A GEOMETRIC HEAT FLOW FOR VECTOR FIELDS

YI LI AND KEFENG LIU

Abstract. This is an announcement of our work [5] on introducing and studying a geometric heat flow to find Killing vector fields on closed Riemannian manifolds with positive sectional curvature. We study its various properties, prove the global existence of the solution of this flow, discuss its convergence and possible applications, and its relation to the Navier-Stokes equations on manifolds. We will study the similar flow in [6] to find holomorphic vector fields on Kähler manifolds.

Contents

1. A geometric heat flow for vector fields

Recently, we have witnessed the power of geometric flows in studying lots of problems in geometry and topology. In this announcement we introduce a geometric heat flow for vector fields on a Riemannian manifold and study its varies properties. The detail paper [5] will appear soon.

Throughout this paper, we adopt the Einstein summation and notions as those in [2]. All manifolds and vector fields are smooth; a

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manifold is said to be *closed* if it is compact and without boundary. We shall often raise and lower indices for tensor fields.

1.1. **Deformation tensor field of a vector field.** Let (M, g) be a closed and orientable Riemannian manifold. To a vector field X we associate its deformation $(0, 2)$-tensor field $\mathbf{Def}(X)$, which is an obstruction of X to be Killing and is locally defined by

$$(\mathbf{Def}(X))_{ij} \doteqdot \frac{\nabla_i X_j + \nabla_j X_i}{2}, \tag{1.1}$$

where ∇ denotes the Levi-Civita connection of g. Equivalently, it is exactly (up to a constant factor) the Lie derivative of g along the vector field X, i.e., $\mathcal{L}_X g$. We say that X is a *Killing vector field* if $\mathbf{Def}(X) = 0$. Consider the L^2-norm of $\mathbf{Def}(X)$:

$$\mathfrak{L}(X) \doteqdot \int_M |\mathbf{Def}(X)|^2 dV, \tag{1.2}$$

where dV stands for the volume form of g and $|\cdot|$ means the norm of $\mathbf{Def}(X)$ with respect to g. Clearly that the critical point X of $\mathfrak{L}$ satisfies

$$\Delta_{\mathrm{LB}} X^i + \nabla^i \mathrm{div}(X) + R^i{}_j X^j = 0. \tag{1.3}$$

Here and henceforth, $\Delta_{\mathrm{LB}} \doteqdot g^{ij}\nabla_i\nabla_j$ is the Laplace-Beltrami operator of g and R_{ij} denotes the Ricci curvature of g.

1.2. **A geometric heat flow for vector fields.** Motivated by (1.3), we introduce a geometric heat flow for vector fields:

$$\partial_t (X_t)^i = \Delta_{\mathrm{LB}}(X_t)^i + \nabla^i \mathrm{div}(X_t) + R^i{}_j (X_t)^j, \quad X_0 = X, \tag{1.4}$$

where X is a fixed vector field on M and $\partial_t \doteqdot \frac{\partial}{\partial t}$ is the time derivative. If we define $\mathrm{Ric}^\sharp$, the $(1, 1)$-tensor field associated to Ric, by

$$g\left(\mathrm{Ric}^\sharp(X), Y\right) \doteqdot \mathrm{Ric}(X, Y),$$

where X, Y are two vector fields, then $\mathrm{Ric}^\sharp$ is an operator on the space of vector fields, denoted by $C^\infty(M, TM)$, and the flow (1.4) can be rewritten as

$$\partial_t X_t = \Delta_{\mathrm{LB}} X_t + \nabla \mathrm{div}(X_t) + \mathrm{Ric}^\sharp(X_t). \tag{1.5}$$

In 1952, Yano (e.g., [12, 13, 14]) showed that a vector field $X = X^i \frac{\partial}{\partial x^i}$ is a Killing vector field if and only if it satisfies

$$\Delta_{\mathrm{LB}} X^i + R^i{}_j X^j = 0, \quad \mathrm{div}(X) = 0. \tag{1.6}$$

His result depends on an integral formula, now called *Yano's integral formula*,

$$0 = \int_M \left[\mathrm{Ric}(X,X) - |\nabla X|^2 + 2|\mathbf{Def}(X)|^2 - |\mathrm{div}(X)|^2\right] dV, \tag{1.7}$$

which holds for any vector field X. This integral formula lets us define so-called the *Bochner-Yano integral* for every vector field X:

$$\mathcal{E}(X) \doteqdot \int_M \left[|\nabla X|^2 + |\mathrm{div}(X)|^2 - \mathrm{Ric}(X,X)\right] dV. \tag{1.8}$$

Consequently, Yano's integral formula implies that $\mathcal{E}(X)$ is always nonnegative and $\mathcal{E}(X) = 2\mathfrak{L}(X)$ for every vector field X. On the other hand, Watanabe [10] proved that X is a Killing vector field if and only if $\mathcal{E}(X) = 0$, and hence if and only if $\mathfrak{L}(X) = 0$.

Yano's equations (1.6) induces a system of equations, called the *Bochner-Yano flow*:

$$\partial_t (X_t)^i = \Delta_{\mathrm{LB}}(X_t)^i + R^i{}_j (X_t)^j, \quad \mathrm{div}(X_t) = 0. \tag{1.9}$$

Notice that Yano's equation (1.6) (resp., Bochner-Yano flow (1.9)) is a special case of our equation (1.3) (resp., our flow (1.4)).

1.3. **Main results.**

Now we can state our main results to the flow (1.4).

Theorem 1.1. (Long-time existence) *Suppose that (M, g) is a closed and orientable Riemannian manifold. Given an initial vector field, the flow (1.4) exists for all time.*

The main method on proving above theorem is the standard approach in PDEs and an application of Sobolev embedding theorem. After establishing the long-time existence, we can study the convergence problem of the flow (1.4).

Theorem 1.2. (Convergence) *Suppose that (M, g) is a closed and orientable Riemannian manifold. If X is a vector field, there exists a unique smooth solution X_t to the flow (1.4) for all time t. As t goes to infinity, the vector field X_t converges uniformly to a Killing vector field X_∞.*

To prove Theorem 1.2, we need some basic properties of the functional $\mathcal{E}$:

$$\begin{aligned}
\mathcal{E}(X_t) &\geq 0, \\
\frac{d}{dt}\mathcal{E}(X_t) &= -2\int_M |\partial_t X_t|^2 dV \quad \leq \quad 0, \\
\mathcal{E}(X_t) &= -\frac{d}{dt}\left(\frac{1}{2}\int_M |X_t|^2 dV\right).
\end{aligned}$$

The first one directly follows from (1.7). Since the flow (1.4) is the gradient flow of the functional $\mathcal{E}$, the rest are clearly. Those properties show that

$$\int_M |\nabla X_t|^2 \, dV \leq \mathcal{E}(X) + \max_{1\leq i,j\leq m}\left(\max_M R_{ij}\right) \cdot \int_M |X|^2 dV,$$

therefore, $\nabla X_t \in L^2(M,g)$. On the other hand $\int_M |X_t|^2 \leq \int_M |X|^2$, we conclude that $||X_t||_{H^1(M,g)} \leq C_1$ for some positive constant C_1 depending only on (M,g) and X. By the regularity of parabolic equations and the flow itself, we obtain $||\partial_t X_t||_{H^\ell(M,g)} \leq C_\ell$ for a positive constant C_ℓ that depends only on (M,g), X, and ℓ, for each integer ℓ. From

$$\int_0^\infty \int_M |\partial_t X_t|^2 dV \leq \frac{1}{2}\mathcal{E}(X),$$

we can find a sequence $\{t_i\}$ of time so that $||\partial_t X_t|_{t=t_i}||_{L^2(M,g)} \to 0$. By Sobolev embedding theorem and the regularity theorem, there exists a smooth vector field X_∞ satisfying

$$\Delta_{\mathrm{LB}} X_\infty + \nabla \mathrm{div}(X_\infty) + \mathrm{Ric}^\sharp(X_\infty) = 0.$$

Remark 1.3. *As Professor Cliff Taubes remarked that Theorem 1.1 and 1.2 also follow from an eigenfunction expansion for the relevant linear operator that defines the flow (1.4), which gives a short proof of those two theorems.*

Theorem 1.2 does *not* give us a nontrivial Killing vector field. For example, if X is identically zero, then by the uniqueness theorem the limit vector field is also identically zero. When the Ricci curvature is negative, Bochner's theorem implies that there is no nontrivial Killing vector field.

To obtain a nonzero Killing vector field, we have the following criterion.

Proposition 1.4. *Suppose that (M,g) is a closed and orientable Riemannian manifold and X is a vector field on M. If X_t is the solution* *of the flow (1.4) with the initial value X, then*

$$\int_0^\infty \mathcal{E}(X_t)dV < \infty. \tag{1.10}$$

Let

$$\mathbf{Err}(X) \doteqdot \frac{1}{2}\int_M |X|^2 dV - \int_0^\infty \mathcal{E}(X_t)dt. \tag{1.11}$$

Therefore $\mathbf{Err}(X) \geq 0$ *and* X_∞ *is nonzero if and only if* $\mathbf{Err}(X) > 0$.

For the moment, we introduce the following notions:

$$\begin{aligned} u_k(t) &\doteqdot \int_M \left|\nabla^k X_t\right|^2 dV, \\ v_k(t) &\doteqdot \int_M \left|\nabla^k \mathrm{div}(X_t)\right|^2 dV. \end{aligned}$$

Here k is a nonnegative integer. An interesting result in this paper is the following estimates:

Theorem 1.5. *Given integers $k \geq 0$ and $\ell \geq 2$. If X_t is a solution of the flow (1.4), then*

$$\left|u_k^{(\ell)}(t)\right| \leq \sum_{i=0}^{k+2\ell} A_{k,i}^{(\ell)} u_i(t) + \sum_{i=0}^{k+2\ell} B_{k,i}^{(\ell)} v_i(t), \tag{1.12}$$

$$\left|v_k^{(\ell)}(t)\right| \leq \sum_{i=0}^{k+2\ell-1} C_{k,i}^{(\ell)} v_i(t) + \sum_{i=0}^{k+2\ell-1} D_{k,i}^{(\ell)} u_i(t) \tag{1.13}$$

for some uniform positive constants $A_{k,i}^{(\ell)}, B_{k,i}^{(\ell)}, C_{k,i}^{(\ell)}$, and $D_{k,i}^{(\ell)}$, depending only on $m, (M,g), i, k$, and ℓ. Here $u_k^{(\ell)}(t)$ and $v_k^{(\ell)}(t)$ denote the ℓ-th derivative of $u_k(t)$ and $v_k(t)$, respectively. For $\ell = 1$, we have more precise estimates:

$$\begin{aligned} |u_k'(t)| &\leq \sum_{i=0}^{k+1} A_{k,i}^{(1)} u_i(t) + \sum_{i=0}^{k+1} B_{k,i}^{(1)} v_i(t), \\ |v_k'(t)| &\leq \sum_{i=0}^{k+1} C_{k,i}^{(1)} v_i(t) + \sum_{i=0}^{k+1} D_{k,i}^{(1)} u_i(t). \end{aligned}$$

Proof. (Sketch) From the flow (1.4), we can prove

$$\begin{aligned}
|u_k'(t)| &\leq 2u_{k+1}(t) + C_k \int_M \left|\nabla^k X_t\right| \left|\nabla^{k+1}\mathrm{div}(X_t)\right| dV \\
&\quad + \sum_{i=0}^{k} C_{k-i}^{(1)} \int_M \left|\nabla^{k-i} X_t\right| \left|\nabla^k X_t\right| dV, \\
|v_k'(t)| &\leq 4v_{k+1}(t) + \sum_{i=0}^{k} C_i^{(2)} \int_M \left|\nabla^{k-i}\mathrm{div}(X_t)\right| \left|\nabla^k \mathrm{div}(X_t)\right| dV \\
&\quad + \sum_{i=0}^{k+1} C_i^{(3)} \int_M \left|\nabla^{k+1-i} X_t\right| \left|\nabla^k \mathrm{div}(X_t)\right| dV,
\end{aligned}$$

for some positive constants $C_k, C_i^{(1)}, C_i^{(2)}$, and $C_i^{(3)}$, depending only on (M,g) and the related indices. Using the notation of $u_k(t)$ and $v_k(t)$, and the Cauchy-Schwarz inequality, we can prove the theorem for $\ell = 1$. Similarly, we can treat with the higher derivatives, but, here, we need a complicated and careful analysis. $\square$

1.4. **The normalization solutions.** When the initial vector field is nonzero, a standard method in PDEs shows that

Proposition 1.6. *Suppose that (M,g) is a closed and orientable Riemannian manifold and X_t solves (1.4) with the initial value X. If X is nonzero, then for any finite t, X_t is nonzero.*

By this proposition, we can normalize X_t to define

$$Y_t \doteqdot \frac{X_t}{[u(t)]^{1/2}}, \quad u(t) \doteqdot u_0(t) = \int_M |X_t|^2 dV. \tag{1.14}$$

Clearly that $\int_M |Y_t|^2 dV = 1$ and Y_t satisfies

$$\partial_t (Y_t)^i = \Delta_{\mathrm{LB}}(Y_t)^i + \nabla^i \mathrm{div}(Y_t) + R^i{}_j (Y_t)^j + \mathcal{E}(Y_t)(Y_t)^i. \tag{1.15}$$

Furthermore, we can prove

Theorem 1.7. *Suppose that (M,g) is a closed and orientable Riemannian manifold. If X_t is the solution of the flow (1.4) with a nonzero initial value X, then the vector field Y_t, defined by (1.14), converges uniformly to a nonzero vector field Y_∞ satisfying $\int_M |Y_\infty|^2 dV = 1$.*

The above theorem tells us even X_t goes to zero, the quotient $X_t/u^{1/2}(t)$ goes to a nonzero vector field. Formally, we have $X_\infty = u_\infty^{1/2} Y_\infty$.

1.5. **A connection to the Navier-Stokes equations.** A surprising observation is that our flow (1.4) is very close to the Navier-Stokes equations [1, 9](without the pressure) on manifolds

$$\partial_t X_t + \nabla_{X_t} X_t = \mathrm{div}(S_t), \quad \mathrm{div}(X_t) = 0, \tag{1.16}$$

where $S_t \doteqdot 2\mathbf{Def}(X_t)$ is the stress tensor of X_t. By an easy computation we can write (1.16) as

$$\partial_t (X_t)^i + (\nabla_{X_t} X_t)^i = \Delta_{\mathrm{LB}}(X_t)^i + \nabla^i \mathrm{div}(X_t) + R^i{}_j X^j, \quad \mathrm{div}(X_t) = 0. \tag{1.17}$$

Compared (1.4) with (1.17), we give a geometric interpolation of the right (or the linear) part of the Navier-Stokes equations on manifolds.

When the Ricci tensor field is identically zero, our flow (1.4) keeps the property that $\mathrm{div}(X_t) = 0$. Indeed, we can prove the following formula:

$$\begin{aligned} \frac{d}{dt}\left(\int_M |\mathrm{div}(X_t)|^2 dV\right) &= -4\int_M |\nabla \mathrm{div}(X_t)|^2 \, dV \\ &\quad - 4\int_M \mathrm{Ric}\left(X_t, \nabla \mathrm{div}(X_t)\right) dV, \end{aligned}$$

from which we see that if $\mathrm{Ric} \equiv 0$ then

$$\frac{d}{dt}\left(\int_M |\mathrm{div}(X_t)|^2 dV\right) = -4\int_M |\nabla \mathrm{div}(X_t)|^2 \, dV \leq 0.$$

Thus the flow (1.4) keeps $\int_M \mathrm{div}|(X_t)|^2 dV = 0$ and hence keeps $\mathrm{div}(X_t) = 0$.

As a consequence of the non-negativity of $\mathcal{E}$ we can prove that

Theorem 1.8. *Suppose that (M, g) is a closed and orientable Riemannian manifold. If X_t is a solution of the Navier-Stokes equations (1.17), then*

$$\frac{d}{dt}\left(\int_M |X_t|^2 dV\right) = -2\mathcal{E}(X_t) \leq 0. \tag{1.18}$$

In particular

$$\int_M |X_t|^2 dV \leq \int_M |X_0|^2 dV. \tag{1.19}$$

The similar result was considered by Wilson [11] for the standard metric on $\mathbb{R}^3$.

2. A conjecture to the flow and its application

Before stating a conjecture to the flow (1.4), we shall look at a simple case that (M, g) is an Einstein manifold with positive sectional curvature and the solution of (1.4) is the sum of the initial vector field and a gradient vector field. That is, we assume

$$R_{ij} = \frac{R}{m} g_{ij}, \quad m \geq 3, \quad X_t = X + \nabla f_t,$$

where f_t are some functions on M. By a theorem of Schur, the scalar curvature R must be a constant. In this case the flow (1.4) is equivalent to

$$\nabla\left(\partial_t f_t - 2\Delta_{\mathrm{LB}} f_t - \frac{2R}{m} f_t\right) = X^\dagger, \tag{2.1}$$

where

$$X^\dagger \doteqdot \Delta_{\mathrm{LB}} X + \nabla(\mathrm{div}(X)) + \mathrm{Ric}^\sharp(X) \tag{2.2}$$

is the vector field associated to X. Clearly that the operator $\dagger$ is not self-adjoint on the space of vector fields, with respect to the L^2-inner product with respect to (M, g).

2.1. Einstein manifolds with positive scalar curvature.

If (M, g) is an m-dimensional Einstein manifold with positive scalar curvature, then we can prove that the limit vector field converges to a nonzero Killing vector field, provided the initial vector field satisfying some conditions. We first give a L^2-estimate for f_t.

Proposition 2.1. *Suppose that (M, g) is an m-dimensional closed and orientable Einstein manifold with positive scalar curvature R, where $m \geq 3$. Let X be a nonzero vector field satisfying $X^\dagger = \nabla\varphi_X$ for some smooth function φ_X on M. Then for any given constant c, the equation*

$$\partial_t f_t = 2\Delta_{\mathrm{LB}} f_t + \frac{2R}{m} f_t + \varphi_X, \quad f_0 = c, \tag{2.3}$$

exists for all time. Moreover,

(i) *we have*

$$\int_M f_t dV = \left[c \cdot \mathrm{Vol}(M, g) + \frac{m}{2R}\int_M \varphi_X dV\right] e^{\frac{2R}{m}t} - \frac{m}{2R}\int_M \varphi_X dV. \tag{2.4}$$

Setting

$$c_X \doteqdot -\frac{m}{2R \cdot \mathrm{Vol}(M, g)}\int_M \varphi_X dV,$$

yields

$$\int_M f_t dV = -\frac{m}{2R}\int_M \varphi_X dV, \quad \textit{if } c = c_X.$$

(ii) *the L^2-norm of f_t is bounded by*

$$||f_t||_2 \le \frac{||\varphi_X||_2}{2\left(\lambda_1 - \frac{R}{m}\right)} + \left[c \cdot \mathrm{Vol}(M,g) - \frac{||\varphi_X||_2}{2\left(\lambda_1 - \frac{R}{m}\right)}\right] e^{-2\left(\lambda_1 - \frac{R}{m}\right)t}, \tag{2.5}$$

where $||\cdot||_2$ means $||\cdot||_{L^2(M,g)}$ the L^2-norm with respect to (M,g), and λ_1 stands for the first nonzero eigenvalue of $9M,g)$.

For a moment, we put

$$a(t) \doteqdot \int_M f_t dV, \quad b(t) \doteqdot \int_M |f_t|^2 dV.$$

Then, the equation (2.3) implies that

$$a'(t) = \frac{2R}{m} a(t) + \int_M \varphi_X dV,$$

and

$$\begin{aligned} b'(t) &= -4\int_M |\nabla f_t|^2 dV + \frac{4R}{m} b(t) + 2\int_M f_t \varphi_X dV \\ &\le -4\left(\lambda_1 - \frac{R}{m}\right) b(t) + 2b^{1/2}(t)||\varphi_X||_{L^2(M,g)}. \end{aligned}$$

By a theorem of Lichnerowicz, we have that $\lambda_1 \ge \frac{R}{m-1} > \frac{R}{m}$. Hence (2.3) and (2.4) follow immediately.

Consequently, we have the following

Theorem 2.2. *Suppose that (M,g) is an m-dimensional closed and orientable Einstein manifold with positive scalar curvature R, where $m \ge 3$. Let X be a nonzero vector field satisfying the following two conditions:*

(i) *$X^\dagger$ is a gradient vector field, and*
(ii) *X is not a gradient vector field.*

Then the flow (1.4) with initial value X converges uniformly to a nonzero Killing vector field.

2.2. **A conjecture and its applications.** By Bochner's theorem, any Killing vector field on a closed and orientable Riemmanian manifold with negative Ricci curvature is trivial. Hence, based on a result in the Einstein case, we propose the following conjecture.

Conjecture 2.3. *Suppose that M is a closed Riemannian manifold with positive sectional curvature. For some initial vector field and a certain Riemannian metric g of positive sectional curvature, the flow (1.4) converges uniformly to a nonzero Killing vector field with respect to g.*

Our study shows that we may need to change to a new metric, which still has positive sectional curvature, to get the nonzero limit which is a Killing vector field with respect to this new metric. For this purpose we have computed variations of the functional $\mathfrak{L}$ or $\mathcal{E}$ relative to the new metric, as well as the Perelman-type functional for our flow.

Obviously a solution of this conjecture immediately answers the following long-standing question of Yau [8].

Question 2.4. *Does there exist an effective $\mathbb{S}^1$-action on a closed manifold with positive sectional curvature?*

Assuming Conjecture 2.3, we can deduce several important corollaries. We first recall the well-known Hopf's conjectures.

Conjecture 2.5. *If M is a closed and even dimensional Riemannian manifold with positive sectional curvature, then the Euler characteristic number of M is positive, i.e., $\chi(M) > 0$.*

Conjecture 2.6. *On $\mathbb{S}^2 \times \mathbb{S}^2$ there is no Riemannian metric with positive sectional curvature.*

For the recent development of Hopf's conjectures, we refer to [7, 8]. A simple argument shows that Conjecture 2.5 and 2.6 follow from Conjecture 2.3.

Corollary 2.7. *Conjecture 2.3 implies Conjecture 2.5.*

Proof. From [4] we know that the Killing vector field X must have zero, and the zero sets consist of finite number of totally geodesic submanifolds $\{M_i\}$ of M with the induced Riemannian metrics. Moreover each M_i is even dimensional and has positive sectional curvature. Hence we have $\chi(M) = \sum_i \chi(M_i)$. By induction, we obtain $\chi(M) > 0$. □

Hsiang and Kleiner [3] showed that if M is a 4-dimensional closed Riemannian manifold with positive sectional curvature, admitting a nonzero Killing vector field, then M is homeomorphic to $\mathbb{S}^4$ or $\mathbb{CP}^2$. Consequently, $\mathbb{S}^2 \times \mathbb{S}^2$ does not admit a Riemannian metric, whose sectional curvature is positive, with a nontrivial Killing vector field. Therefore

Corollary 2.8. *Conjecture 2.3 implies Conjecture 2.6.*

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Department of Mathematics, Harvard University, One Oxford street, Cambridge, MA 02138

E-mail address: `yili@math.harvard.edu`

Department of Mathematics, UCLA, Los Angeles, CA 90095-1555; Center of Mathematical Science, Zhejiang University, Hangzhou, China 310027

E-mail address: `liu@math.ucla.edu`