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HARMONIC COMPLEX STRUCTURES

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ABSTRACT. In this paper, we introduce a new concept so called harmonic complex structure by using harmonic theory for vector bundle-valued differential forms. It is a new structure intermediates between complex structure and Kähler structure. From differential geometric viewpoint, it is a natural generalization of Kähler structure.

1. INTRODUCTION

The harmonic theories for vector bundle-valued differential forms play an important role in harmonic maps and Yang-Mills theories. The main idea is to construct some harmonic theory for vector bundle-valued differential forms such that the harmonic maps or Yang-Mills fields are exact the harmonic forms. Motivated by these ideas, we find, since the almost complex structure is a tangent bundle-valued 1-form, it is very natural to use harmonic theory of vector bundle-valued differential forms to study it. Though this is a very classical thing, until nowadays, we do not find it elsewhere.

Let M be an almost complex manifold. J is an almost complex structure of M, i.e. a smooth section of $\Gamma(T^*M \otimes TM)$ such that $J^2 = -1$ as a endomorphism $J : TM \longrightarrow TM$. Given a Riemannian metric, we can define the Hodge-Laplace operator Δ on $T^*M \otimes TM$. The harmonic complex structure is defined by

$$\Delta J = 0.$$

When M is compact, an almost complex structure J is harmonic complex if and only if for all $X, Y \in \Gamma(TM)$ $\nabla J(X, Y) = \nabla J(Y, X)$ and $Trace \nabla J = 0$. Apparently, a Kähler structure must be harmonic complex. The symmetry also implies that J is integrable (Proposition 2.4). This is the reason that we do not call it harmonic almost complex structure. In fact, Wood [2] defined the harmonic almost complex structure from the viewpoint of energy variation of fiber bundles. But his definition do not imply the integrability. Similar to harmonic maps, we can define the energy of almost complex structures. We also get the Bochner type formula for harmonic complex structures and give several applications. Particularly, we prove that S^6 with standard metric can not admit any harmonic complex structure.

Just as minimal submanifold is the generalization of totally geodesic submanifold, harmonic complex structure is the natural generalization of Kähler structure. We hope that harmonic complex structures can give some new insights about Kähler

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structures and complex structures. We also hope that it can arouse people's interesting as a new geometric structure.

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2. HARMONIC COMPLEX STRUCTURES

This section will be separated in two parts. In first part we will give a brief introduction of harmonic theories for tangent bundle-valued differential forms. In the second part we discuss the harmonic complex structures.

2.1. **Review of harmonic theories for tangent bundle-valued differential forms.** We only work with tangent bundle-valued differential forms. For general vector bundles and more details, we recommend the book of Xin [3].

Let (M, g) be a Riemannian manifold. ∇ is the Levi-Civita connection associated with g. Let TM (resp. T^*M) denote the tangent (resp. cotangent) bundle of M. We denote by $\Gamma(\wedge^p T^*M \otimes TM)$ the set of tangent bundle-valued p-forms over M. The Levi-Civita connection ∇ can be extended canonically to $\Gamma(\wedge^p T^*M \otimes TM)$ by

$$(\nabla_X \omega)(X_1, ..., X_p)$$
$$= \nabla_X(\omega(X_1, ..., X_p)) - \sum_j (\omega(X_1, ..., \nabla_X X_j, ..., X_p)),$$

for any $\omega \in \Gamma(\wedge^p T^*M \otimes TM)$ and $X, X_1, ..., X_p \in \Gamma(TM)$.

We can define the differential operator $d : \Gamma(\wedge^p T^*M \otimes TM) \longrightarrow \Gamma(\wedge^{p+1} T^*M \otimes TM)$. For any $\omega \in \Gamma(\wedge^p T^*M \otimes TM)$ and $X_0, X_1, ..., X_p \in \Gamma(TM)$,

$$d\omega(X_0, ..., X_p) = (-1)^k (\nabla_{X_k} \omega)(X_0, ..., \hat{X}_k, ..., X_p),$$

where $\hat{X}_k$ denotes removing X_k. The co-differential operator $\delta : \Gamma(\wedge^p T^*M \otimes TM) \longrightarrow \Gamma(\wedge^{p-1} T^*M \otimes TM)$ is defined by

$$\delta\omega(X_1, ..., X_{p-1}) = -(\nabla_{e_i} \omega)(e_i, X_1, ..., X_{p-1}),$$

where $\{e_i\}$ is the local orthonormal frame field.

Remark 2.1. It is easy to check that the above differential operator does not satisfy $d^2 = 0$. So there is no Hodge theory for vector bundle-valued differential forms.

Now we can define Hodge-Laplace operator

$$\Delta = d\delta + \delta d.$$

If $\Delta\omega = 0$, we say that ω is harmonic. Similar to differential forms, we also have Bochner techniques for vector bundle-valued differential forms, which play an important role in harmonic map theories (see [3]). If M is compact, from [3], we know that $\Delta\omega = 0$ if and only if $d\omega = 0$ and $\delta\omega = 0$.

2.2. **Harmonic complex structures.** Let M be an compact almost complex manifold. J is the almost complex structure of M.

Definition 2.2. We call that J is a harmonic complex structure if $\Delta J = 0$.

Remark 2.3. By the definition of d and δ, $\Delta J = 0$ if and only if $\nabla J(X, Y) = \nabla J(Y, X)$ for all $X, Y \in \Gamma(TM)$ and $Trace \nabla J = 0$. Recall that a Kähler structure means a hermitian complex structure J such that $\nabla J = 0$ (see [1]). We know immediately that a Kähler structure must be a harmonic complex structure.

Proposition 2.4. A harmonic complex structure must be a complex structure.

Proof. From the definition of dJ, for any $X, Y \in \Gamma(TM)$, we have

$$\begin{aligned} dJ(X,Y) &= (\nabla_X J)(Y) - (\nabla_Y J)(X) \\ &= \nabla_X JY - J(\nabla_X Y) - \nabla_Y JX + J(\nabla_Y X) \\ &= [X, JY] + \nabla_{JY} X - [Y, JX] - \nabla_{JX} Y - J[X,Y] \end{aligned}$$

and

$$\begin{aligned} dJ(JX,JY) &= (\nabla_{JX} J)(JY) - (\nabla_{JY} J)(JX) \\ &= -\nabla_{JX} Y - J(\nabla_{JX} JY) + \nabla_{JY} X + J(\nabla_{JY} JX) \\ &= \nabla_{JY} X - \nabla_{JX} Y - J[JX, JY]. \end{aligned}$$

Hence we have

$$dJ(X,Y) - dJ(JX,JY) = N(J)(X,Y),$$

where N is the Nijenhuis tensor

$$N(J)(X,Y) = [JX, Y] + [X, JY] + J[JX, JY] - J[X,Y].$$

Recall that $\Delta J = 0$ implies $dJ = 0$, by Newlander-Nirenberg theorem (see [1]), we get the proposition.

□

If a Riemannian metric is J invariant, we call it almost-hermitian. A nearly Kähler manifold is an almost-hermitian manifold such that $(\nabla_X J)(X) = 0$ for any $X \in \Gamma(TM)$. The following proposition shows that harmonic complex structures measure the difference between nearly Kähler and Kähler exactly.

Proposition 2.5. If M is almost-hermitian corresponding J. Then M is Kähler if and only if J is a harmonic complex structure and M is nearly Kähler corresponding J.

Proof. The nearly Kähler implies $(\nabla_X J)(Y) = -(\nabla_Y J)(X)$. Combining with $dJ(X,Y) = 0$, one gets $\nabla J = 0$. This shows that M is Kähler. The contrary is trivial. □

3. BOCHNER TYPE FORMULA FOR HARMONIC COMPLEX STRUCTURES AND APPLICATIONS

Similar to the differential forms, we also have the following Weitzenböck formula for tangent bundle-valued differential forms.

Proposition 3.1. ([3]) For any tangent bundle-valued p-form ω, we have

$$\Delta\omega = -\nabla^2\omega + S,$$

where $\nabla^2\omega = \nabla_{e_i}\nabla_{e_i}\omega - \nabla_{\nabla_{e_i e_i}}\omega$ and

$$S(X_1, ..., X_p) = (-1)^k (R(e_i, X_k)\omega)(e_i, X_1, ..., \hat{X_k}, ..., X_p),$$

for any $X_1, ..., X_p \in \Gamma(TM)$. R is the curvature tensor $R(X,Y) = -\nabla_X\nabla_Y + \nabla_Y\nabla_X + \nabla_{[X,Y]}$ and $\{e_i\}$ is the local orthonormal frame field.

Now we use proposition 3.1 to deduce Bochner type formula for harmonic complex structures. Let $\{e_i\}$ is the local orthonormal frame field. We can define the energy density of an almost complex structure J by

$$e(J) = \frac{1}{2} < Je_i, Je_i > .$$

Obviously $e(J)$ is independent on the choosing of $\{e_i\}$. If J is a harmonic complex structure, we have following Bochner type formula.

Proposition 3.2. $\Delta e(J) = |\nabla J|^2 - < R(e_i, e_j)Je_i, Je_j > + < JR(e_i, e_j)e_i, Je_j >$, where $|\nabla J|^2 = |(\nabla_{e_i} J)(e_j)|^2$.

Proof. First we have

$$\begin{aligned} -S(X) &= (R(e_i, X)J)e_i \\ &= ((-\nabla_{e_i}\nabla_X + \nabla_X\nabla_{e_i} + \nabla_{[e_i,X]})J)e_i \\ &= -\nabla_{e_i}((\nabla_X J)e_i) + (\nabla_X J)\nabla_{e_i} e_i + \nabla_X((\nabla_{e_i} J)e_i) \\ &\quad -(\nabla_{e_i} J)\nabla_X e_i + \nabla_{[e_i,X]} Je_i - J\nabla_{[e_i,X]} e_i \\ &= -\nabla_{e_i}(\nabla_X Je_i - J(\nabla_X e_i)) + \nabla_X(J\nabla_{e_i} e_i) - J(\nabla_X \nabla_{e_i} e_i) \\ &\quad +\nabla_X(\nabla_{e_i} Je_i - J(\nabla_{e_i} e_i)) - \nabla_{e_i}(J\nabla_X e_i) + J(\nabla_{e_i}\nabla_X e_i) \\ &\quad +\nabla_{[e_i,X]} Je_i - J\nabla_{[e_i,X]} e_i \\ &= R(e_i, X)Je_i - JR(e_i, X)e_i. \end{aligned}$$

Then

$$< S, J >= - < R(e_i, e_j)Je_i, Je_j > + < J(R(e_i, e_j)e_i), Je_j >,$$

and

$$\begin{aligned} < \nabla^2 J, J > &= < \nabla_{e_i}\nabla_{e_i} J, J >= e_i < \nabla_{e_i} J, J > - < \nabla_{e_i} J, \nabla_{e_i} J > \\ &= \frac{1}{2} e_i e_i < J, J > -|\nabla J|^2 = \Delta e(J) - |\nabla J|^2, \end{aligned}$$

here we choose the normal frame field (i.e. $\nabla_{e_i} e_j|_p = 0$ for a fixed point p). By Weizenböck formula,

$$0 =< \Delta J, J >= - < \nabla^2 J, J > + < S, J >,$$

we get the formula. □

We hope that proposition 3.2 contributes to studying of complex structures and Kähler structures.

Remark 3.3. If J is only an almost complex structure, from the last step of proof of proposition 3.2, we have

$$\Delta e(J)+ < \Delta J, J >= |\nabla J|^2 - < R(e_i, e_j)Je_i, Je_j > + < JR(e_i, e_j)e_i, Je_j > .$$

Corollary 3.4. For a compact almost complex manifold, J is a harmonic complex structure if and only if $\int_M (|\nabla J|^2 - < R(e_i, e_j)Je_i, Je_j > + < JR(e_i, e_j)e_i, Je_j >)dv = 0$.

Corollary 3.5. If M admits a Hermitian harmonic complex structure, then the scale curvature $\leq < R(e_i, e_j)Je_i, Je_j >$. The equal holds if and only if M is Kähler.

Though we do not know whether S^6 has a complex structure, as an application of proposition 3.2, we have

Theorem 3.6. *S^6 with standard metric can not admit any harmonic complex structure.*

Proof. If on the contrary, J is a harmonic complex structure. Locally, we can write $Je_i = J_i^k e_k$. Under the standard metric, the curvature on S^6 can be written as $R_{ijkm} = \delta_{ik}\delta_{jm} - \delta_{jk}\delta_{im}$. Then

$$\begin{aligned} < JR(e_i, e_j)e_i, Je_j > &= < J(R_{iji}^k e_k), Je_j > \\ &= R_{iji}^k < Je_k, Je_j >= R_{ijik} J_k^m J_j^m \\ &= R_{ijij}(J_j^m)^2 = \sum_m (J_j^m)^2 \end{aligned}$$

and

$$\begin{aligned} < R(e_i, e_j)Je_i, Je_j > &= < J_i^k R_{ijk}^l e_l, J_j^m e_m > \\ &= J_i^k J_j^m R_{ijkm} = J_i^i J_j^j - J_i^j J_j^i. \end{aligned}$$

Since J is almost complex structure, we have $traceJ = \sum_i J_i^i = 0$ and $\delta_i^j = |-\delta_i^j| = |\sum_k J_i^k J_k^j| \leq \sum_k \frac{(J_i^k)^2 + (J_k^j)^2}{2}$. So

$$\sum_{i,j} < JR(e_i, e_j)e_i, Je_j >= 6 \sum_{i,j} (J_i^j)^2 > 6$$

and

$$\sum_{i,j} < R(e_i, e_j)Je_i, Je_j >= (\sum_i J_i^i)^2 + 6 = 6.$$

Which is a contradiction to corollary 3.4.

□

Corollary 3.7. The standard metric on S^6 with small perturbation still can not admit any harmonic complex structure.

4. Some properties of Trace

In this section we study the trace of $A \in \Gamma(TM \otimes TM^*) = \Gamma(Hom(TM, TM))$. From the proof of proposition 3.2, we know

$$-S(X) = R(e_i, X)Ae_i - AR(e_i, X)e_i.$$

And

$$\begin{aligned} < (\nabla^2 A)(e_i), e_i > &= < (\nabla_{e_k} \nabla_{e_k} A)e_i, e_i > \\ &= < \nabla_{e_k} \nabla_{e_k} (Ae_i), e_i > \\ &= e_k < \nabla_{e_k} (Ae_i), e_i > \\ &= e_k e_k < Ae_i, e_i > \\ &= \Delta Trace(A), \end{aligned}$$

where choosing the normal frame field. Recall that

$$Trace\Delta A =< (\Delta A)e_i, e_i > .$$

Hence we have

$$Trace\Delta A + \Delta Trace(A) =< AR(e_i, e_j)e_i, e_j > - < R(e_i, e_j)Ae_i, e_j > .$$

The curvature term is

$$\begin{aligned} < AR(e_i, e_j)e_i, e_j > - < R(e_i, e_j)Ae_i, e_j > &= R^k_{iji} < Ae_k, e_j > - < R^l_{ijk} A^k_i e_l, e_j > \\ &= R^k_{iji} A^j_k - R^j_{ijk} A^k_i \\ &= R_{ijik} A^j_k - R_{ijkj} A^k_i \\ &= R_{jijk} A^i_k - R_{ijkj} A^k_i \\ &= R_{ijkj}(A^i_k - A^k_i) \\ &= 0. \end{aligned}$$

Thus we get

Theorem 4.1. $Trace\Delta A + \Delta Trace(A) = 0$.

Corollary 4.2. 1) If M is compact, then $\int_M Trace\Delta A = 0$.
2) For any almost complex structure, we have $Trace\Delta J = 0$.

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