Local Estimates for Some Elliptic Equations Arising from Conformal Geometry

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Abstract

We present some local gradient and Hessian estimates from C^0 estimates for some elliptic equations arising from conformal deformations on the manifolds with totally geodesic boundary, which generalize Sophie Chen's corresponding results.

1 Introduction

Let (M^n, g) be a smooth, compact Riemannian manifold of dimension $n \geq 3$. The Schouten tensor of g is defined by

$$A_g = \frac{1}{n-2}\left(\mathrm{Ric}_g - \frac{R_g}{2(n-1)}g\right),$$

where Ric_g and R_g are the Ricci and scalar curvatures of g, respectively. The k-curvature (or σ_k curvature) is defined to be the k-th elementary symmetric function σ_k of the eigenvalues $\lambda(g^{-1}A_g)$ of $g^{-1}A_g$. If $\tilde{g} = e^{-2u}g$ is a metric conformal to g, the Schouten tensor transforms according to the formula

$$A_{\tilde{g}} = \nabla^2 u + du \otimes du - \frac{1}{2}|\nabla u|^2 g + A_g,$$

where ∇u and $\nabla^2 u$ denote the gradient and Hessian of u with respect to g. Consequently, the problem of conformally deforming a given metric to one with prescribed σ_k-curvature reduces to solving the partial differential equation

$$\sigma_k\Big(\lambda\Big(g^{-1}\Big[\nabla^2 u + du \otimes du - \frac{1}{2}|\nabla u|^2 g + A_g\Big]\Big)\Big) = \psi(x)e^{-2ku}. \tag{1.1}$$

For compact manifolds without boundary, the existence of the solutions to the equation (1.1) has been studied by many authors (see [CGY1], [CGY2], [GW2], [GW3],

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[LL1], [LL2], [GV1], [GV2], [TW], [STW], [GeW], [V2] ect.) since these equations were first introduced by J. A. Viaclovsky [V1]. C^1 and C^2 estimates have also been studied extensively, see [Cn1], [GW1], [GW2], [LL1], [STW], [W2] for local interior estimates and [V2] for global estimates.

Another interesting problem is to study the fully nonlinear equation (1.1) on a compact Riemannian manifold (M^n, g) with boundary ∂M. In [G], Bo Guan studied the existence problem under the Dirichlet boundary condition. There are many poineer works on the Dirichlet problems for fully nonlinear elliptic equtions, see [CNS], [Tr2] ect.. The Neumann problem for (1.1) has been studied by S. S. Chen [Cn2, Cn3], Jin-Li-Li [JLL] and Li-Li [LL3], ect.. Under various conditions, they derive local estimates for solutions and establish some existence results. Before introducing the problem, we need the following definitions.

Define

$$\Gamma_k = \{\Lambda = (\lambda_1, \lambda_2, \dots, \lambda_n) \in \mathbb{R}^n | \quad \sigma_j(\Lambda) > 0, 1 \le j \le k\},$$

and $1 \le k \le n$, where σ_k is the k-th elementary symmetric function defined by

$$\sigma_k(\Lambda) = \sum_{1 \le i_1 < \cdots < i_k \le n} \lambda_{i_1} \cdots \lambda_{i_k}$$

for all $\Lambda = (\lambda_1, \lambda_2, \dots, \lambda_n) \in \mathbb{R}^n$. We also denote $\sigma_0 = 1$. Therefore we have the relation $\Gamma_n \subset \Gamma_{n-1} \subset \cdots \subset \Gamma_1$. For a 2-symmetric form S defined on (M^n, g), $S \in \Gamma_k$ means that the eigenvalues of S, $\lambda(g^{-1}S)$ lie in Γ_k. We also denote $\Gamma_k^- = -\Gamma_k$.

Let $(M^n, g), n \ge 3$, be a smooth compact Riemannian manifold with nonempty smooth boundary ∂M. We denote the mean curvature and the second fundamental form of ∂M by h_g and $L_{\alpha\beta}$, where $\{x^\alpha\}_{1 \le \alpha \le n-1}$ is the local coordinates on the boundary ∂M, and $\frac{\partial}{\partial x^n}$ the unit inner normal with respect to the metric. In this paper we use Fermi coordinates in a boundary neighborhood. In this local coordinates, we take the geodesic in the inner normal direction $\nu = \frac{\partial}{\partial x^n}$ parameterized by arc length, and $(x^1, ..., x^{n-1})$ forms a local chart on the boundary. The metric can be expressed as $g = g_{\alpha\beta} dx^\alpha dx^\beta + (dx^n)^2$. The Greek letters $\alpha, \beta, \gamma, ...$stand for the tangential direction indices, $1 \le \alpha, \beta, \gamma, ... \le n-1$, while the Latin letters $i, j, k, ...$stand for the full indices, $1 \le i, j, k, ... \le n$. In Fermi coordinates, the half ball is defined by $\overline{B}_r^+ = \{x_n \ge 0, \sum_i x_i^2 \le r^2\}$ and the segment on the boundary by $\Sigma_r = \{x_n = 0, \sum_i x_i^2 \le r^2\}$. Under the conformal change of the metric $\widetilde{g} = e^{-2u} g$, the second fundamental form satisfies

$$\widetilde{L}_{\alpha\beta} e^u = \frac{\partial u}{\partial \nu} g_{\alpha\beta} + L_{\alpha\beta}.$$

The boundary is called umbilic if the second fundamental form $L_{\alpha\beta} = \tau_g g_{\alpha\beta}$, where τ_g is the function defined on ∂M. Since the boundary ∂M is connected, by Schur Theorem, $\tau_g = const.$. A totally geodesic boundary is umbilic with $\tau_g = 0$. Note that the umbilicity is conformally invariant. When the boundary is umbilic, the above formula becomes

$$\tau_{\widetilde{g}} e^{-u} = \frac{\partial u}{\partial \nu} + \tau_g.$$

The k-Yamabe problem with umbilic boundary becomes to consider the following equation:

$$\begin{cases} \sigma_k^{1/k}\Big(\lambda\Big(g^{-1}\Big[\nabla^2 u + du \otimes du - \frac{1}{2}|\nabla u|^2 g + A_g\Big]\Big)\Big) = e^{-2u} & \text{in } M, \\ \frac{\partial u}{\partial \nu} = \tau_{\widetilde{g}} e^{-u} - \tau_g & \text{on } \partial M. \end{cases} \tag{1.2}$$

In [Cn2, Cn3] and [JLL], the authors established the a priori estimates and obtained some existence results for (1.2).

In this paper, we will generalize their results to more general equations, which in particular include the equation (1.2). In [GV3], Gursky and Viaclovsky introduced a modified Schouten tensor

$$A_g^t = \frac{1}{n-2}\left(\mathrm{Ric}_g - \frac{tR_g}{2(n-1)} g\right),$$

where $t \in \mathbb{R}$ is a parameter. When $t = 1$, A_g^1 is just the Schouten tensor; $t = n-1$, A_g^{n-1} is the Einstein tensor; while $t = 0$, A_g^0 is the Ricci tensor. Under the conformal change of the metric $\widetilde{g} = e^{-2u} g$, $A_{\widetilde{g}}^t$ satisfies

$$A_{\widetilde{g}}^t = A_g^t + \nabla^2 u + \frac{1-t}{n-2}(\triangle u) g + du \otimes du - \frac{2-t}{2}|\nabla u|^2 g.$$

In [LS] and [SZ], we have studied

$$\sigma_k\left(\lambda\left(g^{-1}\left[A_g^t + \nabla^2 u + \frac{1-t}{n-2}(\triangle u) g + du \otimes du - \frac{2-t}{2}|\nabla u|^2 g\right]\right)\right) = f(x)\, e^{-2ku}$$

for $t \le 1$ or $t \ge n-1$. By use of the parabolic approach, we obtained some existence results. Let (M, g) be a compact, connected Riemannian manifold of dimension $n \ge 3$ with umbilic boundary ∂M, W be a $(0,2)$ symmetric tensor on (M^n, g). Motivated by [Cn1], in this paper we study the following equation

$$\begin{cases} F(g^{-1}W) = f(x, u) & \text{in } M \\ \frac{\partial u}{\partial \nu} = \widetilde{\tau} e^{-u} - \tau & \text{on } \partial M \end{cases} \tag{1.3}$$

where F satisfies some fundamental structure conditions listed later, and τ is the principal curvature of the boundary ∂M. We will establish local a priori estimates for the solutions to the equation (1.3). After that, we will give some applications.

We now describe the fundamental structure conditions for F.

Let Γ be an open convex cone with vertex at the origin satisfying $\Gamma_n \subset \Gamma \subset \Gamma_1$. Suppose that $F(\lambda)$ is a homogeneous symmetric function of degree one in Γ normalized with $F(e) = F((1, \cdots, 1)) = 1$. Moreover, F satisfies the following in Γ:

(A1) F is positive.

(A2) F is concave (i.e., $\frac{\partial^2 F}{\partial\lambda_i \partial\lambda_j}$ is negative semi definite).

(A3) F is monotone (i.e., $\frac{\partial F}{\partial \lambda_i}$ is positive).

(A4) $\frac{\partial F}{\partial \lambda_i} \geq \epsilon \frac{F}{\sigma_1}$, for some constant $\epsilon > 0$, for all i.

We also need an additional condition on the boundary ∂M in the Fermi coordinates:

(B) either $\rho_1 F^{nn} \geq \sum F^{\alpha\alpha}$ or $\rho_2 F^{nn} \leq \sum F^{\alpha\alpha}$, for some $\rho_1, \rho_2 > 0$.

The conditions (A1), (A2), (A3) and (A4) are similar with those in Szu-yu Sophie Chen [Cn2] and [Cn3]. While condition (B) is in some extent different, however, we will show it is reasonable later for the case that $F = \sigma_k^{1/k}$.

Before stating the theorems, we introduce the following notations. Let $f(x, z) : M^n \times \mathbb{R} \to \mathbb{R}$ be given positive function. Let $u = u(x) : M^n \to \mathbb{R}$ be a solution to (1.3). We define

$$\overline{c_{sup}}(r) = sup_{\overline{B}_r^+}(f + |\nabla_x f(x,u)| + |f_z(x,u)| + |\nabla_x^2 f(x,u)| + |\nabla_x f_z(x,u)| + |f_{zz}(x,u)|)$$

or

$$c_{sup}(r) = sup_{B_r}(f + |\nabla_x f(x,u)| + |f_z(x,u)| + |\nabla_x^2 f(x,u)| + |\nabla_x f_z(x,u)| + |f_{zz}(x,u)|),$$

which varies with boundary or interior estimates.

Now we turn to the first equation we are going to discuss: let

$$W = \nabla^2 u + \frac{1-t}{n-2}(\triangle u)\ g + a(x) du \otimes du + b(x)|\nabla u|^2 g + S, \tag{1.4}$$

where t is a constant satisfying $t \leq 1$, S a 2-symmetric form defined on M, and $a(x)$, $b(x)$ are two smooth functions on M. The derivatives are covariant derivatives with respect to the metric g. We have

Theorem 1. *Let F satisfy the structure conditions (A1)-(A4), additional condition (B) in a corresponding cone Γ . Suppose that the boundary ∂M is totally geodesic on the boundary. Let $u(x)$ be a C^4 solution to the equation*

$$\begin{cases} F(g^{-1}(\nabla^2 u + \frac{1-t}{n-2}\triangle u\ g + a(x) du \otimes du + b(x)|\nabla u|^2 g + S)) = f(x,u) & \text{in } \overline{B}_r^+, \\ \frac{\partial u}{\partial x^n} = 0 & \text{on } \Sigma_r. \end{cases} \tag{1.5}$$

Suppose that $|\nabla f| < \Lambda f$, $|f_z| < \Lambda f$ *for some constant* $\Lambda > 0$.

Case (a)

I. If $\Gamma \subset \Gamma_2^+$, $\frac{1-t}{n-2}a(x) - b(x) > \delta_1 > 0$, $\min\left\{(a+b)^2, b^2\right\} \geq \delta_2 > 0$ *and* $t < 1$ *then*

$$sup_{\overline{B}^+_{\frac{r}{2}}}\left(|\nabla^2 u| + |\nabla u|^2\right) \leq C,$$

where C *depends only on* $r, n, \epsilon, \rho_1, \rho_2, \mu, \Lambda, \delta_1, \delta_2, ||a||_{C^2(\overline{B}^+_r)}, ||b||_{C^2(\overline{B}^+_r)}, ||S||_{C^2(\overline{B}^+_r)}, ||g||_{C^3(\overline{B}^+_r)}$ *and* $\overline{c_{sup}}(r)$.

II. If $t = 1, \Gamma \subset \Gamma_2^+$, $-b > \delta_1 > 0$, $\min\left\{(a+b)^2, b^2\right\} \geq \delta_2 > 0$, *and* $a = \text{const.}, b = \text{const.}, S = A$ *(Schouten tensor), then*

$$sup_{\overline{B}^+_{\frac{r}{2}}}\left(|\nabla^2 u| + |\nabla u|^2\right) \leq C,$$

where C *depends only on* $r, n, \epsilon, \rho_1, \rho_2, \mu, \Lambda, \delta_1, \delta_2, ||a||_{C^2(\overline{B}^+_r)}, ||b||_{C^2(\overline{B}^+_r)}, ||S||_{C^2(\overline{B}^+_r)}, ||g||_{C^3(\overline{B}^+_r)}$ *and* $\overline{c_{sup}}(r)$.

Case (b)

I. If we have $\frac{1-t}{n-2}a(x) - b(x) > \delta_1 > 0$, $a(x) + nb(x) < -\delta_3 < 0$,$a(x) \geq 0$ and $t < 1$, *then*

$$sup_{\overline{B}^+_{\frac{r}{2}}}\left(|\nabla^2 u| + |\nabla u|^2\right) \leq C,$$

where C *depends on* $r, n, \epsilon, \rho_1, \rho_2, \mu, \Lambda, \delta_1, \delta_3, ||a||_{C^2(\overline{B}^+_r)}, ||b||_{C^2(\overline{B}^+_r)}, ||S||_{C^2(\overline{B}^+_r)}, ||g||_{C^3(\overline{B}^+_r)}$ *and* $\overline{c_{sup}}(r)$.

II. If we have $t = 1, -b > \delta_1 > 0$, $a + nb < -\delta_3 < 0$, $a \geq 0$ *and* $a = \text{const.}, b = \text{const.}, S = A$ *(Schouten tensor), then*

$$sup_{\overline{B}^+_{\frac{r}{2}}}\left(|\nabla^2 u| + |\nabla u|^2\right) \leq C,$$

where C *depends on* $r, n, \epsilon, \rho_1, \rho_2, \mu, \Lambda, \delta_1, \delta_3, ||a||_{C^2(\overline{B}^+_r)}, ||b||_{C^2(\overline{B}^+_r)}, ||S||_{C^2(\overline{B}^+_r)}, ||g||_{C^3(\overline{B}^+_r)}$ *and* $\overline{c_{sup}}(r)$.

When $t = 1$, the local interior estimates have been discussed by S. Chen in [Cn1]. When $t = 1$, and $a(x) = 1$, $b(x) = -\frac{1}{2}$, the bounary estimates have been obtained by S. Chen [Cn2, Cn3] and Jin-Li-Li [JLL]. If we just focus on the interior estimates for the same equation, we may get

Corollary 1. *Let* F *satisfy the structure conditions (A1)-(A4) in a corresponding cone* Γ. *And* $u(x)$ *be a* C^4 *solution to the equation*

$$F(g^{-1}(\nabla^2 u + \tfrac{1-t}{n-2}\triangle u\ g + a(x)du \otimes du + b(x)|\nabla u|^2 g + S)) = f(x, u) \quad \text{in} \quad M \qquad (1.6)$$

in a local geodesic ball B_r. Suppose that $|\nabla f| < \Lambda f$, $|f_z| < \Lambda f$ for some constant $\Lambda > 0$.
Case (a). If $\Gamma \subset \Gamma_2^+$, $\frac{1-t}{n-2}a(x) - b(x) > \delta_1 > 0$ and $\min\left\{(a+b)^2, b^2\right\} \geq \delta_2 > 0$,then

$$sup_{B_{\frac{r}{2}}}\left(|\nabla^2 u| + |\nabla u|^2\right) \leq C,$$

where C depends only on $r, n, \Lambda, \delta_1, \delta_2, ||a||_{C^2(B_r)}, ||b||_{C^2(B_r)}, ||S||_{C^2(B_r)}, ||g||_{C^3(B_r)}$ and $c_{sup}(r)$.
Case (b). If we have $\frac{1-t}{n-2}a(x) - b(x) > \delta_1 > 0$ and $a(x) + nb(x) < -\delta_3 < 0$. Then we have

$$sup_{B_{\frac{r}{2}}}\left(|\nabla^2 u| + |\nabla u|^2\right) \leq C,$$

where C depends on $r, n, \epsilon, \mu, \Lambda, \delta_1, \delta_3, ||a||_{C^2(B_r)}, ||b||_{C^2(B_r)}, ||S||_{C^2(B_r)}, ||g||_{C^3(B_r)}$ and $c_{sup}(r)$.

The a priori estimate in Theorem 1 and Corollary 1 relies on the signs of $a(x)$ and $b(x)$. In fact, in [STW] the authors give a counterexample to show that there is no regularity if $a(x) = 0$ and $b(x)$ is positive. It is well known that the equation (1.6) has another elliptic branch, namely when the eigenvalues λ lie in the negative cone Γ_k^-. Now we consider the second equation. Let

$$V = \frac{t-1}{n-2}(\triangle u)\ g - \nabla^2 u - a(x)du \otimes du - b(x)|\nabla u|^2 g + S,$$

where t is a constant satisfying $t > n-1$. The derivatives are covariant derivatives with respect to the metric g. We have

Theorem 2. *Let F satisfy the structure conditions (A1)-(A4), additional condition (B) in a corresponding cone Γ. Suppose that the boundary ∂M is totally geodesic on the boundary. Let $u(x)$ be a C^4 solution to the equation*

$$\begin{cases} F(g^{-1}(\frac{t-1}{n-2}(\triangle u)g - \nabla^2 u - a(x)du \otimes du - b(x)|\nabla u|^2 g + S)) = f(x,u) & \text{in } \overline{B}_r^+, \\ \frac{\partial u}{\partial x^n} = 0 & \text{on } \Sigma_r. \end{cases} \tag{1.7}$$

Suppose that $|\nabla f| < \Lambda f$, $|f_z| < \Lambda f$ for some constant $\Lambda > 0$.
Case (a). *If $\Gamma \subset \Gamma_2^+$, $\frac{t-1}{n-2}a(x) + b(x) > \delta_1 > 0$, $\min\left\{(a+b)^2, b^2\right\} \geq \delta_2 > 0$, and $\frac{1}{\rho_2} - \frac{t-1}{n-2} < 0$, then*

$$sup_{\overline{B}^+_{\frac{r}{2}}}\left(|\nabla^2 u| + |\nabla u|^2\right) \leq C,$$

where C depends on $r, n, \epsilon, \rho_1, \rho_2, \mu, \Lambda, \delta_1, \delta_2, ||a||_{C^2(\overline{B}_r^+)}, ||b||_{C^2(\overline{B}_r^+)}, ||S||_{C^2(\overline{B}_r^+)}, ||g||_{C^3(\overline{B}_r^+)}$ and $\overline{c_{sup}}(r)$.

Case (b). *If we have* $a(x)+nb(x)>\delta_3>0, a\,(x)\geq 0$ *and* $\frac{1}{\rho_2}-\frac{t-1}{n-2}<0$, *then*

$$sup_{\overline{B}^+_{\frac{r}{2}}}\left(|\nabla^2 u|+|\nabla u|^2\right)\leq C,$$

where C *depends on* $r, n, \epsilon, \rho_1, \rho_2, \mu, \Lambda, \delta_1, \delta_3, ||a||_{C^2(\overline{B}^+_r)}, ||b||_{C^2(\overline{B}^+_r)}, ||S||_{C^2(\overline{B}^+_r)}, ||g||_{C^3(\overline{B}^+_r)}$ *and* $\overline{c_{sup}}(r)$.

There is following consequence when we just consider the interior estimates especially for the closed manifold without boundary.

Corollary 2. *Let* F *satisfy the structure conditions (A1)-(A4) in a corresponding cone* Γ. *And* $u(x)$ *be a* C^4 *solution to the equation*

$$F(g^{-1}(\tfrac{t-1}{n-2}\,(\triangle u)\,g-\nabla^2 u-a(x)du\otimes du-b(x)|\nabla u|^2 g+S))=f(x,u)\quad \text{in } M \tag{1.8}$$

in a local geodesic ball B_r. *Suppose that* $|\nabla f|<\Lambda f$, $|f_z|<\Lambda f$ *for some constant* $\Lambda>0$.

Case (a). *If* $\Gamma\subset\Gamma_2^+$, $\frac{t-1}{n-2}a(x)+b(x)>\delta_1>0$, $\min\left\{(a+b)^2, b^2\right\}\geq\delta_2>0$, *then*

$$sup_{B_{\frac{r}{2}}}\left(|\nabla^2 u|+|\nabla u|^2\right)\leq C,$$

where C *depends only on* $r, n, \Lambda, \delta_1, \delta_2, ||a||_{C^2(B_r)}, ||b||_{C^2(B_r)}, ||S||_{C^2(B_r)}, ||g||_{C^3(B_r)}$ *and* $c_{sup}(r)$.

Case (b). *If we have* $\frac{t-1}{n-2}a(x)+b(x)>\delta_1>0$, $a(x)+nb(x)>\delta_3>0$, *then*

$$sup_{B_{\frac{r}{2}}}\left(|\nabla^2 u|+|\nabla u|^2\right)\leq C,$$

where C *depends on* $r, n, \Lambda, \delta_1, \delta_3, ||a||_{C^2(B_r)}, ||b||_{C^2(B_r)}, ||S||_{C^2(B_r)}, ||g||_{C^3(B_r)}$ *and* $c_{sup}(r)$.

This paper is organized as follows. We would begin with some background in Section 2. In Section 3, we will discuss the applications which are based on the a priori estimates in Theorem 1 and corollary 1 to get the existent result of k-Yamabe problem. The proof of Theorem 1 and Theorem 2 are in Section 4 and Section 5 respectively.

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2 Preliminaries

In this section, we give some basic facts about homogeneous symmetric functions and show some outcomes by direct calculation under Fermi coordinates.

From Lemma 1 and Lemma 2 below, we can conclude that F satisfies (A1)-(A4).

Lemma 1. *(see [Cn1]) Let Γ be an open convex cone with vertex at the origin satisfying $\Gamma_n^+ \subset \Gamma$, and let $e = (1, \cdots, 1)$ be the identity. Suppose that F is a homogeneous symmetric function of degree one normalized with $F(e) = 1$, and that F is concave in Γ. Then*
(a) $\sum_i \lambda_i \frac{\partial F}{\partial \lambda_i} = F(\lambda)$, for $\lambda \in \Gamma$;
(b) $\sum_i \frac{\partial F}{\partial \lambda_i} \geq F(e) = 1$, for $\lambda \in \Gamma$.

Lemma 2. *(see [Tr2, LT]) Let $G = \left(\frac{\sigma_k}{\sigma_l}\right)^{\frac{1}{k-l}}, 0 \leq l < k \leq n$. Then*
(a) G is positive and concave in Γ_k;
(b) G is monotone in Γ_k, i.e., the matrix $G^{ij} = \frac{\partial G}{\partial W_{ij}}$ is positive definite;
(c) Suppose $\lambda \in \Gamma_k$. For $0 \leq l < k \leq n$, the following is the Newton-Maclaurin inequality

$$k(n-l+1)\sigma_{l-1}\sigma_k \leq l(n-k+1)\sigma_l\sigma_{k-1}.$$

The next lemma is a modification of Lemma 4 in [Cn2], which implies that $\binom{n}{k}^{-\frac{1}{k}} \sigma_k^{\frac{1}{k}}$ satisfies the additional condition (B) with $\rho_1 = \rho_2 = (n-k)$.

Lemma 3. *(see [Cn2]) For $1 \leq k \leq n-1$, if $\lambda \in \Gamma_k$.*
(a) if $\lambda_i \leq 0$, then $\sum_{j \neq i} \frac{\partial \sigma_k}{\partial \lambda_j} \leq (n-k)\frac{\partial \sigma_k}{\partial \lambda_i}$;
(b) if $\lambda_i \geq 0$, then $\sum_{j \neq i} \frac{\partial \sigma_k}{\partial \lambda_j} \geq (n-k)\frac{\partial \sigma_k}{\partial \lambda_i}$.

The following two lemmas will be used in proving Theorem 1 and 2. Let us review some formulae on the boundary under Fermi coordinates (see [Cn3]). The metric is expressed as $g = g_{\alpha\beta}dx^\alpha dx^\beta + (dx^n)^2$. The Christoffel symbols satisfy

$$\Gamma^n_{\alpha\beta} = L_{\alpha\beta}, \Gamma^\beta_{\alpha n} = -L_{\alpha\gamma}g^{\gamma\beta}, \Gamma^n_{\alpha n} = 0, \Gamma^\gamma_{\alpha\beta} = \widetilde{\Gamma}^\gamma_{\alpha\beta},$$

on the boundary, where we denote the tensors and covariant dierivations with respect to the induced metric on the boundary by a tilde (e.g. $\widetilde{\Gamma}^\gamma_{\alpha\beta}$, $\tau_{\tilde{\alpha}\tilde{\beta}}$). When the boundary is umbilic, we have

$$\Gamma^n_{\alpha\beta} = \tau g_{\alpha\beta}, \Gamma^\beta_{\alpha n} = -\tau\delta_{\alpha\beta}, \Gamma^n_{\alpha n} = 0.$$

Lemma 4. *(see [Cn3]) Suppose boundary ∂M is umbilic. Let u satisfy $u_n := \frac{\partial u}{\partial x^n} =$*

$-\tau+\widetilde{\tau}e^{-u}$, *where* $\widetilde{\tau}$ *is constant. Then on the boundary we have*

$$u_{n\alpha}=-\tau_\alpha+\tau u_\alpha-\widetilde{\tau}u_\alpha e^{-u}; \tag{2.1}$$

and

$$\begin{aligned}u_{\alpha\beta n}=&(2\tau-\widetilde{\tau}e^{-u})u_{\alpha\beta}-\tau u_{nn}g_{\alpha\beta}+\widetilde{\tau}u_\alpha u_\beta e^{-u}-\tau_{\tilde{\alpha}\tilde{\beta}}+\tau_\alpha u_\beta+\tau_\beta u_\alpha\\&-\tau_\gamma u_\gamma g_{\alpha\beta}+R_{n\beta\alpha n}(-\tau+\widetilde{\tau}e^{-u})-\tau(-\tau+\widetilde{\tau}e^{-u})^2 g_{\alpha\beta}.\end{aligned} \tag{2.2}$$

Lemma 5. *Suppose the boundary* ∂M *is totally geodesic and* $u_n=0$ *on the boundary. Then we have on the boundary*

$$W_{\alpha\beta n}=\frac{1-t}{n-2}u_{nnn}g_{\alpha\beta}+a_n u_\alpha u_\beta+b_n\left(\Sigma_\gamma u_\gamma^2\right)g_{\alpha\beta}+S_{\alpha\beta n}, \tag{2.3}$$

$$V_{\alpha\beta n}=\frac{t-1}{n-2}u_{nnn}g_{\alpha\beta}-a_n u_\alpha u_\beta-b_n\left(\Sigma_\gamma u_\gamma^2\right)g_{\alpha\beta}+S_{\alpha\beta n}. \tag{2.4}$$

Proof. By the boundary condition we know that $\widetilde{\tau}=\tau=0$. From formulas (2.1) and (2.2) we have $u_{n\alpha}=0$ and $u_{\alpha\beta n}=0$. Then

$$\begin{aligned}W_{\alpha\beta n}&=u_{\alpha\beta n}+\frac{1-t}{n-2}\Sigma_k u_{kkn}g_{\alpha\beta}+au_{\alpha n}u_\beta+au_\alpha u_{\beta n}+a_n u_\alpha u_\beta\\&\quad+2b\Sigma_k u_{kn}u_k g_{\alpha\beta}+b_n\left(\Sigma_k u_k^2\right)g_{\alpha\beta}+S_{\alpha\beta n}\\&=\frac{1-t}{n-2}u_{nnn}g_{\alpha\beta}+a_n u_\alpha u_\beta+b_n\left(\Sigma_\gamma u_\gamma^2\right)g_{\alpha\beta}+S_{\alpha\beta n}.\end{aligned}$$

For V_{ij} we can get the equalities in the same way.

□

3 Applications

We denote $[g]=\{\hat{g}\mid\hat{g}=e^{-2u}g\}$ and $[g]_k=\{\hat{g}\mid\hat{g}\in[g]\cap\Gamma_k^+\}$. We call g is k-admissible if and only if $[g]_k\neq\emptyset$. Now the first Yamabe constant on Riemannian manifold (M^n,g) with nonempty boundary ∂M can be defined as

$$\mathcal{Y}_1[g]=inf_{\widehat{g}\in[g],vol(\widehat{g})=1}\left(\int_M R_{\widehat{g}}+\oint_{\partial M}h_{\widehat{g}}\right).$$

We may define the boundary curvature $\mathcal{B}^k$ for the manifold with umbilic boundary and higher order Yamabe constants $\mathcal{Y}_k[g]$ for $2 \le k < n/2$ as follows (see [Cn3, GLW, S], compare with the definition in [STW]):

$$\mathcal{B}^k = \sum_{i=0}^{k-1} C(n,k,i)\sigma_i \left(g^{-1}A^T\right)\tau^{2k-2i-1},$$

and

$$\mathcal{Y}_k[g] = \begin{cases} inf_{\hat{g}\in[g]_{k-1};vol(\hat{g})=1}\mathcal{F}_k & \text{if } \ [g]_k \neq \emptyset \\ -\infty & \text{if } \ [g]_k = \emptyset \end{cases}$$

where $C(n,k,i) = \frac{(n-i-1)!}{(n-k)!(2k-2i-1)!!}$, $A^T = [A_{\alpha\beta}]$ is the tangential part of the Schouten tensor, τ is a constant satisfying $L_{\alpha\beta} = \tau g_{\alpha\beta}$, and

$$\mathcal{F}_k(\hat{g}) = \int_M \sigma_k(\hat{g}^{-1}A_{\hat{g}}) + \oint_{\partial M} \mathcal{B}^k_{\hat{g}}.$$

By Theorem 1 we can get the following Theorem 3 and 4.

Theorem 3. *([Cn3]) Let $(M, \partial M, g)$ is a compact manifold of dimension $n \ge 3$ with boundary and ∂M is totally geodesic. Suppose that $2 \le k < n/2$ and $Y_1,\ Y_k > 0$. Then there exists a metric $\hat{g} \in [g]$ such that $A_{\hat{g}} \in \Gamma_k$ and $B^k_{\hat{g}} = 0$.*

Proof. Following proof is mainly from [GV1]. Compare with [S], we may prove this theorem by continuity method. Consider a family of equations involving a parameter t,

$$\begin{cases} \sigma^{1/k}(g^{-1}A^t_{\widehat{g}}) = f(x)e^{2u} & \text{in } \ M \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \ \partial M \end{cases} \tag{3.1}$$

where $\widehat{g} = e^{-2u}g,\ f(x) > 0$ and $t \le 1$. Since $\mathcal{Y}_k > 0$, then $[g]_{k-1} \neq \varnothing$. We may assume $g \in [g]_{k-1}$. Therefore the scalar curvature $R_g > 0$. Then there exists $a > -\infty$ so that A^a_g is positive definite. For $t \in [a,1]$, we consider the deformation

$$\begin{cases} \sigma^{1/k}(g^{-1}A^t_{u_t}) = f(x)e^{2u_t} & \text{in } \ M \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \ \partial M \end{cases} \tag{3.2}$$

where $A^t_{u_t} = A^t_{\widehat{g}}$ with $\widehat{g} = e^{-2u_t}g,\ f(x) = \sigma^{1/k}\left(g^{-1}A^a_{u_a}\right) > 0$ and $u_a \equiv 0$ is a solution of (3.2) for $t = a$. Let

$$I = \left\{t \in [a, t_0]\,|\exists \text{ a solution } u_t \in C^{2,\alpha}(M) \text{ of (3.2) with } A_{\hat{g}_t} \in \Gamma_k\right\}.$$

It is easy to prove that the linearized operator $L_t : C^{2,\alpha}(M) \cap \left\{\frac{\partial u}{\partial \nu}|_{\partial M} = 0\right\} \to C^\alpha(M)$ is invertible. This together with the implicit theorem imply that the set I is open.

Since Theorem 1 case (b) implies the C^1 and C^2 estimates of the solution to (3.2) which depend only on the upper bound of u. Since $A^t = A^1 + \frac{1-t}{n-2}\sigma_1(A^1)\,g$, at the maximal point x_0 of u_t, we have $|\nabla u_t| = 0$ and $\nabla^2 u_t\left(x_0\right)$ is negative semi-definite, no matter x_0 being interior or boundary point. Hence,

$$f(x_0)^k e^{2ku(x_0)} = \sigma_k\left(g^{-1}A^t_{u_t}\right) \leq \sigma_k\left(g^{-1}\left(A + \frac{1-t}{n-2}\sigma_1\left(A\right)g\right)\right) \leq C,$$

where we use $\sigma_1\left(A\right) > 0$ and $a \leq t \leq 1$. We then get the upper bound. By the gradient estimate, we may easily get the lower bound of u. Therefore we conclude that $I = [a, 1]$. We thus finish the proof.

□

If (M, g) is a locally conformally flat compact manifold of dimension $n \geq 3$ with umbilic boundary. Then by [E], we may assume that the background metric g is a Yamabe metric with its constant scalar curvature $R > 0$ and the boundary is totally geodasic. Then using the same proof of Theorem 3, we may prove that there exists a metric $\hat{g} \in [g]$ such that $A_{\hat{g}} \in \Gamma_k$ and $\mathcal{B}^k_{\hat{g}} = 0$. By [Cn2], we can get the following existence result.

Theorem 4. *([Cn3]) Let $(M, \partial M, g)$ is a locally conformally flat compact manifold of dimension $n \geq 3$ with umbilic boundary. Suppose that $2 \leq k < n/2$ and $Y_1,\ Y_k > 0$. Then there exists a metric $\hat{g} \in [g]$ such that $\sigma_k(A_{\hat{g}}) = 1$ and $B^k_{\hat{g}} = 0$.*

4 Proof of Theorem 1

(1) Proof in case (a) .

Let $K = \triangle u + a|\nabla u|^2$. Note that $\Gamma \subset \Gamma_1^+$, we can immediately get

$$0 \leq\ tr(W) = (1+n\frac{1-t}{n-2})\triangle u + (a+nb)|\nabla u|^2 + trS \leq (1+n\frac{1-t}{n-2})K - n\delta_1|\nabla u|^2 + C.$$

Then

$$(1 + n\frac{1-t}{n-2})K \geq n\delta_1|\nabla u|^2 - C > -C.$$

Hence, K has lower bound. We also have

$$|\nabla u|^2 \leq \frac{(1 + n\frac{1-t}{n-2})K + C}{n\delta_1}. \tag{4.1}$$

Without loss of generality, we may assume $K > 0$. Otherwise, by the above inequality (4.1), we know that $|\nabla u|^2 \leq C$. Then we have the C^1 estimates. Furthermore, we have $\triangle u \leq C$. And from the condition $\Gamma \subset \Gamma_2^+$, we know that $(trW)^2 - |W|^2 =$

$2\sigma_2(W) > 0$. Therefore $|W| \le Ctr(W) \le C$ which implies $|\nabla^2 u| \le C$. We then get C^2 estimates.

Now by the assumption and (4.1), we have

$$|\nabla u|^2 \le C(K+1), \tag{4.2}$$

where C depends on $||a||_\infty$ and $||b||_\infty$.

By (4.2), we can consequently obtain

$$\triangle u = K - a|\nabla u|^2 \le K + ||a||_\infty |\nabla u|^2 \le C(K+1).$$

From the condition $\Gamma \subset \Gamma_2^+$ again, we know that $|W_{ij}| \le Ctr(W)$ which implies

$$\pm u_{ij} \le C(\triangle u + C(K+1)) + C(K+1),$$

thus

$$|\nabla^2 u| \le C(K+1), \tag{4.3}$$

where C depends only on $||a||_\infty$ and $||b||_\infty$ as well.

As a matter of fact, (4.2) and (4.3) are the fundamental inequalities that we would use once and once again. As a result, the terms u_{ij} and $u_i u_j$ can be controled by $C(K+1)$ later on.

Let $L = \zeta K e^{px_n}$, where $0 \le \zeta \le 1$ is a cutoff function such that $\zeta = 1$ in $B_{\frac{s}{2}}$ and $\zeta = 0$ outside B_s, and also $|\nabla \zeta| \le \frac{C\zeta^{1/2}}{s}, |\nabla^2 \zeta| \le \frac{C}{s^2}$. If we can prove K at the maximum point of L, x_0, is bounded, then K is also bounded in $B_{\frac{s}{2}}$. Furthermore, by (4.2), (4.3), we can get C^1 and C^2 estimates.

To prove $K(x_0)$ is bounded, let us consider an auxiliary function $H = \eta K e^{px_n}$ in a neighborhood $B_r(x_0) \subset B_s$, where $0 \le \eta \le 1$ is a cutoff function depending only on r such that $\eta = 1$ in $B_{\frac{r}{2}}$ and $\eta = 0$ outside B_r. Besides, we may assume F^{ij} at the point x_0 is diagonal. Now let us go on with the proof of case (a) to get C^1 and C^2 estimates.

Step 1. We will prove the maximum point of H, says x_0, must be in the interior of M.

To prove this conclusion we argue by contradiction. That is, we assume H arrives at its maximum point x_0 on the boundary, undertake a direct calculation of H_n, we can show that $H_n|_{x_0} > 0$, which violates the assumption.

Note that η is a function depending only on r, thus at the boundary point we have $\eta_n = 0$. Besides, by use of (2.1), (2.2), (4.2) and (4.3), we can get

$$\begin{aligned} & u_{\alpha\alpha n} + 2au_{\alpha n}u_\alpha + 2au_{nn}u_n + a_n(u_\gamma u_\gamma + u_n u_n) \\ = \;& a_n u_\gamma u_\gamma \geq -C(K+1). \end{aligned}$$

and

$$\begin{aligned} H_n|_{x_0} &= \eta e^{px_n}(K_n + pK) \\ &= \eta e^{px_n}(u_{nnn} + u_{\alpha\alpha n} + 2au_{\alpha n}u_\alpha + 2au_{nn}u_n + a_n(u_\gamma u_\gamma + u_n u_n) + pK) \\ &\geq \eta e^{px_n}(u_{nnn} - C(K+1) + pK) \end{aligned}$$

Now we need the following claim:

Claim. There exist the constants C depending only on $||a||_\infty$, $||b||_\infty$, Λ, t and ε, such that at x_0 we have $u_{nnn} \geq -C(K+1)$,

By the claim we know that if p is large enough, then

$$\begin{aligned} H_n|_{x_0} &= \eta e^{px_n}(u_{nnn} - C(K+1) + pK) \\ &\geq \eta e^{px_n}(-CK - C + pK) \\ &\geq \eta e^{px_n}((p-C)K - C) > 0, \end{aligned}$$

which completes the proof of step 1.

Proof of Claim.

We wish to get the estimates of u_{nnn} from the equation (1.5) itself. Since

$$f_n \;= F^{\alpha\beta}W_{\alpha\beta n} + F^{\alpha n}W_{\alpha nn} + F^{nn}W_{nnn}.$$

Noting that $u_n = 0$ and $\mu = 0$ on the boundary, the formula (2.3) in Lemma 5 combining with (4.2), (4.3) tells us that the first term $W_{\alpha\beta n}$ can be controled by $C(K+1)g_{\alpha\beta} + \frac{1-t}{n-2}u_{nnn}g_{\alpha\beta}$ and C depends only on $||a||_\infty$ and $||b||_\infty$. We restrict the argument at the point x_0. Then the second term vanishes since F^{ij} is diagnal at that point. For the third term, we know that $W_{nnn} = u_{nnn} + \frac{1-t}{n-2}u_{nnn} + 2au_{nn}u_n + 2bu_{kn}u_k + S_{nnn} = u_{nnn} + \frac{1-t}{n-2}u_{nnn} + S_{nnn}$. Since F^{ij} is positive definite, an algebraic fact says that $-(n-2)\,\sigma_1 \leq \lambda_i \leq \sigma_1$. As a result, $|F^{ij}| \leq C\sum F^{ii}$, where C depends only on n.

Without loss of generality, we may assume $u_{nnn} \leq 0$.

Case I : $t < 1$.

$$\begin{aligned} f_n &= F^{\alpha\beta} W_{\alpha\beta n} + F^{nn} W_{nnn} \\ &\leq C \sum F^{\alpha\alpha}(K+1) + \frac{1-t}{n-2} F^{\alpha\beta} u_{nnn} g_{\alpha\beta} + F^{nn} W_{nnn} \\ &\leq F^{nn} u_{nnn} + \sum_{\alpha} F^{\alpha\alpha} (\frac{1-t}{n-2} u_{nnn} + C(K+1)). \end{aligned}$$

Then by the conditions $|\nabla f| < \Lambda f$ and $|f_z| < \Lambda f$, and the condition (B) that either $\frac{\sum F^{\alpha\alpha}}{F^{nn}} \leq \rho_1$ or $\frac{F^{nn}}{\sum F^{\alpha\alpha}} \leq \frac{1}{\rho_2}$, we know either

$$-C \frac{f}{F^{nn}} \leq \left(\frac{1-t}{n-2} \frac{\Sigma F^{\alpha\alpha}}{F^{nn}} + 1 \right) u_{nnn} + C \frac{\Sigma F^{\alpha\alpha}}{F^{nn}} (K+1)$$

or

$$-C \frac{f}{\Sigma F^{\alpha\alpha}} \leq \left(\frac{1-t}{n-2} + \frac{F^{nn}}{\Sigma F^{\alpha\alpha}} \right) u_{nnn} + C (K+1).$$

Besides, by the assumption $u_{nnn} \leq 0$ and the condition (A4) $\frac{\partial F}{\partial \lambda_i} > \varepsilon \frac{F}{\sigma_1}$, then we have either

$$-C (K+1) \leq -C \frac{f}{F^{nn}} \leq u_{nnn} + C (K+1)$$

or

$$-C (K+1) \leq -C \frac{f}{\Sigma F^{\alpha\alpha}} \leq \frac{1-t}{n-2} u_{nnn} + C (K+1).$$

Thus $u_{nnn} \geq -C \frac{n-2}{1-t} (K+1)$ where C depends only on $||a||_\infty$, $||b||_\infty$, Λ, and ε.

Case II : $t = 1$.

Note that

$$W_{\alpha\beta n} = a_n u_\alpha u_\beta + b_n \left(\Sigma_\gamma u_\gamma^2 \right) g_{\alpha\beta} + S_{\alpha\beta n}.$$

Thus under the assumption

$$a = const., b = const., S_{\alpha\beta n} = A_{\alpha\beta n},$$

we have

$$\begin{aligned} f_n|_{x_0} &\leq F^{\alpha\alpha} W_{\alpha\alpha n} + F^{nn} W_{nnn} \\ &= F^{\alpha\alpha} A_{\alpha\alpha n} + \sum F^{nn} W_{nnn} \\ &\leq F^{nn} u_{nnn} + C(K+1)). \end{aligned}$$

The last inequality comes form the calculation in Lemma 8 of [Cn3]: $g^{\beta\gamma} A_{\beta\gamma,n} = 0$. Then the conditions $|\nabla f| < \Lambda f$ and $|f_z| < \Lambda f$, and (A4) $\frac{\partial F}{\partial \lambda_i} > \varepsilon \frac{F}{\sigma_1}$ implies $u_{nnn} \geq -C(K+1)$ where C depends only on $||a||_\infty$, $||b||_\infty$, Λ, and ε.

$\square$

Step 2. Now the C^1 and C^2 estimates will be completed by the rest proof in this part.

At the maximum point of H, x_0, after choosing normal coordinates, we have

$$0 = H_i = e^{px_n}(\eta_i K + \eta K_i + p\delta_{in}K\eta)$$

That is

$$K_i = -(\frac{\eta_i}{\eta} + p\delta_{in})K.$$

We also have

$$\begin{aligned} 0 \quad \geq \quad & H_{ij} = e^{px_n}((\eta_i K + \eta K_i + p\delta_{in}K\eta)p\delta_{jn} + \eta_{ij}K \\ & +\eta K_{ij} + \eta_i K_j + \eta_j K_i + p\delta_{in}K_j\eta + p\delta_{in}K\eta_j). \end{aligned}$$

Note that $|\nabla\eta| \leq \frac{C\eta^{1/2}}{r}, |\nabla^2\eta| \leq \frac{C}{r^2}$, we have

$$0 \geq H_{ij} = e^{px_n}(\eta K_{ij} + \Lambda_{ij}K),$$

where

$$\begin{aligned} \Lambda_{ij} \quad &= \quad \eta_{ij} - p\eta_i\delta_{jn} - p\eta_j\delta_{in} - p^2\eta\delta_{in}\delta_{jn} - \frac{2\eta_i\eta_j}{\eta} \\ &\geq \quad -C\left(p^2+1\right)\delta_{ij} \end{aligned}$$

and C depends only on r.

Denote

$$P^{ij} = F^{ij} + \frac{1-t}{n-2}\left(\Sigma_l F^{ll}\right)\delta^{ij}.$$

By the above inequalities, we have

$$\begin{aligned} 0 &\geq \eta P^{ij}H_{ij}e^{-px_n} \\ &= \eta^2 P^{ij}K_{ij} + \eta\Lambda_{ij}P^{ij}K \\ &\geq \eta^2 P^{ij}\sum_k[u_{ijkk} + 2a(u_{ki}u_{kj} + u_{ijk}u_k) + a_{ij}u_k^2 + 4a_i u_{kj}u_k] \\ &- C\left(\sum_i F^{ii}\right)(1+K) \\ &\geq \eta^2 P^{ij}\sum_k[u_{ijkk} + 2a(u_{ki}u_{kj} + u_{ijk}u_k)] - C(\sum_i F^{ii})(1+K^{3/2}). \end{aligned} \tag{4.4}$$

Now we estimate the terms $\sum_k P^{ij}u_{ijkk}$ and $\sum_k P^{ij}(u_{ki}u_{kj}+u_{ijk}u_k)$ respectively.

$$\begin{aligned}
\sum_k P^{ij}u_{ijkk} &= F^{ij}\sum_k[W_{ijkk}-2a(u_{ik}u_{jk}+u_{ikk}u_j)-2b(u_{lkk}u_l+u_{lk}u_{lk})g_{ij}\\
&\quad -a_{kk}u_iu_j-4a_ku_{ik}u_j-b_{kk}u_l^2g_{ij}-4b_ku_{lk}u_lg_{ij}-S_{ijkk}]\\
&\geq \sum_k f_{kk}+F^{ij}\sum_k[-2a(u_{ik}u_{jk}+u_{kki}u_j)-2b(u_{kkl}u_l+u_{lk}u_{lk})g_{ij}]\\
&\quad -C\sum F^{ii}(1+K^{3/2})\\
&\geq \sum_k f_{kk}+F^{ij}\sum_k[-2au_{ik}u_{jk}-2au_j(-2au_{ki}u_k-a_iu_k^2-(\frac{\eta_i}{\eta}+p\delta_{in})K)\\
&\quad -2bu_{lk}^2g_{ij}-2bu_l(-2au_{kl}u_k-a_lu_k^2-(\frac{\eta_l}{\eta}+p\delta_{ln})K)g_{ij}]\\
&\quad -C\sum F^{ii}(1+K^{3/2}).
\end{aligned}$$

We then get

$$\begin{aligned}
\sum_k P^{ij}u_{ijkk}\geq &\quad \sum_k f_{kk}+F^{ij}\sum_k[-2b\Sigma_l\left(u_{lk}\right)^2g_{ij}-2au_{ik}u_{jk}+4a^2u_ju_{ki}u_k\\
&+4ab\Sigma_l\left(u_lu_{kl}u_k\right)g_{ij}]-C\eta^{-1/2}\sum F^{ii}(1+K^{3/2}). \qquad (4.5)
\end{aligned}$$

We also have

$$\begin{aligned}
2aP^{ij}(u_{ijk}u_k+u_{ki}u_{kj}) = &\ 2aF^{ij}(W_{ijk}u_k-2au_{ik}u_ju_k-2bu_{lk}u_lg_{ij}u_k+u_{ik}u_{jk}\\
&+\frac{1-t}{n-2}\sum_{l,k}u_{lk}^2\delta_{ij}-a_ku_iu_ju_k-b_ku_l^2g_{ij}u_k-S_{ijk}u_k)\\
\geq &\ 2au_kf_k+F^{ij}(-4a^2u_{ik}u_ju_k-4abu_{lk}u_lu_kg_{ij}+2au_{ik}u_{jk})\\
&+\frac{1-t}{n-2}2a\sum_i F^{ii}\sum_{l,k}u_{lk}^2-C\sum_i F^{ii}(1+K^{3/2}). \qquad (4.6)
\end{aligned}$$

From (4.4), (4.5) and (4.6) we therefore have

$$0\geq\left(\sum F^{ii}\right)(2(\frac{1-t}{n-2}a-b)\eta^2|\nabla^2u|^2-C\eta^{3/2}K^{3/2}-C\eta K-C). \qquad (4.7)$$

If there exists a constant $A>0$, such that $|\nabla u|^2(x_0)<A|\triangle u|(x_0)$.

By $|u_{ij}|\leq C(K+1)$, we know that at the point x_0, $|u_{ij}|\leq C(|\triangle u|+1)$. Thus (4.7) becomes

$$0\geq \eta F^{ij}H_{ij}e^{-px_n}\geq\sum F^{ii}(\tfrac{2}{n}(\tfrac{1-t}{n-2}a-b)\eta^2|\triangle u|^2-C\eta^{3/2}|\triangle u|^{3/2}-C\eta|\triangle u|-C).$$

Hence

$$|\triangle u|(x_0)\leq C$$

and

$$K \leq C.$$

Otherwise for any constant $A > 0$ large enough, $|\nabla u|^2(x_0) \geq A|\triangle u|(x_0)$. From $|u_{ij}| \leq C(K+1)$, we know that at the point x_0, $|u_{ij}| \leq C(|\nabla u|^2+1)$. Thus (4.7) becomes

$$0 \geq \sum F^{ii}(2(\tfrac{1-t}{n-2}a-b)\eta^2 u_{il}^2 - C\eta^{3/2}|\nabla u|^3 - C\eta|\nabla u|^2 - C). \tag{4.8}$$

We may assume that W_{ij} is diagonal at the point x_0,

$$W_{ii} = u_{ii} + \frac{1-t}{n-2}\triangle u + au_i^2 + b\Sigma_k u_k^2 + S_{ii},$$

and

$$0 = W_{ij} = u_{ij} + au_iu_j + S_{ij},\ (i \neq j).$$

Since

$$F^{ii}(u_{il} + \frac{1-t}{n-2}\triangle u g_{il} + S_{il})^2 \leq 2F^{ii}[u_{il}^2 + (\frac{1-t}{n-2}\triangle u g_{il} + S_{il})^2],$$

we obtain

$$\begin{aligned}
2\Sigma_{l,i}F^{ii}u_{li}u_{li} \ &\geq \ \Sigma_{l,i}F^{ii}[(u_{il} + \frac{1-t}{n-2}\triangle u g_{il} + S_{il})^2 - 2(\frac{1-t}{n-2}\triangle u g_{il} + S_{il})^2] \\
&\geq \ \Sigma_i F^{ii}[\sum_{j\neq i}(-au_iu_j)^2 + (W_{ii} - au_i^2 - b|\nabla u|^2)^2] \\
&\quad -2\left(\frac{1-t}{n-2}\right)^2 \frac{1}{A^2}\Sigma_i F^{ii}|\nabla u|^4 - C\Sigma_i F^{ii} \\
&\geq \ \Sigma_i F^{ii}[(au_i)^2|\nabla u|^2 + W_{ii}^2 - 2W_{ii}(au_i^2 + b|\nabla u|^2) + b^2|\nabla u|^4 \\
&\quad +2abu_i^2|\nabla u|^2] - 2\left(\frac{1-t}{n-2}\right)^2 \frac{1}{A^2}\Sigma_i F^{ii}|\nabla u|^4 - C\Sigma_i F^{ii} \\
&\geq \ \Sigma_i F^{ii}[(a+b)^2u_i^2 + \sum_{j\neq i} b^2u_j^2]|\nabla u|^2 - 2||a+b||_\infty f|\nabla u|^2 \\
&\quad -2\left(\frac{1-t}{n-2}\right)^2 \frac{1}{A^2}\Sigma_i F^{ii}|\nabla u|^4 - C\Sigma_i F^{ii}.
\end{aligned}$$

By the assumption of the theorem caes (a), $\min\left\{(a+b)^2, b^2\right\} \geq \delta_2 > 0$, we then have

$$2\Sigma_{l,i}F^{ii}u_{li}u_{li} \geq \left(\delta_2 - 2\left(\frac{1-t}{n-2}\right)^2\frac{1}{A^2}\right)\Sigma_i F^{ii}|\nabla u|^4 - C\Sigma_i F^{ii}$$

Therefore, by (4.8) we have

$$0 \geq (\Sigma_i F^{ii})\,[\delta_1\left(\delta_2 - 2\left(\tfrac{1-t}{n-2}\right)^2\tfrac{1}{A^2}\right)\eta^2|\nabla u|^4 - C\eta^{3/2}|\nabla u|^{3/2} - C\eta|\nabla u|^2 - C].$$

Since $A > 0$ large enough, we have

$$|\nabla u|^2(x_0) \le C,$$

therefore

$$K \le C.$$

(2) Proof in case (b).

Since $a(x) + nb(x) < -\delta_3$, by the condition $\Gamma \subset \Gamma_1^+$, we have

$$\begin{aligned} 0 \ \le \ & tr(W) = (1 + n\frac{1-t}{n-2})\triangle u + a|\nabla u|^2 + nb|\nabla u|^2 + trB \\ \le \ & (1 + n\frac{1-t}{n-2})\triangle u - \delta_3|\nabla u|^2 + C. \end{aligned}$$

Then

$$|\nabla u|^2 \le C(\triangle u + 1). \tag{4.9}$$

The proof is similar as the argument in case (a). We take the same auxiliary function $H = \eta(\triangle u + a|\nabla u|^2)e^{px_n} \triangleq \eta K e^{px_n}$, where $\eta(r)$ is a cutoff function as in case (a), p is a large positive constant which is determined in the proof. Without loss of generality, we may assume

$$K = \triangle u + a|\nabla u|^2 >> 1.$$

Since $a(x) \ge 0$, by (4.9), we have

$$\Delta u \le C(K+1)$$

and

$$|\nabla u|^2 \le C(K+1).$$

Therefore by using the same **Claim**, we may prove that the maximum point of H does not achieve on the boundary and it must be in the interior. Suppose that the maximum point of H achieves at x_0, an interior point. Then at this point, we need to note that $|\nabla u|^2, \triangle u$ and K all can be controlled by $C(|\nabla^2 u| + 1)$. By the same computation as in case (a), (4.4), (4.5), (4.6) and (4.7) become

$$0 \ge \eta^2 P^{ij} \sum_k [u_{ijkk} + 2a(u_{ki}u_{kj} + u_{ijk}u_k)] - C(\sum_i F^{ii})(1 + \left|\nabla^2 u\right|^{3/2}), \tag{4.10}$$

$$\begin{aligned} \sum_k P^{ij} u_{ijkk} \ge \quad & \sum_k f_{kk} + F^{ij} \sum_k [-2b\Sigma_l (u_{lk})^2 g_{ij} - 2a u_{ik}u_{jk} + 4a^2 u_j u_{ki} u_k \\ & + 4ab\Sigma_l (u_l u_{kl} u_k) g_{ij}] - C\eta^{-1/2} \sum F^{ii}(1 + \left|\nabla^2 u\right|^{3/2}), \end{aligned} \tag{4.11}$$

$$2aP^{ij}(u_{ijk}u_k + u_{ki}u_{kj}) \geq \; 2au_kf_k + F^{ij}(-4a^2u_{ik}u_ju_k - 4abu_{lk}u_lu_kg_{ij} + 2au_{ik}u_{jk}) + \frac{1-t}{n-2}2a\sum_i F^{ii}\sum_{l,k}u_{lk}^2 - C\sum_i F^{ii}(1+\left|\nabla^2 u\right|^{3/2}), \tag{4.12}$$

and

$$0 \geq \left(\sum F^{ii}\right)(2(\frac{1-t}{n-2}a - b)\left(\eta|\nabla^2 u|\right)^2 - C\left(\eta\left|\nabla^2 u\right|\right)^{3/2} - C\eta\left|\nabla^2 u\right| - C) \tag{4.13}$$

respectively. (4.13) gives $\eta|\nabla^2 u|\,(x_0) \leq C$ and hence the bounds of $K, |\nabla^2 u|$ and $|\nabla u|$.

□

Proof of Corollary 1.
The proof of the case (a) is the corresponding part in Theorem 1 just ignoring the boundary argument. For the case (b), the difference with the corresponding part of Theorem 1 is that we needn't ask $a\,(x) \geq 0$, which is necessary in the boundary argument in Theorem 1. By the same argument as case (b) of Theorem 1, we may get (4.9) and (4.13). Therefore the estimate can be easily obtained.

□

5 Proof of Theorem 2

(1) Proof in case (a) .

Let $H = \eta(\triangle u + a|\nabla u|^2)e^{px_n}$ and $K = \triangle u + a|\nabla u|^2$, where $0 \leq \eta \leq 1$ is a cutoff function as before.

Note that $\Gamma \subset \Gamma_1^+$ and $V = \frac{t-1}{n-2}(\triangle u)\; g - \nabla^2 u - a(x)du \otimes du - b(x)|\nabla u|^2 g + S$, we can immediately get

$$0 \leq \; tr(V) = (n\frac{t-1}{n-2} - 1)\triangle u - (a+nb)|\nabla u|^2 + trS \leq (n\frac{t-1}{n-2} - 1)K - n\delta_1|\nabla u|^2 + C,$$

Hence, $|\nabla u|^2 \leq \frac{(n\frac{t-1}{n-2}-1)K+C}{n\delta_1}$. Thus we have

$$|\nabla u|^2 \leq C(K+1), \tag{5.1}$$

where C depends only on $||a||_\infty, ||b||_\infty$ and δ_1.

By (5.1), we can obtain $\triangle u < C(K+1)$ and

$$|\nabla^2 u| \leq C(K+1). \tag{5.2}$$

Then we may prove that the maximum point of H must be in the interior of M. As case (a) of Theorem 1, we prove this by contradiction. If the maximum point of H, x_0, is on the boundary, then by (2.1), (2.2) and (2.4), firstly we can prove $u_{\alpha\alpha n} + 2au_{\alpha n}u_\alpha + 2au_{nn}u_n + a_n(u_\gamma u_\gamma + u_n u_n) \geq -C(K+1)$. Furthermore, we can get the following Claim as well:

Claim. We can find some positive constant C, such that $u_{nnn}(x_0) \geq -C(K+1)$.

Then from the Claim, we can compute the derivative of H along the inner normal direction x^n and show that $H_n|_{x_0} > 0$ as long as p is chosen large enough, which is contradicts with the assumption that x_0 is a maximum point. Hence, H achieves its maximum at an interior point.

Proof of Claim.

We may assume $u_{nnn} \leq 0$. Choose a normal coordinates at x_0.

(1) If $\frac{\sum F^{\alpha\alpha}}{F^{nn}} \leq \rho_1$:

$$\begin{aligned}
f_n &= F^{\alpha\beta}V_{\alpha\beta n} + F^{nn}V_{nnn} \\
&\leq C\sum F^{\alpha\alpha}(K+1) - \frac{1-t}{n-2}F^{\alpha\beta}u_{nnn}g_{\alpha\beta} + F^{nn}V_{nnn} \\
&\leq F^{nn}(-1-\frac{1-t}{n-2})u_{nnn} + \sum_\alpha F^{\alpha\alpha}(-\frac{1-t}{n-2}u_{nnn} + C(K+1)).
\end{aligned}$$

Then by the conditions $|\nabla f| < \Lambda f$ and $|f_z| < \Lambda f$, we know

$$\begin{aligned}
-C\frac{f}{\Sigma F^{\alpha\alpha}} &\leq \left(-\frac{1-t}{n-2}\frac{\Sigma F^{\alpha\alpha}}{F^{nn}} - 1 - \frac{1-t}{n-2}\right)u_{nnn} + C\frac{\Sigma F^{\alpha\alpha}}{F^{nn}}(K+1) \\
&\leq \left(-1-\frac{1-t}{n-2}\right)u_{nnn} + C(K+1)
\end{aligned}$$

Since $\frac{t-1}{n-2} > 1$ thus $u_{nnn} \geq -C(K+1)$.

(2) If $\frac{F^{nn}}{\sum F^{\alpha\alpha}} \leq \frac{1}{\rho_2}$:

$$\begin{aligned}
f_n &= F^{\alpha\beta}V_{\alpha\beta n} + F^{nn}V_{nnn} \\
&\leq C\sum F^{\alpha\alpha}(K+1) - \frac{1-t}{n-2}F^{\alpha\beta}u_{nnn}g_{\alpha\beta} + F^{nn}V_{nnn} \\
&\leq F^{nn}(-1-\frac{1-t}{n-2})u_{nnn} + \sum_\alpha F^{\alpha\alpha}(-\frac{1-t}{n-2}u_{nnn} + C(K+1)) \\
&\leq -F^{nn}u_{nnn} + \sum_\alpha F^{\alpha\alpha}(-\frac{1-t}{n-2}u_{nnn} + C(K+1)).
\end{aligned}$$

Then by the conditions $|\nabla f| < \Lambda f$ and $|f_z| < \Lambda f$, we know

$$\begin{aligned} -C\frac{f}{\Sigma F^{\alpha\alpha}} &\leq -\left(\frac{1-t}{n-2} + \frac{F^{nn}}{\Sigma F^{\alpha\alpha}}\right) u_{nnn} + C\left(K+1\right) \\ &\leq -\left(\frac{1-t}{n-2} + \frac{1}{\rho_2}\right) u_{nnn} + C\left(K+1\right). \end{aligned}$$

Thus if $\frac{1-t}{n-2} + \frac{1}{\rho_2} < 0$ then $u_{nnn} \geq -C(K+1)$.

$\square$

Now let x_0 be an interior point where H achieves its maximum. At x_0, we have

$$0 = H_i = e^{px_n}(\eta_i K + \eta K_i + p\delta_{in} K\eta),$$

that is

$$K_i = -(\frac{\eta_i}{\eta} + p\delta_{in})K.$$

We also have

$$\begin{aligned} 0 \ \geq \ & H_{ij} = e^{px_n}((\eta_i K + \eta K_i + p\delta_{in}K\eta)p\delta_{jn} + \eta_{ij}K \\ & +\eta K_{ij} + \eta_i K_j + \eta_j K_i + p\delta_{in}K_j\eta + p\delta_{in}K\eta_j). \end{aligned}$$

Note that $|\nabla\eta| \leq \frac{C\eta^{1/2}}{r}, |\nabla^2\eta| \leq \frac{C}{r^2}$, we have

$$0 \geq H_{ij} \triangleq e^{px_n}(\eta K_{ij} + \Lambda_{ij}K),$$

where Λ_{ij} is bounded. If we take

$$Q^{ij} = \frac{t-1}{n-2}(\sum F^{ll})\delta_{ij} - F^{ij},$$

which is also positive, we can obtain

$$0 \geq \eta Q^{ij} H_{ij} e^{-px_n} = -\eta F^{ij} H_{ij} e^{-px_n} + \eta\frac{t-1}{n-2}(\sum F^{ii}) H_{kk} e^{-px_n}.$$

By the same computation as in the case (a) of Theorem 1, we may get

$$0 \geq \sum F^{ii}(2(\frac{t-1}{n-2}a + b)\eta^2|\nabla^2 u|^2 - C\eta^{3/2}K^{3/2} - C\eta K - C). \tag{5.3}$$

As in the case (a) of Theorem 1, we may discuss (5.3) in two cases. If there exists a constant $A > 0$, such that $|\nabla u|^2(x_0) < A|\triangle u|(x_0)$, we may prove

$$|\triangle u|(x_0) \leq C$$

and

$$K \leq C.$$

Otherwise for any constant $A > 0$ large enough, $|\nabla u|^2(x_0) \geq A|\triangle u|(x_0)$. By use of the assumption that $\min\left\{(a+b)^2, b^2\right\} \geq \delta_2 > 0$,we may prove

$$|\nabla u|^2(x_0) \leq C,$$

therefore we have

$$K \leq C.$$

By (5.2), we get the Hessian estimates.

(2) Proof in case (b).

We take the same auxiliary function $H = \eta(\triangle u + a|\nabla u|^2)e^{px_n} \triangleq \eta K e^{px_n}$ as in the case (a), where $0 \leq \eta \leq 1$ is a cutoff function such that $\eta = 1$ in $B_{\frac{r}{2}}$ and $\eta = 0$ outside B_r, and also $|\nabla \eta| \leq \frac{C\eta^{1/2}}{r}, |\nabla^2 \eta| \leq \frac{C}{r^2}$.

Since $a(x) + nb(x) \geq \delta_3$, by the condition $\Gamma \subset \Gamma_1^+$ again, we have

$$\begin{aligned} 0 &\leq tr(V) = (n\frac{t-1}{n-2} - 1)\triangle u - a|\nabla u|^2 - nb|\nabla u|^2 + trS \\ &\leq (n\frac{t-1}{n-2} - 1)\triangle u - \delta_3|\nabla u|^2 + C, \end{aligned}$$

and then

$$|\nabla u|^2 \leq C(\triangle u + 1). \tag{5.4}$$

Without loss of generality, we may assume

$$K = \Delta u + a\left|\nabla u\right|^2 >> 1.$$

Since $a\left(x\right) \geq 0$, by (5.4), we have

$$\Delta u \leq C(K+1) \tag{5.5}$$

and

$$|\nabla u|^2 \leq C(K+1). \tag{5.6}$$

Similar as the proof of the case (b) in Theorem 1, we may prove that the maximum point of H does not appear on the boundary and it must be in the interior. The condition $\frac{1}{\rho_2} - \frac{t-1}{n-2} < 0$ will be used to prove this. Now suppose that the maximum point of H achieves at x_0, an interior point, we may get an inequality just replacing K in (5.3) by $|\nabla^2 u|$

$$0 \geq \sum F^{ii}(2(\frac{t-1}{n-2}a+b)\eta^2|\nabla^2 u|^2 - C(\eta|\nabla^2 u|)^{3/2} - C(\eta|\nabla^2 u|) - C). \qquad (5.7)$$

The coefficient of the highest order term $\frac{t-1}{n-2}a(x)+b(x) > \delta_2 > 0$ since $a(x) \geq 0$ and $a(x)+nb(x) > \delta_3 > 0$. Therefore we can get the bounds of $K, |\nabla^2 u|$ and $|\nabla u|^2$.

□

Proof of Corollary 2.

Here the proof is completely similar as the proof of Corollary 1. In the case (b), we needn't ask $a(x) \geq 0$. We need an additional condition $\frac{t-1}{n-2}a(x)+b(x) > \delta_1 > 0$ to guarantee the coefficient of the highest order term in (5.7) is positive. We remain the detail to the readers.

□

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