On existence of the prescribing k-curvature problem on manifolds with boundary

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Abstract

In this paper we study the problem of conformally deforming a metric to a prescribed k-th order symmetric function of the eigenvalues of the Schouten tensor on compact Riemannian manifolds with totally geodesic boundary. We prove the solvability and the compactness of the solution set for the cases $k \geq n/2$, provided the conformal class admits k-admissible metric. These results had been proved by Gursky and Viaclovsky, Trudinger and Wang for the manifolds without boundary, and by Jin, Li and Li, and S. Chen for the locally conformally flat manifolds with boundary.

1 Introduction

In this paper we study the existence and compactness of the solution set of a prescribing k-curvature problem on manifolds with boundary.

Let (M^n, g) be a smooth, compact Riemannian manifold of dimension $n \geq 3$. The Schouten tensor of g is defined by

$$A_g = \frac{1}{n-2}\left(\mathrm{Ric}_g - \frac{R_g}{2(n-1)}g\right),$$

where Ric and R are the Ricci and scalar curvatures of g, respectively. Let $[g]$ be the set of metrics conformal to g. For $\tilde{g} = e^{-2u}g \in [g]$, we consider the equation

$$\sigma_k^{1/k}\left(\lambda(\widetilde{g}^{-1}A_{\tilde{g}})\right) = f(x), \tag{1.1}$$

where $\sigma_k : R^n \to R$ denotes the k-th elementary symmetric function $(1 \leq k \leq n)$, and $\lambda(g^{-1}A_g)$ the eigenvalues of $g^{-1}A_g$. $\sigma_k(\lambda(g^{-1}A_g))$ is called k-curvature. The

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Schouten tensor transforms according to the formula

$$A_{\tilde{g}} = \nabla^2 u + du \otimes du - \frac{1}{2}|\nabla u|^2 g + A_g,$$

where ∇u and $\nabla^2 u$ denote the gradient and Hessian of u with respect to g. Consequently, (1.1) is equivalent to

$$\sigma_k^{1/k}\Big(\lambda\Big(g^{-1}\Big[\nabla^2 u + du \otimes du - \frac{1}{2}|\nabla u|^2 g + A_g\Big]\Big)\Big) = f(x)e^{-2u}. \qquad (1.2)$$

Let $\Gamma_k \subset R^n$ denote the component of $\{x \in R^n \quad | \quad \sigma_k(x) > 0\}$ containing the positive cone $\{x \in R^n \quad | \quad x_1 > 0, ..., x_n > 0\}$ and $[g]_k = \{\widetilde{g} \in [g] \quad | \quad \lambda(\widetilde{g}^{-1}A_{\tilde{g}}) \in \Gamma_k\}$. We call a metric in $[g]_k$ k-admissible, or simply admissible. And we also call a function u k-admissible, if $e^{-2u}g \in [g]_k$. The k-Yamabe problem is for given (M^n, g) with $g \in \Gamma_k$, finding a solution of (1.2) with $f(x) = constant$. When $k = 1$, it reduces to the classical Yamabe problem. For compact manifolds without boundary, the classical Yamabe problem (i. e. $k = 1$) has been solved by Yamabe [Ya], Trudinger [Tr1], Aubin [Au] and Schoen [S1]. For $k \geq 2$, the existence of the solutions to the k-Yamabe equation ((1.2) for $f(x) = constant$) has been solved for the cases $k = 2$ [CGY1, CGY2, STW], $k = n/2$ [TW2], $k > n/2$ [GV1] [TW1] and for locally conformally flat manifolds [GW2] [LL1] [STW]. The compactness of the solution sets in the above cases are proved.

For compact Riemannian manifold (M^n, g) with nonempty smooth boundary ∂M, there are two classes of boundary condition for the existence problem of equation (1.2). One is the Dirichlet boundary condition, was studied by Bo Guan in [G]. Another is the Neumann problem, has been studied by S. Chen, Jin-Li-Li and Li-Li [Cn1, Cn2, JLL, LL3] ect.. For $k = 1$, there are also several results (e.g. [E], ect.). Under various conditions, they derive local estimates for solutions and establish some existence results.

In this paper we are interested in the case $k \geq n/2$ with the Neumann boundary condition. Under the assumption that the boundary is totally geodesic, we obtain the existence and the compactness of the solutions to the Neumann problem. In fact, we have the following

Theorem 1.1. *Let (M, g) be compact n-dimensional Riemannian manifold with totally geodesic boundary, $n \geq 3$, and assume g is k-admissible with $k > n/2$ and not conformally equivalent to standard hemisphere. Then given any smooth positive function $f \in C^\infty(M)$ there exists a smooth function $u \in C^\infty(M)$ such that the conformal metric $\tilde{g} = e^{-2u}g$ satisfies*

$$\sigma_k^{\frac{1}{k}}(\lambda(\tilde{g}^{-1}A_{\tilde{g}})) = f(x)$$

and with totally geodesic boundary. Additionary, the set of all such solutions is compact in the C^m-topology for any $m \geq 0$.

Theorem 1.2. *Let (M, g) be a compact n-dimensional Riemannian manifold with totally geodesic boundary, and assume g is k-admissible with $k \geq n/2$ and (M, g) is not conformally equivalent to (S_n^+, g_c), where g_c is the standard metric on the hemisphere. Then given any smooth positive function $f \in C^\infty(M)$ there exists smooth function function $u \in C^\infty(M)$ such that the conformal metric $\tilde{g} = e^{-2u} g$ satisfies*

$$\sigma_k^{\frac{1}{k}}(\lambda(\tilde{g}^{-1} A_{\tilde{g}})) = f(x)$$

and with totally geodesic boundary. Besides, the set of all such solutions is compact in the C^m-topology for any $m \geq 0$.

Here Theorem 1.1 generalizes a result in [JLL] where it is assumed that (M, g) is locally conformally flat near ∂M. Theorem 1.2 improves a corresponding result in [Cn1] and [JLL] for the case $k = n/2$, where $f(x) \equiv$ *const.*, and (M, g) is locally conformally flat.

Recall that the second fundamental form L of ∂M with respect to g is defined as

$$L(X, Y) = -g(\nabla_X \nu, Y), \qquad X, Y \in T_x(\partial M),$$

where $T_x(\partial M)$ denotes the tangent space of ∂M at x, ν is the unit inward normal vector field to ∂M in (M, g) and ∇ denotes the Levi-Civita connection with respect to g. A point $x \in \partial M$ is called an umbilic point if $L(X, Y) = \tau_g(x) g(X, Y)$ for all $X, Y \in T_x(\partial M)$. The boundary is called umbilic if every point of ∂M is an umbilic point. A totally geodesic boundary is umbilic with $\tau_g \equiv 0$. Note that the umbilicity is conformally invariant. In fact, we have

$$\widetilde{L}(X, Y) e^u = \frac{\partial u}{\partial \nu} g(X, Y) + L(X, Y) \quad \text{for any } X, Y \in T_x(\partial M),$$

where $\widetilde{L}$ denotes the second fundamental form of ∂M with respect to $\widetilde{g} = e^{-2u} g$. When the boundary is umbilic, the above formula becomes

$$\tau_{\widetilde{g}} e^{-u} = \frac{\partial u}{\partial \nu} + \tau_g.$$

Especially, if (M, g) has totally geodesic boundary and the conformal metric $\tilde{g}$ has totally geodesic boundary as well, then the k-Yamabe problem with totally geodesic boundary becomes to consider the following equation:

$$\begin{cases} \sigma_k^{1/k}\Big(\lambda\Big(g^{-1}\Big[\nabla^2 u + du\otimes du - \frac{1}{2}|\nabla u|^2 g + A_g\Big]\Big)\Big) = f(x)e^{-2u} & \text{in } M, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial M. \end{cases} \tag{1.3}$$

For $k > \frac{n}{2}$, we will along the line of [GV1]. By use of the deformation defined in [GV1], we may get (1.3) when $t = 1$ and the equation for $t = 0$ is easier to analyze. The Leray-Schauder degree, defined in [Li1]([Cn1] for the boundary case) is non-zero. By homotopy-invariance of the degree, the existence of the solution reduces to estabilishing a priori bounds for $t \in [0, 1]$. To prove this, we argue by contradiction. Assuming that there exists a sequence of solutions $\{u_i\}$ for which a C^0-bound fails, we study the blow-up. In section 3, we prove that there are only finite blow-up points. Then, by singularity analysis, we find out that at regular point the super limit of solution u_i is $+\infty$ (section 4). Hence, in section 5, we can get a better rescaled functions w_i. Then by a classic method of gluing two copies of M along the boundary, we derive a $C^{1,1}_{loc}$ function $\widetilde{w}$ on a closed $C^{2,1}$ manifold $\widetilde{M}$. Therefore, by the argument in section 6 and 7 of [GV1], we know $e^{-2w}g$ is in fact the half-plane in Euclidean space, which contradicts with the condition that the manifold is not conformally equivalent to standard hemisphere.

However, when $k = \frac{n}{2}$, the Ricci tensor Ric is only non-negative definite, it is not enough to prove the existence result as the case $k > n/2$. So we need to turn to another method. In [TW2], Trudinger and X.-J. Wang provided another approach. By analyzing the asymptotic behaviour of the solution at singular points, they prove the existence of the solutions to equation (1.2) for manifolds without boundary. We glue two copies of M along the boundary as above, employ the similar argument as [TW2], and give the proof of Theorem 1.2. in section 6.

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2 Deformation and C^1 and C^2 estimates

(1) Deformation

To prove the existence of solution to the equation (1.3), we employ the deformation which defined in [GV2]. More details of the deformation can be consulted in [GV2].

The deformation equations are

$$\begin{cases} \sigma_k^{1/k}\Big(\lambda\Big(g^{-1}\Big[\lambda_k(1-\psi(t))g+\psi(t)A_g+\nabla^2 u+du\otimes du-\frac{1}{2}|\nabla u|^2 g\Big]\Big)\Big) \\ =\psi(t)f(x)e^{-2u}+(1-t)(\int e^{-(n+1)u})^{\frac{2}{n+1}} & \text{in } M, \\ \frac{\partial u}{\partial \nu}=0 & \text{on } \partial M. \end{cases} \tag{2.1}$$

where $\psi\in C^1[0,1]$ satisfies $0\le\psi(t)\le 1, \psi(0)=0$, $\psi(t)=1$ for $t\le\frac{1}{2}$; and $\lambda_k=\binom{n}{k}^{-\frac{1}{k}}vol(M_g)^{\frac{2}{n+1}}$.

As the same as in [GV2], at $t=1$ (2.1) becomes (1.3). While at $t=0$, it becomes into

$$\begin{cases} \sigma_k^{1/k}\Big(\lambda\Big(g^{-1}\Big[\lambda_k g+\nabla^2 u+du\otimes du-\frac{1}{2}|\nabla u|^2 g\Big]\Big)\Big)=(\int e^{-(n+1)u})^{\frac{2}{n+1}} & \text{in } M, \\ \frac{\partial u}{\partial \nu}=0 & \text{on } \partial M. \end{cases}$$

It has been pointed out in [GV2] that the above equation has a unique solution $u(x)\equiv 0$ if $\partial M=\varnothing$. We can show it is also true in our case.

In fact, it is obvious that $u=0$ is a solution. Now we are going to prove the uniqueness.

At the maximum point x_0 of u, no matter x_0 is interior or boundary point, we always have that $\nabla u|_{x_0}=0$, and $\nabla^2 u|_{x_0}$ is nonpositive definite. In fact if x_0 is interior point, it is clear; if x_0 is boundary point, we have $\frac{\partial u}{\partial \nu}|_{x_0}=0$ and $\frac{\partial u}{\partial x^\alpha}|_{x_0}=0$, where $\{x^\alpha\}_{1\le\alpha\le n-1}$ is a local coordinates on the boundary ∂M around x_0. Therefore $\nabla^2 u|_{x_0}$ is nonpositive definite. Now at x_0 we have

$$\begin{aligned} \lambda_k\binom{n}{k}^{1/k} &= \lambda_k\sigma_k^{\frac{1}{k}}(\lambda(g^{-1}\cdot g)) \\ &\ge \sigma_k^{1/k}\Big(\lambda\Big(g^{-1}\Big[\nabla^2 u+du\otimes du-\frac{1}{2}|\nabla u|^2 g+\lambda_k g\Big]\Big)\Big) \\ &= (\int e^{-(n+1)u})^{\frac{2}{n+1}}. \end{aligned}$$

Similarly, at the minimum point of u, it satisfies $\lambda_k\binom{n}{k}^{1/k}\le(\int e^{-(n+1)u})^{\frac{2}{n+1}}$. As a result, $\lambda_k\binom{n}{k}^{1/k}=(\int e^{-(n+1)u})^{\frac{2}{n+1}}$.

By Newton-MacLaurin inequality, we can immediately get $\sigma_k^{1/k}\le\frac{1}{n}\binom{n}{k}^{1/k}\sigma_1$. Hence,

$$\begin{aligned} \lambda_k\binom{n}{k}^{1/k} &= \sigma_k^{1/k}\Big(\lambda\Big(g^{-1}\Big[\lambda_k g+\nabla^2 u+du\otimes du-\frac{1}{2}|\nabla u|^2 g\Big]\Big)\Big) \\ &\le \frac{1}{n}\binom{n}{k}^{1/k}\sigma_1\Big(\lambda\Big(g^{-1}\Big[\lambda_k g+\nabla^2 u+du\otimes du-\frac{1}{2}|\nabla u|^2 g\Big]\Big)\Big) \\ &= \frac{1}{n}\binom{n}{k}^{1/k}\Big(\triangle u+(1-\frac{n}{2})|\nabla u|^2+n\lambda_k\Big). \end{aligned}$$

Then

$$(\frac{n}{2}-1)\int_M |\nabla u|^2 \leq \int_M \triangle u = \int_{\partial M} \frac{\partial u}{\partial \nu} = 0,$$

and $u \equiv const. = 0$.

Thus the operator

$$\begin{aligned}\Psi_t[u] &= \sigma_k^{1/k}\Big(\lambda\Big(g^{-1}\Big[\lambda_k(1-\psi(t))g + \psi(t)A_g + \nabla^2 u + du\otimes du - \frac{1}{2}|\nabla u|^2 g\Big]\Big)\Big) \\ &\quad -\psi(t)f(x)e^{-2u} - (1-t)(\int e^{-(n+1)u})^{\frac{2}{n+1}}\end{aligned}$$

satisfies Leray-Schauder degree $deg(\Psi_0, \mathcal{O}_0, 0) \neq 0$ at $t=0$, where the Leray-Schauder degree is defined by [Li1](see [Cn1] for the boundary case) and $\mathcal{O}_0$ is a neighborhood of the zero solution in $\{u \in C^{4,\alpha}(M) : \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial M\}$. Thus whence we obtain the homotopy-invariance of degree, we can derive that the Leray-Schauder degree is nonzero at $t=1$ which implies equation (1.3) is solvable.

(2) C^1 and C^2 estimates

We use Fermi coordinates in a boundary neighborhood at first. In this local coordinates, we take the geodesic in the inner normal direction $\nu = \frac{\partial}{\partial x^n}$ parameterized by arc length, and $(x^1, ..., x^{n-1})$ forms a local chart on the boundary. The metric can be expressed as $g = g_{\alpha\beta}dx^\alpha dx^\beta + (dx^n)^2$. The Greek letters $\alpha, \beta, \gamma, ...$stand for the tangential direction indices, $1 \leq \alpha, \beta, \gamma, ... \leq n-1$, while the Latin letters $i, j, k, ...$stand for the full indices, $1 \leq i, j, k, ... \leq n$. For Fermi coordinates, see [GrV] for details. In Fermi coordinates, the half ball is defined by $\overline{E}_r^+ = \{x_n \geq 0, \sum_i x_i^2 \leq r^2\}$ and the segment on the boundary by $\Sigma_r = \{x_n = 0, \sum_i x_i^2 \leq r^2\}$. Let us denote the RHS of the equation (2.1) by $h(x,u)$ and define

$$\begin{aligned}\overline{c_{\sup}}(r) &= \sup_{\overline{E}_r^+}(|h| + |\nabla_x h(x,u)| + |h_z(x,u)| + |\nabla_x^2 h(x,u)| \\ &\quad + |\nabla_x h_z(x,u)| + |h_{zz}(x,u)| + \Big|\frac{|\nabla_x h(x,u)|}{\inf_M h}\Big| + \Big|\frac{|h_z(x,u)|}{\inf_M h}\Big|).\end{aligned}$$

Note that the constant $(1-t)(\int e^{-(n+1)u})^{\frac{2}{n+1}}$ is less than $\sigma_k^{1/k}\Big(\lambda\Big(g^{-1}\Big[\lambda_k(1-\psi(t))g + \psi(t)A_g\Big]\Big)\Big)$. According to Theorem 1 in [HS], we have

$$\sup_{\overline{E}_{r/2}^+}\big(|\nabla u|^2 + |\nabla^2 u|\big) \leq C_1 \cdot \overline{c_{sup}}(r) = C_2 \cdot (1 + e^{-2\inf_{\overline{E}_r^+} u}), \tag{2.2}$$

where $C_2 = C_2(n, g, r, f)$.

Now we can immediately get a boundary estimate on the geodesic half ball $B(x,r)=\{y\in M \ \mid \ dist(x,y)<r\}$, since there is a following relationship between the half balls in Fermi coordinates and the half geodesic balls:

$$\overline{E}^{+}_{\rho/\sqrt{2}} \subset B(x,\rho) \subset \overline{E}^{+}_{\sqrt{5}\rho}.$$

In fact, we may assume the fermi coordinate of y is $(y_1,\cdots,y_n)$, z is on the x_n-axis satisfying $dist(z,y)=dist(x_n\text{-axis}, y)$ and let $d=dist(x,y)$, $d_0=dist(x,z)$ and $d_1=dist(z,y)$. Now for any $y\in\overline{E}^{+}_{\rho/\sqrt{2}}$, we have $d_0^2+d_1^2=\sum_\alpha(y_\alpha)^2+(y_n)^2<\rho^2/2$. Thus the triangle formula implies $d^2=dist(x,y)^2\leq(dist(x,z)+dist(x,y))^2\leq 2(d_0^2+d_1^2)<\rho^2$. Therefore $y\in B(x,\rho)$. On the other hand, for any $y\in B(x,\rho)$, we have $d^2<\rho^2$. Thus $d_1^2=dist(z,y)^2\leq dist(x,y)^2=d^2<\rho^2$ and $d_0=dist(z,x)\leq 2dist(x,y)=2d<2\rho$, otherwise $d\geq d_0-d_1>2d-d=d$. Hence, $d_0^2+d_1^2<5\rho^2$ and $y\in\overline{E}^{+}_{\sqrt{5}\rho}$.

Then (2.2) implies

$$\begin{aligned}\sup\nolimits_{B(x_0,r)}\left(|\nabla u|^2+|\nabla^2 u|\right) &\leq \sup\nolimits_{\overline{E}^{+}_{\sqrt{5}r}}\left(|\nabla u|^2+|\nabla^2 u|\right)\\ &\leq C_3\Big(1+e^{-2\inf_{\overline{E}^{+}_{2\sqrt{5}r}} u}\Big)\\ &\leq C_3\Big(1+e^{-2\inf_{B(x_0,2\sqrt{10}r)} u}\Big),\end{aligned} \tag{2.3}$$

where x_0 is a boundary point and $C_3=C_3(n,g,r,f)$.

We can get interior estimate as well. Let

$$\begin{aligned}c_{\sup}(r) \ = \ & \sup_{B(x_0,r)}\big(|h|+|\nabla_x h(x,u)|+|h_z(x,u)|+|\nabla_x^2 h(x,u)|\\ &+|\nabla_x h_z(x,u)|+|h_{zz}(x,u)|+\Big|\frac{|\nabla_x h(x,u)|}{\inf_M h}\Big|+\Big|\frac{|h_z(x,u)|}{\inf_M h}\Big|\big),\end{aligned}$$

where x_0 is an interior point. Then by Corollary 1 in [HS] we have

$$\sup_{B(x_0,r/2)} \left(|\nabla u|^2 + |\nabla^2 u|\right) \leq C_4 \cdot \left(1 + e^{-2\inf_{B(x,r)} u}\right),$$

where $C_4 = C_4(n, g, r, f)$.

Now we may assume that $\inf_M u_i \to -\infty$. Otherwise, the above estimate and Harnack inequality we know it is also upper bounded and completes the proof.

Then there are two possibilities.

(A) One is the blowup subsequence u_{t_i} happens at $t_i \leq 1 - \delta < 1$ for $\delta > 0$. We still denote it by u_i for simplicity. Then, at the maxmum point of u which is either an interior point or a boundary point, we have

$$\delta(\int e^{-(n+1)u_i})^{\frac{2}{n+1}} \leq \sigma_k^{1/k}\left(\lambda\left(g^{-1}\left[\lambda_k(1-\psi(t))g + \psi(t)A_g\right]\right)\right) \leq C_0.$$

Then we can take $\epsilon_i = e^{\inf_M u_i} \triangleq e^{u_i(z_i^0)}$, where $z_j^0 \in M$ is u_j's minimum point. Defining a map:

$$\begin{aligned} \mathcal{T}_i : B(0, c_0) \subset T_{z_i^0}M &\rightarrow B(z_i^0, c_0 \cdot \epsilon_i) \subset M \\ y &\rightarrow \exp_{z_i^0}(\epsilon_i y), \end{aligned}$$

where the metric on tangent space is $\tilde{g}_i = \epsilon_i^{-2}\mathcal{T}_i^* g$ and $B(0, c_0)$ is a geodesic ball in $\exp_{z_i^0}^{-1}(M)$ with radius $c_0 > 0$. Then we can get lower bounded functions in $B(0, c_0)$ on the tangent space $T_{z_i^0}M$: $w_i\,(y) = u_i(\mathcal{T}_i(y)) - \log \epsilon_i \geq 0$.

Furthermore, w_i satisfies

$$\begin{aligned} &\sigma_k^{1/k}\left(\lambda\left(\tilde{g}_i^{-1}\left(\epsilon_i^2\lambda_k(1-\psi(t_i))\tilde{g}_i + \psi(t_i)A_{\tilde{g}_i} + \nabla^2 w_i + dw_i \otimes dw_i - \tfrac{1}{2}|\nabla w_i|^2_{\tilde{g}_i}\tilde{g}_i\right)\right)\right) \\ &= \psi(t_i)f(\mathcal{T}_i\,(y))e^{-2w_i} + \epsilon_i^2(1-t_i)(\textstyle\int e^{-(n+1)u_i})^{\frac{2}{n+1}} \qquad \text{in } B(0, c_0). \end{aligned}$$

Then by the interior and boundary estimates on $B(0, \frac{c_0}{2})$ we can immediately get the upper bound of w_i. We then get the following contradiction:

$$C(n, g, c_0, f) \leq \int_{B(0,\frac{c_0}{4})} e^{-(n+1)w_i} \leq \epsilon_i \int_{B(z_i^0, r/2)} e^{-(n+1)u_i} \leq \epsilon_i (C_0/\delta)^{\frac{n+1}{2}} \to 0.$$

Therefore we have following

Lemma 2.1. (Theorem 2.1 of [GV2]). *For any fixed* $0 < \delta < 1$*, there is a constant* $C = C(\delta, n, g, f)$ *such that any solution of (2.1) with* $t \in [0, 1-\delta]$ *satisfies* $||u||_{C^{4,\alpha}} \leq C$*.*

(B) So without loss of generality, we may assume that u_{t_i} tends to $-\infty$ at the time $t_i \to 1$, where u_{t_i} is the solution of (2.1) at $t = t_i$ which will be denoted by u_i in what follows. Thus equation (2.1) turns to be:

$$\begin{cases} \sigma_k^{1/k}\Big(\lambda\Big(g^{-1}\big(A_g + \nabla^2 u + du \otimes du - \frac{1}{2}|\nabla u|^2 g\big)\Big)\Big) = (1-t)o + f(x)e^{-2u} & \text{in } M, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial M. \end{cases} \tag{2.4}$$

where u is assumed to be k-admissible, and $o \geq 0$ is a constant.

We will get more exact estimate, where we consider the boundary and interior estimate both in geodesic coordinates:

Lemma 2.2. *Let* $u \in C^4(M)$ *be a* k*-admissible solution of (2.1) in* $B(x, r)$ *and* $0 \leq r < 1$*. Then there is a constant* $C = C(n, g, f)$ *such that*

$$\big(|\nabla^2 u| + |\nabla u|^2\big)(x') \leq C\Big(r^{-2} + e^{-2\inf_{B(x,2\sqrt{10}r)} u}\Big). \tag{2.5}$$

for all $x' \in B(x, r)$*.*

Proof. Suppose x is on the boundary ∂M. We define a local diffeomorphism

$$\begin{aligned} \mathcal{O} : B(0, 2\sqrt{10}) \subset T_x M &\to B(x, 2\sqrt{10}r) \\ y &\to \exp_x(ry), \end{aligned}$$

where $B(0, 2\sqrt{10})$ is the geodesic ball in $\exp_x^{-1}(M)$ with the metric $\tilde{g} = r^{-2}\mathcal{O}^* g$.

Let $w(y) = u(\mathcal{O}(y)) - \log r$ in $B(0, 2\sqrt{10})$, then w satisfies

$$\begin{aligned} & \sigma_k^{1/k}\Big(\lambda\Big(\tilde{g}^{-1}\Big[A_{\tilde{g}} + \nabla^2 w + dw \otimes dw - \frac{1}{2}|\nabla w|^2_{\tilde{g}}\tilde{g}\Big]\Big)\Big) \\ = \;& f(\mathcal{O}(y))e^{-2w} + r^2(1-t)o \qquad \text{in } B(0, 2\sqrt{10}). \end{aligned}$$

By a similar argument as (2.3) we can get

$$\begin{aligned} & \sup_{B(x,r)} r^2\big(|\nabla u|^2 + |\nabla^2 u|\big) \\ \leq \;& \sup_{B(0,1)} \big(|\nabla w|^2 + |\nabla^2 w|\big) \end{aligned}$$

$$
\begin{aligned}
&\leq C \cdot \Big(e^{-2 \inf_{B(0,2\sqrt{10})} w} + 1 \Big) \\
&\leq C \cdot \Big(e^{-2 \inf_{B(x,2\sqrt{10}r)} u} r^2 + 1 \Big).
\end{aligned}
$$

Therefore,

$$\sup_{B(x,r)} |\nabla^2 u|(x) + |\nabla u|^2(x) \leq C\Big(r^{-2} + e^{-2\inf_{B(x,2\sqrt{10}r)} u} \Big),$$

where $C = C(n, g, f)$.

When x is an interior point, the interior estimate in [GV1] says

$$\sup_{B(x,r)} |\nabla^2 u|(x) + |\nabla u|^2(x) \leq C\Big(r^{-2} + e^{-2\inf_{B(x,2r)} u} \Big),$$

where $C = C(n, g, f)$.

Hence we can conclude the estimate (2.5), no matter x is a boundary or interior point. □

3 Finite blowup points

In this section, we are going to prove that there are finite blowup points for prescribing k-curvature problem, where $k > n/2$. By [GVW1], we have $Ric_g \geq \frac{(2k-n)(n-1)}{n(k-1)} \sigma_1(A_g) g$ for k-admissible metric g.

Lemma 3.1 below has been proved by P. Guan and G. Wang for the interior point (Proposition 3.6 in [GW1]), we only focus on the proof for the boundary point.

Lemma 3.1. *There exists ς and $C = (\varsigma, n, g, f)$ such that any solution $u \in C^2(B(x, \rho))$ with $\int_{B(x,\rho)} e^{-nu} < \varsigma$ satisfies $\inf_{B(x,\rho/2)} u_i \geq \log \rho - C$.*

Proof. Without loss of generality, we may assume $x \in \partial M$. Let us argue by contradiction. If there exists sequences $\{\rho_i\}$ and $\{u_i\}$ satisfying

$$\int_{B(x,\rho_i)} e^{-nu_i} \to 0 \quad \text{and} \quad \inf_{B(x,\rho_i/2)} u_i - \log \rho_i \to -\infty.$$

Then define a local diffeomorphism map:

$$
\begin{aligned}
\mathcal{T}_i : B(0,1) \subset \overline{\mathbb{R}}^+_n &\to B(x, \rho_i) \\
y &\to \exp_x(\rho_i y),
\end{aligned}
$$

where $B(0, 1)$ means the unit geodesic ball in $\exp_x^{-1}(M)$ under the metric $\tilde{g}_i = \rho_i^{-2} \mathcal{T}_i^* g$ and $\overline{\mathbb{R}}^+_n$ half-plane in $\mathbb{R}^n$.

Let $w_i(y) = u_i(\mathcal{T}_i(y)) - \log \rho_i$ in $B(0,1)$. Then the assumption becomes

$$\int_{B(0,1)} e^{-nw_i} dV_{g_{w_i}} = \int_{B(0,1)} \rho_i^n \frac{1}{\rho_i^n} e^{-nu_i} dV_{g_{u_i}} \to 0 \quad \text{and} \quad \inf_{B(0,1/2)} w_i \to -\infty, \tag{3.1}$$

where $B(0,1/2)$ means the geodesic ball in $\exp_x^{-1}(M)$ under the metric $\tilde{g}_i = \rho_i^{-2} \mathcal{T}_i^* g$.

Moreover, w_i satisfies the same type equation as (2.4).

$$\begin{cases} \sigma_k^{1/k}\Big(\lambda\Big(\tilde{g}_i^{-1}\Big[A_{\tilde{g}_i} + \nabla^2 w_i + dw_i \otimes dw_i - \frac{1}{2}|\nabla w_i|^2_{\tilde{g}_i}\tilde{g}_i\Big]\Big)\Big) & \\ = f(\mathcal{T}_i(y))e^{-2w_i} + \rho_i^2(1-t_i)o & \text{in } B(0,1), \\ \frac{\partial w_i}{\partial \tilde{\nu}} = 0 & \text{on } B(0,1) \cap \partial \overline{\mathbb{R}}_n^+, \end{cases}$$

where $\tilde{\nu}$ is the inner normal vector of $B(0,1) \cap \partial \overline{\mathbb{R}}_n^+$.

Let $h_i(r) = r(\frac{3}{4} - r) \sup_{B(0,r)} e^{-w_i}, r \in (0, \frac{3}{4})$, where $B(0,r)$ denotes the geodesic ball in $\exp_x^{-1}(M)$ with radius r. Suppose $\max_r h_i(r) = h_i(r_i^0) = r_i^0(\frac{3}{4} - r_i^0)e^{-w_i(z_i^0)}$ and $s_i^0 = \frac{2}{3}(\frac{3}{4} - r_i^0)$.

Since the function $r(\frac{3}{4} - r)$ is concave and $h_i(r_i^0) \geq h_i(r_i^0 + s_i^0)$, we have

$$\sup_{B(0, r_i^0 + s_i^0)} e^{-w_i} \leq 3e^{-w_i(z_i^0)}.$$

Now denote $e^{w_i(z_i^0)}$ by ϵ_i. From (3.1)

$$3\epsilon_i^{-1} \geq \sup_{B(0, r_i^0 + s_i^0)} e^{-w_i} \geq \sup_{B(0,1/2)} e^{-w_i} \to +\infty,$$

we can get $\epsilon_i \to 0$.

Denote the set $\{y \in B(0,1) \mid dist(z_i^0, y) < s_i^0\}$ by $B(z_i^0, s_i^0)$, and note that $B(z_i^0, s_i^0) \subset B(0, r_i^0 + s_i^0)$, we have

$$\sup_{B(z_i^0, s_i^0)} e^{-w_i} \leq 3\epsilon_i^{-1}. \tag{3.2}$$

Define a mapping:

$$\begin{aligned} \mathcal{U}_i : B(0, s_i^0/\epsilon_i) \subset T_{z_i^0}(T_x M) &\rightarrow B(z_i^0, s_i^0) \\ y &\rightarrow \exp_{z_i^0}(\epsilon_i y), \end{aligned}$$

where the metric on tangent space is $\breve{g}_i = \epsilon_i^{-2} \mathcal{U}_i^* \tilde{g}_i$ and $B(0, s_i^0/\epsilon_i)$ is the geodesic ball in $\exp_{z_i^0}^{-1}(B(z_i^0, s_i^0)) \Big(\subset T_{z_i^0}(T_x M)\Big)$.

Moreover, consider a sequence of functions: $v_i(y) = w_i(\mathcal{U}_i(y)) - \log \epsilon_i$. Then v_i satisfies

$$\begin{aligned}&\sigma_k^{1/k}\Big(\lambda\Big(\breve{g}_i^{-1}\Big[A_{\breve{g}_i} + \nabla^2 v_i + dv_i \otimes dv_i - \tfrac{1}{2}|\nabla v_i|^2_{\breve{g}_i}\breve{g}_i\Big]\Big)\Big)\\ &= f(\mathcal{T}_i \cdot \mathcal{U}_i\,(y))e^{-2v_i} + \epsilon_i^2\rho_i^2(1-t_i)o \qquad\qquad \text{in } B(0, s_i^0/\epsilon_i),\end{aligned}$$

which is the same type equation as (2.4). By (3.2), we have $v_i \geq -\log 3$ in $B(0, s_i^0/\epsilon_i)$. We may check that there is a constant c_1 such that $s_i^0/\epsilon_i = \frac{2}{3}(\frac{3}{4} - r_i^0)/\epsilon_i > c_1$. In fact, $r_i^0(\frac{3}{4} - r_i^0)\epsilon_i^{-1} = h_i(r_i^0) \geq h_i(1/2) \to +\infty$ then $\lim_{i\to\infty} s_i^0/\epsilon_i = +\infty$.

Then applying Lemma 2.2 on $B(0, c_1)$ which is a geodesic half-ball, we can get an upper bound of v_i on $B(0, c_1/2)$. Thus

$$\begin{aligned} C_1(n, g, f, c_1) &\leq \int_{B(0,c_1/2)} e^{-nv_i} \leq \int_{B(0,s_i^0/\epsilon_i)} e^{-nv_i}\\ &= \int_{B(z_i^0,s_i^0)} e^{-nw_i} dV_{g_{w_i}} \leq \int_{B(0,1)} e^{-nw_i} dV_{g_{w_i}},\end{aligned}$$

which contradicts the fact $\int_{B(0,1)} e^{-nw_i} dV_{g_{w_i}} \to 0$. □

Similar as [GV2], for a given point $x \in M$, we define the mass of x by

$$m(\{u_i\}; x) = \lim_{r\to 0} \overline{\lim_{i\to\infty}} \int_{B(x,r)} e^{-nu_i}.$$

We also denote $\Sigma[\{u_i\}] = \{x \quad | \quad m(\{u_i\}; x) \neq 0\}$.

By use of the ϵ-regularity result and the volume comparison theorem, we may get following propositions. See [Gur1] for their proofs.

Proposition 3.1(Lemma 2.3 of [Gur1])**.** *Given a sequence of smooth solutions of (2.4) with* $\inf_M u_i \to -\infty$*. Then there exists a positive constant* $\mu(n, g, f, \varsigma)$ *such that either* $m(\{u_i\}; x) = 0$ *or* $m(\{u_i\}; x) \geq \mu$*.*

Proposition 3.2(§2 of [Gur1]). *Suppose* u_i *is a blowup sequence of (2.4), then*
(1) $\Sigma[\{u_i\}]$ *is nonempty;*
(2) $\Sigma[\{u_i\}]$ *is finite.*

Corollary 3.1(Corollary 4.7 of [GV1]). *Let* u_i *be a solution of (2.4), then*
(1) *when* $u_i(x_i)$ *converges to* $-\infty$*, any accumulated point of* x_i *must belong to* $\Sigma[\{u_i\}]$*.*
(2) *for any* $x_0 \in \Sigma[\{u_i\}]$*, there exists a sequence points* x_i *such that* $\lim_{i\to\infty} x_i = x_0$ *and* $\lim_{i\to\infty} u_i(x_i) = -\infty$*.*

Proof. (1) Let $u_i(x_i)$ be a blowup sequence converging to $-\infty$. Since the manifold M is compact, then there is a convergent subsequence x_{i_k}. Suppose $\lim_{k\to\infty} x_{i_k} = x_0$.

We assert that x_0 must be in $\Sigma[\{u_i\}]$. Otherwise, by Corollary 3.1, there exist constants r_0 and J such that $\inf_{B(x,r_0)} u_i \geq -C$ where $i \geq J$. Besides, we can find K such that $x_{i_k} \in B(x, r_0)$ for any $k \geq K$. Therefore, $u_j(x_{i_k}) \geq -C$ when $j \geq J$ and $k \geq K$, which contradicts the fact that $\lim_{k\to\infty} u_{i_k}(x_{i_k}) = -\infty$.

(2) Otherwise, there exist a neighborhood of x_0, for example $B(x_0, r_0)$, and a constant C such that $\inf_{B(x_0,r_0)} u_i \geq -C$. Then

$$m(\{u_i\}; x_0) = \lim_{r\to 0} \limsup_{i\to\infty} \int_{B(x_0,r)} e^{-nu_i} \leq \lim_{r\to 0} \int_{B(x_0,r)} e^{nC} = 0,$$

which contradicts that $x_0 \in \Sigma[\{u_i\}]$. □

4 Tend to $+\infty$ in regular set

In this section, we will prove that at any regular point $x \in M\backslash\Sigma[\{u_i\}]$, $\limsup_{i\to\infty} u_i(x) = +\infty$, where Ric_g is still larger than $\frac{(2k-n)(n-1)}{n(k-1)}\sigma_1(A_g)g$ when g is k-admissible ($k > n/2$) . We will prove it by contradiction.

In what follows we suppose that there is a regular point x' such that $\limsup_{i\to\infty} u_i(x') < +\infty$.

Lemma 4.1. *We can find a subsequence $u_{i_k} \to u \in C^\infty(M \setminus \Sigma[\{u_i\}])$, where the convergence is in C^m on compact sets away from $\Sigma[\{u_i\}]$.*

Proof. Suppose K is a compact subset in $M \setminus \Sigma[\{u_i\}]$. Then we can find compact sets K' and $\tilde{K}'$, such that $K \cup \{x'\} \subset K' \subset \tilde{K}'$. Since compact set $\tilde{K}'$ is covered by finite open sets, then apply Lemma 2.2, we may get $\sup_{K'} |\nabla u_i| \leq C_1$ and $\sup_K u_i \leq \sup_{K'} u_i \leq \inf_{K'} u_i + C_2 \leq \limsup_{i\to\infty} u_i(x') + C_2 \leq C_3$. Therefore, by regularity theory, there is a subsequence u_{i_k} convergent uniformly to u on the compact set K. □

For simplicity we still denote the subsequence by u_i in Lemma 4.1 and derive a contradiction about the limit u.

Lemma 4.2 (Proposition 3.3. of [GV1]). *Let $u \in C^2(M)$ and assume $g_u = e^{-2u}g$ has non-negative scalar curvature. Suppose there is a ball $B(x,\rho) \subset M$ and constants $\alpha_0 > 0$ and $A_0 > 0$ with*

$$\int_{B(x,\rho)} e^{\alpha_0 u} dV_g \leq B_0. \tag{4.1}$$

Then there is a constant $C = C(n, g, \rho, \alpha_0, B_0)$, such that

$$\max_M u \leq C. \tag{4.2}$$

Proof. The proof for compact manifolds without boundary is given in [GV1]. Here we present the proof for manifolds with boundary case. We denote R_u the scalar curvature of g_u, then

$$\frac{1}{2(n-1)}R_u e^{-2u} = \frac{1}{2(n-1)}R + \triangle u - \frac{n-2}{2}|\nabla u|^2.$$

By the condition of k-admissible ($k > n/2$), we know that both R_u and R are positive.

If we let $v = e^{-\frac{n-2}{2}u}$, then

$$\frac{n-2}{2}\frac{R_u}{2(n-1)}v^{\frac{4}{n-2}+1} = -\triangle v + \frac{R(n-2)}{4(n-1)}v,$$

hence

$$\triangle v \leq C_0 v. \tag{4.3}$$

Let $\varepsilon < \frac{2\alpha_0}{n-2}$. Multiply with $v^{-2\varepsilon-1}$ on both sides of (4.3) and integral by parts, we have

$$\begin{aligned} C_0\int_M v^{-2\varepsilon} &\geq \int_M v^{-2\varepsilon-1}\triangle v = (1+2\varepsilon)\int_M v^{-2\varepsilon-2}|\nabla v|^2 + \int_{\partial M} v^{-2\varepsilon-1}\frac{\partial v}{\partial \nu} \\ &= \frac{1+2\varepsilon}{\varepsilon^2}\int_M |\nabla(v^{-\varepsilon})|^2. \end{aligned}$$

Then, by the lower bound of the first eigenvalue η_1 for Neumann boundary condition (see [LY]), we see that $\int_M v^{-2\varepsilon}$ can be controlled by a constant depending on $vol\big(B(x,\rho)\big)$ and the bound of $\int_{B(x,\rho)} v^{-\varepsilon} < B_0^{\frac{n-2}{2\alpha_0}\varepsilon} vol\big(B(x,\rho)\big)^{1-\frac{(n-2)\varepsilon}{2\alpha_0}} \triangleq A_0$:

$$\begin{aligned} \int_M v^{-2\varepsilon} &\leq \frac{(\int_M v^{-\varepsilon})^2}{vol(M)} + \frac{1}{\eta_1}\int_M |\nabla(v^{-\varepsilon})|^2 \leq \frac{(\int_M v^{-\varepsilon})^2}{vol(M)} + \frac{C_0\frac{\varepsilon^2}{1+2\varepsilon}}{\eta_1}\int_M v^{-2\varepsilon} \\ &\leq \frac{(\int_{B(x,\rho)} v^{-\varepsilon} + (\int_{B(x,\rho)^c} v^{-2\varepsilon})^{\frac{1}{2}} vol(B(x,\rho)^c)^{\frac{1}{2}})^2}{vol(M)} + \frac{C_0\varepsilon^2}{\eta_1(1+2\varepsilon)}\int_M v^{-2\varepsilon} \\ &\leq \frac{A_0^2 + A_0\theta\int_{B(x,\rho)^c} v^{-2\varepsilon} + \frac{A_0}{\theta}vol(B(x,\rho)^c) + vol(B(x,\rho)^c)\int_{B(x,\rho)^c} v^{-2\varepsilon}}{vol(M)} \\ &\quad + \frac{C_0\varepsilon^2}{\eta_1(1+2\varepsilon)}\int_M v^{-2\varepsilon} \\ &\leq C_1(A_0,\theta) + (\frac{A_0\theta + vol(B(x,\rho)^c)}{vol(M)} + \frac{C_0\varepsilon^2}{\eta_1(1+2\varepsilon)})\int_M v^{-2\varepsilon}. \end{aligned}$$

Then by choosing a suitable ε and θ such that

$$\frac{A_0\theta + vol\big(B(x,\rho)^c\big)}{vol(M)} + \frac{C_0\varepsilon^2}{\eta_1(1+2\varepsilon)}$$

is strictly less than 1. Then we can get the upper bound of $\int_M v^{-2\varepsilon}$ which depends on g, A_0, ρ and α_0. In other words, when ε is small enough, (4.1) implies a global integral upper bound of $v^{-2\varepsilon}$.

Then we complete the proof via Green representation Theorem. Note that when $s < \frac{n}{n-1}$, the Green function G and its gradient $|\nabla G|$ is L^s integrable. If we denote $\frac{1}{1-\frac{1}{s}}$ by s', then we want to find a good enough $L^{s'}$ integrable function: Let $w = v^{-\frac{2\varepsilon}{s'}}$, then $w \in L^{s'}(M)$, and

$$\begin{aligned} \triangle w &= -\frac{2\varepsilon}{s'} v^{-2\frac{\varepsilon}{s'}-1} \triangle v + \frac{2\varepsilon}{s'}(\frac{2\varepsilon}{s'}+1) v^{\frac{2\varepsilon}{s'}-2} |\nabla v|^2 \\ &\geq -C_2 \frac{2\varepsilon}{s'} w. \end{aligned}$$

Consequently, by Green representation theorem we have

$$\begin{aligned} w(a) &= -\int_M G(a,\cdot)\triangle w + \int_{\partial M} \frac{\partial G(a,\cdot)}{\partial \nu} w \\ &\leq C_3 \int_M G(a,\cdot) w + \int_{\partial M} G(a,\cdot) \frac{\partial w}{\partial \nu} \\ &\leq C_4 (||G(a,\cdot)||_{L^s} ||w||_{L^{s'}} + ||\nabla G(a,\cdot)||_{L^s} ||w||_{L^{s'}}) \\ &\leq C_5(n, g, A_0, \rho, \alpha_0). \end{aligned}$$

This gives (4.2). □

Proposition 4.1 (Proposition 4.6. of [GV1]). *There is a neighborhood $B(x_0, \bar{\rho})$ of $x_0 \in \Sigma[\{u_i\}]$ and constant $C(n, g, f, \bar{\rho}, \varsigma)$, such that for $x \in B(x_0, \bar{\rho}) \setminus \{x_0\}$ it satisfies*

$$u(x) \geq \log d_g(x, x_0) - C.$$

Proof. Since u_i is bounded above in some neighborhood U of regular point, so

$$\int_U e^{\alpha u_i} \leq C_1,$$

Where α is a constant.

Then according to Lemma 4.2.

$$\max_M u_i \leq C_2.$$

Let $u(x) = \limsup_{k\to\infty} u_{i_k}$ in M, then the limit satisfies

$$\sup_{M\setminus\Sigma[\{u_i\}]} u \leq C_2$$

Then by the Volume Comparison Theorem, and Fatou Lemma,

$$\int_M e^{-nu} \leq \limsup_{k\to\infty} \int_M e^{-nu_{i_k}} = vol(g_{u_{i_k}}) \leq v_0.$$

Hence, there exists $\bar{\rho}$ small enough such that

$$\int_{B(x_0,2\bar{\rho})} e^{-nu} \leq \varsigma/2,$$

which ς is appeared in Lemma 3.1..

Then for any point $x \in B(x_0, \bar{\rho}) \setminus \{x_0\}$, $B(x, d_g(x, x_0)/2) \subset B(x_0, 2\bar{\rho})$ and

$$\int_{B(x,\frac{1}{2}d_g(x,x_0))} e^{-nu} \leq \varsigma/2.$$

Note that u is a $C^2(B(x, \frac{1}{2}d(x, x_0)))$ solution of (2.4). Hence by Lemma 3.1.

$$u(x) \geq \inf_{B(x,\frac{1}{2}d(x,x_0))} u(x) \geq \log d_g(x, x_0) - C.$$

□

Similar as the case of the manifolds without boundary ([GV1]), for the manifolds with nonempty boundary, we can get following propositions. Their proofs are also similar as in [GV1], we just only notice the boundary condition $\frac{\partial u}{\partial \nu} = 0$ when we take integral on the boundary. We omit their proofs.

Proposition 4.2 (Theorem 3.5. of [GV1]). *Let $u \in C^{1,1}_{loc}(\overline{A(\frac{1}{2}r_1, 2r_2)})$, where $x_0 \in M$ and $A(\frac{1}{2}r_1, 2r_2)$ denotes the annulus $B(x_0, 2r_2)/\overline{B(x_0, \frac{1}{2}r_1)}$, with $0 < r_1 < r_2$. Assume $\frac{\partial u}{\partial \nu} = 0$ on the boundary ∂M and $g_u = e^{-2u}g$ satisfies $Ric(g_u) - 2\delta\sigma_1(A_u)g \geq 0$ almost everywhere in $A(\frac{1}{2}r_1, 2r_2)$ for some $0 \leq \delta < \frac{1}{2}$. Define $\alpha_\delta = \frac{n-2}{1-2\delta}\delta \leq 0$ and $p = n+2\alpha_\delta \geq n$. Then given any $\alpha \geq \alpha_\delta$, there are constants $C = C((\alpha-\alpha_\delta)^{-1}, n, g) > 0$ such that*

$$\int_{A(r_1,r_2)} |\nabla u|^p e^{\alpha u} \leq C\Big(\int_{A(\frac{1}{2}r_1,2r_2)} e^{\alpha u}|Ric|^{\frac{p}{2}} + \frac{1}{r_1^p}\int_{A(\frac{r_1}{2},r_2)} e^{\alpha u} + \frac{1}{r_2^p}\int_{A(r_1,2r_2)e^{\alpha u}}\Big).$$

Corollary 4.1(Corollary 3.9. of [GV1]). *Let $u \in C^{1,1}_{loc}(M)$ satisfies $\frac{\partial u}{\partial \nu} = 0$ on the boundary ∂M. Also assume $g_u = e^{-2u}g$ is k-admissible with ($k > n/2$). Suppose δ satisfies $0 < \delta \leq \min\left\{\frac{1}{2}, \frac{(2k-n)(n-1)}{2n(k-1)}\right\}$. And define $\alpha_\delta = \frac{n-2}{1-2\delta}\delta$. Then for any $\alpha > \alpha_\delta$, there exists a constant $C = C(\delta, n, g, \alpha)$ such that*

$$||e^{(\alpha/p)u}||_{C^{\gamma_0}} \leq C||e^{(\alpha/p)u}||_{L^p},$$

where $\gamma_0 = \frac{2\alpha_\delta}{n+2\alpha_\delta}$.

Proposition 4.3(Proposition 4.5. of [GV1]). *Suppose* $x_0 \in \Sigma[\{u_i\}]$ *and* u_i *is a blowup sequence near* x_0*. Then given any* $\theta > 0$ *there exists neighborhood* U *of* x_0 *and a constant* $C = C(\theta, n, g, f)$ *such that the function* $u = \limsup_{i\to\infty} u_i$ *satisfies*

$$u(x) \leq (2-\theta)\log d_g(x, x_0) + C,$$

for all $x \neq x_0$ *in* U*.*

Now from Proposition 4.1 and Proposition 4.3 we get a contradiction. This implies that the assumption $\limsup_i u_i(x') < +\infty$ for some regular point x' is impossible. Thus we have the following

Proposition 4.4. (1) $\limsup_i u_i = +\infty$ *in* $M/\Sigma[\{u_i\}]$.
(2) *There is a subsequence* u_{i_k} *tending uniformly to* $+\infty$ *on compact set* $K \subset M/\Sigma[\{u_i\}]$.

Proof of (2). Note that $\limsup_i u_i(x') = +\infty$, we may suppose $\lim_k u_{i_k}(x') = +\infty$.

There are compact sets K' and $\tilde{K}'$, such that $K \cup \{x'\} \subset K' \subset \tilde{K}'$. Applying Lemma 2.2 on $\tilde{K}'$, then $\sup_{K'} |\nabla u_i| \leq C_1$ and $\inf_K u_i \geq \inf_{K'} u_i \geq \sup_{K'} u_i - C_2$

Then for any fixed $N \in \mathbb{N}$, since $\lim_k u_{i_k}(x') = +\infty$ we can find $J \in \mathbb{N}$, such that $u_{i_k}(x') > N + C_2$ when $k > J$. Hence, $\inf_K u_{i_k} \geq u_{i_k}(x') - C_2 > N$. $\square$

In the rest of the proof we will consider the subsequence u_{i_k} chosen above (still denoted by u_i) and a non-empty set $\Sigma_0[\{u_{i_k}\}] \subset \Sigma[\{u_i\}]$ which will be denoted by Σ_0 for simplicity.

5 Complete the proof of Theorem 1.1.

Note that u_i satisfies $\lim_i u_i(x') = +\infty$, for some regular point x' in $M \setminus \Sigma_0$. Let

$$w_i(x) = u_i(x) - u_i(x').$$

We will show that w_i converges to a $C^{1,1}_{loc}$-limit in $M \setminus \Sigma_0$.

Proposition 5.1. *We have*

(1) $S_{w_i} = Ric_{w_i} - 2\delta\sigma_1(A_{w_i})g$ *is positive semi-definite.*

(2) Let $M_r = M \setminus \cup_{x_k \in \Sigma} B(x_k, r)$*, where* $r > 0$ *small enough. Then we can find* $J = J(r)$ *and constant* $C = C(n, g, f)$ *such that*

$$|\nabla^2 w_i| + |\nabla w_i|^2 \leq Cr^{-2}$$

for all $x \in M_r$ and $i > J = J(r)$.

(3)The sequence w_i has a global upper bound $\max_M w_i \leq C$ and a L^∞ bound in M_r.

Proof. (1) $S_{w_i} = Ric_{w_i} - 2\delta\sigma_1(A_{w_i})g = Ric_{u_i} - 2\delta\sigma_1(A_{u_i})g \geq 0$.

(2) By Proposition 4.4 (2), we can find $J(r) \in \mathbb{N}$ such that $e^{-2\inf_{M_{r/2}} u_i} \leq r^{-2}$ when $i > J$. Therefore, for any $x \in M_r$ and $i > J$, we have

$$\begin{aligned}
& \sup_{B(x,r/4)} \left(|\nabla^2 w_i| + |\nabla w_i|^2\right) \\
= & \sup_{B(x,r/4)} \left(|\nabla^2 u_i| + |\nabla u_i|^2\right) \\
\leq & C_1\Big(r^{-2} + e^{-2\inf_{M_{\sqrt{10}r/2}} u_i}\Big) \\
\leq & 2C_1 r^{-2}.
\end{aligned}$$

Since M_r is compact, by finite covering argument, we know that there is a constant $C = C(n, g, f)$ such that

$$|\nabla^2 w_i| + |\nabla w_i|^2 \leq Cr^{-2}$$

for all $x \in M_r$ and $i > J = J(r)$.

(3) We may assume that r is small enough and M_r contains x'. By (2) $|\nabla w_i|^2(x) \leq 2C_1 r^{-2}$ in M_r. Then

$$\sup_{M_r} w_i \leq \inf_{M_r} w_i + C_2 \leq w_i(x') + C_2 = C_2,$$

where C_2 depends on n, g, f and r. By Lemma 4.2., we get a global upper bound $\max_M w_i \leq C$. For the lower bound, we have

$$\inf_{M_r} w_i \geq \sup_{M_r} w_i - C_2 \geq w_i(x') - C_2 = -C_2.$$

□

Then Arzela-Ascoli theorem implies that a subsequence of w_i (denoted by w_i again) converges on compact sets $K \subset M \setminus \Sigma_0$ in $C^{1,\alpha}(K)$ where $\alpha \in (0, 1)$. Hence, from Rademacher theorem, $\nabla^2 w$ is well defined almost everywhere. Thus we can obtain the following corollary immediately.

Corollary 5.1. (1) *The limit $w = \lim_i w_i$ is in $C^{1,1}_{loc}(M \setminus \Sigma_0)$.*

(2) *$S_w = Ric_w - 2\delta\sigma_1(A_w)g$ is positive semi-definite almost every where in M.*

Now we may consider a doubling manifold $\widetilde{M}$ of M by gluing two copies of M along the boundary ∂M. With the given smooth Riemannian metric g on M, there is a standard metric $\widetilde{g}$ on $\widetilde{M}$ induced from g. When ∂M is totally geodesic in (M, g), then $\widetilde{g}$ is $C^{2,1}$ on $\widetilde{M}$, see [E] for instance.

If we denote corresponding double of Σ_0 by $\widetilde{\Sigma_0}$, then we can extend w to a $C^{1,1}_{loc}(\widetilde{M} \setminus \widetilde{\Sigma_0})$ function $\widetilde{w}$ as follows:

Near the boundary we take Fermi Coordinates, $\widetilde{w}$ is then defined as

$$\widetilde{w}(x_1, \cdots, x_n) = \begin{cases} w(x_1, \cdots, x_n), & x_n \geq 0, \\ w(x_1, \cdots, -x_n), & x_n \leq 0. \end{cases}$$

Since ∇w is locally Lipschitz, then $\nabla \widetilde{w}$ is the same. In fact, take a geodesic convex neighborhood $\widetilde{B}(x, \widetilde{r})$ centered at any $x \in \partial M$. We may assume p and q are two points with $x_n(p) \geq 0$ and $x_n(q) \leq 0$. Then the geodesic connecting p and q is contained in $\widetilde{B}(x, \widetilde{r})$ and must pass across the boundary ∂M. Thus there exists a point z in $\widetilde{B}(x, \widetilde{r}) \cap \partial M$ such that $\widetilde{dist}(p, q) = \widetilde{dist}(p, z) + \widetilde{dist}(z, q)$ where the distance function under metric $\widetilde{g}$ denote by $\widetilde{dist}(\cdot, \cdot)$. Since $\nabla_{\frac{\partial}{\partial x^i}} w$ $(1 \leq i \leq n)$ is a locally Lipschitz function we know that there exist a constant L such that $|\nabla_{\frac{\partial}{\partial x^i}} w(p) - \nabla_{\frac{\partial}{\partial x^i}} w(z)| \leq L \cdot dist(p, z)$ and $|\nabla_{\frac{\partial}{\partial x^i}} \widetilde{w}(z) - \nabla_{\frac{\partial}{\partial x^i}} \widetilde{w}(q)| \leq L \cdot \widetilde{dist}(z, q)$. Therefore,

$$\begin{aligned} & |\nabla_{\frac{\partial}{\partial x^i}} \widetilde{w}(p) - \nabla_{\frac{\partial}{\partial x^i}} \widetilde{w}(q)| \\ \leq \; & |\nabla_{\frac{\partial}{\partial x^i}} \widetilde{w}(p) - \nabla_{\frac{\partial}{\partial x^i}} \widetilde{w}(z)| + |\nabla_{\frac{\partial}{\partial x^i}} \widetilde{w}(z) - \nabla_{\frac{\partial}{\partial x^i}} \widetilde{w}(q)| \\ \leq \; & L\Big(\widetilde{dist}(p, z) + \widetilde{dist}(z, q)\Big) \\ = \; & L \cdot \widetilde{dist}(p, q). \end{aligned}$$

It is obviously that $\widetilde{w}$ is a C^1 function. Now we may conclude that$\widetilde{w} \in C^{1,1}_{loc}(\widetilde{M} \setminus \widetilde{\Sigma_0})$. Then the following Corollary is immediately.

Corollary 5.2. (1) *The limit* $\widetilde{w} = \lim_i \widetilde{w}_i$ *is in* $C^{1,1}_{loc}(\widetilde{M} \setminus \widetilde{\Sigma_0})$.

(2) $S_{\widetilde{w}} = Ric_{\widetilde{w}} - 2\delta\sigma_1(A_{\widetilde{w}})\widetilde{g}$ *is positive semi-definite almost every where in* $(\widetilde{M}, \widetilde{g})$.

By a similar proof as it in §6 and §7 of [GV1], we can get the following proposition.

Proposition 5.2. (1) *There exists an isometry*

$$\Phi : (\widetilde{M}_{reg}, e^{-2\widetilde{w}}\widetilde{g}) \to (\mathbb{R}^n, g_{Euc}),$$

where g_{Euc} *is the Euclidean metric and* $\widetilde{M}_{reg} = \widetilde{M} \setminus \widetilde{\Sigma_0}$.

(2) $(M_{reg}, e^{-2w}g)$ *is isometric to the half-plane in Euclidean space, where* $M_{reg} = M \setminus \Sigma_0$.

Thus, Theorem 1.1. follows immediately, since that (M, g) is assumed to be not conformally equivalent to a standard hemisphere.

6 Proof of Theorem 1.2.

Note that when $k = n/2$, we can not find a positive δ such that $\mathrm{Ric_g} \geq \delta\sigma_1(\mathrm{A_g})\mathrm{g}$. Therefore the proof of Theorem 1.1 can not go through in the case of $k = n/2$. Nevertheless, any k-admissible solution w ($k \geq n/2$) on (M, g) satisfies another crucial inequality (see [TW2])

$$W_{nn} + \frac{1}{n-2}\sum_{k=1}^{n} W_{kk} \geq 0, \tag{6.1}$$

where $W_{ij} = w_{ij} + w_i w_j - \frac{g_{ij}}{2}(\Sigma_{k=1}^n w_k)^2 + (A_g)_{ij}$.

Let u_i be a sequence of k-admissible solutions to equation (1.2). In [TW2], Trudinger and Wang consider the rescaled k-admissible solutions $w_j = u_j - \sup_M u_j$ and prove the rescaled sequence w_j converges in $W^{1,p}$ (for any $1 < p < \frac{n}{n-1}$)) to an admissible function w. Roughly speaking, if $\bar{x}$ is a blowup point of w, then inequality (6.1) implies one side of the estimate for the limit function w near $\bar{x}$:

$$w(x) \leq 2\log d(x, \bar{x}) + C. \tag{6.2}$$

Furthermore, they prove

$$w(x) = 2\log d(x, \bar{x}) + o(1), \tag{6.3}$$

where $o(1) \to 0$ when $d(x, \bar{x}) \to 0$. From (6.3), one can show that each blowup point is isolated, which implies the number of blowup points is finite. Combining the fact that $\mathrm{Ric_g} \geq 0$, by the volume comparison theorem, one can show as in [GV1, TW1] the ratio of the volume of the geodesic ball of radius r in the metric $e^{-2w}g$ with that of the Euclidean ball, is non-increasing. Therefore w has exactly one blowup point 0 and the manifold $(M \setminus \{0\}, e^{-2w}g)$ is isometric to the Euclidean space which is contradiction with the assumption. Therefore there is a unform L^∞ bound for solutions and so the set of solutions is compact.

Now, similarly, when dealing with manifold with boundary we expect to prove the conformal metric $e^{-2w}g$ is in fact Euclidean metric on half-plane and get a contradiction, where w is the limit function of the rescaled sequence $w_j = u_j - \sup_M u_j$ on manifold with boundary.

To this end, we will double the manifold (M, g). Specifically speaking, given a smooth Riemannian metric g on M, there is a standard metric $\widehat{g}$ on $\widehat{M}$ induced from g, which is glued by two copies of M along the boundary ∂M. When ∂M is totally geodesic in (M, g), then $\widehat{g}$ is $C^{2,1}$ on $\widehat{M}$, see [E].

Then we extend the functions w_j to a function $\hat{w}_j$ on $\widehat{M}$ as follows:

$$\hat{w}_j(x_1, \cdots, x_n) = \begin{cases} w_j(x_1, \cdots, x_n), & x_n \geq 0, \\ w_j(x_1, \cdots, -x_n), & x_n \leq 0, \end{cases}$$

where we take Fermi Coordinates near the boundary as before. We firstly verify that $\hat{w}_j$ satisfies the preliminary Lemmas in section 2 of [TW2]. However, from the boundary condition we can see that $\hat{w}_j$ are in fact C^2 $k-$admissible functions on $(\widehat{M}, \widehat{g})$. Let us calculate under Fermi coordinates:

$$\begin{aligned} &\lim_{x_n \to 0^+} \frac{\partial \hat{w}_j}{\partial x^n}(x_1, \cdots, x_n) = \frac{\partial w_j}{\partial x^n}(x_1, \cdots, x_{n-1}, 0) \\ = \;& 0 = -\frac{\partial w_j}{\partial x^n}(x_1, \cdots, x_{n-1}, 0) = \lim_{x_n \to 0^-} \frac{\partial \hat{w}_j}{\partial x^n}(x_1, \cdots, x_n), \end{aligned}$$

and

$$\lim_{x_n \to 0^+} \frac{\partial^2 \hat{w}_j}{\partial (x^n)^2}(x_1, \cdots, x_n) = \lim_{x_n \to 0^-} \frac{\partial^2 \hat{w}_j}{\partial (x^n)^2}(x_1, \cdots, x_n).$$

Thus immediately from the $k-$admissible property of w_j we know that $\hat{w}_j$ are $k-$admissible and sub-harmonic with some elliptic operator.

As a matter of fact, we may extend the definition of k-admissible and sub-harmonic(super-harmonic) in the viscosity sense (see [TW2] for details).

We call a metric $\tilde{g} = \mathcal{X} g$ is $k-$admissible if $\mathcal{X}$ is lower semi-continuous, does not equal to ∞, and there exists a sequence of $k-$admissible functions $\mathcal{X}_m \in C^2(M)$ such that $\mathcal{X}_m \to \mathcal{X}$ almost everywhere in M.

We say a function v is super-harmonic with respect to a elliptic operator $\mathcal{L}$ if (i) v is lower semicontinuous (l.s.c.); (ii) v does not equal to ∞ in any open set; (iii) for any open subset $D \subset M$ and any function $h \in C^2(\bar{D})$ satisfying $\mathcal{L}(h) = 0$ in D and $h \leq v$ on ∂D, we have $h \leq v$ in D. Subharmonic functions are defined as the negative of super-harmonic ones (See page 131 of [HKM]).

As a result, if $\hat{w}_j$ is $k-$admissible, then the function $\hat{v}_j = e^{-\frac{n-2}{2}\hat{w}_j}$ is super-harmonic with respect to the conformal laplace operator $\mathcal{L} \triangleq \triangle_g - \frac{n-2}{4(n-1)} R_g$. The corresponding maximal (minimal) radial functions are also super-harmonic (subharmonic), where the maximal and minimal radial functions in $B_R(x_0)$ are defined by

$$\widetilde{\hat{v}}(x) = \inf\{\hat{v}(y) : y \in \partial B_r(x_0), r = d(x, x_0)\}$$

and

$$\widetilde{\hat{w}}(x) = \sup\{\hat{w}(y) : y \in \partial B_r(x_0), r = d(x, x_0)\},$$

respectively, where $r \leq R$.

Now that $\hat{w}_j$ ($\hat{v}_j$) and $\widetilde{\hat{w}}_j$ ($\widetilde{\hat{v}}_j$) maintain the subharmonic (super-harmonic) property, thus Lemma 2.1, Lemma 2.2, Corollary 2.1 and hence Corollary 2.2 and Corollary 2.3 in [TW2] still hold for $\hat{v}$:

Lemma 6.1(Corollary 2.2 of [TW2]). *Let $\hat{v}$ be super-harmonic with respect to the conformal Laplacian operator $\mathcal{L}$. Then the maximal radial function $\widetilde{\hat{v}}(r)$ is locally uniformly Hölder continuous away from 0, with Hölder exponent $\alpha \in (0, 1/n)$.*

Lemma 6.2(Corollary 2.3 of [TW2]). *Let $\hat{v}_j$ be a sequence of super-harmonic functions which converges to $\hat{v}$ in $L^1(B_r(0))$. Then the corresponding maximal radial functions $\widetilde{\hat{v}}_j(r)$ converges locally uniformly to $\widetilde{\hat{v}}(r)$.*

Proof of Theorem 1.2:

By use of the argument in [TW2] and [GV1], we only sketch the proof here. Suppose $x_{0,j}$ is a blow up sequence of u_j and $\bar{x} = \lim_{j\to\infty} x_{0,j}$. Let x_0^j be the maximum point of u_j. Since $e^{-2\sup u_j} f(x_0^j) = e^{-2u_j(x_0^j)} f(x_0^j) \leq C(\triangle u_j + A_g)(x_0^j) \leq C$ and thus $\sup_M u_j > -\infty$, $x_{0,j}$ is also a blow up sequence of $w_j = u_j - \sup_M u_j$, and $\hat{w}_j(x_{0,j}) \to -\infty$, $\hat{v}_j(x_{0,j}) \to +\infty$.

Now we are going to prove the limit of $\hat{w} = \lim_{j\to\infty} \hat{w}_j$ satisfies (6.3). We begin with two observations:

(1) If we denote $\hat{v}_j(x_{0,j})^{\frac{1}{n-2}}$ by R_j and $\frac{1}{1-2^{-\frac{1}{n-2}}}$ by A_0, then when j is large enough in $B(x_{0,j}, A_0 R_j^{-1})$ there must be some local maximum points of $\hat{v}_j$, named by x_j. Furthermore, x_j is the local maximum points of $\hat{v}_j$ in $B(x_j, \frac{1}{2}\hat{v}_j(x_j)^{-\frac{1}{n-2}})$ yet (see Lemma 3.2. in [TW2] for details).

(2) Note that maximal and minimal radial functions depend only on distance to the center. Thus we may denote that

$$\widetilde{\hat{w}}_j(r) = \sup\{\hat{w}_j(y) : y \in \partial B_r(x_j)\},$$

and

$$\widetilde{\hat{w}}(r) = \sup\{\hat{w}(y) : y \in \partial B_r(\bar{x})\}.$$

In virtue of Lemma 6.1 and Lemma 6.2, we can get $\widetilde{\hat{w}}(r) = \lim\limits_{j\to\infty} \widetilde{\hat{w}}_j(r)$.

Then by a similar argument in section 3 of [TW2], we can see that $\hat{w}$ satisfies (6.3) in $(\widehat{M}, \widehat{g})$ and singular points are isolated. Furthermore, since the Ricci curvature of

$(\widehat{M}, \widehat{g})$ is still non-negative, by the Volume Comparison Theorem, there is at most one end and away from the singular points; the metric $e^{-2\hat{w}}\widehat{g}$, a doubling of $e^{-2w}g$, is in fact a Euclidean one (see section 7 of [GV1] for details). Finally, restricting the argument to manifold M, we can see $(M \setminus \{\bar{x}\}, e^{-2w}g\ )$ is just the half-plane in Euclidean space, which contradicts with the assumption. Therefore there is a unform L^∞ bound for solutions and so the set of solutions is compact. This completes the proof of Theorem 1.2. $\square$

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