{2.23}$$

By (2.9), we have

$$\|r^{k+1}-r^k\|_{L^\infty} \le \left(\frac{1}{2}\right)^{k+1}, \tag{2.24}$$

which implies that $\sum_{k=1}^{\infty} \left[r^{k+1}(t,\theta) - r^k(t,\theta)\right]$ is uniformly convergent in the strip $(0,T]\times\mathbb{R}^m$. Therefore, $\lim_{k\to\infty} r^k(t,\theta)$ gives the unique local solution of the Cauchy problem (2.5). Thus, the proof Lemma 2.1 is completed. □

Lemma 2.2 *Suppose that*

$$f \in C^1, \quad r_0 \in L^\infty$$

and let $M \triangleq \|r_0\|_{L^\infty}$*. Suppose furthermore that* $r(t,\theta)$ *is the solution of Cauchy problem (2.5) on the strip* Π_T*, then it holds that*

$$\|r(t,\theta)\|_{L^\infty(\Pi_T)} \leq M. \tag{2.25}$$

Proof. It follows from the proof of Lemma 2.1 that

$$\|r(t,\theta)\|_{L^\infty(\Pi_T)} \leq (m+1)M \triangleq K. \tag{2.26}$$

Introduce

$$w(t,\theta) = r(t,\theta) - M - \frac{K}{L^2}\left(|\theta|^2 + CLe^t\right), \tag{2.27}$$

where C and L are positive constants to be determined. By (2.27),

$$r_t = w_t + \frac{CK}{L}e^t, \quad \triangle r = \triangle w + \frac{2Km}{L^2}. \tag{2.28}$$

On the other hand,

$$\sum_{j=1}^{m}(f_j(r)r)_{\theta_j} = \sum_{j=1}^{m}(g_j(r))_{\theta_j} = \sum_{j=1}^{m} g_j'(r)r_{\theta_j} = \sum_{j=1}^{m} g_j'(r)\left(w_{\theta_j} + \frac{2K}{L^2}\theta_j\right). \tag{2.29}$$

Thus,

$$w_t + \sum_{j=1}^{m} g_j'(r)w_{\theta_j} + \frac{2K}{L^2}\sum_{j=1}^{m} g_j'(r)\theta_j + \frac{CK}{L}e^t - \frac{2Km}{L^2} = \triangle w. \tag{2.30}$$

Choose sufficiently large C such that

$$w(0,\theta) = r_0(\theta) - M - \frac{K}{L^2}(|\theta|^2 + CL) < 0, \quad \forall\, \theta \in \mathbb{R}^m, \tag{2.31}$$

and

$$\begin{cases} w(t,\pm L,\theta_2,\cdots,\theta_m) = r(t,\pm L,\theta_2,\cdots,\theta_m)M - \frac{K}{L^2}\left[(L^2+\theta_2^2+\cdots+\theta_m^2) + CLe^t\right] < 0, \\ w(t,\theta_1,\pm L,\cdots,\theta_m) = r(t,\theta_1,\pm L,\cdots,\theta_m) - M - \frac{K}{L^2}\left[(\theta_1^2+L^2+\cdots+\theta_m^2) + CLe^t\right] < 0, \\ \cdots\cdots \\ w(t,\theta_1,\cdots,\theta_{m-1},\pm L) = r(t,\theta_1,\cdots,\theta_{m-1},\pm L) - M - \frac{K}{L^2}\left[(\theta_1^2+\cdots+\theta_{m-1}^2+|L|^2) + CLe^t\right] < 0 \end{cases} \tag{2.32}$$

for all $t \in [0,T]$.

In what follows, we prove that, for any $(t,\theta) \in (0,T) \times (-L,L)^m$, it holds that

$$w(t,\theta) < 0. \tag{2.33}$$

In fact, if (2.33) is not true, then we can define $\bar{t}$ by

$$\bar{t} = \inf_{t\in(0,T]}\{t \,|\, w(t,\theta) = 0 \ \text{ for some } \ \theta \in (-L,L)^m\}. \tag{2.34}$$

It is easy to see that there exists a point, denoted by $\bar{\theta} \in (-L, L)^m$, such that

$$w(\bar{t}, \bar{\theta}) = 0, \quad w_{\theta_1}(\bar{t}, \bar{\theta}) = 0, \quad \cdots, \quad w_{\theta_m}(\bar{t}, \bar{\theta}) = 0 \tag{2.35}$$

and

$$w_{\theta_i \theta_i}(\bar{t}, \bar{\theta}) \leq 0, \quad \forall\, i \in \{1, \cdots, m\}. \tag{2.36}$$

By (2.35)-(2.36), it follows from (2.30) that

$$w_t(\bar{t}, \bar{\theta}) + \frac{2K}{L^2} \sum_{j=1}^{m} g_j'(r(\bar{t}, \bar{\theta}))\bar{\theta}_j + \frac{CK}{L} e^{\bar{t}} - \frac{2Km}{L^2} \leq 0. \tag{2.37}$$

Noting

$$\|g_j'(\bullet)\|_{L^\infty} < \infty \quad \text{and} \quad (\bar{t}, \theta_j) \in (0, T] \times (-L, L), \tag{2.38}$$

we can choose a sufficiently large C such that

$$\frac{2K}{L^2} \sum_{j=1}^{m} g_j'(r(\bar{t}, \bar{\theta}))\bar{\theta}_j + \frac{CK}{L} e^{\bar{t}} - \frac{2Km}{L^2} > 0. \tag{2.39}$$

Combining (2.37) and (2.39)

$$w_t(\bar{t}, \bar{\theta}) < 0. \tag{2.40}$$

On the other hand, by the definition of $(\bar{t}, \bar{\theta})$ it holds that

$$w_t(\bar{t}, \bar{\theta}) = \lim_{\triangle t \to 0} \frac{w(\bar{t}, \bar{\theta}) - w(\bar{t} - \triangle t, \bar{\theta})}{\triangle t} \geq 0, \tag{2.41}$$

which is a contradiction. This proves (2.33).

Noting (2.27) and (2.33) and letting $L \to \infty$ gives

$$r(t, \theta) \leq M, \quad \forall\, (t, \theta) \in \Pi_T. \tag{2.42}$$

Similarly, letting

$$w(t, \theta) = r(t, \theta) + M + \frac{K}{L^2} \left(|\theta|^2 + CLe^t\right), \tag{2.43}$$

we can prove

$$r(t, \theta) \geq -M, \quad \forall\, (t, \theta) \in \Pi_T. \tag{2.44}$$

Combining (2.42) and (2.44) leads to

$$\|r(t, \theta)\|_{L^\infty(\Pi_T)} \leq M \tag{2.45}$$

Thus, the proof of Lemma 2.2 is completed. $\square$

By Lemma 2.1 and Lemma 2.2, we have

Theorem 2.1 *If $f \in C^1$ and $r_0 \in L^\infty$, then the Cauchy problem (2.5) admits a unique global smooth solution on $[0, \infty) \times \mathbb{R}^m$.*

Now we turn to consider the Cauchy problem (2.1) (i.e., (2.1a)-(2.1b)), (2.3). We have

Theorem 2.2 *Under the assumptions of Theorem 1.1, the Cauchy problem (2.1), (2.3) admits a unique global smooth solution on $[0, \infty) \times \mathbb{R}^m$.*

Proof. Noting (1.4), by maximum principle we obtain that, on the existence domain of smooth solution, it holds that

$$r(t, \theta) > 0. \tag{2.46}$$

On the one hand, we observe that, under the condition (2.46), the equation (2.1) can be reduced to the system (2.1a)-(2.1b); on the other hand, we notice that, once $r = r(t, \theta)$ is solved from the Cauchy problem (A.17), then the equation (2.1b) becomes linear. Therefore, Theorem 2.2 follows Theorem 2.1 directly. □

Obviously, Theorem 1.1 follows Theorem 2.2 directly.

3 Time-asymptotic behavior of smooth solutions — Proof of Theorem 1.2

In this section, we first investigate the time-asymptotic behavior of solution of the Cauchy problem (1.2), (1.3) in the case that $f_i(\nu) \equiv c_i$, and then based on this, we prove Theorem 1.2.

Notice that, in the present situation, the equation (2.1a) can be rewritten as

$$\frac{\partial r}{\partial t} + \sum_{i=1}^{m} c_i \frac{\partial r}{\partial \theta_i} = \Delta r.$$

Let

$$\tau = t, \ \ \eta_i = \theta_i - c_i t,$$

then

$$\frac{\partial}{\partial \tau} = \frac{\partial}{\partial t}\frac{\partial t}{\partial \tau} + \sum_{i=1}^{m} \frac{\partial}{\partial \theta_i}\frac{\partial \theta_i}{\partial \tau} = \frac{\partial}{\partial t} + \sum_{i=1}^{m} c_i \frac{\partial}{\partial \theta_i}.$$

Therefore, without loss of generality, we turn to consider the time-asymptotic behavior of solution of the Cauchy problem

$$\begin{cases} u_t - \Delta u = 0 & \text{in } \ (0, \infty) \times \mathbb{R}^n, \\ u(x, 0) = u_0(x) & \text{in } \ \mathbb{R}^n, \end{cases} \tag{3.1}$$

where $u = u(t,x)$ is the unknown function, the initial data $u_0(x) \in C(T^n)$ is a periodic function with period, say, $\mathbb{T}^n = \{x = (x_1, \cdots, x_n) \mid -l_i \le x_i \le l_i\}$.

Theorem 3.1 *If the initial data $u_0(x)$ satisfies Dirichlet conditions, then it holds that*

$$u(t,x) \longrightarrow \overline{u_0} \quad \text{as } t \nearrow \infty, \tag{3.2}$$

where $\overline{u_0}$ stands for the mean value of $u_0(x)$, which is defined by

$$\overline{u_0} = \frac{1}{\text{vol}\{\mathbb{T}^n\}} \int_{\mathbb{T}^n} u_0(x)dx. \tag{3.3}$$

Proof. Recall Dirichlet conditions: if the periodic function $u_0(x)$ satisfies Dirichlet conditions, then

- $u_0(x)$ has a finite number of extrema in any given interval;
- $u_0(x)$ has a finite number of discontinuities in any given interval;
- $u_0(x)$ is absolutely integrable over a period;
- $u_0(x)$ is bounded.

By the theory of Fourier series, under Dirichlet conditions, $u_0(x)$ is equal to the sum of its Fourier series at each point where $u_0(x)$ is continuous; moreover, the behavior of the Fourier series at points of discontinuity is determined as well. Therefore, it holds that

$$\begin{aligned} u_0(x) = & \sum_{m_1=0}^{\infty} \cdots \sum_{m_n=0}^{\infty} A^0_{m_1 \cdots m_n} \cos \frac{m_1 \pi}{l_1} x_1 \cdots \cos \frac{m_n \pi}{l_n} x_n + \\ & \sum_{i=1}^{n} \left\{ \sum_{m_1=0}^{\infty} \cdots \sum_{m_i=1}^{\infty} \cdots \sum_{m_n=0}^{\infty} A^1_{m_1 \cdots m_i \cdots m_n} \cos \frac{m_1 \pi}{l_1} x_1 \cdots \sin \frac{m_i \pi}{l_i} x_i \cdots \cos \frac{m_n \pi}{l_n} x_n \right\} + \\ & \sum_{i=1}^{n-1} \left\{ \sum_{j=i+1}^{n} \left\{ \sum_{m_1=0}^{\infty} \cdots \sum_{m_i=1}^{\infty} \cdots \sum_{\substack{m_j=1 \\ i<j}}^{\infty} \cdots \sum_{m_n=0}^{\infty} A^2_{m_1 \cdots m_i \cdots m_j \cdots m_n} \times \right. \right. \\ & \left. \left. \cos \frac{m_1 \pi}{l_1} x_1 \cdots \sin \frac{m_i \pi}{l_i} x_i \cdots \sin \frac{m_j \pi}{l_j} x_j \cdots \cos \frac{m_n \pi}{l_n} x_n \right\} \right\} + \cdots + \\ & \sum_{m_1=1}^{\infty} \cdots \sum_{m_n=1}^{\infty} A^n_{m_1 \cdots m_n} \sin \frac{m_1 \pi}{l_1} x_1 \cdots \sin \frac{m_n \pi}{l_n} x_n, \end{aligned} \tag{3.4}$$

where $A^j_{m_1\cdots m_n}$ $(j=0,1,2\cdots n)$ stand for the Fourier coefficients which are given by

$$
\begin{cases}
A^0_{m_1\cdots m_n} & = \dfrac{\lambda_{m_1\cdots m_n}}{l_1\cdots l_n}\displaystyle\int_{-l_1}^{l_1}\cdots\int_{-l_n}^{l_n} u_0(x)\cos\frac{m_1\pi}{l_1}x_1\cdots\cos\frac{m_n\pi}{l_n}x_n dx_1\cdots dx_n,\\
A^1_{m_1\cdots m_i\cdots m_n} & = \dfrac{\lambda_{m_1\cdots m_n}}{l_1\cdots l_n}\displaystyle\int_{-l_1}^{l_1}\cdots\int_{-l_i}^{l_i}\cdots\int_{-l_n}^{l_n} u_0(x)\times\\
& \quad \cos\dfrac{m_1\pi}{l_1}x_1\cdots\sin\dfrac{m_i\pi}{l_i}x_i\cdots\cos\dfrac{m_n\pi}{l_n}x_n dx_1\cdots dx_i\cdots dx_n,\\
A^2_{m_1\cdots m_i\cdots m_j\cdots m_n} & = \dfrac{\lambda_{m_1\cdots m_n}}{l_1\cdots l_n}\displaystyle\int_{-l_1}^{l_1}\cdots\int_{-l_i}^{l_i}\cdots\int_{-l_j}^{l_j}\cdots\int_{-l_n}^{l_n} u_0(x)\times\\
& \quad \cos\dfrac{m_1\pi}{l_1}x_1\cdots\sin\dfrac{m_i\pi}{l_i}x_i\cdots\sin\dfrac{m_j\pi}{l_j}x_j\cdots\cos\dfrac{m_n\pi}{l_n}x_n dx_1\cdots dx_i\cdots dx_j\cdots dx_n,\\
\cdots\cdots & \\
A^n_{m_1\cdots m_n} & = \dfrac{\lambda_{m_1\cdots m_n}}{l_1\cdots l_n}\displaystyle\int_{-l_1}^{l_1}\cdots\int_{-l_n}^{l_n} u_0(x)\sin\frac{m_1\pi}{l_1}x_1\cdots\sin\frac{m_n\pi}{l_n}x_n dx_1\cdots dx_n,
\end{cases}
\tag{3.5}
$$

in which

$$
\begin{cases}
\lambda_{m_1\cdots m_n} = \dfrac{1}{2^n}, \quad \text{if } m_1=m_2=\cdots=m_n=0,\\
\lambda_{m_1\cdots m_n} = \dfrac{1}{2^{n-1}}, \quad \text{if } m_{i_1}=m_{i_2}=\cdots=m_{i_{n-1}}=0, \ \ m_{i_n}\neq 0,\\
\cdots\cdots\\
\lambda_{m_1\cdots m_n} = 1, \quad \text{if } m_1=m_2=\cdots=m_n\neq 0.
\end{cases}
\tag{3.6}
$$

It is easy to see that

$$
\begin{aligned}
u_0(x) \quad = \quad & \frac{1}{2^n l_1 \cdots l_n} \int_{-l_1}^{l_1} \cdots \int_{-l_n}^{l_n} u_0(x) dx_1 \cdots dx_n + \sum_{i=1}^{n} \left\{ \sum_{m_i=1}^{\infty} A^0_{0\cdots m_i \cdots 0} \cos \frac{m_i \pi}{l_i} x_i \right\} + \\
& \sum_{i=1}^{n-1} \left\{ \sum_{j=i+1}^{n} \left\{ \sum_{m_i=1}^{\infty} \sum_{\substack{m_j=1 \\ i<j}}^{\infty} A^0_{0\cdots m_i \cdots m_j \cdots 0} \cos \frac{m_i \pi}{l_i} x_i \cos \frac{m_j \pi}{l_j} x_j \right\} \right\} + \\
& \sum_{i=1}^{n-2} \left\{ \sum_{j=i+1}^{n-1} \left\{ \sum_{k=j+1}^{n} \left\{ \sum_{m_i=1}^{\infty} \sum_{\substack{m_j=1 \\ i<j}}^{\infty} \sum_{\substack{m_k=1 \\ i<j<k}}^{\infty} A^0_{0\cdots m_i \cdots m_j \cdots m_k \cdots 0} \times \right.\right.\right. \\
& \left.\left.\left. \cos \frac{m_i \pi}{l_i} x_i \cos \frac{m_j \pi}{l_j} x_j \cos \frac{m_k \pi}{l_k} x_k \right\} \right\} \right\} + \cdots + \\
& \sum_{m_1=1}^{\infty} \cdots \sum_{m_n=1}^{\infty} A^0_{m_1 \cdots m_n} \cos \frac{m_1 \pi}{l_1} x_1 \cdots \cos \frac{m_n \pi}{l_n} x_n + \\
& \sum_{i=1}^{n} \left\{ \sum_{m_1=0}^{\infty} \cdots \sum_{m_i=1}^{\infty} \cdots \sum_{m_n=0}^{\infty} A^1_{m_1 \cdots m_n} \cos \frac{m_1 \pi}{l_1} x_1 \cdots \sin \frac{m_i \pi}{l_i} x_i \cdots \cos \frac{m_n \pi}{l_n} x_n \right\} + \\
& \sum_{i=1}^{n-1} \left\{ \sum_{j=i+1}^{n} \left\{ \sum_{m_1=0}^{\infty} \cdots \sum_{m_i=1}^{\infty} \cdots \sum_{\substack{m_j=1 \\ i<j}}^{\infty} \cdots \sum_{m_n=0}^{\infty} A^2_{m_1 \cdots m_n} \times \right.\right. \\
& \left.\left. \cos \frac{m_1 \pi}{l_1} x_1 \cdots \sin \frac{m_i \pi}{l_i} x_i \cdots \sin \frac{m_j \pi}{l_j} x_j \cdots \cos \frac{m_n \pi}{l_n} x_n \right\} \right\} + \\
& \sum_{i=1}^{n-2} \left\{ \sum_{j=i+1}^{n-1} \left\{ \sum_{k=j+1}^{n} \left\{ \sum_{m_1=0}^{\infty} \cdots \sum_{m_i=1}^{\infty} \cdots \sum_{\substack{m_j=1 \\ i<j}}^{\infty} \cdots \sum_{\substack{m_k=1 \\ i<j<k}}^{\infty} \cdots \sum_{m_n=0}^{\infty} A^3_{m_1 \cdots m_n} \times \right.\right.\right. \\
& \left.\left.\left. \cos \frac{m_1 \pi}{l_1} x_1 \cdots \sin \frac{m_i \pi}{l_i} x_i \cdots \sin \frac{m_j \pi}{l_j} x_j \cdots \sin \frac{m_k \pi}{l_k} x_k \cdots \cos \frac{m_n \pi}{l_n} x_n \right\} \right\} \right\} + \cdots + \\
& \sum_{m_1=1}^{\infty} \cdots \sum_{m_n=1}^{\infty} A^n_{m_1 \cdots m_n} \sin \frac{m_1 \pi}{l_1} x_1 \cdots \sin \frac{m_n \pi}{l_n} x_n.
\end{aligned} \tag{3.7}
$$

Noting that, for the fundament solution $K = K(t, x)$, it holds that

$$
\int_{\mathbb{R}^n} K(t, x) dx = \int_{\mathbb{R}^n} \left(\frac{1}{4\pi t} \right)^{\frac{n}{2}} \exp \left\{ -\frac{|x|^2}{4t} \right\} dx = 1, \tag{3.8}
$$

we obtain

$$
\begin{aligned}
u(t,x) &= \int_{\mathbb{R}^n} u_0(\xi)K(t,x-\xi)d\xi \\
&= \int_{\mathbb{R}^n} u_0(x-\xi)K(t,\xi)d\xi \\
&= \int_{\mathbb{R}^n} u_0(x-\xi)\left(\frac{1}{4\pi t}\right)^{\frac{n}{2}} \exp\left\{-\frac{|\xi|^2}{4t}\right\} d\xi \\
&= \frac{1}{2^n l_1\cdots l_n}\int_{-l_1}^{l_1}\cdots\int_{-l_n}^{l_n} u_0(x)dx_1\cdots dx_n + \sum_{i=1}^{n} A^i \\
&= \overline{u_0(x)} + \sum_{i=1}^{n} A^i,
\end{aligned}
\tag{3.9}
$$

where

$$
\begin{aligned}
A^0 &= \left(\frac{1}{4\pi t}\right)^{\frac{n}{2}} \sum_{i=1}^{n}\left\{\sum_{m_i=1}^{\infty} A^0_{0\cdots m_i\cdots 0}\int_{\mathbb{R}^n}\exp\left\{-\frac{\xi_1^2+\cdots+\xi_n^2}{4t}\right\}\cos\frac{m_i\pi}{l_i}(x_i-\xi_i)d\xi_1\cdots d\xi_n\right\}+ \\
&\left(\frac{1}{4\pi t}\right)^{\frac{n}{2}} \sum_{i=1}^{n-1}\left\{\sum_{j=i+1}^{n}\left\{\sum_{m_i=1}^{\infty}\sum_{\substack{m_j=1\\ i<j}}^{\infty} A^0_{0\cdots m_i\cdots m_j\cdots 0}\int_{\mathbb{R}^n}\exp\left\{-\frac{\xi_1^2+\cdots+\xi_n^2}{4t}\right\}\times\right.\right. \\
&\left.\left.\cos\frac{m_i\pi}{l_i}(x_i-\xi_i)\cos\frac{m_j\pi}{l_j}(x_j-\xi_j)d\xi_1\cdots d\xi_n\right\}\right\}+ \\
&\left(\frac{1}{4\pi t}\right)^{\frac{n}{2}} \sum_{i=1}^{n-2}\left\{\sum_{j=i+1}^{n-1}\left\{\sum_{k=j+1}^{n}\left\{\sum_{m_i=1}^{\infty}\sum_{\substack{m_j=1\\ i<j}}^{\infty}\sum_{\substack{m_k=1\\ i<j<k}}^{\infty} A^0_{0\cdots m_i\cdots m_j\cdots m_k\cdots 0}\times\right.\right.\right. \\
&\left.\left.\left.\int_{\mathbb{R}^n}\exp\left\{-\frac{\xi_1^2+\cdots+\xi_n^2}{4t}\right\}\ \cos\frac{m_i\pi}{l_i}(x_i-\xi_i)\cos\frac{m_j\pi}{l_j}(x_j-\xi_j)\cos\frac{m_k\pi}{l_k}(x_k-\xi_k)\right\}\right\}\right\}+\cdots+ \\
&\left(\frac{1}{4\pi t}\right)^{\frac{n}{2}} \sum_{m_1=1}^{\infty}\cdots\sum_{m_n=1}^{\infty} A^0_{m_1\cdots m_n}\int_{\mathbb{R}^n}\exp\left\{-\frac{\xi_1^2+\cdots+\xi_n^2}{4t}\right\}\times \\
&\cos\frac{m_1\pi}{l_1}(x_1-\xi_1)\cdots\cos\frac{m_n\pi}{l_n}(x_n-\xi_n)d\xi_1\cdots d\xi_n,
\end{aligned}
\tag{3.10}
$$

$$
\begin{aligned}
A^1 &= \left(\frac{1}{4\pi t}\right)^{\frac{n}{2}} \sum_{i=1}^{n}\left\{\sum_{m_1=0}^{\infty}\cdots\sum_{m_i=1}^{\infty}\cdots\sum_{m_n=0}^{\infty} A^1_{m_1\cdots m_n}\int_{\mathbb{R}^n}\exp\left\{-\frac{\xi_1^2+\cdots+\xi_n^2}{4t}\right\}\times\right. \\
&\left.\cos\frac{m_1\pi}{l_1}(x_1-\xi_1)\cdots\sin\frac{m_i\pi}{l_i}(x_i-\xi_i)\cdots\cos\frac{m_n\pi}{l_n}(x_n-\xi_n)d\xi_1\cdots d\xi_n\right\},
\end{aligned}
\tag{3.11}
$$

$$
\begin{aligned}
A^2 &= \left(\frac{1}{4\pi t}\right)^{\frac{n}{2}} \sum_{i=1}^{n-1} \left\{ \sum_{j=i+1}^{n} \left\{ \sum_{m_1=0}^{\infty} \cdots \sum_{m_i=1}^{\infty} \cdots \sum_{\substack{m_j=1 \\ i<j}}^{\infty} \cdots \sum_{m_n=0}^{\infty} A^2_{m_1\cdots m_n} \int_{\mathbb{R}^n} \exp\left\{-\frac{\xi_1^2+\cdots+\xi_n^2}{4t}\right\} \times \right.\right. \\
&\left.\left. \cos\frac{m_1\pi}{l_1}(x_1-\xi_1)\cdots \sin\frac{m_i\pi}{l_i}(x_i-\xi_i)\cdots \sin\frac{m_j\pi}{l_j}(x_j-\xi_j)\cdots \cos\frac{m_n\pi}{l_n}(x_n-\xi_n)d\xi_1\cdots d\xi_n \right\}\right\},
\end{aligned} \tag{3.12}
$$

$$
\begin{aligned}
A^3 &= \left(\frac{1}{4\pi t}\right)^{\frac{n}{2}} \sum_{i=1}^{n-2} \left\{ \sum_{j=i+1}^{n-1} \left\{ \sum_{k=j+1}^{n} \left\{ \sum_{m_1=0}^{\infty} \cdots \sum_{m_i=1}^{\infty} \cdots \sum_{\substack{m_j=1 \\ i<j}}^{\infty} \cdots \sum_{\substack{m_k=1 \\ i<j<k}}^{\infty} \cdots \sum_{m_n=0}^{\infty} A^3_{m_1\cdots m_n} \times \right.\right.\right. \\
& \int_{\mathbb{R}^n} \exp\left\{-\frac{\xi_1^2+\cdots+\xi_n^2}{4t}\right\} \cos\frac{m_1\pi}{l_1}(x_1-\xi_1)\cdots \sin\frac{m_i\pi}{l_i}(x_i-\xi_i)\cdots \\
&\left.\left.\left. \sin\frac{m_j\pi}{l_j}(x_j-\xi_j)\cdots \sin\frac{m_k\pi}{l_k}(x_k-\xi_k)\cdots \cos\frac{m_n\pi}{l_n}(x_n-\xi_n)d\xi_1\cdots d\xi_n \right\}\right\}\right\}
\end{aligned} \tag{3.13}
$$

and so on, particularly,

$$
\begin{aligned}
A^n &= \left(\frac{1}{4\pi t}\right)^{\frac{n}{2}} \sum_{m_1=1}^{\infty} \cdots \sum_{m_n=1}^{\infty} A^n_{m_1\cdots m_n} \int_{\mathbb{R}^n} \exp\left\{-\frac{\xi_1^2+\cdots+\xi_n^2}{4t}\right\} \times \\
& \sin\frac{m_1\pi}{l_1}(x_1-\xi_1)\cdots \sin\frac{m_n\pi}{l_n}(x_n-\xi_n)d\xi_1\cdots d\xi_n.
\end{aligned} \tag{3.14}
$$

.

On the one hand,

$$
\begin{aligned}
&\left(\frac{1}{4\pi t}\right)^{\frac{n}{2}} \int_{\mathbb{R}} \exp\left\{-\frac{y^2}{4t}\right\} \cos\frac{n\pi}{l}(x-y)dy \\
= \;& \left(\frac{1}{4\pi t}\right)^{\frac{n}{2}} \int_{\mathbb{R}} \exp\left\{-\frac{y^2}{4t}\right\} \left[\cos\frac{n\pi}{l}x\cos\frac{n\pi}{l}y + \sin\frac{n\pi}{l}x\sin\frac{n\pi}{l}y\right] dy \\
= \;& \left(\frac{1}{4\pi t}\right)^{\frac{n}{2}} \cos\frac{n\pi}{l}x \int_{\mathbb{R}} \exp\left\{-\frac{y^2}{4t}\right\} \cos\frac{n\pi}{l}ydy + \left(\frac{1}{4\pi t}\right)^{\frac{n}{2}} \sin\frac{n\pi}{l}x \int_{\mathbb{R}} \exp\left\{-\frac{y^2}{4t}\right\} \sin\frac{n\pi}{l}ydy \\
= \;& \left(\frac{1}{4\pi t}\right)^{\frac{n}{2}} \cos\frac{n\pi}{l}x \int_{\mathbb{R}} \exp\left\{-\frac{y^2}{4t}\right\} \left[\exp\left\{\frac{n\pi}{l}yi\right\} + \exp\left\{-\frac{n\pi}{l}yi\right\}\right] dy + \\
& \frac{1}{2i}\left(\frac{1}{4\pi t}\right)^{\frac{n}{2}} \sin\frac{n\pi}{l}x \int_{\mathbb{R}} \exp\left\{-\frac{y^2}{4t}\right\} \left[\exp\left\{\frac{n\pi}{l}yi\right\} - \exp\left\{-\frac{n\pi}{l}yi\right\}\right] dy \\
= \;& \left(\frac{1}{4\pi t}\right)^{\frac{n}{2}} \cos\frac{n\pi}{l}x \int_{\mathbb{R}} \exp\left\{-\frac{y^2}{4t} + \frac{n\pi}{l}yi\right\} dy + \\
& \left(\frac{1}{4\pi t}\right)^{\frac{n}{2}} \cos\frac{n\pi}{l}x \int_{\mathbb{R}} \exp\left\{-\frac{y^2}{4t} - \frac{n\pi}{l}yi\right\} dy + \\
& \left(\frac{1}{4\pi t}\right)^{\frac{n}{2}} \sin\frac{n\pi}{l}x \int_{\mathbb{R}} \exp\left\{-\frac{y^2}{4t} + \frac{n\pi}{l}yi\right\} dy - \\
& \frac{1}{2i}\left(\frac{1}{4\pi t}\right)^{\frac{n}{2}} \sin\frac{n\pi}{l}x \int_{\mathbb{R}} \exp\left\{-\frac{y^2}{4t} - \frac{n\pi}{l}yi\right\} dy.
\end{aligned}
\tag{3.15}
$$

On the other hand,

$$
\begin{aligned}
&\int_{\mathbb{R}} \exp\left\{-(ay^2+2by+c)\right\} dy \\
&\qquad = \int_{\mathbb{R}} \exp\left\{-\frac{1}{a}\left[(ay+b)^2+ac-b^2\right]\right\} dy \\
&\qquad = \exp\left\{\frac{b^2-ac}{a}\right\} \int_{\mathbb{R}} \exp\left\{-\frac{1}{a}(ay+b)^2\right\} dy \\
&\qquad = \left(\frac{4}{a}\right)^{\frac{1}{2}} \exp\left\{\frac{b^2-ac}{a}\right\} \int_0^{\infty} \exp(-y^2)dy \\
&\qquad = \left(\frac{\pi}{a}\right)^{\frac{1}{2}} \exp\left\{\frac{b^2-ac}{a}\right\} \quad (a>0,\ ac-b^2>0).
\end{aligned}
\tag{3.16}
$$

Thus

$$
\left(\frac{1}{4\pi t}\right)^{\frac{1}{2}} \int_{\mathbb{R}} \exp\left\{-\frac{y^2}{4t}\right\} \cos\frac{n\pi}{l}(x-y)dy = \exp\left\{-\left(\frac{n\pi}{l}\right)^2 t\right\} \cos\frac{n\pi}{l}x. \tag{3.17}
$$

Similarly,

$$
\left(\frac{1}{4\pi t}\right)^{\frac{1}{2}} \int_{\mathbb{R}} \exp\left\{-\frac{y^2}{4t}\right\} \sin\frac{n\pi}{l}(x-y)dy = \exp\left\{-\left(\frac{n\pi}{l}\right)^2 t\right\} \sin\frac{n\pi}{l}x. \tag{3.18}
$$

So

$$
\begin{aligned}
A^0 = & \sum_{i=1}^{n}\left\{\sum_{m_i=1}^{\infty} A^0_{0\cdots m_i\cdots 0}\left(\frac{1}{4\pi t}\right)^{\frac{n-1}{2}}\int_{\mathbb{R}^{n-1}}\exp\left\{-\frac{\xi_1^2+\cdots+\widehat{\xi_i^2}+\cdots+\xi_n^2}{4t}\right\}d\xi_1\cdots\widehat{d\xi_i}\cdots d\xi_n \times\right. \\
& \left.\left(\frac{1}{4\pi t}\right)^{\frac{1}{2}}\int_{\mathbb{R}}\exp\left\{-\frac{\xi_i^2}{4t}\right\}\cos\frac{m_i\pi}{l_i}(x_i-\xi_i)d\xi_i\right\}+ \\
& \sum_{i=1}^{n-1}\left\{\sum_{j=i+1}^{n}\left\{\sum_{m_i=1}^{\infty}\sum_{\substack{m_j=1\\ i<j}}^{\infty} A^0_{0\cdots m_i\cdots m_j\cdots 0}\left(\frac{1}{4\pi t}\right)^{\frac{n-2}{2}}\int_{\mathbb{R}^{n-2}}\exp\left\{-\frac{\xi_1^2+\cdots+\widehat{\xi_i^2}+\cdots+\widehat{\xi_j^2}+\cdots+\xi_n^2}{4t}\right\}\times\right.\right. \\
& d\xi_1\cdots\widehat{d\xi_i}\cdots\widehat{d\xi_j}\cdots d\xi_n\left(\frac{1}{4\pi t}\right)^{\frac{1}{2}}\int_{\mathbb{R}}\exp\left\{-\frac{\xi_i^2}{4t}\right\}\cos\frac{m_i\pi}{l_i}(x_i-\xi_i)d\xi_i \times \\
& \left.\left.\left(\frac{1}{4\pi t}\right)^{\frac{1}{2}}\int_{\mathbb{R}}\exp\left\{-\frac{\xi_j^2}{4t}\right\}\cos\frac{m_j\pi}{l_j}(x_j-\xi_j)d\xi_j\right\}\right\}+ \\
& \sum_{i=1}^{n-2}\left\{\sum_{j=i+1}^{n-1}\left\{\sum_{k=j+1}^{n}\left\{\sum_{m_i=1}^{\infty}\sum_{\substack{m_j=1\\ i<j}}^{\infty}\sum_{\substack{m_k=1\\ i<j<k}}^{\infty} A^0_{0\cdots m_i\cdots m_j\cdots m_k\cdots 0}\left(\frac{1}{4\pi t}\right)^{\frac{n-3}{2}}\times\right.\right.\right. \\
& \int_{\mathbb{R}^{n-3}}\exp\left\{-\frac{\xi_1^2+\cdots+\widehat{\xi_i^2}+\cdots+\widehat{\xi_j^2}+\cdots+\widehat{\xi_k^2}+\cdots+\xi_n^2}{4t}\right\}d\xi_1\cdots\widehat{d\xi_i}\cdots\widehat{d\xi_j}\cdots\widehat{d\xi_k}\cdots d\xi_n\times \\
& \left(\frac{1}{4\pi t}\right)^{\frac{1}{2}}\int_{\mathbb{R}}\exp\left\{-\frac{\xi_i^2}{4t}\right\}\cos\frac{m_i\pi}{l_i}(x_i-\xi_i)d\xi_i\left(\frac{1}{4\pi t}\right)^{\frac{1}{2}}\int_{\mathbb{R}}\exp\left\{-\frac{\xi_j^2}{4t}\right\}\cos\frac{m_j\pi}{l_j}(x_j-\xi_j)d\xi_j\times \\
& \left.\left.\left.\left(\frac{1}{4\pi t}\right)^{\frac{1}{2}}\int_{\mathbb{R}}\exp\left\{-\frac{\xi_j^2}{4t}\right\}\cos\frac{m_j\pi}{l_k}(x_k-\xi_k)d\xi_k)\right\}\right\}\right\}+\cdots+ \\
& \sum_{m_1=1}^{\infty}\cdots\sum_{m_n=1}^{\infty} A^0_{m_1\cdots m_n}\left(\frac{1}{4\pi t}\right)^{\frac{1}{2}}\int_{\mathbb{R}}\exp\left\{-\frac{\xi_1^2}{4t}\right\}\cos\frac{m_1\pi}{l_1}(x_1-\xi_1)d\xi_1\times \\
& \cdots(\frac{1}{4\pi t})^{\frac{1}{2}}\int_{\mathbb{R}}\exp\left\{-\frac{\xi_n^2}{4t}\right\}\cos\frac{m_n\pi}{l_n}(x_n-\xi_n)d\xi_n \\
= & \sum_{i=1}^{n}\left\{\sum_{m_i=1}^{\infty} A^0_{0\cdots m_i\cdots 0}\exp\left\{-\left(\frac{m_i\pi}{l_i}\right)^2 t\right\}\cos\frac{m_i\pi}{l_i}x_i\right\}+ \\
& \sum_{i=1}^{n-1}\left\{\sum_{j=i+1}^{n}\left\{\sum_{m_i=1}^{\infty}\sum_{\substack{m_j=1\\ i<j}}^{\infty} A^0_{0\cdots m_i\cdots m_j\cdots 0}\exp\left\{-\left[\left(\frac{m_i\pi}{l_i}\right)^2+\left(\frac{m_j\pi}{l_j}\right)^2\right]t\right\}\cos\frac{m_i\pi}{l_i}x_i\cos\frac{m_j\pi}{l_j}x_j\right\}\right\}+ \\
& \sum_{i=1}^{n-2}\left\{\sum_{j=i+1}^{n-1}\left\{\sum_{k=j+1}^{n}\left\{\sum_{m_i=1}^{\infty}\sum_{\substack{m_j=1\\ i<j}}^{\infty}\sum_{\substack{m_k=1\\ i<j<k}}^{\infty} A^0_{0\cdots m_i\cdots m_j\cdots m_k\cdots 0}\times\right.\right.\right. \\
& \left.\left.\left.\exp\left\{-\left[\left(\frac{m_i\pi}{l_i}\right)^2+\left(\frac{m_j\pi}{l_j}\right)^2+\left(\frac{m_k\pi}{l_k}\right)^2\right]t\right\}\cos\frac{m_i\pi}{l_i}x_i\cos\frac{m_j\pi}{l_j}x_j\cos\frac{m_k\pi}{l_k}x_k\right\}\right\}\right\}+\cdots+ \\
& \sum_{m_1=1}^{\infty}\cdots\sum_{m_n=1}^{\infty} A^0_{m_1\cdots m_n}\exp\left\{-\left[\left(\frac{m_1\pi}{l_1}\right)^2+\cdots+\left(\frac{m_n\pi}{l_n}\right)^2\right]t\right\}\cos\frac{m_1\pi}{l_1}x_1\cdots\cos\frac{m_n\pi}{l_n}x_n,
\end{aligned}
\tag{3.19}
$$

$$
\begin{aligned}
A^1 &= \sum_{i=1}^{n}\Bigg\{\sum_{m_1=0}^{\infty}\cdots\sum_{m_i=1}^{\infty}\cdots\sum_{m_n=0}^{\infty}A^1_{m_1\cdots m_n}\left(\frac{1}{4\pi t}\right)^{\frac{1}{2}}\int_{\mathbb{R}}\exp\left\{-\frac{\xi_1^2}{4t}\right\}\cos\frac{m_1\pi}{l_1}(x_1-\xi_1)d\xi_1\cdots \\
&\quad \left(\frac{1}{4\pi t}\right)^{\frac{1}{2}}\int_{\mathbb{R}}\exp\left\{-\frac{\xi_i^2}{4t}\right\}\sin\frac{m_i\pi}{l_i}(x_i-\xi_i)d\xi_i\cdots\left(\frac{1}{4\pi t}\right)^{\frac{1}{2}}\int_{\mathbb{R}}\exp\left\{-\frac{\xi_n^2}{4t}\right\}\cos\frac{m_n\pi}{l_n}(x_n-\xi_n)d\xi_n\Bigg\} \\
&= \sum_{i=1}^{n}\Bigg\{\sum_{m_1=0}^{\infty}\cdots\sum_{m_i=1}^{\infty}\cdots\sum_{m_n=0}^{\infty}A^1_{m_1\cdots m_n}\exp\left\{-\left[\left(\frac{m_1\pi}{l_1}\right)^2+\cdots+\left(\frac{m_n\pi}{l_n}\right)^2\right]t\right\}\times \\
&\quad \cos\frac{m_1\pi}{l_1}x_1\cdots\sin\frac{m_i\pi}{l_i}x_i\cdots\cos\frac{m_n\pi}{l_n}x_n\Bigg\},
\end{aligned}
\tag{3.20}
$$

$$
\begin{aligned}
A^2 &= \sum_{i=1}^{n-1}\Bigg\{\sum_{j=i+1}^{n}\Bigg\{\sum_{m_1=0}^{\infty}\cdots\sum_{m_i=1}^{\infty}\cdots\sum_{\substack{m_j=1\\ i<j}}^{\infty}\cdots\sum_{m_n=0}^{\infty}A^2_{m_1\cdots m_n}\times \\
&\quad \left(\frac{1}{4\pi t}\right)^{\frac{1}{2}}\int_{\mathbb{R}}\exp\left\{-\frac{\xi_1^2}{4t}\right\}\cos\frac{m_1\pi}{l_1}(x_1-\xi_1)d\xi_1\cdots\left(\frac{1}{4\pi t}\right)^{\frac{1}{2}}\int_{\mathbb{R}}\exp\left\{-\frac{\xi_i^2}{4t}\right\}\sin\frac{m_i\pi}{l_i}(x_i-\xi_i)d\xi_i\cdots \\
&\quad \left(\frac{1}{4\pi t}\right)^{\frac{1}{2}}\int_{\mathbb{R}}\exp\left\{-\frac{\xi_j^2}{4t}\right\}\sin\frac{m_j\pi}{l_j}(x_j-\xi_j)d\xi_j\cdots\left(\frac{1}{4\pi t}\right)^{\frac{1}{2}}\int_{\mathbb{R}}\exp\left\{-\frac{\xi_n^2}{4t}\right\}\cos\frac{m_n\pi}{l_n}(x_n-\xi_n)d\xi_n\Bigg\}\Bigg\} \\
&= \sum_{i=1}^{n-1}\Bigg\{\sum_{j=i+1}^{n}\Bigg\{\sum_{m_1=0}^{\infty}\cdots\sum_{m_i=1}^{\infty}\cdots\sum_{\substack{m_j=1\\ i<j}}^{\infty}\cdots\sum_{m_n=0}^{\infty}A^2_{m_1\cdots m_n}\exp\left\{-\left[\left(\frac{m_1\pi}{l_1}\right)^2+\cdots+\left(\frac{m_n\pi}{l_n}\right)^2\right]t\right\}\times \\
&\quad \cos\frac{m_1\pi}{l_1}x_1\cdots\sin\frac{m_i\pi}{l_i}x_i\cdots\sin\frac{m_j\pi}{l_j}x_j\cdots\cos\frac{m_n\pi}{l_n}x_n\Bigg\}\Bigg\},
\end{aligned}
\tag{3.21}
$$

$$
\begin{aligned}
A^3 \;=\; & \sum_{i=1}^{n-2}\left\{\sum_{j=i+1}^{n-1}\left\{\sum_{k=j+1}^{n}\left\{\sum_{m_1=0}^{\infty}\cdots\sum_{m_i=1}^{\infty}\cdots\sum_{\substack{m_j=1\\ i<j}}^{\infty}\cdots\sum_{\substack{m_k=1\\ i<j<k}}^{\infty}\cdots\sum_{m_n=0}^{\infty} A^3_{m_1\cdots m_n}\right.\right.\right.\right. \times \\
& \left(\frac{1}{4\pi t}\right)^{\frac{1}{2}}\int_{\mathbb{R}}\exp\left\{-\frac{\xi_1^2}{4t}\right\}\cos\frac{m_1\pi}{l_1}(x_1-\xi_1)d\xi_1\cdots\left(\frac{1}{4\pi t}\right)^{\frac{1}{2}}\int_{\mathbb{R}}\exp\left\{-\frac{\xi_i^2}{4t}\right\}\sin\frac{m_i\pi}{l_i}(x_i-\xi_i)d\xi_i\cdots \\
& \left(\frac{1}{4\pi t}\right)^{\frac{1}{2}}\int_{\mathbb{R}}\exp\left\{-\frac{\xi_j^2}{4t}\right\}\sin\frac{m_j\pi}{l_j}(x_j-\xi_j)d\xi_j\cdots\left(\frac{1}{4\pi t}\right)^{\frac{1}{2}}\int_{\mathbb{R}}\exp\left\{-\frac{\xi_i^2}{4t}\right\}\sin\frac{m_k\pi}{l_k}(x_k-\xi_k)d\xi_k\cdots \\
& \left(\frac{1}{4\pi t}\right)^{\frac{1}{2}}\int_{\mathbb{R}}\exp\left\{-\frac{\xi_n^2}{4t}\right\}\cos\frac{m_n\pi}{l_n}(x_n-\xi_n)d\xi_n \\
= \; & \sum_{i=1}^{n-2}\left\{\sum_{j=i+1}^{n-1}\left\{\sum_{k=j+1}^{n}\left\{\sum_{m_1=0}^{\infty}\cdots\sum_{m_i=1}^{\infty}\cdots\sum_{\substack{m_j=1\\ i<j}}^{\infty}\cdots\sum_{\substack{m_k=1\\ i<j<k}}^{\infty}\cdots\sum_{m_n=0}^{\infty} A^3_{m_1\cdots m_n}\right.\right.\right.\right. \times \\
& \exp\left\{-\left[\left(\frac{m_1\pi}{l_1}\right)^2+\cdots+\left(\frac{m_n\pi}{l_n}\right)^2\right]t\right\}\times \\
& \left.\left.\left.\cos\frac{m_1\pi}{l_1}x_1\cdots\sin\frac{m_i\pi}{l_i}x_i\cdots\sin\frac{m_j\pi}{l_j}x_j\cdots\sin\frac{m_k\pi}{l_k}x_k\cdots\cos\frac{m_n\pi}{l_n}x_n\right\}\right\}\right\}
\end{aligned}
\tag{3.22}
$$

and so on, in particular,

$$
\begin{aligned}
A^n \;=\; & \sum_{m_1=1}^{\infty}\cdots\sum_{m_n=1}^{\infty} A^n_{m_1\cdots m_n}\left(\frac{1}{4\pi t}\right)^{\frac{1}{2}}\int_{\mathbb{R}}\exp\left\{-\frac{\xi_1^2}{4t}\right\}\sin\frac{m_1\pi}{l_1}(x_1-\xi_1)d\xi_1\cdots \\
& \left(\frac{1}{4\pi t}\right)^{\frac{1}{2}}\int_{\mathbb{R}}\exp\left\{-\frac{\xi_n^2}{4t}\right\}\sin\frac{m_n\pi}{l_n}(x_n-\xi_n)d\xi_n \\
= \; & \sum_{m_1=1}^{\infty}\cdots\sum_{m_n=1}^{\infty} A^n_{m_1\cdots m_n}\exp\left\{-\left[\left(\frac{m_1\pi}{l_1}\right)^2+\cdots+\left(\frac{m_n\pi}{l_n}\right)^2\right]t\right\}\times \\
& \sin\frac{m_1\pi}{l_1}x_1\cdots\sin\frac{m_n\pi}{l_n}x_n.
\end{aligned}
\tag{3.23}
$$

Noting

$$
\exp\left\{-\left[\left(\frac{m_1\pi}{l_1}\right)^2+\cdots+\left(\frac{m_n\pi}{l_n}\right)^2\right]t\right\}\longrightarrow 0 \qquad as \quad t\longrightarrow\infty, \tag{3.24}
$$

we obtain the desired (3.2) from (3.9) and (3.19)-(3.24) immediately. Thus, the proof of Theorem 3.1 is completed. □

We now prove Theorem 1.2.

Proof of Theorem 1.2. By Theorem 1.1, the Cauchy problem (1.2)-(1.3) admits a global smooth solution $x = x(t, \theta_1, \cdots, \theta_n)$, and then by Theorem 3.1, the solution $x = x(t, \theta)$ satisfies (1.5). (1.5) implies that hypersurfaces converge to a sphere with radius $\bar{r}_0$ in the C^∞-topology as t goes to the infinity. Thus, the proof of Theorem 1.2 is completed. □

4 Conclusions and open problems

In this paper we introduce a new geometric flow with rotational invariance. This flow is described by, formally a system of hyperbolic partial differential equations with viscosity, essentially a coupled system of hyperbolic-parabolic partial differential equations with rotational invariance, which possesses very interesting geometric properties and dynamical behavior. We prove that, under this kind of new flow, an arbitrary smooth closed contractible hypersurface in the Euclidean space $\mathbb{R}^{n+1}$ $(n \geq 1)$ converges to $\mathbb{S}^n$ in the C^∞-topology as t goes to the infinity. As mentioned before, this result covers the well-known theorem of Gage and Hamilton in [4] for the curvature flow of plane curves and the famous result of Huisken in [5] on the flow by mean curvature of convex surfaces, respectively. In fact, more applications of this flow to differential geometry and physics can be expected.

In the present paper, we only investigate the evolution of closed contractible hypersurfaces in the Euclidean space $\mathbb{R}^n$ $(n \geq 2)$ under the flow equation (1.2), there are some fundamental and interesting problems. In particular, the following open problems seems to us more interesting and important: (i) use the flow equation (2.1) to investigate the deformation of a closed m-dimensional sub-manifold $x_0 = x_0(\theta_1, \cdots, \theta_m)$; (ii) find a suitable way to extend the results presented in this paper to the case of Riemannian manifolds in stead of the Euclidean space $\mathbb{R}^n$; (iii) introduce the theory of viscous shock waves to investigate geometric problems. These problems are worthy to study in the future.

Appendix

In this appendix, we investigate the time-asymptotic behavior of solution of the following Cauchy problem

$$\begin{cases} \dfrac{\partial u}{\partial t} + \displaystyle\sum_{i=1}^{n} \dfrac{\partial (f_i(u)u)}{\partial \theta_i} = \Delta u & \text{in } (0,\infty) \times \mathbb{R}^n, \\ u(0,\theta) = u_0(\theta) & \text{in } \mathbb{R}^n, \end{cases} \tag{A.1}$$

where u is the unknown scalar function, $f_i(u)$ $(i = 1, \cdots, n)$ are smooth functions, u_0 is a periodic function which stands for the initial data.

Lemma A.1 *If* $u_0(\theta)$ *is a smooth periodic function with period* $\mathbb{T}^n = \{\theta = (\theta_1, \theta_2, \cdots\cdots\theta_n) \mid -l_i \leq \theta_i \leq l_i\}$ *, then* $u(t,x)$ *is also periodic with period* $\mathbb{T}^n$ *for any* t*.*

Proof. Since $u_0(\theta) = u(0,\theta)$ is periodic with period $\mathbb{T}^n$, it holds that

$$u(0,\theta) = u_0(\theta) = u_0(\theta + \mathbb{T}^n) = u(0, \theta + \mathbb{T}^n). \tag{A.2}$$

Taking into account the property of the uniqueness of the solution $u(t,\theta)$, gives

$$u(t,\theta) = u(t, \theta + \mathbb{T}^n). \tag{A.3}$$

This proves Lemma A.1. $\square$

Let Ω be a smooth domain in $\mathbb{R}^n$ and consider the parabolic operator

$$Lu = \sum_{i,j=1}^{n} a_{ij}(t,x)\frac{\partial^2 u}{\partial\theta_i\partial\theta_j} + \sum_{i=1}^{n} b_i(t,\theta)\frac{\partial u}{\partial\theta_i} + c(t,\theta)u \tag{A.4}$$

with smooth and bounded coefficients and a nondegenerate matrix (a_{ij}).

Lemma A.2 (Harnack's inequality) *Suppose that $u(t,\theta) \in C^2((0,T)\times\Omega)$ is a solution of $\frac{\partial u}{\partial t} - Lu \geq 0$ in $(0,T)\times\Omega$, suppose furthermore that $u(t,\theta) \geq 0$ in $(0,T)\times\Omega$. Then for any given compact subset D of Ω and each $\tau \in (0,T)$, there exists a positive constant C_1 depending only on D, τ and the coefficients of L, such that*

$$\sup_K u(t-\tau,\theta) \leq C_1 \inf_K u(t,\theta). \tag{A.5}$$

Lemma A.3 *Suppose that $u(t,\theta)$ is a periodic solution of the following equation*

$$\frac{\partial u}{\partial t} + \sum_{i=1}^{n} \frac{\partial(f_i(u)u)}{\partial\theta_i} = \Delta u, \quad \forall\ (t,\theta) \in (0,\infty)\times\mathbb{R}^n \tag{A.6}$$

with a period, say, $\mathbb{T}^n = \{\theta = (\theta_1, \theta_2, \cdots\cdots\theta_n) \mid -l_i \leq \theta_i \leq l_i\}$, where $f_i \in L^\infty$ $(i \in N)$. Then there exists a positive constant C_2 depending only on $\mathbb{T}^n$,τ and f_i, such that

$$\|u(t-\tau,\theta)\|_{L^\infty(\mathbb{T}^n)} \leq C_2\|u(t,\theta)\|_{L^1(\mathbb{T}^n)}. \tag{A.7}$$

Proof. The proof will be divided into two cases.

Case I: $u(t,\theta) > 0$

By Harnack inequality, i.e., Lemma A.2, there exists a positive constant C_3 depending only on $\mathbb{T}^n$, τ and f_i such that

$$\sup_{\mathbb{T}^n} u(t-\tau,\theta) \leq C \inf_{\mathbb{T}^n} u(t,\theta). \tag{A.8}$$

By the mean-value theorem, there exists a point $\theta_0 \in \mathbb{T}^n$ such that

$$\overline{u(t,\theta)} \triangleq \frac{1}{|\mathbb{T}^n|}\int_{\mathbb{T}^n} u(t,\theta)d\theta = \frac{1}{|\mathbb{T}^n|}|\mathbb{T}^n|u(t,\theta_0) \geq \inf_{\mathbb{T}^n} u(t,\theta), \tag{A.9}$$

Therefore, using (A.8), we have

$$\begin{aligned}\tfrac{1}{|\mathbb{T}^n|}\|u(t,\theta)\|_{L^1(\mathbb{T}^n)} &= \overline{u(t,\theta)} \geq \inf_{\mathbb{T}^n} u(t,\theta)\\ &\geq \frac{1}{C}\sup_{\mathbb{T}^n} u(t-\tau,\theta) \geq \frac{1}{C}\|u(t-\tau,\theta)\|_{L^\infty(\mathbb{T}^n)}.\end{aligned} \tag{A.10}$$

So

$$\|u(t-\tau,\theta)\|_{L^\infty(\mathbb{T}^n)} \leq C_3\|u(t,\theta)\|_{L^1(\mathbb{T}^n)}, \tag{A.11}$$

where $C_3 = \frac{C_1}{|\mathbb{T}^n|}$.

Case II: General case

Let

$$u(t,\theta)^+ = \max\{u(t,\theta),0\}, \quad u(t,\theta)^- = -\min\{u(t,\theta),0\}. \tag{A.12}$$

Then

$$u(t,\theta) = u(t,\theta)^+ - u(t,\theta)^- \tag{A.13}$$

and

$$u(0,\theta) = u(0,\theta)^+ - u(0,\theta)^-. \tag{A.14}$$

By Theorem 2.1, it is easy to see that the solution $u(t,\theta)$ is unique. Substituting (A.13) into (A.6) gives

$$\frac{\partial u^+}{\partial t} + \sum_{i=1}^n \frac{\partial(f_i(u)u^+)}{\partial\theta_i} - \Delta u^+ - \left\{\frac{\partial u^-}{\partial t} + \sum_{i=1}^n \frac{\partial(f_i(u)u^-)}{\partial\theta_i} + \Delta u^-\right\} = 0. \tag{A.15}$$

We turn to investigate the following Cauchy problems

$$\begin{cases} \dfrac{\partial u^+}{\partial t} + \displaystyle\sum_{i=1}^n \frac{\partial(f_i(u)u^+)}{\partial\theta_i} = \Delta u^+ & \text{in } (0,\infty)\times\mathbb{R}^n,\\ u^+(0,\theta) = u_0^+(\theta) & \text{in } \mathbb{R}^n \end{cases} \tag{A.16}$$

and

$$\begin{cases} \dfrac{\partial u^-}{\partial t} + \displaystyle\sum_{i=1}^n \frac{\partial(f_i(u)u^-)}{\partial\theta_i} = \Delta u^- & \text{in } (0,\infty)\times\mathbb{R}^n,\\ u^-(0,\theta) = u_0^-(\theta) & \text{in } \mathbb{R}^n. \end{cases} \tag{A.17}$$

By making use of the method of the proof of Theorem 2.1, we can easily prove the Cauchy problems (A.16) and (A.17) admit the unique non-negative solution, respectively. Noticing (A.11), we have

$$\sup_{\mathbb{T}^n} u(t-\tau,\theta) \leq \sup_{\mathbb{T}^n} u^+(t-\tau,\theta) \leq C_3\int_{\mathbb{T}^n} u^+(t,\theta)d\theta \tag{A.18}$$

and

$$\inf_{\mathbb{T}^n} u(t-\tau,\theta) \geq -\sup_{\mathbb{T}^n} u^-(t-\tau,\theta) \geq -C_3\int_{\mathbb{T}^n} u^-(t,\theta)d\theta. \tag{A.19}$$

Consequently,

$$\begin{aligned}\|u(t-\tau,\theta)\|_{L^\infty(\mathbb{T}^n)} &\leq C_3\int_{\mathbb{T}^n}(u^+(t,\theta)+u^-(t,\theta))d\theta \\ &\leq C_3\int_{\mathbb{T}^n}|u(t,\theta)|d\theta \leq C_3\|u(t,\theta)\|_{L^1(\mathbb{T}^n)}.\end{aligned} \tag{A.20}$$

This is the desired estimate (A.7). Thus, the proof of Lemma A.3 is completed. □

Let us recall some notions which will be used later. Let $C^\infty_{per}(\mathbb{T}^n)$ be the space of $\mathbb{T}^n$-periodic functions in $C^\infty(\mathbb{R}^n)$, and

$$\begin{cases} W^{1,\infty}_{per,loc}(\mathbb{R}\times\mathbb{T}^n) & \triangleq \{u(t,\theta)\mid u(t,\theta)\in W^{1,\infty}_{loc}(\mathbb{R}^{1+n}) \text{ and } u(t,\theta) \text{ is } \mathbb{T}^n-\text{periodic in } \theta\}, \\ H^1_{per}(\mathbb{T}^n) & \triangleq \left\{u(t,\theta)\mid u(t,\theta)\in \overline{C^\infty_{per}(\mathbb{T}^n)}\bigcap H^1(\mathbb{T}^n)\right\}, \quad \|\bullet\|_{H^1_{per}(\mathbb{T}^n)}\triangleq\|\bullet\|_{H^1(\mathbb{T}^n)}. \end{cases} \tag{A.21}$$

Noticing (2.10) (*i.e.*, $g_i(u)=f_i(u)u$), we can rewrite (A.6) as

$$\frac{\partial u}{\partial t}+\mathrm{div}_\theta g(u)=\Delta u. \tag{A.22}$$

In a manner similar to [1], we can prove the following result on stationary solutions of (A.22).

Proposition A.1 *Let*

$$g(v(t,\theta))\in W^{1,\infty}_{per,loc}(\mathbb{R}\times\mathbb{T}^n)^n, \quad \mathrm{div}_\theta g(v(t,\theta))\in L^\infty_{loc}(\mathbb{R}^{1+n}).$$

Suppose that there exist real numbers $C_0>0, m\geq 0$ and $l\in[0,\frac{n+2}{n-2})$ for $n\geq 3$, such that

$$|g_i'(v(t,\theta))|\leq C_0(1+|v|^m) \tag{A.23}$$

and

$$|\mathrm{div}_\theta g(v(t,\theta))|\leq C_0(1+|v|^l) \tag{A.24}$$

for all $(t,\theta)\in\mathbb{R}\times\mathbb{T}^n$. Suppose furthermore that the couple (m,l) satisfies at least one of the following conditions

$$m=0, \tag{A.25}$$

$$l\in[0,1), \tag{A.26}$$

$$l<\min\left\{\frac{n+2}{n},2\right\} \quad \text{and} \quad \text{there exists } t_0\in\mathbb{R} \text{ such that } \mathrm{div}_\theta g(v(t_0,\theta))=0 \text{ for all } \theta\in T^n. \tag{A.27}$$

Then for any fixed $p\in\mathbb{R}$, there exists a unique solution $v=v(p,\bullet)\in H^1_{per}(\mathbb{T}^n)$ of the problem

$$-\Delta v(p,\theta)+\mathrm{div}_\theta g(v(p,\theta))=0, \quad \overline{v(p,\bullet)}=p. \tag{A.28}$$

Moveover, $v(p,\bullet)$ satisfies the growth property: if $p > q$, then

$$v(p,\theta) > v(q,\theta),\ \forall\ \theta \in \mathbb{T}^n. \tag{A.29}$$

Remark A.1 *Usually, the problem (A.28) is called "cell problem".*

We now state the main result in this section.

Theorem A.1 *Suppose that $u_0(\theta) \in L^\infty_{per}(\mathbb{T}^n)$,*

$$u = u(t,\theta) \in C([0,\infty), L^1(\mathbb{T}^n)) \bigcap L^\infty([0,\infty)\times\mathbb{T}^n) \bigcap L^2_{loc}([0,\infty), H^1_{per}(\mathbb{T}^n))$$

be the unique solution of the Cauchy problem

$$\begin{cases} \dfrac{\partial u}{\partial t} + \mathrm{div}_\theta g(u) = \Delta u \quad in\ (0,\infty)\times\mathbb{T}^n, \\ u(\theta,0) = u_0(\theta) \quad\ in\ \mathbb{T}^n, \end{cases} \tag{A.30}$$

and $v(\overline{u_0},\theta) \in H^1_{per}(\mathbb{T}^n)$ is the solution of the associated cell problem (A.28), where

$$\overline{u_0} \triangleq \frac{1}{\mathrm{vol}\{\mathbb{T}^n\}} \int_{\mathbb{T}^n} u_0(x)dx.$$

Suppose furthermore that $g(v(t,\theta)) \in W^{1,\infty}_{per,loc}(\mathbb{R}\times\mathbb{T}^n)^n$, $\partial_{\theta_i} g_i'(u(t,\theta)) \in L^\infty_{loc}(R\times\mathbb{T}^n)$ and the assumptions of Proposition A.1 are satisfied. Suppose finally that there exist constants $\beta_1,\beta_2 \in \mathbb{R}$ such that

$$v(\beta_1,\theta) \le u_0(\theta) \le v(\beta_2,\theta). \tag{A.31}$$

Then it holds that

$$\|u(t,\theta) - v(\overline{u_0},\theta)\|_{L^\infty(\mathbb{T}^n)} \to 0 \quad as\ \ t\to\infty. \tag{A.32}$$

Before proving Theorem A.1, we introduce the following notations:

$$\mathbf{M}(t,\theta) \triangleq \sup_{\tau \ge t} u(\tau,\theta), \tag{A.33}$$

$$\mathbf{P}(t) \triangleq \inf\{p \mid v(p,\theta) \ge \mathbf{M}(t,\theta)\ for\ \theta\in\mathbb{T}^n\} \tag{A.34}$$

and

$$\mathbf{N}(p,t) \triangleq \{\theta \in \mathbb{T}^n \mid v(p,\theta) < \mathbf{M}(t,\theta)\}. \tag{A.35}$$

By maximum principle, we can find that if $u = u(t,\theta)$ is a solution of (A.22) with the initial data satisfying (A.30), then

$$v(\beta_1,\theta) \le u(t,\theta) \le v(\beta_2,\theta), \quad \forall\ (t,\theta) \in [0,\infty)\times\mathbb{T}^n.$$

Moreover, it is easy to show that $\mathbf{P}(t)$ is a bounded non-increasing function of t. Therefore, we may set

$$\mathbf{P} \stackrel{\triangle}{=} \lim_{t\to\infty} \mathbf{P}(t).$$

In order to prove Theorem A.1, we need the following lemma.

Lemma A.4 *Suppose that $u = u(t, \theta)$ is a solution of*

$$\frac{\partial u}{\partial t} + \sum_{i=1}^{n} \frac{\partial (f_i(u)u)}{\partial \theta_i} = \Delta u \tag{A.36}$$

and $v(q, \theta)$ is the solution of the associated cell problem (A.28). Then for any given positive constant ε, there exist $t_0 \in \mathbb{R}$ and sequences $\{t_n\}$ and $\{\tau_n\}$ satisfying $\tau_n \geq t_n > t_0$ and $\lim_{n\to\infty} t_n = \infty$, and $\theta_n \in \mathbf{N}(t_1, t_n)$, such that

$$|\omega_n(0, \theta_n)| \leq \varepsilon, \tag{A.37}$$

where

$$\omega_n(t, \theta) \stackrel{\triangle}{=} v(\mathbf{P}(t_n), \theta) - u(\tau_n + t, \theta) \quad (t \in [0, 1], \theta \in \mathbb{T}^n). \tag{A.38}$$

Proof. Since $v(t, \theta)$ is a continuous function of t, for any given positive constant $\varepsilon > 0$, there exists a positive constant δ such that

$$\|v(t, \theta) - v(\mathbf{P}, \theta)\|_{L^\infty(\mathbb{T}^n)} \leq \frac{1}{3}\varepsilon, \tag{A.39}$$

provided that $|t - \mathbf{P}| \leq \delta$. Choose $t_0 \in \mathbb{R}$ such that

$$|\mathbf{P} - \mathbf{P}(t)| \leq \delta \quad for \quad t \geq t_0. \tag{A.40}$$

By (A.39), we have

$$\|v(\mathbf{P}, \theta) - v(\mathbf{P}(t), \theta)\|_{L^\infty(\mathbb{T}^n)} \leq \frac{1}{3}\varepsilon. \tag{A.41}$$

Let $t_1 \in \mathbb{R}$ satisfy

$$|t_1 - \mathbf{P}| \leq \delta \quad for \quad t_1 < \mathbf{P}. \tag{A.42}$$

By (A.39) again, we get

$$\|v(t_1, \theta) - v(\mathbf{P}, \theta)\|_{L^\infty(\mathbb{T}^n)} \leq \frac{1}{3}\varepsilon. \tag{A.43}$$

Combining (A.41) and (A.43) gives

$$\begin{aligned} \|v(t_1, \theta) - v(\mathbf{P}(t), \theta)\|_{L^\infty(\mathbb{T}^n)} &\leq \|v(t_1, \theta) - v(\mathbf{P}, \theta)\|_{L^\infty(\mathbb{T}^n)} + \|v(\mathbf{P}, \theta) - v(\mathbf{P}(t), \theta)\|_{L^\infty(\mathbb{T}^n)} \\ &\leq \frac{1}{3}\varepsilon + \frac{1}{3}\varepsilon = \frac{2}{3}\varepsilon. \end{aligned} \tag{A.44}$$

Hence, for $t \geq t_0$ and $\theta \in \mathbf{N}(t_1, t)$, it holds that

$$v(\mathbf{P}(t), \theta) - \frac{2}{3}\varepsilon \leq v(t_1, \theta) \leq \mathbf{M}(t, \theta) \leq v(\mathbf{P}(t), \theta). \tag{A.45}$$

Let $\{t_n\}$ be a sequence and satisfy

$$\lim_{n\to\infty} t_n = \infty.$$

For $\theta_n \in \mathbf{N}(t_1, t_n)$, there exists τ_n $(\geq t_n)$ such that

$$|u(\tau_n, \theta_n) - \mathbf{M}(t_n, \theta_n)| \leq \frac{1}{3}\varepsilon. \tag{A.46}$$

Consequently, for large n we have $t_n > t_0$, and then we obtain from (A.45) that

$$v(\mathbf{P}(t_n), \theta_n) - \frac{2}{3}\varepsilon \leq v(t_1, \theta_n) \leq \mathbf{M}(t, \theta_n) \leq v(\mathbf{P}(t_n), \theta_n). \tag{A.47}$$

Combining (A.46) and (A.47) yields

$$\omega_n(0, \theta_n) = |v(\mathbf{P}(t_n), \theta_n) - u(\tau_n, \theta_n)| \leq \varepsilon. \tag{A.48}$$

This proves the desired (A.37). □

We now prove Theorem A.1 .

Proof of Theorem A.1. Without loss of generality, we may choose τ_n and t_n, with $\tau_n \geq t_n$, it follows that ω_n is a non-negative function. By (A.28) and (A.31), ω_n is a non-negative solution of the following equation

$$\frac{\partial \omega_n}{\partial t} + \mathrm{div}_\theta \left\{ \int_0^1 g'[\tau v(\mathbf{P}(t_n), \theta) + (1-\tau) u(\tau_n + t, \theta)] d\tau \omega_n \right\} = \Delta \omega_n \quad \text{in } (0, \infty) \times \mathbb{T}^n. \tag{A.49}$$

Since $v(t, \theta)$ is a solution of the associated cell problem (A.28), we can choose $K > 0$ such that

$$-K \leq v(t_1, \theta) \leq v(t_2, \theta) \leq K. \tag{A.50}$$

So we have

$$\left\| \int_0^1 g'[\tau v(\mathbf{P}(t_n), \theta) + (1-\tau) u(\tau_n + t, \theta)] d\tau \right\|_{L^\infty([0,1]\times\mathbb{T}^n)} \leq \|g'\|_{L^\infty([-K,K]\times\mathbb{T}^n)}. \tag{A.51}$$

According to Lemma A.2, there exists a constant C only depending on $\mathbb{T}^n$ and $\|\partial_u g\|_{L^\infty([-K,K]\times\mathbb{T}^n)}$ such that

$$\sup_{\mathbb{T}^n} \omega_n(-\alpha, \theta) \leq C \inf_{\mathbb{T}^n} \omega_n(0, \theta). \tag{A.52}$$

It follows from Lemma A.4 that

$$0 \leq v(\mathbf{P}(t_n), \theta) - u(\tau_n - \alpha, \theta) \leq C\varepsilon. \tag{A.53}$$

Thus, there exists a sequence η_n such that

$$\|u(\eta_n,\theta)-v(\mathbf{P},\theta)\|_{L^\infty(\mathbb{T}^n)}\to 0 \quad as \;\; n\to\infty. \tag{A.54}$$

On the one hand, integrating equation (A.31) over all $\mathbb{T}^n$ leads to

$$\int_{\mathbb{T}^n} u(t,\theta)d\theta=\int_{\mathbb{T}^n} u(0,\theta)d\theta, \tag{A.55}$$

which implies that the total mass of solutions is conserved for all time. On the other hand, letting $n\to\infty$ and combining (A.28) and (A.54) gives

$$\overline{u_0}=\mathbf{P}. \tag{A.56}$$

If u_1,u_2 are solutions of (A.36), we can obtain the L^1 contraction property by similar method in [15],

$$\|u_1(t)-u_2(t)\|_{L^1(\mathbb{T}^n)}\le\|u_1(s)-u_2(s)\|_{L^1(\mathbb{T}^n)} \quad for \;\; 0\le s\le t. \tag{A.57}$$

Because $v(\overline{u_0},\theta)$ is a stationary solution of (A.31), we choose $u_1=u, u_2=v(\overline{u_0},\theta), s=\eta_n, t\ge\eta_n$, and then we have

$$\|u(t,\theta)-v(\overline{u_0},\theta)\|_{L^1(\mathbb{T}^n)}\to 0 \quad as \;\; t\to\infty. \tag{A.58}$$

By Lemma A.3 and (A.58), we obtain

$$\|u(t,\theta)-v(\overline{u_0},\theta)\|_{L^\infty(\mathbb{T}^n)}\to 0 \quad as \;\; t\to\infty. \tag{A.59}$$

Thus, the proof of Theorem A.1 is completed. □

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