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CLOSED GEODESICS AND VOLUME GROWTH OF RIEMANNIAN MANIFOLDS

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Abstract. In this paper, we study the relation between the existence of closed geodesics and the volume growth of open Riemannian manifolds with nonnegative curvature.

1. Introduction

Let M^n be an n-dimensional complete, noncompact Riemannian manifold with sectional curvature $K_M \geq 0$. Let

$$\alpha_M = \lim_{r\to\infty} \frac{Vol(B(p,r))}{\omega_n r^n},$$

where $Vol(B(p,r))$ is the volume of geodesic ball in M^n with radius r around p and ω_n denotes the volume of unit ball in R^n. From [6] we know that α_M is independent of the choosing of base point p. By the Bishop-Gromov volume comparison theorem, we have $0 \leq \alpha_M \leq 1$ and M^n is isometric to R^n if and only if $\alpha_M = 1$.

The main goal of this paper is to prove the following theorem.

Theorem 1.1. *Let M^n be a complete noncompact manifold with nonnegative section curvature. If M^n contains a closed geodesic, then the volume growth $\alpha_M = 0$. In other words, if $\alpha_M > 0$, then M^n does not contain any closed geodesic.*

We may see theorem 1.1 in an intuitive manner: To an open manifold with nonnegative section curvature, the closed geodesic will make the manifold shrink.

By Cheeger-Gromoll's soul theorem (see [2]), if the soul of M^n is not a point, then M^n must contain at least one closed geodesic. If the soul is one point, M^n still may have many closed geodesics. The following is a simple example.

Example 1.2. Let $M^2 = C_+ \cup S_1^2$ be a cylinder $C_+ = S^1 \times [0,\infty) = \{(x,y,z)|x^2 + y^2 = 1, z \geq 0\}$ glued to the lower hemi-sphere $S_-^2 = \{(x,y,z)|x^2+y^2+z^2 = 1, z \leq 0\}$. Then the soul of M^2 is a point, but M^2 admits infintely many closed geodesics.

In fact, our theorem is more significant when the soul is one point. In this case, the volume growth gives a sufficient condition of the nonexistence of closed geodesics, while this is not a trivial thing.

Remark 1.3. In what follows, we always assume that manifolds are complete noncompact with nonnegative sectional curvature.

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2. Proof of Theorem 1.1

The proof of theorem 1.1 is based on the following two lemmas.

Lemma 2.1. *Let $\sigma(t)$ be a closed geodesic of M^n with canonical parameter of the arc such that $\sigma(0) = \sigma(b) = p$, $\sigma^{'}(0) = \sigma^{'}(b)$, where b is the length of $\sigma(t)$. For any ray $\gamma(t)$ starts at p, we have $\alpha = \angle(\sigma^{'}(0), \gamma^{'}(0)) = \pi/2$.*

Proof. Let l be the length of $\sigma(t)$ from $\sigma(0)$ to $\sigma(l)$. By the Toponogv comparison theorem [1], we have

$$t^2 + l^2 - 2tl\cos\alpha \geq d^2(\sigma(l), \gamma(t)),$$

thus

$$\cos\alpha \leq \frac{t^2 + l^2 - d^2(\sigma(l), \gamma(t))}{2tl},$$

where $d(.,.)$ is the distance function. Recalling the condition of Toponogv comparison theorem [1], one only needs $l < \infty$. Let $l = b$, then $t = d(\sigma(b), \gamma(t))$. Thus

$$\cos\alpha \leq \frac{b}{2t}.$$

Let $t \longrightarrow \infty$, then

$$\cos\alpha \leq 0,$$

so

$$\alpha \geq \pi/2.$$

Considering $\sigma(b-t)$, we obtain

$$\pi - \alpha \geq \pi/2.$$

Hence

$$\alpha = \pi/2.$$

□

Remark 2.2. Lemma 2.1 can also be deduced by analytic method. For example, see theorem 1.10 of [2]. But our proof is more directly.

Next lemma is due to Ordway, Stephens and Yang [6]. It shows that α_M is determined by "the volume of rays".

Lemma 2.3. *Let $\Sigma = \{\nu \in S_pM | exp_p(t\nu) \text{ is a ray}, t \geq 0\}$. S_pM is unit sphere in T_pM. Set*

$$C(\Sigma) = \{q \in M | q = exp_p(t\nu), \nu \in \Sigma, t \geq 0\}$$

and

$$B(\Sigma, r) = B(p, r) \bigcap C(\Sigma).$$

Then we have

$$\alpha_M = \lim_{r \to \infty} \frac{Vol(B(\Sigma, r))}{\omega_n r^n}.$$

The proof of lemma 2.3 is based on Bishop-Gromov volume comparison theorem. For details, one may see [6].

Now we can prove theorem 1.1.

Proof. If M^n contains a closed geodesic, by lemma 2.1, we have $mes(\Sigma) = 0$ (induced measure of unit sphere). By Fubini theorem, for any $r > 0$ we have

$$mes(exp^{-1}(B(\Sigma, r))) = 0.$$

Since exp is C^∞, by Sard theorem [3], for any $r > 0$ we have

$$Vol(B(\Sigma, r)) = 0.$$

Then by lemma 2.3, we have $\alpha_M = 0$. □

3. An application of Theorem 1.1

Combining with Cheeger-Gromoll's soul theorem (see [2]), we get an another proof of Marenich and Toponogov's following beautiful theorem (see [5]).

Theorem 3.1. *If $\alpha_M > 0$, then M^n is diffeomorphic to R^n.*

Proof. If M^n is not diffeomorphic to R^n, by Cheeger-Gromoll's soul theorem, the soul (is a totally geodesic submanifold) of M^n is not a point. Then the soul must contain a closed geodesic (since any compact manifold contains at least one closed geodesic [4]). It is also the closed geodesic of M^n. Which is a contradiction to theorem 1.1. □

Remark 3.2. By a different method, theorem 3.1 is also a consequence of Perelman's celebrated flat strip theorem (cf. [7]).

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