GRAVITATIONAL ANOMALY CANCELLATION AND MODULAR INVARIANCE

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Abstract. In this paper, by combining modular forms and characteristic forms, we obtain general anomaly cancellation formulas of any dimension. For $4k+2$ dimensional manifolds, our results include the gravitational anomaly cancellation formulas of Alvarez-Gaumé and Witten in dimensions 2, 6 and 10 ([2]) as special cases. In dimension $4k+1$, we derive anomaly cancellation formulas for index gerbes. In dimension $4k+3$, we obtain certain results about eta invariants, which are interesting in spectral geometry.

1. Introduction

In [2], it is shown that in certain parity-violating gravity theory in $4k+2$ dimensions, when Weyl fermions of spin-$\frac{1}{2}$ or spin-$\frac{3}{2}$ or self-dual antisymmetric tensor field are coupled to gravity, perturbative anomalies occur. Alvarez-Gaumé and Witten calculate the anomalies and show that there are cancellation formulas for these anomalies in dimensions $2, 6, 10$. Let $\widehat{I}_{1/2}, \widehat{I}_{3/2}$ and $\widehat{I}_A$ be the spin-$\frac{1}{2}$, spin-$\frac{3}{2}$ and antisymmetric tensor anomalies respectively. By direct computations, Alvarez-Gaumé and Witten find anomaly cancellation formulas in dimensions 2, 6, 10 respectively,

$$-\widehat{I}_{1/2}+\widehat{I}_A=0, \tag{1.1}$$

$$21\widehat{I}_{1/2}-\widehat{I}_{3/2}+8\widehat{I}_A=0, \tag{1.2}$$

and

$$-\widehat{I}_{1/2}+\widehat{I}_{3/2}+\widehat{I}_A=0. \tag{1.3}$$

These anomaly cancellation formulas can tell us how many fermions of different types should be coupled to the gravity to make the theory anomaly free. Alvarez, Singer and Zumion [1] reproduce the above anomalies in a different way by using the family index theorem instead of Feynman diagram methods.

When perturbative anomalies cancel, this means that the effective action is invariant under gauge and coordinate transformations that can be reached continuously from the identity. In [27], Witten introduced the global anomaly by asking whether the effective action is invariant under gauge and coordinate transformations that are not continuously connected to the identity. Witten's work suggests that the global anomaly should be related to the holonomy of a natural connection on the determinant line bundle of the family Dirac operators.

From the topological point of view, anomaly measures the nontriviality of the determinant line bundle of a family of Dirac operators. The perturbative anomaly detects the real first Chern class of the determinant line bundle while the global anomaly detects the integral first Chern class beyond the real information (cf. [16]).

For a family of Dirac operators on an even dimensional closed manifold, the determinant line bundle over the parametrizing space carries the Quillen metric as well as the Bismut-Freed connection compatible with the Quillen metric such that the curvature of the Bismut-Freed connection is the two form component of the Atiyah-Singer family index theorem [10, 11]. The curvature of the Bismut-Freed connection is the representative (up to a constant) of the real first Chern class of the determinant line bundle. In this paper, by developing modular invariance of certain characteristic forms, we derive cancellation formulas for the curvatures of determinant line bundles of family signature operators and family tangent twisted Dirac operators on $4k+2$ dimensional manifolds (see Theorem 2.2.1 and Theorem 2.2.2). When $k = 0, 1, 2$, i.e. in dimensions 2, 6, and 10, our cancellation formulas just give the Alvarez-Gaumé-Witten cancellation formulas (1.1)-(1.3) (see Theorem 2.2.5 and its proof).

For global anomaly, in [11], Bismut and Freed prove the holonomy theorem suggsted by Witten. Later, to detect the integral information of the first Chern class of the determinant line bundle, Freed uses Sullivan's $\mathbf{Z}/k$ manifolds [16]. In this paper, we also give cancellation formulas for the holonomies (with respect to the Bismut-Freed connections) of determinant line bundles of family signature operators and family tangent twisted Dirac operators on $4k + 2$ dimensional manifolds for torsion loops which appear in the data of $\mathbf{Z}/k$ surfaces (Theorem 2.2.3 and 2.2.4).

The general anomaly cancellation formulas in dimension $4k$ have been studied in [21]. One naturally asks if there are similar results in odd dimensions.

For a family of Dirac operators on an odd dimensional manifold, Lott ([23]) constructed an abelian gerbe-with-connection whose curvature is the three form component of the Atiyah-Singer families index theorem. This gerbe is called the index gerbe, which is a higher analogue of the determinant line bundle. As Lott remarks in his paper that the curvature of such gerbes are also certain nonabelian gauge anomaly from a Hamitonian point of view. In this paper, we derive anomaly cancellation formulas for the curvatures of index gerbes of family odd signature operators and family tangent twisted Dirac operators on $4k+1$ dimensional manifolds (see Theorem 2.3.1, Theorem 2.3.2 and Corollary 2.3.1-2.3.3). Moreover, based on a result of Ebert [14], we can also derive anomaly cancellation formulas on the de Rham cohomology level (but not on the form level), which does not involve the family odd signature operators (see Theorem 2.3.3, Theorem 2.3.4 and Corollary 2.3.1-2.3.3). We hope there is some physical meaning related to our cohomological anomaly cancellation formulas.

In dimension $4k + 3$, we derive some results for the reduced η-invariants of family odd signature operators and family tangent twisted Dirac operators, which are interesting in spectral geometry (Theorem 2.4.1, Theorem 2.4.2 and Corollary 2.4.1-2.4.3). Moreover, function of the form

$$\exp\{2\pi\sqrt{-1}(\text{linear combination of reduced eta invariants})\}$$

has appeared in physics ([13]) as phase of effective action of M-theory in 11 dimension. We hope our results for reduced η-invariants can also find applications in physics.

We obtain our anomaly cancellation formulas by combining the family index theory and modular invariance of characteristic forms. Gravitational and gauge anomaly cancellations are very important in physics because they can keep the consistency of certain quantum field theories. It is quite interesting to notice that these cancellation formulas are consequences of the modular properties of characteristic forms which are rooted in elliptic genera.

2. Results

In this section, we will first prepare some geometric settings in Section 2.1 and then present our results in Section 2.2-2.4. The proofs of the theorems in Section 2.2-2.4 will be given in Section 3.

2.1. **Geometric Settings.** Following [9], we define some geometric data on a fiber bundle as follows. Let $\pi : M \to Y$ be a smooth fiber bundle with compact fibers Z and connected base Y. Let TZ be the vertical tangent bundle of the fiber bundle and g^Z be a metric on TZ. Let T^HM be a smooth subbundle of TM such that $TM = T^HM \oplus TZ$. Assume that TY is endowed with a metric g^Y. We lift the metric of TY to T^HM and by assuming that T^HM and TZ are orthogonal, TM is endowed with a metric which we denote $g^Y \oplus g^Z$. Let ∇^L be the Levi-Civita connection of TM for the metric $g^Y \oplus g^Z$ and P_Z denote the orthogonal projection from TM to TZ. Let ∇^Z denote the connection on TZ defined by the relation $U \in TM, V \in TZ, \nabla^Z_U V = P_Z \nabla^L_U V$. ∇^Z preserves the metric g^Z. Let $R^Z = \nabla^{Z,2}$ be the curvature of ∇^Z.

Let E, F be two Hermitian vector bundles over M carrying Hermitian connections ∇^E, ∇^F respectively. Let $R^E = \nabla^{E,\,2}$ (resp. $R^F = \nabla^{F,\,2}$) be the curvature of ∇^E (resp. ∇^F). If we set the formal difference $G = E - F$, then G carries an induced Hermitian connection ∇^G in an obvious sense. We define the associated Chern character form as (cf. [29])

$$\mathrm{ch}(G, \nabla^G) = \mathrm{tr}\left[\exp\left(\frac{\sqrt{-1}}{2\pi}R^E\right)\right] - \mathrm{tr}\left[\exp\left(\frac{\sqrt{-1}}{2\pi}R^F\right)\right].$$

For any complex number t, let

$$\Lambda_t(E) = \mathbf{C}|_M + tE + t^2\Lambda^2(E) + \cdots, \quad S_t(E) = \mathbf{C}|_M + tE + t^2S^2(E) + \cdots$$

denote respectively the total exterior and symmetric powers of E, which live in $K(M)[[t]]$. The following relations between these two operations ([3], Chap. 3) hold,

$$S_t(E) = \frac{1}{\Lambda_{-t}(E)}, \qquad \Lambda_t(E - F) = \frac{\Lambda_t(E)}{\Lambda_t(F)}. \tag{2.1}$$

The connections ∇^E, ∇^F naturally induce connections on S_tE, Λ_tE, etc. Moreover, if $\{\omega_i\}$, $\{{\omega_j}'\}$ are formal Chern roots for Hermitian vector bundles E, F respectively, then [15, Chap. 1]

$$\mathrm{ch}\left(\Lambda_t(E), \nabla^{\Lambda_t(E)}\right) = \prod_i (1 + e^{\omega_i}t). \tag{2.2}$$

We have the following formulas for Chern character forms,

$$\mathrm{ch}\left(S_t(E), \nabla^{S_t(E)}\right) = \frac{1}{\mathrm{ch}\left(\Lambda_{-t}(E), \nabla^{\Lambda_{-t}(E)}\right)} = \frac{1}{\prod_i (1 - e^{\omega_i}t)}\ , \tag{2.3}$$

$$\mathrm{ch}\left(\Lambda_t(E - F), \nabla^{\Lambda_t(E-F)}\right) = \frac{\mathrm{ch}\left(\Lambda_t(E), \nabla^{\Lambda_t(E)}\right)}{\mathrm{ch}\left(\Lambda_t(F), \nabla^{\Lambda_t(F)}\right)} = \frac{\prod_i (1 + e^{\omega_i}t)}{\prod_j (1 + e^{{\omega_j}'}t)}\ . \tag{2.4}$$

If W is a real Euclidean vector bundle over M carrying a Euclidean connection ∇^W, then its complexification $W_{\mathbf{C}} = W \otimes \mathbf{C}$ is a complex vector bundle over M carrying a canonically induced Hermitian metric from that of W, as well as a Hermitian connection $\nabla^{W_{\mathbf{C}}}$ induced from ∇^W.

If E is a vector bundle (complex or real) over M, set $\widetilde{E} = E - \mathrm{dim}E$ in $K(M)$ or $KO(M)$.

Let $q = e^{2\pi\sqrt{-1}\tau}$ with $\tau \in \mathbf{H}$, the upper half complex plane. Let $T_{\mathbf{C}}Z$ be the complexification of TZ. Set

$$\Theta_1(T_{\mathbf{C}}Z) = \bigotimes_{n=1}^{\infty} S_{q^n}(\widetilde{T_{\mathbf{C}}Z}) \otimes \bigotimes_{m=1}^{\infty} \Lambda_{q^m}(\widetilde{T_{\mathbf{C}}Z}), \tag{2.5}$$

$$\Theta_2(T_{\mathbf{C}}Z) = \bigotimes_{n=1}^{\infty} S_{q^n}(\widetilde{T_{\mathbf{C}}Z}) \otimes \bigotimes_{m=1}^{\infty} \Lambda_{-q^{m-\frac{1}{2}}}(\widetilde{T_{\mathbf{C}}Z}), \tag{2.6}$$

which are elements in $K(M)[[q^{\frac{1}{2}}]]$.

$\Theta_1(T_{\mathbf{C}}Z)$ and $\Theta_2(T_{\mathbf{C}}Z)$ admit formal Fourier expansion in $q^{1/2}$ as

$$\Theta_1(T_{\mathbf{C}}Z) = A_0(T_{\mathbf{C}}Z) + A_1(T_{\mathbf{C}}Z)q^{1/2} + \cdots, \tag{2.7}$$

$$\Theta_2(T_{\mathbf{C}}Z) = B_0(T_{\mathbf{C}}Z) + B_1(T_{\mathbf{C}}Z)q^{1/2} + \cdots, \tag{2.8}$$

where the A_j's and B_j's are elements in the semi-group formally generated by complex vector bundles over M. Moreover, they carry canonically induced connections denoted by ∇^{A_j} and ∇^{B_j} respectively, and let $\nabla^{\Theta_1(T_{\mathbf{C}}Z)}$, $\nabla^{\Theta_2(T_{\mathbf{C}}Z)}$ be the induced connections with $q^{1/2}$-coefficients on Θ_1, Θ_2 from the ∇^{A_j}, ∇^{B_j}.

The four Jacobi theta functions are defined as follows (cf. [12]):

$$\theta(v,\tau) = 2q^{1/8}\sin(\pi v)\prod_{j=1}^{\infty}\left[(1-q^j)(1-e^{2\pi\sqrt{-1}v}q^j)(1-e^{-2\pi\sqrt{-1}v}q^j)\right], \tag{2.9}$$

$$\theta_1(v,\tau) = 2q^{1/8}\cos(\pi v)\prod_{j=1}^{\infty}\left[(1-q^j)(1+e^{2\pi\sqrt{-1}v}q^j)(1+e^{-2\pi\sqrt{-1}v}q^j)\right], \tag{2.10}$$

$$\theta_2(v,\tau) = \prod_{j=1}^{\infty}\left[(1-q^j)(1-e^{2\pi\sqrt{-1}v}q^{j-1/2})(1-e^{-2\pi\sqrt{-1}v}q^{j-1/2})\right], \tag{2.11}$$

$$\theta_3(v,\tau) = \prod_{j=1}^{\infty}\left[(1-q^j)(1+e^{2\pi\sqrt{-1}v}q^{j-1/2})(1+e^{-2\pi\sqrt{-1}v}q^{j-1/2})\right]. \tag{2.12}$$

They are all holomorphic functions for $(v,\tau) \in \mathbf{C}\times\mathbf{H}$, where $\mathbf{C}$ is the complex plane and $\mathbf{H}$ is the upper half plane.

Define two q-series (see Section 3 for details)

$$\delta_2(\tau) = -\frac{1}{8}(\theta_1(0,\tau)^4 + \theta_3(0,\tau)^4), \quad \varepsilon_2(\tau) = \frac{1}{16}\theta_1(0,\tau)^4\theta_3(0,\tau)^4. \tag{2.13}$$

They have the following Fourier expansions in $q^{1/2}$:

$$\delta_2(\tau) = -\frac{1}{8} - 3q^{1/2} - 3q + \cdots, \quad \varepsilon_2(\tau) = q^{1/2} + 8q + \cdots.$$

When the dimension of the fiber is $8m+1, 8m+2$ or $8m+3$, define virtual complex vector bundles $b_r(T_{\mathbf{C}}Z)$ on M, $0\le r\le m$, via the equality

$$\Theta_2(T_{\mathbf{C}}Z)\equiv\sum_{r=0}^{m} b_r(T_{\mathbf{C}}Z)(8\delta_2)^{2m+1-2r}\varepsilon_2^r \mod q^{\frac{m+1}{2}}\cdot K(M)[[q^{\frac{1}{2}}]]. \tag{2.14}$$

When the dimension of the fiber is $8m-1, 8m-2$ or $8m-3$, define virtual complex vector bundles $z_r(T_{\mathbf{C}}Z)$ on M, $0\le r\le m$, via the equality

$$\Theta_2(T_{\mathbf{C}}Z)\equiv\sum_{r=0}^{m} z_r(T_{\mathbf{C}}Z)(8\delta_2)^{2m-2r}\varepsilon_2^r \mod q^{\frac{m+1}{2}}\cdot K(M)[[q^{\frac{1}{2}}]]. \tag{2.15}$$

It's not hard to show that each $b_r(T_{\mathbf{C}}Z), 0\le r\le m$, is a canonical linear combination of $B_j(T_{\mathbf{C}}Z), 0\le j\le r$. This is also true for $z_r(T_{\mathbf{C}}Z)$'s. These $b_r(T_{\mathbf{C}}Z)$'s and $z_r(T_{\mathbf{C}}Z)$'s carry canonically induced metrics and connections.

From (2.14) and (2.15), it's not hard to calculate that

$$b_0(T_{\mathbf{C}}Z)=-\mathbf{C},\ b_1(T_{\mathbf{C}}Z)=T_{\mathbf{C}}Z+\mathbf{C}^{24(2m+1)-\dim Z} \tag{2.16}$$

and

$$z_0(T_{\mathbf{C}}Z)=\mathbf{C},\ z_1(T_{\mathbf{C}}Z)=-T_{\mathbf{C}}Z-\mathbf{C}^{48m-\dim Z}. \tag{2.17}$$

2.2. **Determinant Line Bundles and Anomaly Cancellation Formulas.** Suppose the dimension of the fiber is $2n$ and the dimension of the base Y is p. Assume that TZ is oriented. Let T^*Z be the dual bundle of TZ.

Let $E=\bigoplus_{i=0}^{2n}E^i$ be the smooth infinite-dimensional $\mathbf{Z}$-graded vector bundle over Y whose fibre over $y\in Y$ is $C^\infty(Z_y,\Lambda_{\mathbf{C}}(T^*Z)|_{Z_y})$, i.e.

$$C^\infty(Y,E^i)=C^\infty(M,\Lambda_{\mathbf{C}}(T^*Z)),$$

where $\Lambda_{\mathbf{C}}(T^*Z)$ is the complexified exterior algebra bundle of TZ.

For $X\in TZ$, let $c(X),\widehat{c}(X)$ be the Clifford actions on $\Lambda_{\mathbf{C}}(T^*Z)$ defined by $c(X)=X^*-i_X, \widehat{c}(X)=X^*+i_X$, where $X^*\in T^*Z$ corresponds to X via g^Z.

Let $\{e_1,e_2,\cdots,e_{2n}\}$ be an oriented orthogonal basis of TZ. Set

$$\Omega=(\sqrt{-1})^n c(e_1)\cdots c(e_{2n}).$$

Then Ω is a self-adjoint element acting on $\Lambda_{\mathbf{C}}(T^*Z)$ such that $\Omega^2=\mathrm{Id}|_{\Lambda_{\mathbf{C}}(T^*Z)}$.

Let dv_Z be the Riemannian volume form on fibers Z associated to the metric g^Z (dv_Z is actually a section of $\Lambda^{\dim Z}_{\mathbf{C}}(T^*Z)$). Let $\langle\ ,\ \rangle_{\Lambda_{\mathbf{C}}(T^*Z)}$ be metric on $\Lambda_{\mathbf{C}}(T^*Z)$ induced by g^Z. Then E has a Hermitian metric h^E such that for $\alpha,\alpha'\in C^\infty(Y,E)$ and $y\in Y$,

$$\langle\alpha,\alpha'\rangle_{h^E}(y)=\int_{Z_y}\langle\alpha,\alpha'\rangle_{\Lambda_{\mathbf{C}}(T^*Z)}\,dv_{Z_y}.$$

Let d^Z be the exterior differentiation along fibers. d^Z can be considered as an element of $C^\infty(Y,\mathrm{Hom}(E^\bullet,E^{\bullet+1}))$. Let d^{Z*} be the formal adjoint of d^Z with respect to the inner product $\langle\ ,\ \rangle_{h^E}$. Define the family signature operator (c.f.[24]) D^Z_{sig} to be

$$D^Z_{sig}=d^Z+d^{Z*}:C^\infty(M,\Lambda_{\mathbf{C}}(T^*Z))\to C^\infty(M,\Lambda_{\mathbf{C}}(T^*Z)). \tag{2.18}$$

The $\mathbf{Z}_2$-grading of D^Z_{sig} is given by the $+1$ and -1 eigenbundles of Ω. Clearly, for each $y\in Y$,

$$(D^Z_{sig})_y:C^\infty(Z_y,\Lambda_{\mathbf{C}}(T^*Z)|_y)\to C^\infty(Z_y,\Lambda_{\mathbf{C}}(T^*Z)|_y)$$

is the signature operator for the fiber Z_y.

Further assume that TZ is spin. Following [9], the family Dirac operators are defined as follows.

Let O be the $SO(2n)$ bundle of oriented orthogonal frames in TZ. Since TZ is spin, the $SO(2n)$ bundle $O \xrightarrow[\varrho]{} M$ lifts to a $Spin(2n)$ bundle

$$O' \xrightarrow[\sigma]{} O \xrightarrow[\varrho]{} M$$

such that σ induces the covering projection $Spin(2n) \to SO(2n)$ on each fiber. Assume $F, F_\pm$ denote the Hermitian bundles of spinors

$$F = O' \times_{Spin(2n)} S_{2n},\ F_\pm = O' \times_{Spin(2n)} S_{\pm,2n},$$

where $S_{2n} = S_{+,2n} \oplus S_{-,2n}$ is the space of complex spinors. The connection ∇^Z on O lifts to a connection on O'. $F, F_\pm$ are then naturally endowed with a unitary connection, which we simply denote by ∇.

Let V be a l-dimensional complex Hermitian bundle on M. Assume that V is endowed with a unitary connection ∇^V whose curvature is R^V. The Hermitian bundle $F \otimes V$ is naturally endowed with a unitary connection which we still denote by ∇.

Let $H^\infty, H^\infty_\pm$ be the sets of C^∞ sections of $F \otimes V, F_\pm \otimes V$ over M. $H^\infty, H^\infty_\pm$ are viewed as the sets of C^∞ sections over Y of infinite dimensional bundles which are still denoted by $H^\infty, H^\infty_\pm$. For $y \in Y$, $H^\infty_y, H^\infty_{y,\pm}$ are the sets of C^∞ sections over Z_y of $F \otimes V, F_\pm \otimes V$.

The elements of TZ acts by Clifford multiplication on $F \otimes V$. Suppose $\{e_1, e_2, \cdots e_{2n}\}$ is a local orthogonal basis of TZ. Define the family Dirac operator twisted by V to be $D^Z \otimes V = \sum_{i=1}^{2n} e_i \nabla_{e_i}$. Let $(D^Z \otimes V)_\pm$ denote the restriction of $D^Z \otimes V$ to $H^\infty_\pm$. For each $y \in Y$,

$$(D^Z \otimes V)_y = \begin{bmatrix} 0 & (D^Z \otimes V)_{y,-} \\ (D^Z \otimes V)_{y,+} & 0 \end{bmatrix} \in \mathrm{End}^{odd}(H^\infty_{y,+} \oplus H^\infty_{y,-}) \tag{2.19}$$

is the twisted Dirac operator on the fiber Z_y.

The family signature operator is a twisted family Dirac operator. Actually, we have $D^Z_{sig} = D^Z \otimes F$ (cf. [8]).

Let $\mathcal{L}_{D^Z \otimes V} = \det(\mathrm{Ker}(D^Z \otimes V)_+)^* \otimes \det(\mathrm{Ker}(D^Z \otimes V)_-)$ be the determinant line bundle of the family operator $D^Z \otimes V$ over Y ([26, 10]). The nontriviality of $\mathcal{L}_{D^Z \otimes V}$ is certain anomaly in physics.

The determinant line bundle carries the Quillen metric $g^{\mathcal{L}_{D^Z \otimes V}}$ as well as the Bismut-Freed connection $\nabla^{\mathcal{L}_{D^Z \otimes V}}$ compatible to $g^{\mathcal{L}_{D^Z \otimes V}}$, the curvature $R^{\mathcal{L}_{D^Z \otimes V}}$ of which is equal to the two-form component of the Atiyah-Singer families index theorem [10, 11]. $\frac{1}{2\pi} R^{\mathcal{L}_{D^Z \otimes V}}$ is a representative of the local anomaly.

For the global anomaly, in [11], Bismut and Freed give a heat equation proof of the holonomy theorem in the form suggested by Witten in [27]. To detect information for the integral first Chern class of $\mathcal{L}_{D^Z \otimes V}$, Freed uses $\mathbf{Z}/k$ manifolds in [16]. $\mathbf{Z}/k$ manifold is introduced by Sullivan in his studies of geometric topology. A closed $\mathbf{Z}/k$ manifold (c.f. [16]) consists of (1) a compact manifold Q with boundary; (2) a closed manifold P; (3) a decomposition $\partial Q = \coprod_{i=1}^{k} (\partial Q)_i$ of the boundary of Q into k disjoint manifolds and diffeomorphisms $\alpha_i : P \to (\partial Q)_i$. The identification space $\overline{Q}$, formed by attaching Q to P by α_i is more properly called $\mathbf{Z}/k$ manifolds. $\overline{Q}$ is singular at identification points. If Q and P are compatibly oriented, then $\overline{Q}$ carries a fundamental class $[\overline{Q}] \in H_*(\overline{Q}, \mathbf{Z}/k)$. In [16], the first Chern class of the determinant line bundle over $\overline{\Sigma} \to Y$ is evaluated for all Z/k surfaces and all maps to detect the rest information other than the real information.

For local anomalies, we have the following cancellation formula formulas.

Theorem 2.2.1. *If the fiber is $8m+2$ dimensional, then the following local anomaly cancellation formula holds,*

$$R^{\mathcal{L}_{D^Z_{sig}}} - 8\sum_{r=0}^{m} 2^{6m-6r} R^{\mathcal{L}_{D^Z \otimes b_r(T_{\mathbf{C}}Z)}} = 0. \tag{2.20}$$

Theorem 2.2.2. *If the fiber be $8m-2$ dimensional, then the following local anomaly cancellation formula holds,*

$$R^{\mathcal{L}_{D^Z_{sig}}} - \sum_{r=0}^{m} 2^{6m-6r} R^{\mathcal{L}_{D^Z \otimes z_r(T_{\mathbf{C}}Z)}} = 0. \tag{2.21}$$

For global anomalies, we have the following cancellation formulas concerning the holonomies.

Theorem 2.2.3. *If the fiber is $8m+2$ dimensional, (Σ, S) is a $\mathbf{Z}/k$ surface and $f : \overline{\Sigma} \to Y$ is a map, then*

$$\begin{aligned} &\frac{\sqrt{-1}}{2\pi}\mathrm{lnhol}_{\mathcal{L}_{D^Z_{sig}}}(S) - 8\sum_{r=0}^{m} 2^{6m-6r}\frac{\sqrt{-1}}{2\pi}\mathrm{lnhol}_{\mathcal{L}_{D^Z \otimes b_r(T_{\mathbf{C}}Z)}}(S) \\ \equiv &c_1(f^*(\mathcal{L}_{D^Z_{sig}}))[\overline{\Sigma}] - 8\sum_{r=0}^{m} 2^{6m-6r} c_1\left(f^*\left(\mathcal{L}_{D^Z \otimes b_r(T_{\mathbf{C}}Z)}\right)\right)[\overline{\Sigma}] \mod 1, \end{aligned} \tag{2.22}$$

where we view $\mathbf{Z}/k \cong \mathbf{Z}[1/k]/\mathbf{Z} \subset \mathbf{Q}/\mathbf{Z}$.

Theorem 2.2.4. *If the fiber is $8m-2$ dimensional, (Σ, S) is a $\mathbf{Z}/k$ surface and $f : \overline{\Sigma} \to Y$ is a map, then*

$$\begin{aligned} &\frac{\sqrt{-1}}{2\pi}\mathrm{lnhol}_{\mathcal{L}_{D^Z_{sig}}}(S) - \sum_{r=0}^{m} 2^{6m-6r}\frac{\sqrt{-1}}{2\pi}\mathrm{lnhol}_{\mathcal{L}_{D^Z \otimes z_r(T_{\mathbf{C}}Z)}}(S) \\ \equiv &c_1(f^*(\mathcal{L}_{D^Z_{sig}}))[\overline{\Sigma}] - \sum_{r=0}^{m} 2^{6m-6r} c_1\left(f^*\left(\mathcal{L}_{D^Z \otimes z_r(T_{\mathbf{C}}Z)}\right)\right)[\overline{\Sigma}] \mod 1, \end{aligned} \tag{2.23}$$

where we view $\mathbf{Z}/k \cong \mathbf{Z}[1/k]/\mathbf{Z} \subset \mathbf{Q}/\mathbf{Z}$.

Putting $m = 0$ and $m = 1$ in the above theorems and using (2.16) as well as (2.17), we have

Corollary 2.2.1. *If the fiber is 2 dimensional, then the following local anomaly cancellation formula holds,*

$$R^{\mathcal{L}_{D^Z_{sig}}} + 8R^{\mathcal{L}_{D^Z}} = 0. \tag{2.24}$$

If (Σ, S) is a $\mathbf{Z}/k$ surface and $f : \overline{\Sigma} \to Y$ is a map, then

$$\begin{aligned} &\frac{\sqrt{-1}}{2\pi}\mathrm{lnhol}_{\mathcal{L}_{D^Z_{sig}}}(S) + 8\frac{\sqrt{-1}}{2\pi}\mathrm{lnhol}_{\mathcal{L}_{D^Z}}(S) \\ \equiv &c_1\left(f^*(\mathcal{L}_{D^Z_{sig}})\right)[\overline{\Sigma}] + 8c_1\left(f^*\left(\mathcal{L}_{D^Z}\right)\right)[\overline{\Sigma}] \qquad \mod 1. \end{aligned} \tag{2.25}$$

Corollary 2.2.2. *If the fiber is* 6 *dimensional, then the following local anomaly cancellation formula holds,*

$$R^{\mathcal{L}_{D^Z_{sig}}}+R^{\mathcal{L}_{D^Z\otimes T_{\mathbf{C}}Z}}-22R^{\mathcal{L}_{D^Z}}=0. \tag{2.26}$$

If (Σ,S) *is a* $\mathbf{Z}/k$ *surface and* $f:\overline{\Sigma}\to Y$ *is a map, then*

$$\begin{aligned}&\frac{\sqrt{-1}}{2\pi}\mathrm{lnhol}_{\mathcal{L}_{D^Z_{sig}}}(S)+\frac{\sqrt{-1}}{2\pi}\mathrm{lnhol}_{\mathcal{L}_{D^Z\otimes T_{\mathbf{C}}Z}}(S)-22\frac{\sqrt{-1}}{2\pi}\mathrm{lnhol}_{\mathcal{L}_{D^Z}}(S)\\ \equiv& c_1(f^*(\mathcal{L}_{D^Z_{sig}}))[\overline{\Sigma}]+c_1\left(f^*\left(\mathcal{L}_{D^Z\otimes T_{\mathbf{C}}Z}\right)\right)[\overline{\Sigma}]-22c_1\left(f^*\left(\mathcal{L}_{D^Z}\right)\right)[\overline{\Sigma}] \mod 1.\end{aligned} \tag{2.27}$$

Corollary 2.2.3. *If the fiber is* 10 *dimensional, then the the following local anomaly cancellation formula holds,*

$$R^{\mathcal{L}_{D^Z_{sig}}}-8R^{\mathcal{L}_{D^Z\otimes T_{\mathbf{C}}Z}}+16R^{\mathcal{L}_{D^Z}}=0. \tag{2.28}$$

If (Σ,S) *is a* $\mathbf{Z}/k$ *surface and* $f:\overline{\Sigma}\to Y$ *is a map, then*

$$\begin{aligned}&\frac{\sqrt{-1}}{2\pi}\mathrm{lnhol}_{\mathcal{L}_{D^Z_{sig}}}(S)-8\frac{\sqrt{-1}}{2\pi}\mathrm{lnhol}_{\mathcal{L}_{D^Z\otimes T_{\mathbf{C}}Z}}(S)+16\frac{\sqrt{-1}}{2\pi}\mathrm{lnhol}_{\mathcal{L}_{D^Z}}(S)\\ \equiv& c_1(f^*(\mathcal{L}_{D^Z_{sig}}))[\overline{\Sigma}]-8c_1\left(f^*\left(\mathcal{L}_{D^Z\otimes T_{\mathbf{C}}Z}\right)\right)[\overline{\Sigma}]+16c_1\left(f^*\left(\mathcal{L}_{D^Z}\right)\right)[\overline{\Sigma}] \mod 1.\end{aligned} \tag{2.29}$$

Our anomaly cancellation formulas actually imply the Alvarez-Gaumé and Witten anomaly cancellation formulas.

Theorem 2.2.5. *In dimensions* $2,6,10$*, our anomaly cancellation formulas (2.24), (2.26) and (2.28) give the gravitational anomaly cancellation formulas (1.1)-(1.3) of Alvarez-Gaumé and Witten.*

The proof this theorem will also be given in Section 3.

2.3. **Index Gerbes and Anomaly Cancellation Formulas.** Now we still assume that TZ is oriented but the dimension of the fiber is $2n+1$, i.e. we consider odd dimensional fibers. We still adopt the geometric settings in Section 2.1.

Let $\{e_1,e_2,\cdots,e_{2n+1}\}$ be an oriented orthogonal basis of TZ. Set

$$\Gamma=(\sqrt{-1})^{n+1}c(e_1)\cdots c(e_{2n+1}).$$

Then Γ is a self-adjoint element acting on $\Lambda_{\mathbf{C}}(T^*Z)$ such that $\Gamma^2=\mathrm{Id}|_{\Lambda_{\mathbf{C}}(T^*Z)}$.

Define the family odd signature operator B^Z_{sig} to be

$$B^Z_{sig}=\Gamma d^Z+d^Z\Gamma:C^\infty(M,\Lambda^{even}_{\mathbf{C}}(T^*Z))\to C^\infty(M,\Lambda^{even}_{\mathbf{C}}(T^*Z)). \tag{2.30}$$

For each $y\in Y$,

$$(B^Z_{sig})_y:C^\infty(Z_y,\Lambda^{even}_{\mathbf{C}}(T^*Z)|_y)\to C^\infty(Z_y,\Lambda^{even}_{\mathbf{C}}(T^*Z)|_y) \tag{2.31}$$

is the odd signature operator B_{even} for the fiber Z_y in [5].

Now assume that TZ is spin and still let V be a l-dimensional complex Hermitian bundle with the unitary connection ∇^V. One can still define the family Dirac operator $D^Z\otimes V$ similar as the even dimensional fiber case. The only difference is that now the spinor bundle F' associated to TZ is not $\mathbf{Z}_2$-graded. Let H^∞ be the set of C^∞ sections of $F'\otimes V$ over M. H^∞ is viewed as the set of C^∞ sections over Y of infinite dimensional bundles which are still denoted by H^∞. For $y\in Y$, H^∞_y is the set of C^∞ sections over Z_y of $F'\otimes V$. For each $y\in Y$,

$$(D^Z\otimes V)_y\in\mathrm{End}(H^\infty_y)$$

is the twisted Dirac operator on the fiber Z_y.

The family odd signature operator is a twisted family Dirac operator. Actually, we have $B^Z_{sig} = D^Z \otimes F'$ (c.f. [17]).

As a higher analogue of the determinant line bundle, Lott ([23]) constructs the index gerbe $\mathcal{G}^{D^Z \otimes V}$ with connection on Y for the family twisted odd Dirac operator $D^Z \otimes V$, the curvature $R^{\mathcal{G}_{D\otimes V}}$ (a closed 3-form on Y) of which is equal to the the three-form component of the Atiyah-Singer families index theorem. As remarked in [23], the curvature of the index gerbe is certain nonabelian gauge anomaly in physics ([15], cf. [23]).

We have the following anomaly cancellation formulas for index gerbes.

Theorem 2.3.1. *If fiber is $8m+1$ dimensional, then the following anomaly cancellation formula holds,*

$$R^{\mathcal{G}_{B^Z_{sig}}} - 8\sum_{r=0}^{m} 2^{6m-6r} R^{\mathcal{G}_{D\otimes b_r(T_{\mathbf{C}}Z)}} = 0. \tag{2.32}$$

Theorem 2.3.2. *If the fiber is $8m-3$ dimensional, then the following anomaly cancellation formula holds,*

$$R^{\mathcal{G}_{B^Z_{sig}}} - \sum_{r=0}^{m} 2^{6m-6r} R^{\mathcal{G}_{D\otimes z_r(T_{\mathbf{C}}Z)}} = 0. \tag{2.33}$$

If ω is a closed differential form on Y, denote the cohomology class ω represents in the de Rham cohomology of Y by $[\omega]$.

We have the following cancellation formulas for cohomology anomalies.

Theorem 2.3.3. *If the fiber is $8m+1$ dimensional, then the following anomaly cancellation formula in cohomology holds,*

$$\sum_{r=0}^{m} 2^{6m-6r} [R^{\mathcal{G}_{D\otimes b_r(T_{\mathbf{C}}Z)}}] = 0. \tag{2.34}$$

Theorem 2.3.4. *If the fiber is $8m-3$ dimensional, then the following anomaly cancellation formula in cohomology holds,*

$$\sum_{r=0}^{m} 2^{6m-6r} [R^{\mathcal{G}_{D\otimes z_r(T_{\mathbf{C}}Z)}}] = 0. \tag{2.35}$$

Putting $m=0$ and $m=1$ in the above theorems and using (2.16) as well as (2.17), we have

Corollary 2.3.1. *If the fiber is 1 dimensional, i.e. for the circle bundle case, the following anomaly cancellation formula holds,*

$$R^{\mathcal{G}_{B^Z_{sig}}} + 8R^{\mathcal{G}_{D^Z}} = 0. \tag{2.36}$$

The cohomology anomaly,

$$[R^{\mathcal{G}_{D^Z}}] = 0. \tag{2.37}$$

Corollary 2.3.2. *If the fiber is 5 dimensional, then the following anomaly cancellation formula holds,*

$$R^{\mathcal{G}_{B^Z_{sig}}} + R^{\mathcal{G}_{D^Z\otimes T_{\mathbf{C}}Z}} - 21R^{\mathcal{G}_{D^Z}} = 0. \tag{2.38}$$

The cohomology anomaly,

$$[R^{\mathcal{G}_{D^Z\otimes T_{\mathbf{C}}Z}}]-21[R^{\mathcal{G}_{D^Z}}]=0. \tag{2.39}$$

Corollary 2.3.3. *If the fiber is* 9 *dimensional, then the following anomaly cancellation formula holds,*

$$R^{\mathcal{G}_{B^Z_{sig}}}-8R^{\mathcal{G}_{D^Z\otimes T_{\mathbf{C}}Z}}+8R^{\mathcal{G}_{D^Z}}=0. \tag{2.40}$$

The cohomology anomaly,

$$[R^{\mathcal{G}_{D^Z\otimes T_{\mathbf{C}}Z}}]-[R^{\mathcal{G}_{D^Z}}]=0. \tag{2.41}$$

2.4. **Results for η-invariants.** For $y\in Y$, let $\eta_y(D^Z\otimes V)(s)$ be the eta function associated with $(D^Z\otimes V)_y$. Define ([4])

$$\overline{\eta}_y(D^Z\otimes V)(s)=\frac{\eta_y(D^Z\otimes V)(s)+\ker(D^Z\otimes V)_y}{2}. \tag{2.42}$$

Denote $\overline{\eta}_y(D^Z\otimes V)(0)$ (a function on Y) by $\overline{\eta}(D^Z\otimes V)$.

We still adopt the setting of family odd signature operators and family twisted Dirac operators on a family of odd manifolds in Section 2.3. We have the following theorems on the reduced η-invariants.

Theorem 2.4.1. *If the fiber is* $8m+3$ *dimensional, then*

$$\exp\left\{2\pi\sqrt{-1}\left(\overline{\eta}(B^Z_{sig})-8\sum_{r=0}^{m}2^{6m-6r}\overline{\eta}(D^Z\otimes b_r(T_{\mathbf{C}}Z))\right)\right\} \tag{2.43}$$

is a constant function on Y.

Theorem 2.4.2. *If the fiber is* $8m-1$ *dimensional, then*

$$\exp\left\{2\pi\sqrt{-1}\left(\overline{\eta}(B^Z_{sig})-\sum_{r=0}^{m}2^{6m-6r}\overline{\eta}(D^Z\otimes z_r(T_{\mathbf{C}}Z))\right)\right\} \tag{2.44}$$

is a constant function on Y.

Putting $m=0$ and $m=1$ in the above theorems and using (2.16) as well as (2.17), we have

Corollary 2.4.1. *If the fiber is* 3 *dimensional, then*

$$\exp\left\{2\pi\sqrt{-1}\left(\overline{\eta}(B^Z_{sig})+8\overline{\eta}(D^Z)\right)\right\} \tag{2.45}$$

is a constant function on Y.

Corollary 2.4.2. *If the fiber is* 7 *dimensional, then*

$$\exp\left\{2\pi\sqrt{-1}\left(\overline{\eta}(B^Z_{sig})+\overline{\eta}(D^Z\otimes T_{\mathbf{C}}Z)-23\overline{\eta}(D^Z)\right)\right\} \tag{2.46}$$

is a constant function on Y.

Corollary 2.4.3. *If the fiber is* 11 *dimensional, then*

$$\exp\left\{2\pi\sqrt{-1}\left(\overline{\eta}(B^Z_{sig})-8\overline{\eta}(D^Z\otimes T_{\mathbf{C}}Z)+24\overline{\eta}(D^Z)\right)\right\} \tag{2.47}$$

is a constant function on Y.

3. Proofs

In this section, we prove the theorems stated in Section 2.

3.1. **Preliminaries.** Let

$$SL_2(\mathbf{Z}) := \left\{ \left(\begin{array}{cc} a & b \\ c & d \end{array} \right) \middle| a, b, c, d \in \mathbf{Z},\ ad - bc = 1 \right\}$$

as usual be the modular group. Let

$$S = \left(\begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right), \quad T = \left(\begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array} \right)$$

be the two generators of $SL_2(\mathbf{Z})$. Their actions on $\mathbf{H}$ are given by

$$S : \tau \to -\frac{1}{\tau}, \quad T : \tau \to \tau + 1.$$

Let

$$\Gamma_0(2) = \left\{ \left(\begin{array}{cc} a & b \\ c & d \end{array} \right) \in SL_2(\mathbf{Z}) \middle| c \equiv 0 \pmod 2 \right\},$$

$$\Gamma^0(2) = \left\{ \left(\begin{array}{cc} a & b \\ c & d \end{array} \right) \in SL_2(\mathbf{Z}) \middle| b \equiv 0 \pmod 2 \right\}$$

be the two modular subgroups of $SL_2(\mathbf{Z})$. It is known that the generators of $\Gamma_0(2)$ are T, ST^2ST and the generators of $\Gamma^0(2)$ are STS, T^2STS.(cf. [12]).

If we act theta-functions by S and T, the theta functions obey the following transformation laws (cf. [12]),

(3.1)

$$\theta(v, \tau + 1) = e^{\frac{\pi\sqrt{-1}}{4}}\theta(v, \tau), \quad \theta(v, -1/\tau) = \frac{1}{\sqrt{-1}} \left(\frac{\tau}{\sqrt{-1}} \right)^{1/2} e^{\pi\sqrt{-1}\tau v^2} \theta(\tau v, \tau)\ ;$$

$$\theta_1(v, \tau+1) = e^{\frac{\pi\sqrt{-1}}{4}}\theta_1(v, \tau), \quad \theta_1(v, -1/\tau) = \left(\frac{\tau}{\sqrt{-1}} \right)^{1/2} e^{\pi\sqrt{-1}\tau v^2} \theta_2(\tau v, \tau)\ ; \tag{3.2}$$

$$\theta_2(v, \tau + 1) = \theta_3(v, \tau), \quad \theta_2(v, -1/\tau) = \left(\frac{\tau}{\sqrt{-1}} \right)^{1/2} e^{\pi\sqrt{-1}\tau v^2} \theta_1(\tau v, \tau)\ ; \tag{3.3}$$

$$\theta_3(v, \tau + 1) = \theta_2(v, \tau), \quad \theta_3(v, -1/\tau) = \left(\frac{\tau}{\sqrt{-1}} \right)^{1/2} e^{\pi\sqrt{-1}\tau v^2} \theta_3(\tau v, \tau)\ . \tag{3.4}$$

Definition 3.1. *Let Γ be a subgroup of $SL_2(\mathbf{Z})$. A modular form over Γ is a holomorphic function $f(\tau)$ on $\mathbf{H} \cup \{\infty\}$ such that for any*

$$g = \left(\begin{array}{cc} a & b \\ c & d \end{array} \right) \in \Gamma\ ,$$

the following property holds

$$f(g\tau) := f(\frac{a\tau + b}{c\tau + d}) = \chi(g)(c\tau + d)^l f(\tau),$$

where $\chi : \Gamma \to \mathbf{C}^*$ is a character of Γ and l is called the weight of f.

If Γ is a modular subgroup, let $\mathcal{M}_{\mathbf{R}}(\Gamma)$ denote the ring of modular forms over Γ with real Fourier coefficients. Writing simply $\theta_j = \theta_j(0, \tau),\ 1 \le j \le 3$, we introduce four (the second two have already appeared in Section 2.1) explicit modular forms (cf. [20], [21]),

$$\delta_1(\tau) = \frac{1}{8}(\theta_2^4 + \theta_3^4), \quad \varepsilon_1(\tau) = \frac{1}{16}\theta_2^4\theta_3^4\ ,$$

$$\delta_2(\tau) = -\frac{1}{8}(\theta_1^4 + \theta_3^4), \quad \varepsilon_2(\tau) = \frac{1}{16}\theta_1^4\theta_3^4 \ ,$$

They have the following Fourier expansions in $q^{1/2}$:

$$\delta_1(\tau) = \frac{1}{4} + 6q + 6q^2 + \cdots \ , \quad \varepsilon_1(\tau) = \frac{1}{16} - q + 7q^2 + \cdots \ ,$$

$$\delta_2(\tau) = -\frac{1}{8} - 3q^{1/2} - 3q + \cdots \ , \quad \varepsilon_2(\tau) = q^{1/2} + 8q + \cdots \ ,$$

where the "$\cdots$" terms are the higher degree terms, all of which have integral coefficients. They also satisfy the transformation laws (cf. [20], [21]),

$$\delta_2\left(-\frac{1}{\tau}\right) = \tau^2\delta_1(\tau) \quad , \quad \varepsilon_2\left(-\frac{1}{\tau}\right) = \tau^4\varepsilon_1(\tau). \tag{3.5}$$

Let $\widehat{A}(TZ, \nabla^Z)$ and $L(TZ, \nabla^Z)$ be the Hirzebruch characteristic forms defined respectively by (cf. [29]) for (TZ, ∇^Z):

$$\begin{aligned} \widehat{A}(TZ, \nabla^Z) &= \det{}^{1/2}\left(\frac{\frac{\sqrt{-1}}{4\pi}R^Z}{\sinh\left(\frac{\sqrt{-1}}{4\pi}R^Z\right)}\right), \\ \widehat{L}(TZ, \nabla^Z) &= \det{}^{1/2}\left(\frac{\frac{\sqrt{-1}}{2\pi}R^Z}{\tanh\left(\frac{\sqrt{-1}}{4\pi}R^Z\right)}\right). \end{aligned} \tag{3.6}$$

If ω is a differential form, denote the j-component of ω by $\omega^{(j)}$.

3.2. **Proofs of Theorem 2.2.1, 2.2.3, 2.2.5, 2.3.1, 2.3.3 and 2.4.1.** Suppose the dimension of TZ be $8m+1, 8m+2$ or $8m+3$. For the vertical tangent bundle TZ, set

$$P_1(\nabla^Z, \tau) := \left\{\widehat{L}(TZ, \nabla^Z)\mathrm{ch}\left(\Theta_1(T_{\mathbf{C}}Z), \nabla^{\Theta_1(T_{\mathbf{C}}Z)}\right)\right\}^{(8m+4)} \tag{3.7}$$

and

$$P_2(\nabla^Z, \tau) := \left\{\widehat{A}(TZ, \nabla^Z)\mathrm{ch}\left(\Theta_2(T_{\mathbf{C}}Z), \nabla^{\Theta_2(T_{\mathbf{C}}Z)}\right)\right\}^{(8m+4)}. \tag{3.8}$$

Proposition 3.1. *$P_1(\nabla^Z, \tau)$ is a modular form of weight $4m+2$ over $\Gamma_0(2)$; $P_2(\nabla^Z, \tau)$ is a modular form of weight $4m+2$ over $\Gamma^0(2)$.*

Proof. In terms of the theta functions, by the Chern-weil theory, the following identities hold,

$$P_1(\nabla^Z, \tau) = \left\{\det{}^{\frac{1}{2}}\left(\frac{R^Z}{2\pi^2}\frac{\theta'(0,\tau)}{\theta(\frac{R^Z}{2\pi^2}, \tau)}\frac{\theta_1(\frac{R^Z}{2\pi^2}, \tau)}{\theta_1(0,\tau)}\right)\right\}^{(8m+4)}, \tag{3.9}$$

$$P_2(\nabla^Z, \tau) = \left\{\det{}^{\frac{1}{2}}\left(\frac{R^Z}{4\pi^2}\frac{\theta'(0,\tau)}{\theta(\frac{R^Z}{4\pi^2}, \tau)}\frac{\theta_2(\frac{R^Z}{4\pi^2}, \tau)}{\theta_2(0,\tau)}\right)\right\}^{(8m+4)}. \tag{3.10}$$

Applying the transformation laws of the theta functions, we have

$$P_1\left(\nabla^Z, -\frac{1}{\tau}\right) = 2^{4m+2}\tau^{4m+2}P_2(\nabla^Z, \tau), \ P_1(\nabla^Z, \tau+1) = P_1(\nabla^Z, \tau). \tag{3.11}$$

Because the generators of $\Gamma_0(2)$ are T, ST^2ST and the generators of $\Gamma^0(2)$ are STS, T^2STS, the proposition follows easily. □

Lemma 3.1 (cf. [21]). *One has that $\delta_1(\tau)$ (resp. $\varepsilon_1(\tau)$) is a modular form of weight 2 (resp. 4) over $\Gamma_0(2)$, $\delta_2(\tau)$ (resp. $\varepsilon_2(\tau)$) is a modular form of weight 2 (resp. 4) over $\Gamma^0(2)$, while $\delta_3(\tau)$ (resp. $\varepsilon_3(\tau)$) is a modular form of weight 2 (resp. 4) over $\Gamma_\theta(2)$ and moreover $\mathcal{M}_{\mathbf{R}}(\Gamma^0(2)) = \mathbf{R}[\delta_2(\tau), \varepsilon_2(\tau)]$.*

We then apply Lemma 3.1 to $P_2(\nabla^Z, \tau)$ to get that

$$P_2(\nabla^Z,\tau) = h_0(T_{\mathbf{C}}Z)(8\delta_2)^{2m+1} + h_1(T_{\mathbf{C}}Z)(8\delta_2)^{2m-1}\varepsilon_2 + \cdots + h_m(T_{\mathbf{C}}Z)(8\delta_2)\varepsilon_2^m. \tag{3.12}$$

Comparing (2.14), we can see that

$$h_r(T_{\mathbf{C}}Z) = \left\{\widehat{A}(TZ,\nabla^Z)\mathrm{ch}(b_r(T_{\mathbf{C}}Z))\right\}^{(8m+4)},\ 0 \le r \le m.$$

By (3.5), (3.11) and (3.12), we have

$$P_1(\nabla^Z,\tau) = 2^{4m+2}[h_0(T_{\mathbf{C}}Z)(8\delta_1)^{2m+1} + h_1(T_{\mathbf{C}}Z)(8\delta_1)^{2m-1}\varepsilon_1 + \cdots + h_m(T_{\mathbf{C}}Z)(8\delta_1)\varepsilon_1^m]. \tag{3.13}$$

Comparing the constant term of the above equality, we see that

$$\{\widehat{L}(TZ,\nabla^Z)\}^{(8m+4)} = 8\sum_{r=0}^{m} 2^{6m-6r}\{\widehat{A}(TZ,\nabla^Z)\mathrm{ch}(b_r(T_{\mathbf{C}}Z))\}^{(8m+4)}. \tag{3.14}$$

In the following, we will deal with the even case and odd case respectively.

3.2.1. *The case of even dimensional fibers.* We have the following Bismut-Freed theorem on the curvature of the determinant line bundle with Bismut-Freed connection.

Theorem 3.2.1 (Bismut-Freed, [11]).

$$R^{\mathcal{L}_{D^Z\otimes V}} = 2\pi\sqrt{-1}\left\{\int_Z \widehat{A}(TZ,\nabla^Z)\mathrm{ch}(V,\nabla^V)\right\}^{(2)}. \tag{3.15}$$

To detect mod k information of the first Chern class of the determinant line bundle, Freed has the following result.

Theorem 3.2.2 (Freed, [16]). *If (Σ, S) is a $\mathbf{Z}/k$ surface and $f : \overline{\Sigma} \to Y$ is a map, then*

$$\begin{aligned} &c_1\left(f^*\left(\mathcal{L}_{D^Z\otimes V}\right)\right)[\overline{\Sigma}] \\ =&\frac{1}{k}\frac{\sqrt{-1}}{2\pi}\int_\Sigma f^*\left(\int_Z \widehat{A}(TZ,\nabla^Z)\mathrm{ch}(V,\nabla^V)\right) + \frac{\sqrt{-1}}{2\pi}\mathrm{lnhol}_{\mathcal{L}_{D^Z\otimes V}}(S) \mod 1, \end{aligned} \tag{3.16}$$

where we view $\mathbf{Z}/k \cong \mathbf{Z}\left[1/k\right]/\mathbf{Z} \subset \mathbf{Q}/\mathbf{Z}$.

If TZ is of dimension $8m+2$, integrating both sides of (3.14) along the fiber, we have

$$\left\{\int_Z \widehat{L}(TZ,\nabla^Z)\right\}^{(2)} - 8\sum_{r=0}^{m} 2^{6m-6r}\left\{\int_Z \widehat{A}(TZ,\nabla^Z)\mathrm{ch}(b_r(T_{\mathbf{C}}Z))\right\}^{(2)} = 0. \tag{3.17}$$

By Theorem 3.2.1, we get

(3.18)

$$
\begin{aligned}
&R^{\mathcal{L}_{D^Z_{sig}}} - 8\sum_{r=0}^{m} 2^{6m-6r} R^{\mathcal{L}_{D^Z\otimes b_r(T_{\mathbf{C}}Z)}}\\
=&2\pi\sqrt{-1}\left\{\int_Z \widehat{L}(TZ,\nabla^Z)\right\}^{(2)} - 8\sum_{r=0}^{m} 2^{6m-6r} 2\pi\sqrt{-1}\left\{\int_Z \widehat{A}(TZ,\nabla^Z)\mathrm{ch}(b_r(T_{\mathbf{C}}Z))\right\}^{(2)}\\
=&0.
\end{aligned}
$$

Therefore Theorem 2.2.1 follows.

Similarly, Freed's Theorem 3.2.2 and (3.17) give us

(3.19)

$$
\begin{aligned}
&c_1(f^*(\mathcal{L}_{D^Z_{sig}}))[\Sigma] - 8\sum_{r=0}^{m} 2^{6m-6r} c_1\left(f^*\left(\mathcal{L}_{D^Z\otimes b_r(T_{\mathbf{C}}Z)}\right)\right)[\Sigma]\\
&-\left(\frac{\sqrt{-1}}{2\pi}\mathrm{lnhol}_{\mathcal{L}_{D^Z_{sig}}}(S) - 8\sum_{r=0}^{m} 2^{6m-6r}\frac{\sqrt{-1}}{2\pi}\mathrm{lnhol}_{\mathcal{L}_{D^Z\otimes b_r(T_{\mathbf{C}}Z)}}(S)\right) \equiv 0 \mod 1
\end{aligned}
$$

and so Theorem 2.2.3 follows.

To prove Theorem 2.2.5, it's not hard to see from (32), (38) and (56) in [2] that, up to a same constant,

$$
\widehat{I}_{1/2} = \left\{\int_Z \widehat{A}(TZ,\nabla^Z)\right\}^{(2)} = R^{\mathcal{L}_{D^Z}},
$$

$$
\widehat{I}_{3/2} = \left\{\int_Z \widehat{A}(TZ,\nabla^Z)(\mathrm{ch}(T_{\mathbf{C}}Z,\nabla^Z)-1)\right\}^{(2)} = R^{\mathcal{L}_{D^Z\otimes T_{\mathbf{C}}Z}} - R^{\mathcal{L}_{D^Z}}
$$

and

$$
\widehat{I}_A = -\frac{1}{8}\left\{\int_Z \widehat{L}(TZ,\nabla^Z)\right\}^{(2)} = -\frac{1}{8}R^{\mathcal{L}_{D^Z_{sig}}},
$$

where in the fiber bundle $Z \to M \to Y$, Z is a $4k+2$ dimensional spin manifold and Y is the quotient space of the space of metrics on Z by the action of certain subgroup of $\mathrm{Diff}(M)$.

In dimension 2, by (2.24),

$$
-\widehat{I}_{1/2} + \widehat{I}_A = -R^{\mathcal{L}_{D^Z}} - \frac{1}{8}R^{\mathcal{L}_{D^Z_{sig}}} = -\frac{1}{8}(R^{\mathcal{L}_{D^Z_{sig}}} + 8R^{\mathcal{L}_{D^Z}}) = 0.
$$

Therefore (1.1) follows.

In dimension 6, by (2.26),

$$
\begin{aligned}
&21\widehat{I}_{1/2} - \widehat{I}_{3/2} + 8\widehat{I}_A\\
=&21R^{\mathcal{L}_{D^Z}} - (R^{\mathcal{L}_{D^Z\otimes T_{\mathbf{C}}Z}} - R^{\mathcal{L}_{D^Z}}) - R^{\mathcal{L}_{D^Z_{sig}}}\\
=&22R^{\mathcal{L}_{D^Z}} - R^{\mathcal{L}_{D^Z\otimes T_{\mathbf{C}}Z}} - R^{\mathcal{L}_{D^Z_{sig}}}\\
=&0.
\end{aligned}
$$

Therefore (1.2) follows.

In dimension 10, by (2.28),

$$
\begin{aligned}
&-\widehat{I}_{1/2}+\widehat{I}_{3/2}+\widehat{I}_A\\
=&-R^{\mathcal{L}_{D^Z}}+(R^{\mathcal{L}_{D^Z\otimes T_{\mathbf{C}}Z}}-R^{\mathcal{L}_{D^Z}})-\frac{1}{8}R^{\mathcal{L}_{D^Z_{sig}}}\\
=&-\frac{1}{8}(16R^{\mathcal{L}_{D^Z}}-8R^{\mathcal{L}_{D^Z\otimes T_{\mathbf{C}}Z}}+R^{\mathcal{L}_{D^Z_{sig}}})\\
=&0.
\end{aligned}
$$

Therefore (1.3) follows.

3.2.2. *The case of odd dimensional fibers.* Lott has the following theorem for the curvature of index gerbes.

Theorem 3.2.3 (Lott, [23])**.**

$$
R^{\mathcal{G}_{D^Z\otimes V}}=\left\{\int_Z\widehat{A}(TZ,\nabla^Z)\mathrm{ch}(V,\nabla^V)\right\}^{(3)}. \tag{3.20}
$$

If TZ is of dimension $8m+1$, integrating both sides of (3.14) along the fiber, we get

$$
\left\{\int_Z\widehat{L}(TZ,\nabla^Z)\right\}^{(3)}-8\sum_{r=0}^m 2^{6m-6r}\left\{\int_Z\widehat{A}(TZ,\nabla^Z)\mathrm{ch}(b_r(T_{\mathbf{C}}Z))\right\}^{(3)}=0. \tag{3.21}
$$

Note that we have $\widehat{A}(TZ,\nabla^Z)\mathrm{ch}(F',\nabla^{F'})=\widehat{L}(TZ,\nabla^Z)$ ([17]). So by Theorem 3.2.3 and (3.21), we have

$$
\begin{aligned}
&R^{\mathcal{G}_{B^Z_{sig}}}-8\sum_{r=0}^m 2^{6m-6r}R^{\mathcal{G}_{D^Z\otimes b_r(T_{\mathbf{C}}Z)}}\\
=&\left\{\int_Z\widehat{L}(TZ,\nabla^Z)\right\}^{(3)}-8\sum_{r=0}^m 2^{6m-6r}\left\{\int_Z\widehat{A}(TZ,\nabla^Z)\mathrm{ch}(b_r(T_{\mathbf{C}}Z))\right\}^{(3)}\\
=&0.
\end{aligned} \tag{3.22}
$$

Therefore Theorem 2.3.1 follows.

On the family odd signature operators, there is the following theorem:

Theorem 3.2.4 (Ebert, [14])**.** *The family index of the odd signature operator on an oriented bundle $M\to Y$ with odd dimensional fibers is trivial, i.e,* $\mathrm{ind}(B^Z_{sig})=0\in K^1(Y)$.

The following theorem on the odd Chern form for a family of self-adjoint Dirac operators is due to Bismut and Freed.

Theorem 3.2.5 (Bismut-Freed, [11])**.** $\int_Z\widehat{A}(TZ,\nabla^Z)\mathrm{ch}(V,\nabla^V)$ *represents the odd Chern character of* $\mathrm{ind}(D^Z\otimes V)$.

Combining Theorem 3.2.4 and 3.2.5, we see that $[\int_Z\widehat{L}(TZ,\nabla^Z)]$ is zero in de Rham cohomology. In particular, by Theorem 3.2.3, $\left[R^{\mathcal{G}_{B^Z_{sig}}}\right]=0$. Therefore, (3.22) implies that

$$
\sum_{r=0}^m 2^{6m-6r}[R^{\mathcal{G}_{D^Z\otimes b_r(T_{\mathbf{C}}Z)}}]=0. \tag{3.23}
$$

So Theorem 2.3.3 follows.

If d is a real number, let $\{d\}$ denote the image of d in $\mathbf{R}/\mathbf{Z}$. As noted in [4, 6], $\overline{\eta}_y(D^Z\otimes V)(0)$ has integer jumps and therefore $\{\overline{\eta}((D^Z\otimes V))\}$ is a C^∞ function of on Y with values in $\mathbf{R}/\mathbf{Z}$ ([4, 6]). For odd dimensional fibers, we have the following Bismut-Freed theorem for the reduced η-invariants.

Theorem 3.2.6 (Bismut-Freed, [11])**.**

$$d\{\overline{\eta}(D^Z\otimes V)\} = \left\{\int_Z \widehat{A}(TZ,\nabla^Z)\mathrm{ch}(V,\nabla^V)\right\}^{(1)}. \tag{3.24}$$

If TZ is of $8m+3$ dimensional, integrating both sides of (3.14) along the fiber, we get

$$\left\{\int_Z \widehat{L}(TZ,\nabla^Z)\right\}^{(1)} - 8\sum_{r=0}^{m}2^{6m-6r}\left\{\int_Z \widehat{A}(TZ,\nabla^Z)\mathrm{ch}(b_r(T_{\mathbf{C}}Z))\right\}^{(1)} = 0. \tag{3.25}$$

Then the Bismut-Freed Theorem 3.2.6 gives us

$$\begin{aligned}&d\{\overline{\eta}(B^Z_{sig})\} - 8\sum_{r=0}^{m}2^{6m-6r}d\{\overline{\eta}(D^Z\otimes b_r(T_{\mathbf{C}}Z))\}\\ =&\left\{\int_Z \widehat{L}(TZ,\nabla^Z)\right\}^{(1)} - 8\sum_{r=0}^{m}2^{6m-6r}\left\{\int_Z \widehat{A}(TZ,\nabla^Z)\mathrm{ch}(b_r(T_{\mathbf{C}}Z))\right\}^{(1)}\\ =&0.\end{aligned} \tag{3.26}$$

Therefore we obtain

$$d\left(\{\overline{\eta}(B^Z_{sig})\} - 8\sum_{r=0}^{m}2^{6m-6r}\{\overline{\eta}(D^Z\otimes b_r(T_{\mathbf{C}}Z))\}\right) = 0. \tag{3.27}$$

Since Y is connected,

$$\{\overline{\eta}(B^Z_{sig})\} - 8\sum_{r=0}^{m}2^{6m-6r}\{\overline{\eta}(D^Z\otimes b_r(T_{\mathbf{C}}Z))\}$$

must be a constant function on Y. Therefore it's not hard to see that Theorem 2.4.1 follows.

3.3. **Proofs of Theorem 2.2.2, 2.2.4, 2.3.2, 2.3.4 and 2.4.2.** The proofs are similar to the proofs of Theorem 2.2.1. 2.2.3, 2.3.1, 2.3.3 and 2.4.1.

Let the dimension of TZ be $8m-1, 8m-2$ or $8m-3$. For the vertical tangent bundle TZ, set

$$Q_1(\nabla^Z,\tau) := \left\{\widehat{L}(TZ,\nabla^Z)\mathrm{ch}\left(\Theta_1(T_{\mathbf{C}}Z),\nabla^{\Theta_1(T_{\mathbf{C}}Z)}\right)\right\}^{(8m)}, \tag{3.28}$$

$$Q_2(\nabla^Z,\tau) := \left\{\widehat{A}(TZ,\nabla^Z)\mathrm{ch}\left(\Theta_2(T_{\mathbf{C}}Z),\nabla^{\Theta_2(T_{\mathbf{C}}Z)}\right)\right\}^{(8m)}. \tag{3.29}$$

Similar to Proposition 3.1, we have

$$Q_1(\nabla^Z,\tau) = \left\{\det{}^{\frac{1}{2}}\left(\frac{R^Z}{2\pi^2}\frac{\theta'(0,\tau)}{\theta(\frac{R^Z}{2\pi^2},\tau)}\frac{\theta_1(\frac{R^Z}{2\pi^2},\tau)}{\theta_1(0,\tau)}\right)\right\}^{(8m)}, \tag{3.30}$$

$$Q_2(\nabla^Z,\tau)=\left\{\det{}^{\frac{1}{2}}\left(\frac{R^Z}{4\pi^2}\frac{\theta'(0,\tau)}{\theta(\frac{R^Z}{4\pi^2},\tau)}\frac{\theta_2(\frac{R^Z}{4\pi^2},\tau)}{\theta_2(0,\tau)}\right)\right\}^{(8m)}. \tag{3.31}$$

Also $Q_1(\nabla^Z,\tau)$ is a modular form of weight $4m$ over $\Gamma_0(2)$ and $Q_2(\nabla^Z,\tau)$ is a modular form of weight $4m$ over $\Gamma^0(2)$. Moreover,

$$Q_1\left(\nabla^Z,-\frac{1}{\tau}\right)=2^{4m}\tau^{4m}Q_2(\nabla^Z,\tau),\ Q_1(\nabla^Z,\tau+1)=Q_1(\nabla^Z,\tau). \tag{3.32}$$

Similar to (3.12) and (3.13), by using Lemma 3.1 and (3.32), we have

$$\begin{aligned}Q_2(\nabla^Z,\tau)=&\left\{\widehat{A}(TZ,\nabla^Z)\mathrm{ch}(z_0(T_{\mathbf{C}}Z))\right\}^{(8m)}(8\delta_2)^{2m}\\&+\left\{\widehat{A}(TZ,\nabla^Z)\mathrm{ch}(z_1(T_{\mathbf{C}}Z))\right\}^{(8m)}(8\delta_2)^{2m-2}\varepsilon_2\\&+\cdots+\left\{\widehat{A}(TZ,\nabla^Z)\mathrm{ch}(z_m(T_{\mathbf{C}}Z))\right\}^{(8m)}\varepsilon_2^m,\end{aligned} \tag{3.33}$$

and

$$\begin{aligned}Q_1(\nabla^Z,\tau)=&2^{4m}\left[\left\{\widehat{A}(TZ,\nabla^Z)\mathrm{ch}(z_0(T_{\mathbf{C}}Z))\right\}^{(8m)}(8\delta_1)^{2m}\right.\\&\left.+\cdots+\left\{\widehat{A}(TZ,\nabla^Z)\mathrm{ch}(z_m(T_{\mathbf{C}}Z))\right\}^{(8m)}\varepsilon_1^m\right].\end{aligned} \tag{3.34}$$

Comparing the constant term of the above equality, we see that

$$\{\widehat{L}(TZ,\nabla^Z)\}^{(8m)}=\sum_{r=0}^{m}2^{6m-6r}\{\widehat{A}(TZ,\nabla^Z)\mathrm{ch}(z_r(T_{\mathbf{C}}Z))\}^{(8m)}. \tag{3.35}$$

Then one can integrate both sides of (3.35) along the fiber and combine the theorems of Bismut-Freed, Freed, Lott and Ebert to obtain Theorem 2.2.2, 2.2.4, 2.3.2, 2.3.4 and 2.4.2.

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