SOME REMARKS ON CIRCLE ACTION ON MANIFOLDS

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Abstract. This paper contains several results concerning circle action on almost-complex and smooth manifolds. More precisely, we show that, for an almost-complex manifold M^{2mn}(resp. a smooth manifold N^{4mn}), if there exists a partition $\lambda = (\lambda_1, \cdots, \lambda_u)$ of weight m such that the Chern number $(c_{\lambda_1} \cdots c_{\lambda_u})^n[M]$ (resp. Pontrjagin number $(p_{\lambda_1} \cdots p_{\lambda_u})^n[N]$) is nonzero, then *any* circle action on M^{2mn} (resp. N^{4mn}) has at least $n+1$ fixed points. When an even-dimensional smooth manifold N^{2n} admits a semi-free action with isolated fixed points, we show that N^{2n} bounds, which generalizes a well-known fact in the free case. We also provide a topological obstruction, in terms of the first Chern class, to the existence of semi-free circle action with *nonempty* isolated fixed points on almost-complex manifolds. The main ingredients of our proofs are Bott's residue formula and rigidity theorem.

1. Introduction and main results

Unless otherwise stated, all the manifolds (smooth or almost-complex) mentioned in the paper are closed, connected and oriented. For almost-complex manifolds, we take the canonical orientations induced from the almost-complex structures. We denote by superscripts the corresponding *real* dimensions of such manifolds. When M is a smooth (resp. almost-complex) manifold, we say M has an S^1-action if M admits a circle action which preserves the smooth (resp. almost-complex) structure.

Given a manifold M and an S^1-action, the study of the fixed point set M^{S^1} is an important topic in geometry and topology. In ([10], p.338), Kosniowski proposed the following conjecture, which relates the number of fixed points to the dimension of the manifold.

Conjecture 1.1 (Kosniowski)**.** Suppose that M^{2n} is a unitary S^1-manifold with isolated fixed points. If M is not a boundary then this action has at least $[\frac{n}{2}]+1$ fixed points.

Remark 1.2. A unitary S^1-manifold means that M has a weakly almost-complex structure and S^1 acts on M preserving this structure.

Recently, Pelayo and Tolman showed that ([15], Theorem 1), if a symplectic manifold (M^{2n}, ω) has an symplectic S^1-action and the weights induced from the isotropy representations on the fixed points satisfy some subtle condition, then this action has at least $n+1$ fixed points.

Remark 1.3. If the action is Hamiltonian, this result is well-known. The reason is that the fixed points are exactly the critical points of the corresponding momentum map (a perfect Morse-Bott function) and the even-dimensional Betti numbers of M are all positive. The conclusion then follows from the Morse inequality.

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Inspired by the techniques from [15], we will show the following theorem in Section 3, which is our first main result.

Theorem 1.4. (1) *Suppose M^{2mn} is an almost-complex manifold. If there exists a partition $\lambda = (\lambda_1, \cdots, \lambda_u)$ of weight m such that the corresponding Chern number $(c_{\lambda_1} \cdots c_{\lambda_u})^n[M]$ is nonzero, then for* any *S^1-action on M, it must have at least $n+1$ fixed points.*

(2) *Suppose N^{4mn} is a smooth manifold. If there exists a partition $\lambda = (\lambda_1, \cdots, \lambda_u)$ of weight m such that the corresponding Pontrjagin number $(p_{\lambda_1} \cdots p_{\lambda_u})^n[M]$ is nonzero, then for* any *S^1-action on N, it must have at least $n+1$ fixed points.*

Corollary 1.5. (1) If the Chern number $c_m^n[M]$ is nonzero, then for any S^1-action on almost-complex manifold M^{2mn}, it has at least $n+1$ fixed points. In particular, if $c_1^n[M]$ is nonzero, then any S^1-action on almost-complex manifold M^{2n} has at least $n+1$ fixed points.

(2) If the Pontrjagin number $p_m^n[N]$ is nonzero, then for any S^1-action on smooth manifold N^{4mn}, it has at least $n+1$ fixed points. In particular, if $p_1^n[N]$ is nonzero, then any S^1-action on N^{4n} has at least $n+1$ fixed points.

It is a well-known fact that, if a smooth manifold N^n has a free S^1-action, then N^n bounds, i.e., N^n can be realized as the oriented boundary of some smooth, oriented, $(n+1)$-dimensional manifold with boundary. Using the language of cobordism theory, $[N^n] = 0 \in \Omega_*^{SO}$, where Ω_*^{SO} is the oriented cobordism ring. In particular, all the Pontrjagin numbers and Stiefel-Whitney numbers vanish. This well-known fact is not difficult to prove: N^n is the total space of the principal S^1-bundle over the quotient manifold N^n/S^1, of which the structure group is $S^1 = SO(2)$. Then we can extend the action of $SO(2)$ to the 2-disk D^2 to get the associated D^2-bundle $N^n \times_{SO(2)} D^2$, of which the boundary is exactly N^n.

We recall that a circle action is called *semi-free* if it is free outside the fixed point set or equivalently, the isotropic subgroup of any point is either trivial or the whole circle. In ([14], Theorem 1.1), the authors showed that the Pontrjagin numbers of manifolds admitting a semi-free action with isolated fixed points are all zero. Our following result is a generalization of both the well-known fact mentioned above and ([14], Theorem 1.1).

Theorem 1.6. *If an even-dimensional smooth manifold N^{2n} admits a semi-free S^1-action with isolated fixed points, then N^{2n} bounds. In particular, all the Pontrjagin numbers and Stiefel-Whitney numbers vanish.*

This result tells us that Pontrjagin numbers and Stiefel-Whitney numbers are numerical obstructions to the existence of semi-free actions with isolated fixed points on smooth manifolds. However, in contrast to the smooth case, when an almost-complex manifold (M^{2n}, J) has a semi-free S^1-action with *nonempty* isolated fixed points, the Chern numbers of (M^{2n}, J) don't vanish (see Lemma 3.5). One may ask, in the almost-complex case, whether there still exist some topological obstructions to the existence of semi-free S^1-actions. We know that the first Chern class plays an important role in complex (almost-complex) manifolds. The following result provides such an obstruction to the existence of semi-free S^1-action on almost-complex manifolds.

Theorem 1.7. *Let (M^{2n}, J) be an almost-complex manifold admitting a semi-free S^1-action with nonempty isolated fixed points. Then the first Chern class $c_1(M) \in H^2(M; \mathbb{Z})$ is either primitive or twice a primitive element.*

Remark 1.8. For almost-complex manifolds, in the semi-free case with isolated fixed points, the only known examples are $(\mathbb{C}P)^n$, equipped with the diagonal S^1-action. Note that these examples are even Hamiltonian S^1-actions on symplectic manifolds. In fact, Hattori [5] showed that, if a symplectic manifold (M^{2n}, ω) admits a Hamiltonian, semi-free S^1-action with isolated fixed points, then the cohomology ring and the Chern classes of (M^{2n}, ω) are the same as $(\mathbb{C}P^1)^n$. While in the almost-complex case, much less is known. It would be very interesting to find some more topological obstructions.

In Section 2, we will review the Bott's residue formula and prove a rigidity proposition. Then in Section 3, the three subsections will be devoted to the proofs of Theorems 1.4, 1.6 and 1.7 respectively.

2. Preliminaries

2.1. **Bott's residue formula.**

2.1.1. *almost-complex case.* Let (M^{2n}, J) be an almost-complex manifold with a circle action with isolated fixed points, say $\{P_1, \cdots, P_r\}$. In each fixed point P_i, there are well-defined n integer weights $k_1^{(i)}, \cdots, k_n^{(i)}$ (not necessarily distinct) induced from the isotropy representation of this S^1-action on the holomorphic tangent space $T_{p_i}M$ in the sense of J. Note that these $k_1^{(i)}, \cdots, k_n^{(i)}$ are *nonzero* as the fixed points are isolated. Let $f(x_1, \cdots, x_n)$ be a symmetric polynomial in the variables $x_1, \cdots, x_n$. Then $f(x_1, \cdots, x_n)$ can be written in an essentially unique way in terms of the elementary symmetric polynomials $\widetilde{f}(e_1, \cdots, e_n)$, where $e_i = e_i(x_1, \cdots, x_n)$ is the i-th elementary symmetric polynomial of $x_1, \cdots, x_n$.

Now we can state a version of the Bott residue formula $\big($cf. [4] or ([2], p.598)$\big)$ which reduces the computations of Chern numbers of (M^{2n}, J) to $\{k_j^{(i)}\}$, as follows.

Theorem 2.1 (Bott residue formula)**.** *With above notations understood and moreover suppose the degree of $f(x_1, \cdots, x_n)$ is not greater than n ($deg(x_i) = 1$). Then*

$$\sum_{i=1}^{r} \frac{f(k_1^{(i)}, \cdots, k_n^{(i)})}{\prod_{j=1}^{n} k_j^{(i)}} = \widetilde{f}(c_1, \cdots, c_n) \cdot [M], \tag{2.1}$$

where c_i is the i-th Chern class of (M^{2n}, J) and $[M]$ is the fundamental class of M induced from J.

Remark 2.2. If the degree of $f(x_1, \cdots, x_n)$ is less than n, then the left-hand side of (2.1) vanishes. If the action has no fixed points (though not necessarily free) it follows that all Chern numbers are zero.

2.1.2. *smooth case.* Let N^{2n} be a smooth manifold with a circle action with isolated fixed points, say $\{P_1, \cdots, P_r\}$. In each fixed point P_i, the tangent space $T_{p_i}N$ splits as an S^1-module induced from the isotropy representation as follows

$$T_{p_i}N = \bigoplus_{j=1}^{n} V_j^{(i)},$$

where each $V_j^{(i)}$ is a real 2-plane. We choose an isomorphism of $\mathbb{C}$ with $V_j^{(i)}$ relative to which the representation of S^1 on $V_j^{(i)}$ is given by $e^{\sqrt{-1}\theta} \mapsto e^{\sqrt{-1}k_j^{(i)}\theta}$ with $k_j^{(i)} \in \mathbb{Z} - \{0\}$. We can

assume the rotation numbers $k_1^{(i)}, \cdots, k_n^{(i)}$ be chosen in such a way that the usual orientations on the summands $V_j^{(i)} \cong \mathbb{C}$ induce the given orientation on $T_{p_i}N$. Note that these $k_1^{(i)}, \cdots, k_n^{(i)}$ are uniquely defined up to even number of sign changes. In particular, their product $\prod_{j=1}^n k_j^{(i)}$ is well-defined.

Let $f(x_1^2, \cdots, x_n^2)$ be a symmetric polynomial in the variables $x_1^2, \cdots, x_n^2$. Let $\sigma_i = \sigma_i(x_1^2, \cdots, x_n^2)$ be the i-th elementary symmetric polynomial in the variables $x_1^2, \cdots, x_n^2$. Then $f(x_1^2, \cdots, x_n^2)$ can be written in an essentially unique way in terms of $\sigma_1, \cdots, \sigma_n$, say $\widetilde{f}(\sigma_1, \cdots, \sigma_n)$. Then we have

Theorem 2.3 (Bott residue formula)**.** *With above notations understood and moreover suppose the degree of $f(x_1^2, \cdots, x_n^2)$ is not greater than n ($deg(x_i) = 1$). Then*

$$\sum_{i=1}^{r} \frac{f((k_1^{(i)})^2, \cdots, (k_n^{(i)})^2)}{\prod_{j=1}^n k_j^{(i)}} = \widetilde{f}(p_1, \cdots, p_n) \cdot [N], \tag{2.2}$$

where p_i is the i-th Pontrjagin class of N and $[N]$ is the fundamental class of N determined by the orientation.

Remark 2.4. Since $\deg(f(x_1^2, \cdots, x_n^2)) \leq n$, what possible appear in $\widetilde{f}(p_1, \cdots, p_n)$ are $p_1, \cdots, p_{[\frac{n}{2}]}$. $\widetilde{f}(p_1, \cdots, p_n) \cdot [N]$ is nonzero only if n is even. If $\deg(f(x_1^2, \cdots, x_n^2)) < n$, then the left-hand side of (2.2) vanishes. If the action has no fixed points (though not necessarily free), it follows that all Pontrjagin numbers are zero.

2.2. **A rigidity result.** In this subsection we want to prove a special rigidity result for circle actions on almost-complex manifolds with isolated fixed points. For more details on the rigidity of elliptic complexes, see [12] and [13].

Let (M^{2n}, J) be an almost-complex manifold with first Chern class $c_1 \in H^2(M; \mathbb{Z})$ divisible by a positive integer $d > 1$. Then there exists a holomorphic line bundle L over M such that $L^{\otimes d} = K$, where K is the canonical line bundle, all in the sense of J. We consider the holomorphic Euler number of M with coefficients in the line bundle L, $\chi(M, L)$, which is a genus ([6]) with respect to the characteristic power series

$$\frac{x}{1 - e^{-x}} \cdot e^{-\frac{x}{d}}. \tag{2.3}$$

$\chi(M, L)$ is the index of the Todd operator twisted by L.

Now suppose we have an S^1-action on (M^{2n}, J). Consider the d-fold covering $S^1 \to S^1$ with $\mu \mapsto \lambda = \mu^d$. Then μ acts on M and K through λ. This action can be lifted to L. Then for any $g \in S^1$, we can define the equivariant index $\chi(g; M, L)$, which is a finite Laurent series in g.

Now suppose this circle action on M has isolated fixed points. Using the notations in Section 2.1.1, we have

Proposition 2.5. Suppose the first Chern class of M is divisible by $d > 1$. Then the rational function

$$\sum_{i=1}^{r} \frac{g^{\frac{\sum_{j=1}^n k_j^{(i)}}{d}}}{\prod_{j=1}^n (1 - g^{k_j^{(i)}})}$$

is identically equal to 0, where g is an indeterminate.

Proof. Suppose $g \in S^1$ is a topological generator. Then the fixed points of the action g are exactly $\{P_1, \cdots, P_r\}$. Note that the characteristic power series corresponding to $\chi(M, L)$ is (2.3), then the Lefschetz fixed point formula of Atiyah-Bott-Segal-Singer ([2], p.562) tells us that

$$\chi(g; M, L) = \sum_{i=1}^{r} \prod_{j=1}^{n} \frac{g^{\frac{k_j^{(i)}}{d}}}{1 - g^{k_j^{(i)}}}.$$

The rigidity result of almost-complex manifolds on the level of d (cf. p.43 and p.58 of [7] or appendix III of [8]) tells us that, for any topological generator $g \in S^1$,

$$\chi(g; M, L) \equiv \chi(M, L).$$

Since the topological generators in S^1 are dense, we have an identity

$$\chi(M, L) \equiv \sum_{i=1}^{r} \prod_{j=1}^{n} \frac{g^{\frac{k_j^{(i)}}{d}}}{1 - g^{k_j^{(i)}}}$$

for any indeterminate g.

For any $k_j^{(i)} \in \mathbb{Z} - \{0\}$, we have

$$\lim_{g \to \infty} \frac{g^{\frac{k_j^{(i)}}{d}}}{1 - g^{k_j^{(i)}}} = 0.$$

Therefore,

$$\sum_{i=1}^{r} \prod_{j=1}^{n} \frac{g^{\frac{k_j^{(i)}}{d}}}{1 - g^{k_j^{(i)}}} \equiv 0$$

for any indeterminate g, which completes the proof. □

3. Proof of main results

3.1. **Proof of Theorem 1.4.** Now suppose (M^{2mn}, J) is an almost-complex manifold with some Chern number $(c_{\lambda_1} \cdots c_{\lambda_u})^n[M] \neq 0$. Note that any S^1-action on M must have at least one fixed point, otherwise all the Chern numbers of M vanish by Remark 2.2. If the fixed point set of the S^1-action is not isolated, then at least one connected component is a submanifold of positive dimension. In this case there are infinitely many fixed points. To complete the proof of the first part of Theorem 1.4, it suffices to consider the S^1-actions with nonempty isolated fixed points.

Like the notations in Section 2.1.1, we assume the isolated fixed points are $\{P_1, \cdots, P_r\}$. In each fixed point P_i we have mn integer weights $k_1^{(i)}, k_2^{(i)} \cdots, k_{mn}^{(i)}$. Given any partition $\lambda = (\lambda_1, \cdots, \lambda_u)$ of weight m, We define

$$c_\lambda(i) := \prod_{t=1}^{u} \Big(\sum_{1 \leq j_1 < j_2 < \cdots < j_{\lambda_t} \leq mn} k_{j_1}^{(i)} k_{j_2}^{(i)} \cdots k_{j_{\lambda_t}}^{(i)} \Big).$$

Let

$$\{c_\lambda(i) \mid 1 \leq i \leq r\} = \{s_1, \cdots, s_l\} \subset \mathbb{Z}$$

and define

$$A_t := \sum_{\substack{1\le i\le r\\ c_\lambda(i)=s_t}} \frac{1}{\prod_{j=1}^{mn} k_j^{(i)}}, \qquad 1\le t\le l.$$

Lemma 3.1. *If the Chern number $(c_{\lambda_1}\cdots c_{\lambda_u})^n[M]\neq 0$, then at least one of A_t is nonzero.*

Proof. Suppose $A_t=0$ for all $t=1,\cdots,l$. Then Bott residue formula (2.1) tells us

$$(c_{\lambda_1}\cdots c_{\lambda_u})^n[M]=\sum_{t=1}^{l}(s_t)^n\cdot A_t=0.$$

□

The following lemma is inspired by ([15], Lemma 8).

Lemma 3.2. *If $r\le n$, then $A_t=0$ for all $t=1,\cdots,l$.*

Proof. Taking

$$f(x_1,\cdots,x_{mn})=\big[\prod_{t=1}^{u}\big(\sum_{1\le j_1<\cdots<j_{\lambda_t}\le mn} x_{j_1}x_{j_2}\cdots x_{j_{\lambda_t}}\big)\big]^i,\ i=0,1,\cdots,r-1,$$

in (2.1), we have

$$\begin{cases} A_1+A_2+\cdots+A_l=0\\ s_1A_1+s_2A_2+\cdots+s_lA_l=0\\ \vdots\\ (s_1)^{r-1}A_1+(s_2)^{r-1}A_2+\cdots+(s_l)^{r-1}A_l=0 \end{cases} \tag{3.1}$$

Note that $l\le r$ and $s_1,\cdots,s_l$ are mutually distinct, which means the coefficient matrix of the first l lines of (3.1) is the nonsingular Vandermonde matrix. Hence the only possibility is

$$A_1=\cdots=A_l=0.$$

□

Combining Lemma 3.1 with Lemma 3.2 will lead to the proof of the first part of Theorem 1.4. The proof of the second part is similar and so we omit it.

Remark 3.3. It is not surprise that Bott's residue formula we used here is similar to the Atiyah-Bott-Berline-Vergne localization formula used in [15]. In fact it turns out that Bott's residue formula can be put into the framework of the equivariant cohomology theory ([1], [3]). But Bott's original formula is more suitable for our purpose. Note that our sufficient condition (vanishing of some characteristic number) guaranteeing an explicit lower bound of the number of fixed points relies *only* on the manifold itself. While the sufficient condition in ([15], Theorem 1) relies on the data near the fixed points of the action.

3.2. **Proof of Theorem 1.6.** In this section we assume that N^{2n} has a circle action with isolated fixed points and keep the notations of Section 2.1.2 in mind.

The following proposition says, if the action is semi-free, then $[N^{2n}]$ is at most a torsion element in the oriented cobordism ring Ω_*^{SO}.

Proposition 3.4 (Pantilie-Wood)**.** Suppose N^{2n} has a semi-free S^1-action with isolated fixed points. Then all the Pontrjagin numbers of N^{2n} vanish. Equivalently,

$$[N^{2n}] = 0 \in \Omega_*^{SO} \otimes \mathbb{Q}.$$

The proof of ([14], Theorem 1.1) also uses Bott residue formula, in the language of differential geometry. Here we give a quite direct topological proof, although the essential is the same.

Proof. When n is odd, this proposition obviously holds for dimensional reason.

Suppose n is even, say $2q$. As noted in Section 2.1.2, in each fixed point P_i, the weights $k_1^{(i)}, \cdots, k_n^{(i)}$ are unique up to even number of sign changes. Since the action is semi-free, all these $k_j^{(i)}$ are ± 1. Let ρ_0 (resp. ρ_1) be the number of fixed points with even (resp. odd) -1.

Take $f = 1$ in (2.2), we have

$$\rho_0 - \rho_1 = 0, \tag{3.2}$$

which means the number of the fixed points are even.

Let $\lambda = (\lambda_1, \cdots, \lambda_l)$ be a partition of q. According to (2.2), the corresponding Pontrjagin number $p_\lambda[N] = p_{\lambda_1} \cdots p_{\lambda_l}[N]$ equals to

$$\binom{2q}{\lambda_1} \cdots \binom{2q}{\lambda_l} (\rho_0 - \rho_1) = 0.$$

This completes the proof of this proposition. □

From ([9], Lemma 2.2) we have

$$[N^{2n}] = \sum_{P_i} [\mathbb{C}P^n\big|_{P_i}] \in \Omega_*^{SO}, \tag{3.3}$$

where $\mathbb{C}P^n\big|_{P_i}$ is the n-dimensional complex projective space associated to the fixed point P_i and given a suitable orientation.

When n is odd, $[\mathbb{C}P^n] = 0 \in \Omega_*^{SO}$ (cf. [8], p.1) and therefore $[N^{2n}] = 0$.

When n is even, say $2q$, Porposition 3.4 and (3.3) imply

$$\sum_{P_i} [\mathbb{C}P^{2q}\big|_{P_i}] = 0 \in \Omega_*^{SO} \otimes \mathbb{Q}. \tag{3.4}$$

It is well-known that $[\mathbb{C}P^{2q}]$ is *not* a torsion element. From (3.2) we have known the number of the fixed points $\{P_i\}$ are even. Hence the only possibility that (3.4) holds is that half of the orientations of such $\mathbb{C}P^{2q}\big|_{P_i}$ are canonical and half are opposite to the canonical orientation, which means the right-hand side, and therefore the left-hand side of (3.3) are zero. This completes the proof of Theorem 1.6.

3.3. **Proof of Theorem 1.7.** In this subsection, (M^{2n}, J) is an almost-complex manifold with a semi-free circle action with isolated fixed points. Let ρ_t be the number of fixed points of the circle action with exactly t negative weights. In fact these ρ_t are all related to each other ([16], Lemma 3.1)

$$\rho_t = \rho_0 \cdot \binom{n}{t}, \qquad 0 \leq t \leq n.$$

This fact can also be derived from a rigidity result (cf. [11], Theorem 3.2). This fact means the isolated fixed point set is nonempty if and only if $\rho_0 > 0$.

The following lemma shows that the first Chern class of (M^{2n}, J) is nonzero.

Lemma 3.5. *The Chern number $c_1 c_{n-1}[M]$ is equal to $\rho_0 \cdot n \cdot 2^n$. In particular, if the isolated fixed points set is nonempty, then $c_1(M)$ is nonzero.*

Proof. In each fixed point P_i, the n weights $k_1^{(i)}, \cdots, k_n^{(i)}$ are all ± 1. If the number of -1 among $k_1^{(i)}, \cdots, k_n^{(i)}$ is t, then it is easy to check

$$\frac{e_1(k_1^{(i)}, \cdots, k_n^{(i)}) e_{n-1}(k_1^{(i)}, \cdots, k_n^{(i)})}{\prod_{j=1}^n k_j^{(i)}} = (n-2t)^2.$$

By Bott's residue formula (2.1) we have

$$c_1 c_{n-1}[M] = \sum_{t=0}^{n} \rho_t (n-2t)^2 = \rho_0 \sum_{t=0}^{n} \binom{n}{t} (n-2t)^2 = \rho_0 \cdot n \cdot 2^n.$$

□

Now we can prove our last main result, Theorem 1.7.

Proof. Since $c_1(M) \neq 0$, we can assume $c_1(M) = d \cdot x$, where d is a positive integer and $x \in H^2(M; \mathbb{Z})$ is a primitive element. It suffices to show, if $d > 1$, then d must be 2.

Using Proposition 2.5 we have

$$\begin{aligned} 0 &\equiv \sum_{t=0}^{n} \rho_t g^{\frac{n-2t}{d}} \frac{(-g)^t}{(1-g)^n} \\ &= \rho_0 \frac{g^{\frac{n}{d}}}{(1-g)^n} \sum_{t=0}^{n} \binom{n}{t} (-1)^t g^{\frac{(d-2)t}{d}}. \end{aligned}$$

If the isolated fixed point set is nonempty, then $\rho_0 > 0$. In this case, the last expression is identically zero if and only if $d = 2$. □

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