DEGREES OF PERIODS

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Abstract. We introduce the concept of degree to classify the periods in the sense of Kontsevich and Zagier. Some properties of degree are proved. Using this notion we give some new understanding of some problems in transcendental number theory. The zeta function of a period is defined and some its interesting properties are given.

1. Introduction

In the wonderful exposition [2], Kontsevich and Zagier defined the concept of period: integral of a rational function over a domain bounded by polynomial inequalities with rational coefficients. By its definition the set of periods is countable and includes all algebraic numbers. Moreover, it is a ring, the sum and product of two periods are still periods. Many important transcendental numbers arising from modular forms, L-functions, hypergeometric functions, etc are periods. On the other hand, from the point of view of algebraic geometry, periods are integrals of closed algebraic differential forms over relative algebraic chains (cf.[1] and [2]).

The Galois theory plays a fundamental role in algebraic number theory. What can we do something for transcendental number theory? From Grothendieck's motive point of view, period is a suitable category for building a Galois theory (called motive Galois group) (cf. [1]).

The periods are also intended to bridge the gap between the algebraic numbers and the transcendental numbers. They are natural objects whether from the point of view of number theory or algebraic geometry.

The main purpose of the paper is to try to classify these periods under suitable category. The main tool is the concept of degree introduced by the author. We find that this concept can give some theoretic solutions to some problems in transcendental number theory. For example, we prove that the sum of two transcendental periods with different degrees is a transcendental number. We also define the zeta function for a period and prove some interesting properties.

2. definition of a period

Let us recall the definition of a period [2].

Definition 2.1. A period is a complex number whose real and imaginary parts are absolutely convergent multiple integrals

$$\int_{\Sigma} R$$

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where Σ is a domain in $\mathbb{R}^n$ given by polynomial inequalities with rational coefficients and R is a rational function with rational coefficients.

In above definition one can replace "rational coefficients" by "algebraic coefficients" by introducing more variables. Because the integral of any real function is equal to the area under its graph, any period can be written as the volume of a domain defined by polynomial inequalities with rational coefficients. So we can rewrite the definition as

Definition 2.2. A period is a complex number whose real and imaginary parts are absolutely convergent multiple integrals

$$\int_{\Sigma} dx_1...dx_n$$

where Σ is a domain in $\mathbb{R}^n$ given by polynomial inequalities with algebraic coefficients.

For simplicity, in what follows we always use definition 2.2 as the definition of a period.

The set of periods is clearly countable. It is a ring and includes all algebraic numbers. For instance, let p be an algebraic number, then

$$p = \int_{0\leq x\leq p} dx.$$

Many interesting transcendental numbers also are periods.

Example 2.3. (1)

$$\pi = \iint_{x^2+y^2\leq 1} dxdy.$$

(2)

$$\log(q) = \iint_{1\leq x\leq q, xy\leq 1, y\geq 0} dxdy,$$

where q is a positive algebraic number.

(3) All $\zeta(s)$ (s is positive integers) are periods [2]. $\zeta(s)$ is Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s}.$$

Recall that (cf. [3]) $\zeta(2k) = \frac{2^{2k-1}}{(2k)!} B_k \pi^{2k}$ where B_k is the Bernoulli number.

(4) Some values of the gamma function

$$\Gamma(s) = \int_0^{\infty} t^{s-1} e^{-t} dt$$

at rational values, $\Gamma(p/q)^q$ ($p, q \in \mathbb{N}$) are periods [2].

(5) Let

$$E_k(z) = \frac{1}{2} \sum_{m,n\in\mathbb{Z};(m,n)=1} \frac{1}{(mz+n)^k}$$

be the Eisenstein series of weight k. If $z_0 \in \overline{\mathbb{Q}}$, then $\pi^k E_k(z_0)$ is a period [2].

Though there are numerous non-period transcendental numbers, we have not a simple criterion for testing them. So the first essential problem is to find one concrete transcendental number which is not a period.

It seems that (conjecturally in [2]) the Euler constant

$$\gamma = \sum_{n\to\infty} (1 + \frac{1}{2} + ... + \frac{1}{n} - \log n) = 0.5772156...$$

and basis of natural logarithms

$$e = \sum_{n\to\infty}(1+\frac{1}{n})^n = 2.7182818...$$

are not periods.

3. DEGREE OF A PERIOD

Since so many transcendental numbers are periods. How to differentiate them? To deal with this problem, we introduce the following concept.

Definition 3.1. If p is a real period, we define the degree of p as the minimal dimension of the domain Σ such that

$$p = \int_\Sigma 1$$

where Σ is a domain in Euclid space given by polynomial inequalities with algebraic coefficients.

For any complex period $p = p_1 + ip_2$, we define $\deg(p) = max(\deg(p_1), \deg(p_2))$.

If p is not a period, we may define the $\deg(p) = \infty$. Thus we can extend the degree to whole complex number field $\mathbb{C}$.

By the definition, $\deg(0) = 0$ and $\deg(p) = 1$ if and only if p is an non-zero algebraic number. It is obviously that $\deg(\pi) = \deg(\log(n)) = 2, n \in \mathbb{Z}, n > 1$.

Let $\mathbb{P}$ denotes the set of all periods. Let $P_k = \{p \in \mathbb{P} | \deg(p) = k\}$, then $\mathbb{P} = \bigcup_{k=0}^{\infty} P_k$. Thus we give a classification for all periods.

The following two propositions are the basic properties of degrees.

Proposition 3.2. *Let p_1, p_2 be two periods, then* $\deg(p_1p_2) \leq \deg(p_1) + \deg(p_2)$ *and* $\deg(p_1 + p_2) \leq max(\deg(p_1), \deg(p_2))$.

Proof. First we consider the real case. Assume that $\deg(p_1) = k, \deg(p_2) = l$, then there exists two domains $\Sigma_1 \subseteq \mathbb{R}^k, \Sigma_2 \subseteq \mathbb{R}^l$ both bounded by polynomial inequalities with algebraic coefficients such that

$$p_1 = \int_{\Sigma_1} dx_1...dx_k, p_2 = \int_{\Sigma_2} dy_1...dy_l.$$

One has

$$p_1p_2 = \int_{\Sigma_1\times\Sigma_2} dx_1...dx_kdy_1...dy_l,$$

where $\Sigma_1 \times \Sigma_2 \subset \mathbb{R}^k \times \mathbb{R}^l = \mathbb{R}^{k+l}$ also bounded by polynomial inequalities with algebraic coefficients. So $\deg(p_1p_2) \leq \deg(p_1) + \deg(p_2)$.

Suppose that $k \leq l$, then

$$\deg(p_1) + \deg(p_2) = \int_{\Sigma_k\times\Delta} dx_1...dx_l + \int_{\Sigma_1} dx_1...dx_l,$$

where Δ is the $l-k$-times product of $[0, 1]$. Hence $\deg(p_1 + p_2) \leq max(\deg(p_1), \deg(p_2))$.

For the complex case, let $p_1 = a_1 + ib_1, p_2 = a_2 + ib_2$, where a_1, a_2, b_1, b_2 are real periods. One gets

$$\begin{aligned}
\deg(p_1p_2) &= \deg(a_1a_2 - b_1b_2 + i(a_1b_2 + a_2b_1)) \\
&= max(\deg(a_1a_2 - b_1b_2), \deg(a_1b_2 + a_2b_1)) \\
&\leq max(max(\deg(a_1a_2), \deg(b_1b_2)), max(\deg(a_1b_2), \deg(a_2b_1))) \\
&= max(\deg(a_1a_2), \deg(b_1b_2), \deg(a_1b_2), \deg(a_2b_1)) \\
&\leq max(\deg(a_1) + \deg(a_2), \deg(b_1) + \deg(b_2), \deg(a_1) + \deg(b_2), \\
&\quad \deg(a_2) + \deg(b_1)) \\
&= \deg(p_1) + \deg(p_2))
\end{aligned}$$

and

$$\begin{aligned}
\deg(p_1 + p_2) &= max(\deg(a_1 + a_2), \deg(b_1 + b_2)) \\
&\leq max(max(\deg(a_1), \deg(a_2)), max(\deg(b_1), \deg(b_2))) \\
&= max(\deg(a_1), \deg(a_2), \deg(b_1), \deg(b_2)) \\
&= max(max(\deg(a_1), \deg(b_1)), max(\deg(a_2), \deg(b_2))) \\
&= max(\deg(p_1), \deg(p_2)).
\end{aligned}$$

□

Generally, we can not get $\deg(p_1 + p_2) = max(\deg(p_1), \deg(p_2))$. The simplest example is $p_1 = \pi, p_2 = 1 - \pi$. The following examples also show that the equality $\deg(p_1p_2) = \deg(p_1) + \deg(p_2)$ is not true generally.

Example 3.3. 1): Consider

$$\xi = \iiint_{x^2+y^2\leq 1, 0\leq z(x^2+y^2+1)\leq 1} dxdydz = \iint_{x^2+y^2\leq 1} \frac{dxdy}{x^2 + y^2 + 1}$$

$$= \int_0^{2\pi} \int_0^1 \frac{rdrd\theta}{r^2 + 1} = \pi \log 2.$$

$\deg(\xi) \leq 3$. But $\deg(\pi) + \deg(\log 2) = 4$.

2): Consider

$$\eta = \iiint_{x^2+y^2\leq 1, 0\leq z((x^2+y^2)^2+1)\leq 4} dxdydz = \iint_{x^2+y^2\leq 1} \frac{4dxdy}{(x^2 + y^2)^2 + 1}$$

$$= \int_0^{2\pi} \int_0^1 \frac{4rdrd\theta}{r^4 + 1} = \pi^2.$$

$\deg(\eta) \leq 3$. But $2\deg(\pi) = 4$.

Proposition 3.4. *If p is a nonzero algebraic number and p_1 is any non-zero period, then* $\deg(p + p_1) = \deg(p_1) = \deg(pp_1)$.

Proof. The first equality follows from $\deg(p_1) = \deg(-p+p+p_1) \leq \deg(p+p_1) \leq \deg(p_1)$. For the real case, $\deg(pp_1) = \deg(p_1)$ is obviously from the definition. In complex case,

let $p = a + ib, p_1 = a_1 + ib_1, a, b \in \overline{\mathbb{Q}}$, a_1 and b_1 are any real periods. We have

$$\begin{aligned}\deg(pp_1) &= \deg(aa_1 - bb_1 + i(ba_1 + ab_1)) \\ &= max(\deg(aa_1 - bb_1)\deg(ba_1 + ab_1)) \\ &\leq max(max(\deg(aa_1), \deg(bb_1)), max(\deg(ba_1), \deg(ab_1))) \\ &= max(\deg(aa_1), \deg(bb_1), \deg(ba_1), \deg(ab_1)).\end{aligned}$$

Since $p \neq 0$, the last equation equals $\deg(p_1)$. So $\deg(pp_1) \leq \deg(p_1)$. But p is any nonzero algebraic number, so one has $\deg(p_1) = \deg(\frac{1}{p}pp_1) \leq \deg(pp_1)$. Hence $\deg(p_1) = \deg(pp_1)$.

□

Denote $\mathbb{P}_k = \{p \in \mathbb{P} | \deg(p) \leq k\}$, $\mathbb{P}_k + \mathbb{P}_l = \{p_k + p_l | p_k \in \mathbb{P}_k, p_l \in \mathbb{P}_l\}$, $\mathbb{P}_k\mathbb{P}_l = \{p_k p_l | p_k \in \mathbb{P}_k, p_l \in \mathbb{P}_l\}$. Then $\mathbb{P}_k + \mathbb{P}_l \subseteq \mathbb{P}_{max(k,l)}$ and $\mathbb{P}_k\mathbb{P}_l \subseteq \mathbb{P}_{k+l}$. $\mathbb{P}_k$ has a good graded characteristic. It is a additive group but in general (except $k = 1$) not a ring. Proposition 3.4 tells us that $\mathbb{P}_k$ is a $\mathbb{P}_1$-module, i.e. $\overline{\mathbb{Q}}$-module.

If we consider the map $d : \mathbb{P} \times \mathbb{P} \to \mathbb{Z}$ by

$$(p_1, p_2) \mapsto \deg(p_1 - p_2).$$

It obviously satisfies

- $d(p_1, p_2) = 0$ if and only if $p_1 = p_2$.
- $d(p_1, p_2) = d(p_2, p_1)$.
- $d(p_1, p_2) \leq d(p_1, p_3) + d(p_3, p_2)$.

So d defines a metric on $\mathbb{P}$. The $\mathbb{P}_k$ is a ball of radius k and center at 0.

4. Some results of periods with low degrees and related problems

Using the decomposition properties of rational functions with one variable, we can get the precise forms of some periods with degrees ≤ 2.

Theorem 4.1. *Let p be a real period with $\deg(p) \leq 2$. If it can be written as $p = \int R(x)dx$ for some rational function $R(x)$. Then it has the form $a\arctan\xi + b\log\eta + c$, where $a, b, c, \xi, \eta \in \overline{\mathbb{Q}}$.*

Proof. Because any rational function can decompose as following four types

$$\frac{A}{x-a}, \frac{A}{(x-a)^n}, \frac{Bx+C}{x^2+bx+c}, \frac{Bx+C}{(x^2+bx+c)^n}$$

where $A, B, C, a, b, c \in \overline{\mathbb{Q}}$ and $n \geq 2$. By elementary integral theory, in every type the integral value has the form $a\arctan\xi + b\log\eta + c$. □

It seems very difficult to determine the degree of a given period. We present following three problems.

Problem 1: Give a concrete period such that the degree ≥ 3.

Problem 2: Let p_1, p_2 be two non-algebraic periods. Does $\deg(p_1 p_2) \geq 2$?

Problem 3: Determine the precise forms of all periods with degrees = 2.

5. Apply the degree to transcendence

In general, determining the transcendence of the sum of two transcendental numbers is a very difficult problem. For example, the transcendence of $e+\pi$ is a longstanding problem in number theory. But if the transcendental numbers are periods. We have some theoretic solutions.

Theorem 5.1. *Let p_1, p_2 be two transcendental periods. If $\deg(p_1) \neq \deg(p_2)$, then both p_1/p_2 and $p_1 + p_2$ are transcendental numbers.*

Proof. If $p_1/p_2 = a$ is algebraic, by Proposition 3.4 one have $\deg(p_1) = \deg(p_2 a) = \deg(p_2)$. Which is a contradiction.

Since $\deg(p_1) \neq \deg(p_2)$, we may assume that $\deg(p_1) < \deg(p_2)$. By Proposition 3.2 one has $\deg(p_2) = \deg(-p_1 + p_1 + p_2) \leq max(\deg(p_1), \deg(p_1 + p_2)) = \deg(p_1 + p_2) \leq max(\deg(p_1), \deg(p_2)) = \deg(p_2)$. So we have $\deg(p_1 + p_2)) = \deg(p_2)$. Hence $p_1 + p_2$ are transcendental. □

More generally, we have following result about linearly independence

Theorem 5.2. *Let p_1, p_2 be any two complex numbers. If $\deg(p_1) \neq \deg(p_2)$, then p_1 and p_2 are linearly independent over $\overline{\mathbb{Q}}$.*

Proof. If one is not a period, the theorem is obviously true. We may assume that both are periods. If p_1 and p_2 are linearly dependent, let $ap_1 + bp_2 = c, a, b, \in \overline{\mathbb{Q}} \setminus 0, c \in \overline{\mathbb{Q}}$. Then $\deg(p_1) = \deg(\frac{c}{a} - \frac{c}{b}p_2) = \deg(p_2)$. Which is a contradiction. □

It is obviously that above results can extend to arbitrary periods. That is, if $1 < \deg(p_1) < \deg(p_2) < ... < \deg(p_k)$, then $p_1 + p_2 + ... + p_k$ is transcendental. If $1 \leq \deg(p_1) < \deg(p_2) < ... < \deg(p_k) \leq \infty$, then $p_1, p_2, ..., p_k$ are linearly independent over $\overline{\mathbb{Q}}$.

It was conjectured in [2] that the basis of the natural logarithms e is not a period. i.e. $\deg(e) = \infty$. This implies that $e+\pi$ is a transcendental number. Using Theorem 5.2 we can improve this as

Corollary 5.3. *To prove that $e + \pi$ is a transcendental number, one only needs to prove that $\deg(e) \geq 3$.*

6. zeta functions of periods

Let p be a period. We consider the following zeta function for p

$$\zeta_p(t) = \exp(\sum_{m=1}^{\infty} \frac{t^m \deg(p^m)}{m}), 0 \leq t < 1.$$

It is the analogue of Weil's zeta function for algebraic variety over finite fields. We find that $\zeta_p(t)$ has some interesting properties.

Theorem 6.1. (1) $\zeta_0(t) = 1$. *If p is a non-zero algebraic number, then $\zeta_p(t) = \frac{1}{1-t}$.*

(2) $\zeta_{p_1 p_2}(t) \leq \zeta_{p_1}(t)\zeta_{p_2}(t)$.

(3) $\zeta_p(t) \leq \exp(\frac{t \deg(p)}{1-t})$.

(4) *If $\deg(p_1) \leq \deg(p_2)$, then $\zeta_{p_1+p_2}(t) \leq \exp(\frac{t \deg(p_2)}{1-t})$.*

Proof. Since $\deg(0) = 0$ and $\deg(p) = 1$ for non-zero algebraic number p. (1) is directly. From Proposition 3.2, we have

$$\begin{aligned}\zeta_{p_1p_2}(t) &= \exp(\sum_{m=1}^{\infty}\frac{t^m\deg(p_1^mp_2^m)}{m})\\ &\leq \exp(\sum_{m=1}^{\infty}\frac{t^m(\deg(p_1^m)+\deg(p_2^m))}{m})\\ &= \zeta_{p_1}(t)\zeta_{p_2}(t),\end{aligned}$$

and

$$\zeta_p(t) \leq \exp(\sum_{m=1}^{\infty}t^m\deg(p)) = \exp(\frac{t\deg(p)}{1-t}).$$

If $\deg(p_1) \leq \deg(p_2)$,

$$\begin{aligned}\zeta_{p_1+p_2}(t) &= \exp(\sum_{m=1}^{\infty}\frac{t^m\deg((p_1+p_2)^m)}{m})\\ &\leq \exp(\sum_{m=1}^{\infty}\frac{t^m\deg(p_1^kp_2^{m-k})}{m})\\ &\leq \exp(\sum_{m=1}^{\infty}\frac{t^m(k\deg(p_1)+(m-k)\deg(p_2))}{m})\\ &\leq \exp(\sum_{m=1}^{\infty}\frac{t^mm\deg(p_2)}{m})\\ &= \exp(\frac{t\deg(p_2)}{1-t}).\end{aligned}$$

In the second step we choose k such that $\deg(p_1^kp_2^{m-k}) = \max\{\deg(p_1^ip_2^{m-i}), 0 \leq i \leq m\}$. □

Problem: Let p be a non-algebraic period. Is $\zeta_p(t)$ a transcendental function?

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