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$U(1)$-decoupling, KK and BCJ relations in $\mathcal{N}=4$ SYM

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Abstract: By using the BCFW recursion relation of $\mathcal{N}=4$ super Yang-Mills theory, we proved the color reflection, $U(1)$-decoupling, Kleiss-Kuijf and Bern-Carrasco-Johansson relations for color-ordered amplitudes of $\mathcal{N}=4$ SYM theory. This proof verified the conjectured BCJ relations of matter fields. The proof depended only on general properties of super-amplitudes. We showed also that color reflection relation and $U(1)$-decoupling relation are special cases of KK relations.

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1. Introduction

The calculation of scattering amplitudes is important in quantum field theory. Generally one can use Feynman diagrams to compute the scattering amplitudes, but the number of Feynman diagrams increases dramatically with the increasing of external particles, therefore it is hard to handle the calculation both theoretically and practically. Many efficient methods have been proposed to solve this problem, especially under the practical demanding of newly particle experiments, such as LHC in Geneva. Among these methods there is BCFW on-shell recursion relation[1, 2], which was inspired by Witten's twistor program [3]. BCFW recursion relation is a powerful method to obtain simpler and more compact expressions for tree-level amplitudes. Besides this, it is also a very powerful tool to prove many useful relations of amplitudes. The supersymmetric BCFW recursion relation has also been written down recently [4, 5, 6, 7] and has been applied to many places such as [8].

Through the calculation of amplitudes, a number of useful relations have been found for color-order tree amplitudes of gluons. These include the color-order reflection relation, the $U(1)$-decoupling relation, the Kleiss-Kuijf relations [9] and the BCJ-relations [10]. The Kleiss-Kuijif relations reduce the number of independent color-ordered amplitudes of n-point gluons to $(n-2)!$. The BCJ-relations reduce the number of independent color-ordered amplitudes further to $(n-3)!$. The KK relations have been proved by field theory in [11] and again proved beautifully, together with BCJ-relations, using string theory method in [12, 13](see further works [14, 15, 16]). Another proof of all these relations has been given in [17] using BCFW recursion relation, where only general properties of scattering amplitudes in S-matrix program are used.

In this short note we extend the work of [17] to $\mathcal{N}=4$ SYM theory. More accurately, starting from the color reflection relation of three-point super-amplitudes, which can be directly verified in S-matrix program ([18, 19]), plus supersymmetric BCFW recursion relation, we will prove following four properties: (1) color reflection relation for general n; (2) $U(1)$-decoupling relation; (3) KK relations and (4) BCJ relations. Since we can expand the super-field into on-shell component fields through the η^A series, after getting the relations of super-amplitudes of $\mathcal{N}=4$ SYM theory we can recover the relations of gluon amplitudes and BCJ relations of matter fields from super-amplitude expanding. This verifies the conjecture in [20].

The paper is organized as follows, in section two we briefly review some basic facts of $\mathcal{N}=4$ SYM theory, including the BCFW recursion relation of SYM version. In section three and four we prove the color reflection relation and $U(1)$-decoupling relation. In section five, we prove the KK relations. Before giving the general proof we show that color reflection relation and $U(1)$-decoupling relation are just special cases of KK relations. In section six, we show that there is a primary relation in general formula of BCJ relations and other BCJ relations can be generated from this primary one combined with KK relations. Conclusion and discussions are presented in the last section.

2. Review

It is well known that tree-level amplitude of pure gluons is identical to the one obtained from $\mathcal{N}=4$ supersymmetric Yang-Mills theory. For $\mathcal{N}=4$ SYM theory, we can group all components into following on-shell superfield [6, 21]

$$\Phi(p,\eta) = G^+(p) + \eta^A\psi^+_A(p) + \frac{1}{2}\eta^A\eta^B S_{AB}(p) + \frac{1}{3!}\eta^A\eta^B\eta^C\epsilon_{ABCD}\psi^{D-}(p) + \frac{1}{4!}\eta^A\eta^B\eta^C\eta^D\epsilon_{ABCD}G^-(p) \tag{2.1}$$

with Grassmann coordinates $\eta^A, A=1,2,3,4$. Using superfield, the amplitude can be written as functions of $(\lambda_i,\widetilde{\lambda}_i,\eta_i^A)$. For example, the n-point super-MHV amplitude is given by Nair's formula[22] as

$$\mathcal{A}(\lambda,\widetilde{\lambda},\eta) = \frac{\delta^4(\sum_i\lambda_i\widetilde{\lambda}_i)\delta^8(\sum_i\lambda_i\eta_i^A)}{\langle 1\ 2\rangle\langle 2\ 3\rangle\ldots\langle n\ 1\rangle}\ , \tag{2.2}$$

where the Grassmann variables η_i appear in super-delta function to indicate the super energy-momentum conservation $\sum_i\lambda_i\eta_i=0$. To obtain the corresponding scattering amplitudes for various components we just need to expand above expression with respect to η^A. More concretely, after expanding into series of η^A we get

$$\mathcal{A} = \sum_{a_1..a_n=0}^{4} \mathcal{A}_{a_1...a_n} \prod_{i=1}^{n} \eta_i^{a_i} \ , \tag{2.3}$$

where $\mathcal{A}_{a_1...a_n}$ gives amplitude with particular field configuration indicated by $(a_1, a_2, ..., a_n)$. In this case, if we have the relations of super-amplitudes for $\mathcal{N} = 4$ SYM theory, we can expand super-amplitudes into component fields by η^A series, then we obtain relations for gluon and matter fields.

We will prove many relations of super-amplitudes by induction starting from relations of three-point super-amplitudes. By direct verification we know that the general three-point super-amplitudes, including the MHV and $\overline{\text{MHV}}$ super-amplitudes, have the property $\mathcal{A}(1,2,3) = -\mathcal{A}(3,2,1)$. We can observe this property directly from the component amplitudes. For components of pure gluon we have

$$A(1^-,2^-,3^+) = \frac{\langle 1\ 2\rangle^3}{\langle 2\ 3\rangle \langle 3\ 1\rangle} \ , \quad A(1^+,2^+,3^-) = \frac{[1\ 2]^3}{[2\ 3]\,[3\ 1]} \ . \tag{2.4}$$

Let us take $A(1^-,2^-,3^+)$ for example, after using cyclic relation we have

$$A(3^+,2^-,1^-) = A(2^-,1^-,3^+) = \frac{\langle 2\ 1\rangle^3}{\langle 1\ 3\rangle \langle 3\ 2\rangle} = -\frac{\langle 1\ 2\rangle^3}{\langle 2\ 3\rangle \langle 3\ 1\rangle} = -A(1^-,2^-,3^+) \ , \tag{2.5}$$

where we have used the property that $\langle i\ j\rangle = -\langle j\ i\rangle$. The same argument applies to $\overline{\text{MHV}}_3$ where $[i\ j] = -[j\ i]$ is used, thus we have $A(1,2,3) = -A(3,2,1)$. For components of two-fermion and one gluon we have

$$A(1_f^-,2_g^-,3_f^+) = \frac{\langle 1\ 2\rangle^3 \langle 3\ 2\rangle}{\langle 1\ 2\rangle \langle 2\ 3\rangle \langle 3\ 1\rangle} \ , \quad A(1_f^-,2_g^+,3_f^+) = \frac{[1\ 2]^3\,[3\ 2]}{[1\ 2]\,[2\ 3]\,[3\ 1]} \ . \tag{2.6}$$

Similarly we can write down the amplitudes of $A(3_f^+,2_g^-,1_f^-)$ and $A(3_f^+,2_g^+,1_f^-)$, by comparing to the ones given above, and after paying attention to the case that there is one minus sign from η_i exchanging, we can see again the above relations preserved. The left case is amplitudes of two-scalar and gluon,

$$A(1_s,2_g^-,3_s) = \frac{\langle 1\ 2\rangle^2 \langle 3\ 2\rangle^2}{\langle 1\ 2\rangle \langle 2\ 3\rangle \langle 3\ 1\rangle} \ , \quad A(1_s,2_g^+,3_s) = \frac{[1\ 2]^2\,[3\ 2]^2}{[1\ 2]\,[2\ 3]\,[3\ 1]} \ . \tag{2.7}$$

This is similar to the pure gluon case. We reverse the color order $(1,2,3) \to (3,2,1)$ and would see that $A(1,2,3) = -A(3,2,1)$. Thus we confirm the relation of three-point super-amplitudes

$$\mathcal{A}(1,2,3) = -\mathcal{A}(3,2,1) \ . \tag{2.8}$$

Relation (2.8) can be considered as color reflection relation or $U(1)$-decoupling relation of three-point. It is the fundamental relation and starting from this relation we can prove relations of general amplitudes by induction.

In order to prove the $U(1)$-decoupling, KK and BCJ relations in $\mathcal{N}=4$ SYM theory we need also the BCFW recursion relation of supersymmetric version[4, 5, 6, 7], which is given by

$$\begin{aligned}&\mathcal{A}(\{\eta_1,\lambda_1,\tilde{\lambda}_1\},\{\eta_2,\lambda_2,\tilde{\lambda}_2\},\eta_i)=\\&\sum_{L,R}\int d^4\eta\ \mathcal{A}(\{\eta_1(z_P),\lambda_1(z_P),\tilde{\lambda}_1\},\eta,\eta_L)\frac{1}{P^2}\mathcal{A}(\{\eta_2,\lambda_2,\tilde{\lambda}_2(z_P)\},\eta,\eta_R)\end{aligned} \tag{2.9}$$

with (1,2)-shifting. The amplitudes in terms of BCFW expansion are tree level on-shell super-amplitudes of lower point, while η_L and η_R are Grassmann variables attached to left and right sub-super-amplitudes. Note that η_1 has also been shifted, $\eta_1(z)=\eta_1+z\eta_2$, while others are unchanged, to keep the super energy-momentum conservation $\sum_i \lambda_i(z)\eta_i(z)=0$ if we shift $\lambda_1=\lambda_1+z\lambda_2$. The super-space integration is over shifted $\eta=\eta(z)$, and in $\mathcal{N}=4$ SYM theory, this integration can always be carried out by super-delta function from super-amplitudes, without doing actual calculation.

One important observation of (2.9) is that $\hat{P}(z)$ depends only on the sum of momenta and has nothing to do with color order. The same fact exists for $\hat{\eta}(z)$, which is decided by the manner of η_1 shifting and also has nothing to do with the color order. In this case we can group terms of the same channel in BCFW expansion, which usually have different color-order sub-super-amplitudes, without changing the $\eta(z)$ integration. Another important property is that when taking the (i,j)-shifting, super-amplitudes of $\mathcal{N}=4$ SYM have a good behavior $\mathcal{A}^{\mathcal{N}=4}(z)\to 0$ with $z\to\infty$ no matter which helicities of the shifted momenta are[6], i.e., there is no boundary contribution. The boundary behavior is very important when applying the BCFW recursion relation, and there are many works on the boundary behavior[23, 24, 25, 26, 27, 28, 29]. Luckily in $\mathcal{N}=4$ SYM theory we do not have bad deformation at all, thus we do not need to consider the details of (i,j)-shifting. We should also emphasize that for $\mathcal{N}=4$ super-amplitudes, when the shifted momenta are adjacent, we have $\mathcal{A}^{\mathcal{N}=4}(z)\to 1/z$ when $z\to\infty$, while the shifted momenta are not adjacent, we have $\mathcal{A}^{\mathcal{N}=4}(z)\to 1/z^2$ when $z\to\infty$. This fact helps a lot when proving the BCJ relations in gauge theory, as well as in gravity where they are just the bonus relations [6, 30].

3. The color reflection relation

The general expression for color reflection relation of super-amplitudes can be written as

$$\mathcal{A}(1,2,\ldots,n)=(-)^n\mathcal{A}(n,n-1,\ldots,1)\ . \tag{3.1}$$

Since color reflection relation of three-point super-amplitudes is satisfied, we want to prove the general case by induction starting from three-point case. Using the BCFW recursion relation of $\mathcal{N}=4$ SYM theory and (1,n)-shifting we have

$$\mathcal{A}(1,2,\ldots,n)=\int d^4\eta_{\hat{P}}\sum_{i=2}^{n-2}\mathcal{A}(\hat{1},\ldots,i,-\hat{P}_i;\eta_{\hat{P}})\frac{1}{P_i^2}\mathcal{A}(\hat{P}_i,i+1,\ldots,\hat{n};\eta_{\hat{P}})\ ,$$

then we use $(i+1)$ and $(n-i+1)$-point color reflection relations to re-write these sub-super-amplitudes as

$$\begin{aligned}\mathcal{A}(\hat{1},\ldots,i,-\hat{P}_i;\eta_{\hat{P}}) &= (-)^{i+1}\mathcal{A}(-\hat{P}_i,i,\ldots,\hat{1};\eta_{\hat{P}})\ ,\\ \mathcal{A}(\hat{P}_i,i+1,\ldots,\hat{n};\eta_{\hat{P}}) &= (-)^{n-i+1}\mathcal{A}(\hat{n},\ldots,i+1,\hat{P}_i;\eta_{\hat{P}})\ ,\end{aligned}$$

where variable z of shifted momenta in sub-super-amplitudes has been replaced by z_P so that every momentum in sub-super-amplitudes is on-shell, and we are able to use color reflection relation of lower-point. Putting sub-super-amplitudes of the left hand side on the right hand side and vice versa, we have

$$\begin{aligned}\mathcal{A}(1,2,\ldots,n) &= \int d^4\eta_{\hat{P}}\sum_{i=2}^{n-2}(-)^{n-i+1}\mathcal{A}(\hat{n},\ldots,i+1,\hat{P}_i;\eta_{\hat{P}})\frac{1}{P_i^2}(-)^{i+1}\mathcal{A}(-\hat{P}_i,i,\ldots,\hat{1};\eta_{\hat{P}})\\ &= (-)^n\mathcal{A}(n,n-1,\ldots,1)\ .\end{aligned}$$

In this step we have used the factor that $\eta_{\hat{P}}$ depends on the sum of η_i and does not change under color reflection, so that we can write these terms back to get color reversed super-amplitudes and finish the proof.

4. The $U(1)$-decoupling relation

4.1 The four-point case

Let us start with the simplest case of $U(1)$-decoupling relation, which is given by

$$\mathcal{A}(1,2,3,4)+\mathcal{A}(1,3,4,2)+\mathcal{A}(1,4,2,3)=0\ . \tag{4.1}$$

Let us take $(1,2)$ to do the shifting. As mentioned before, in $\mathcal{N}=4$ SYM theory, no matter what the helicity configuration is, there is always a good deformation for given pair (i,j). When doing the BCFW expansion of SYM version we should also shift corresponding η_i to keep the super-energy-momentum conservation. The η shifting depends on the way taking $\langle 1|2]$-shifting or $\langle 2|1]$-shifting, but since the super-space integration is only formally kept in the steps of demonstration, it is not necessary caring about the details of shifting. With the choice of $(1,2)$ pair we have following contributions for various amplitudes. Firstly for $\mathcal{A}(1,2,3,4)$ we have

$$\mathcal{A}(1,2,3,4) = \int d^4\eta_{\hat{P}}\mathcal{A}(\hat{2},3,-\hat{P}_{23};\eta_{\hat{P}})\frac{1}{P_{23}^2}\mathcal{A}(\hat{P}_{23},4,\hat{1};\eta_{\hat{P}})\ . \tag{4.2}$$

Then for $\mathcal{A}(1,3,4,2)$ we have

$$\mathcal{A}(1,3,4,2) = \int d^4\eta_{\hat{P}}\mathcal{A}(\hat{1},3,-\hat{P}_{13};\eta_{\hat{P}})\frac{1}{P_{13}^2}\mathcal{A}(\hat{P}_{13},4,\hat{2};\eta_{\hat{P}})\ , \tag{4.3}$$

and finally for $\mathcal{A}(1,4,2,3)$ we have two pole contributions given as

$$\mathcal{A}(1,4,2,3) = \int d^4\eta_{\hat{P}}\left[\mathcal{A}(\hat{1},4,\hat{P}_{23};\eta_{\hat{P}})\frac{1}{P_{23}^2}\mathcal{A}(-\hat{P}_{23},\hat{2},3;\eta_{\hat{P}})+\mathcal{A}(3,\hat{1},-\hat{P}_{13};\eta_{\hat{P}})\frac{1}{P_{13}^2}\mathcal{A}(\hat{P}_{13},4,\hat{2};\eta_{\hat{P}})\right] \tag{4.4}$$

Now let us compare various terms. The first term of (4.4) is almost same as the one in (4.2) with only difference in the order of the factor $\mathcal{A}(\hat{1},4,\hat{P}_{23})$ v.s. $\mathcal{A}(4,\hat{1},\hat{P}_{23})$, and has the same sub-super-amplitude $\mathcal{A}(-\hat{P}_{23},\hat{2},3)$, thus the sum of these two terms is proportional to $\mathcal{A}(-\hat{P}_{23},\hat{2},3)[\mathcal{A}(\hat{P}_{23},4,\hat{1})+\mathcal{A}(\hat{1},4,\hat{P}_{23})]$. Remembering the following result for color reflection relation

$$\mathcal{A}(1,2,...,n)=(-)^n\mathcal{A}(n,n-1,...,1)\ , \tag{4.5}$$

we know immediately that $\mathcal{A}(\hat{P}_{23},4,\hat{1})+\mathcal{A}(\hat{1},4,\hat{P}_{23})=0$, i.e., sum of these two terms is zero. Similarly the contribution from the second term of (4.4) plus the contribution from (4.3) is zero. We can see that adding them together we get zero and reproduce the relation (4.1).

4.2 n-point case

We want to demonstrate the $U(1)$-decoupling relation by induction. We will prove that if $U(1)$-decoupling relation are true for all tree level super-amplitudes less than n-point, then the n-point relation must also be true.

The n-point $U(1)$-decoupling relation for $\mathcal{N}=4$ SYM can be written as

$$\sum_{\sigma\in\ Cyclic}\mathcal{A}(1,\sigma(2,3,\ldots,n))=0\ , \tag{4.6}$$

where leg 1 is fixed. By using cyclic relation we can always fix leg 2 instead of leg 1 and rewrite this relation in a second form

$$\sum_{\sigma}\mathcal{A}(2,\sigma(1,3,\ldots,n))=0\ , \tag{4.7}$$

where the sum of σ is over ordered permutations of $\{1\}\cup\{3,\ldots,n\}$. Using BCFW recursion relation of $\mathcal{N}=4$ SYM and $(2,n)$-shifting, we can write down all terms of BCFW expansion. In order to show the relation among various recursion terms clearly, here we will introduce the split sign $|$ to express one term of BCFW expansion, i.e.,

$$\mathcal{A}(1,\ldots,i|i+1,\ldots,n)\equiv\int d^4\eta\ \mathcal{A}(1,\ldots,i,-P;\eta,\eta_L)\frac{1}{P^2}\mathcal{A}(P,i+1,\ldots,n;\eta,\eta_R)\ . \tag{4.8}$$

When there are more than one split sign in one amplitude, we mean the sum of each corresponding term, for example,

$$\begin{aligned}\mathcal{A}(1,2|3|4,5,6)\ \equiv&\int d^4\eta\ \mathcal{A}(1,2,-P_{12};\eta,\eta_L)\frac{1}{P_{12}^2}\mathcal{A}(P_{12},3,4,5,6;\eta,\eta_R)\\&+\int d^4\eta\ \mathcal{A}(1,2,3,-P_{123};\eta,\eta_L)\frac{1}{P_{123}^2}\mathcal{A}(P_{123},4,5,6;\eta,\eta_R)\ .\end{aligned} \tag{4.9}$$

Using this notation, we can write down all terms of BCFW expansion under $(2,n)$-shifting. Explicitly we have a special case,

$$\mathcal{A}(1,2,3,\ldots,n)=\mathcal{A}(1,\hat{2}|3|\ldots|n-1,\hat{n})+\mathcal{A}(\hat{2},3|4|\ldots,|\hat{n},1)\ , \tag{4.10}$$

and other general cases in which leg 2 and leg n are adjacent, such as

$$\mathcal{A}(1,n,2,\ldots,n-1)=\mathcal{A}(\hat{2},3|4|\ldots|n-2|n-1|1,\hat{n})\ , \tag{4.11}$$

$$\mathcal{A}(1,n-1,n,2,\ldots,n-2)=\mathcal{A}(\hat{2},3|4|\ldots|n-2|1|n-1,\hat{n})\ , \tag{4.12}$$

and so on. Terms of one certain super-amplitude can be divided into two parts, characterized by the splits before leg 1 and after leg 1.

We will show that sum of all these terms equals to zero. Take the second term in the special case (4.10) and all terms split before leg 1 in general cases, and arrange them as follows,

$$\begin{aligned}
&\mathcal{A}(\hat{2},3|1,4,5,\ldots,n-2,n-1,\hat{n})\\
&+\mathcal{A}(\hat{2},3|4|1,5,\ldots,n-2,n-1,\hat{n})\\
&+\mathcal{A}(\hat{2},3|4|5|1,\ldots,n-2,n-1,\hat{n})\\
&+\ldots\\
&+\mathcal{A}(\hat{2},3|4|5|6|\ldots|n-2|1,n-1,\hat{n})\\
&+\mathcal{A}(\hat{2},3|4|5|6|\ldots|n-2|n-1|1,\hat{n})\\
&+\mathcal{A}(\hat{2},3|4|5|6|\ldots|n-2|n-1|\hat{n},1)\ ,
\end{aligned}$$

then we can group all terms of the same channel $P_{2,\ldots,k}$, i.e., terms of the same vertical line, into one part. Using cyclic relation to fix leg 1 of all the right hand side sub-super-amplitudes, we have

$$\int d^4\eta_{\hat{P}}\mathcal{A}(\hat{2},3,\ldots,k,-\hat{P})\frac{1}{P^2}\sum_{\sigma\in\ Cyclic}\mathcal{A}(1,\sigma(k+1,\ldots,\hat{n},\hat{P}))\ . \tag{4.13}$$

It is clearly seen that sum of the same channel $P_{2,\ldots,k}$ is zero because of $(n-k+2)$-point $U(1)$-decoupling relation. The same argument holds for the sum of the first term in (4.10) and all terms split after leg 1 in general cases. Then we have summed up all terms and gotten a zero result, which proved the general $U(1)$-decoupling relation.

5. The KK relations

Again to get some sense of these relations, let us see some special cases. The case of $n=3$ is simplest and it is given by

$$\mathcal{A}(1,3,\{2\})=(-)\mathcal{A}(1,2,3)\ . \tag{5.1}$$

As we have mentioned before, this relation can also be considered as color reflection relation or $U(1)$-decoupling relation of three-point. In fact, the color reflection relation and $U(1)$-decoupling relation are both special cases of KK relations, connected by cyclic relation of amplitudes. We will prove the general KK relations by induction later, and firstly let us get some sense of these special cases.

The general KK relations of $\mathcal{N} = 4$ SYM are given by[9, 12]

$$\mathcal{A}(1, \{\alpha\}, n, \{\beta\}) = (-1)^{n_\beta} \sum_{\sigma \in OP(\{\alpha\},\{\beta^T\})} \mathcal{A}(1, \sigma, n) \ . \quad (5.2)$$

The sum is over all ordered permutations of set $\alpha \bigcup \beta^T$, where the relative ordering in each set α and β^T, which is the reversed ordering of set β, is preserved. The n_β is the number of elements in set $\{\beta\}$. If set $\{\beta\}$ is an empty set, KK relations become $\mathcal{A} = \mathcal{A}$ identity.

5.1 Color reflection relation as a special case

Considering the special case that $\{\alpha\}$ is an empty set, we have

$$\mathcal{A}(1, n, \{\beta_1, \ldots, \beta_m\}) = (-)^m \mathcal{A}(1, \beta_m, \ldots, \beta_1, n) = (-)^{m+2} \mathcal{A}(1, \beta_m, \ldots, \beta_1, n) \ . \quad (5.3)$$

Using cyclic relation we have $\mathcal{A}(1, n, \beta_1, \ldots, \beta_m) = \mathcal{A}(n, \beta_1, \ldots, \beta_m, 1)$, together with above result we get the wanted $(m + 2)$-point color reflection relation.

5.2 $U(1)$-decoupling relation as a special case

$U(1)$-decoupling relation is also a special case of KK relations if we consider that set $\{\alpha\}$ or $\{\beta\}$ has only one element. In the case that set $\{\beta\}$ has one element, we have

$$\mathcal{A}(1, \{\alpha_1, \ldots, \alpha_k\}, n, \beta) = -\sum_{i=0}^{k} \mathcal{A}(1, \alpha_1, \ldots, \alpha_i, \beta, \alpha_{i+1}, \ldots, \alpha_k, n) \ . \quad (5.4)$$

Using cyclic relation we could fix leg β instead of leg 1, then we have

$$\sum_{\sigma \in \ Cyclic} \mathcal{A}(\beta, \sigma(n, 1, \alpha_1, \ldots, \alpha_k)) = 0 \ . \quad (5.5)$$

This is the $(k + 3)$-point $U(1)$-decoupling relation.

Considering set $\{\alpha\}$ with only one element, we have

$$\mathcal{A}(1, \alpha, n, \{\beta_1, \ldots, \beta_m\}) = (-)^m \sum_{i=0}^{m} \mathcal{A}(1, \beta_m, \ldots, \beta_{i+1}, \alpha, \beta_i, \ldots, \beta_1, n) \ . \quad (5.6)$$

Using the $(m + 3)$-point color reflection relation for right hand side, we have

$$\mathcal{A}(1, \alpha, n, \{\beta_1, \ldots, \beta_m\}) = (-)^{2m+3} \sum_{i=0}^{m} \mathcal{A}(n, \beta_1, \ldots, \beta_i, \alpha, \beta_{i+1}, \ldots, \beta_m, 1) \ . \quad (5.7)$$

Using cyclic relation we can again rewrite this relation in standard form, which is nothing but $(m+3)$-point $U(1)$-decoupling relation with α fixed and the cyclic ordering of $(1, n, \beta_1, \ldots, \beta_m)$.

5.3 The proof of general case

After considering above special cases, let us come into more complex situation, the general KK relations with both set $\{\alpha\}$ and $\{\beta\}$ having more than one element. The idea of demonstration is simply the same as we have done before, using BCFW recursion relation of SYM version to expand super-amplitudes into sum of sub-super-amplitudes. The three-point case can be verified directly. Let us assume that the super-amplitude is $\mathcal{A}(1,\{\alpha_1,...,\alpha_k\},n,\{\beta_1,...,\beta_m\})$, then if we take $(1,n)$-shifting, by BCFW recursion relation of $\mathcal{N}=4$ SYM theory we could write the left hand side of KK relations (5.2) as

$$\begin{aligned}&\mathcal{A}(1,\{\alpha_1,...,\alpha_k\},n,\{\beta_1,...,\beta_m\}) \\ =&\int d^4\eta_{\hat{P}}\left[\sum_{i=0}^{k}\sum_{j=0}^{m}\mathcal{A}(\beta_{j+1},...,\beta_m,\hat{1},\alpha_1,...,\alpha_i,\hat{P}_{ij};\eta_{\hat{P}})\frac{1}{P_{ij}^2}\mathcal{A}(-\hat{P}_{ij},\alpha_{i+1},...,\alpha_k,\hat{n},\beta_1,...,\beta_j;\eta_{\hat{P}})\right]_{(i,j)\neq(0,m),(k,0)}\end{aligned} \tag{5.8}$$

where two cases $(i=0,j=m)$ and $(i=k,j=0)$ should be excluded from the summation. Now we use the induction for each sub-super-amplitude

$$\mathcal{A}(\beta_{j+1},...,\beta_m,\hat{1},\alpha_1,...,\alpha_i,\hat{P}_{ij};\eta_{\hat{P}}) = (-)^{m-j}\sum_{\sigma_{ij}}\mathcal{A}(\hat{1},\sigma_{ij},\hat{P}_{ij};\eta_{\hat{P}})\ , \tag{5.9}$$

$$\mathcal{A}(-\hat{P}_{ij},\alpha_{i+1},...,\alpha_k,\hat{n},\beta_1,...,\beta_j;\eta_{\hat{P}}) = (-)^{j}\sum_{\widetilde{\sigma}_{ij}}\mathcal{A}(-\hat{P}_{ij},\widetilde{\sigma}_{ij},\hat{n};\eta_{\hat{P}})\ . \tag{5.10}$$

The BCFW recursion of SYM version for right hand side of KK relations (5.2) is

$$\int d^4\eta_{P_c}\sum_{\sigma_c}\sum_{c=1}^{m+k-1}\mathcal{A}(\hat{1},\gamma_1,\ldots,\gamma_c,\hat{P}_c;\eta_{\hat{P}_c})\frac{1}{P_c^2}\mathcal{A}(-\hat{P}_c,\gamma_{c+1},\ldots,\gamma_{m+k},\hat{n};\eta_{\hat{P}_c})\ , \tag{5.11}$$

where $\sigma_c=\{\gamma_1,\ldots,\gamma_{m+k}\}=OP\{\alpha\}\cup\{\beta^T\}$.

It is easy to see that for given $\sigma_{ij},\widetilde{\sigma}_{ij}$, the combination of set $\{1,\sigma_{ij},\widetilde{\sigma}_{ij},n\}$ is one allowed permutation σ_c of expression (5.11). Also the index i,j specify a particular BCFW-splitting of σ_c. In other words, we have shown that each term in (5.8) will be found in (5.11). More specifically, for each term in $\sum_{i=0}^{k}\sum_{j=0}^{m}\sum_{\sigma_{ij}}\sum_{\tilde{\sigma}_{ij}}$ there is one corresponding term in $\sum_{c=1}^{m-k+1}\sum_{\sigma_c}$ and vice versa. Considering one fixed split in (5.11), i.e.,

$$\int d^4\eta_{P_c}\sum_{\sigma}\mathcal{A}(\hat{1},\gamma_1,\ldots,\gamma_c,\hat{P}_c;\eta_{\hat{P}_c})\frac{1}{P_c^2}\mathcal{A}(-\hat{P}_c,\gamma_{c+1},\ldots,\gamma_{m+k},\hat{n};\eta_{\hat{P}_c})\ , \tag{5.12}$$

the number of γ in the left hand side is c. Then we should re-group terms of BCFW expansion of (5.8) as follows. Take the front i' elements of $\{\alpha\}$ and the front j' elements of $\{\beta^T\}$ which satisfies $i'+j'=c$, their ordered permutations give σ_{ij} if we identify $i=i', j=m-j'$. Since i can take the value from 0 to k and j from 0 to m, there are many combinations of (i',j') which satisfy $i'+j'=c$. We should group terms of BCFW expansion according to c, i.e., we group terms in (5.8) which satisfy $i'+j'=i+m-j=c$, and this transfers $\sum_{i=0}^{k}\sum_{j=0}^{m}$ to $\sum_{c=1}^{m+k-1}$ and σ_{ij} to $\sum_{i=0}^{c} OP\{\alpha_1,\ldots,\alpha_i\}\cup\{\beta_m,\ldots,\beta_{i+m-c+1}\}$. For one certain c, this is just set $\{\gamma_1,\ldots,\gamma_c\}$ in (5.12), where γ takes the value from front i' elements of $\{\alpha\}$ and front j' elements of $\{\beta^T\}$ which satisfies $i'+j'=c$. Then terms of BCFW expansion of left hand side and right hand side match to each other. Thus if we can show the total number of terms is same for both (5.8) and (5.11), we have proved the identity.

To count terms it is easy to see that there are $C_{i+m-j}^{i}=\frac{(i+m-j)!}{i!(m-j)!}$ terms at the right hand side of equation (5.9), while there are $C_{j+k-i}^{j}=\frac{(j+k-i)!}{j!(k-i)!}$ terms at the right hand side of equation (5.10). Thus the total number of terms of (5.8) is

$$-2\frac{(m+k)!}{m!k!}+\sum_{i=0}^{k}\sum_{j=0}^{m}\frac{(i+m-j)!}{i!(m-j)!}\frac{(j+k-i)!}{j!(k-i)!}\ , \tag{5.13}$$

where $-2\frac{(m+k)!}{m!k!}$ counts the two excluded cases. The right hand side of KK-relations will have

$$\frac{(k+m)!}{k!m!}(k+m-1) \tag{5.14}$$

terms after using the BCFW recursion relation to expand each super-amplitude into $(k+m-1)$ terms as in (5.11). These two numbers match up as it should be, which can be easily checked in Mathematica.

6. The BCJ relations

6.1 Direct verification of four-point case

Let us again start with the simplest case of BCJ relations, i.e., the $n=4$ case. There are two independent relations:

$$s_{23}\mathcal{A}(1,2,3,4)=s_{13}\mathcal{A}(1,3,4,2)\ ,\qquad s_{12}\mathcal{A}(1,2,3,4)=s_{13}\mathcal{A}(1,4,2,3)\ , \tag{6.1}$$

where we use the notation $s_{ij}=(k_i+k_j)^2$. By BCFW recursion relation of $\mathcal{N}=4$ SYM theory we know that super-amplitude of four-point can be expressed as products of two super-amplitudes of three-point. Due to the three-point kinematics one of MHV$_3$ and $\overline{\text{MHV}}_3$ should vanishes, so there are only MHV amplitudes of four-point case. Since there is a simple function for all MHV amplitudes[22]:

$$\mathcal{A}(1,\ldots,n;\eta_1,\ldots,\eta_n)=\frac{\delta^4(\sum p)\delta^{2\mathcal{N}}(\sum_i\lambda_i\eta_i)}{\langle 1\ 2\rangle\langle 2\ 3\rangle\cdots\langle n\ 1\rangle}\ , \tag{6.2}$$

thus we can use this to directly verify relation (6.1). Take the first relation of (6.1) for example, by writing $s_{ij}=\langle i\ j\rangle\,[i\ j]$ we have

$$s_{23}\mathcal{A}(1,2,3,4)-s_{13}\mathcal{A}(1,3,4,2)=\frac{\langle 4|2|3]+\langle 4|1|3]}{\langle 1\ 2\rangle\langle 3\ 4\rangle\langle 4\ 1\rangle\langle 4\ 2\rangle}\delta^4(\sum p)\delta^8(\sum_i\lambda_i\eta_i)\ , \tag{6.3}$$

and the numerator equals to zero because of energy-momentum conservation. With simple calculation we can verify other relations similarly.

6.2 The fundamental relation

In $\mathcal{N}=4$ SYM theory we write the general formula of BCJ relations as[10, 20]

$$\mathcal{A}(1,2,\{4,5,...,m\},3,\{m+1,m+2,...,n\}) = \sum_{\sigma_i\in POP}\mathcal{A}(1,2,3,\sigma_i)\mathcal{F}\ . \tag{6.4}$$

The primary one is the one with $m=4$ and others could be derived by repeatedly using this one and KK relations. For example considering BCJ relations of five-point case[10, 20], we have the primary one

$$0 = -s_{24}\mathcal{A}(1,2,4,3,5)+(s_{14}+s_{45})\mathcal{A}(1,2,3,4,5)+s_{14}\mathcal{A}(1,2,3,5,4)\ . \tag{6.5}$$

We want to show that relation (6.5) is the essential one, i.e., from this we can derive all other equations. Let us try to derive other relations, for example,

$$\begin{aligned}\mathcal{A}(1,4,2,3,5) &= \frac{-s_{12}s_{45}\mathcal{A}(1,2,3,4,5)+s_{25}(s_{14}+s_{24})\mathcal{A}(1,4,3,2,5)}{s_{35}s_{24}}\ ,\\ \mathcal{A}(1,2,4,3,5) &= \frac{s_{45}(s_{12}+s_{24})\mathcal{A}(1,2,3,4,5)-s_{25}s_{14}\mathcal{A}(1,4,3,2,5)}{s_{35}s_{24}}\ .\end{aligned} \tag{6.6}$$

Starting from (6.5) and related KK relations

$$s_{24}\mathcal{A}(1,2,4,3,5) = (s_{14}+s_{45})\mathcal{A}(1,2,3,4,5)+s_{14}\mathcal{A}(1,2,3,5,4)\ , \tag{6.7}$$

$$\mathcal{A}(1,2,3,5,4) = -\mathcal{A}(1,2,3,4,5)-\mathcal{A}(1,2,4,3,5)-\mathcal{A}(1,4,2,3,5)\ , \tag{6.8}$$

we can derive

$$(s_{24}+s_{14})\mathcal{A}(1,2,4,3,5) = s_{45}\mathcal{A}(1,2,3,4,5)-s_{14}\mathcal{A}(1,4,2,3,5)\ . \tag{6.9}$$

The advantage of this relation is that $1,5$ have been put at the beginning and end position. Using this one, we can reduce the basis from $(n-2)!$ to $(n-3)!$. Exchanging $2,4$ in (6.9) we obtain

$$(s_{24}+s_{12})\mathcal{A}(1,4,2,3,5) = s_{25}\mathcal{A}(1,4,3,2,5)-s_{12}\mathcal{A}(1,2,4,3,5)\ . \tag{6.10}$$

Combining above one with (6.9) we can get the wanted relations (6.6) immediately.

For another example we consider the following BCJ relation

$$s_{24}s_{13}\mathcal{A}(1,2,4,5,3) = -s_{34}s_{51}\mathcal{A}(1,2,3,4,5)-s_{14}(s_{13}+s_{35})\mathcal{A}(1,2,3,5,4)\ . \tag{6.11}$$

To show this, we write down (6.5) and the one given by the $3\leftrightarrow 4$ changing, i.e.,

$$s_{24}\mathcal{A}(1,2,4,3,5) = (s_{14}+s_{45})\mathcal{A}(1,2,3,4,5)+s_{14}\mathcal{A}(1,2,3,5,4)\ , \tag{6.12}$$

$$s_{23}\mathcal{A}(1,2,3,4,5) = (s_{13}+s_{35})\mathcal{A}(1,2,4,3,5)+s_{13}\mathcal{A}(1,2,4,5,3)\ . \tag{6.13}$$

Thus we have

$$\begin{aligned}
\mathcal{A}(1,2,4,5,3) &= \frac{s_{23}\mathcal{A}(1,2,3,4,5)-(s_{13}+s_{35})\mathcal{A}(1,2,4,3,5)}{s_{13}} \\
&= \frac{s_{23}\mathcal{A}(1,2,3,4,5)}{s_{13}} - \frac{(s_{13}+s_{35})}{s_{13}}\frac{(s_{14}+s_{45})\mathcal{A}(1,2,3,4,5)+s_{14}\mathcal{A}(1,2,3,5,4)}{s_{24}} \\
&= \mathcal{A}(1,2,3,4,5)\frac{s_{24}s_{23}-(s_{13}+s_{35})(s_{14}+s_{45})}{s_{13}s_{24}} - \mathcal{A}(1,2,3,5,4)\frac{s_{14}(s_{13}+s_{35})}{s_{13}s_{24}} \\
&= \mathcal{A}(1,2,3,4,5)\frac{s_{24}s_{23}-(s_{24}-s_{51})(s_{13}-s_{51})}{s_{13}s_{24}} - \mathcal{A}(1,2,3,5,4)\frac{s_{14}(s_{13}+s_{35})}{s_{13}s_{24}} \\
&= \mathcal{A}(1,2,3,4,5)\frac{-s_{51}s_{34}}{s_{13}s_{24}} - \mathcal{A}(1,2,3,5,4)\frac{s_{14}(s_{13}+s_{35})}{s_{13}s_{24}} \ .
\end{aligned}$$

Other BCJ relations should be derived similarly from the primary one and KK relations. Thus in the proof we could only consider the primary one $m = 4$.

6.3 The primary formula of BCJ relation

For the special case of BCJ relations when $m = 4$, the general formula (6.4) is given by

$$\begin{aligned}
s_{24}\mathcal{A}(1,2,4,3,5,6,...,n) &= \mathcal{A}(1,2,3,4,5,6,...,n)(s_{41}+s_{45}+...+s_{4n}) \\
&\quad + \sum_{i=5}^{n}\mathcal{A}(1,2,3,5,...,i,4,i+1,...,n)(s_{41}+\sum_{k=i+1}^{n}s_{4k}) \ . \qquad (6.14)
\end{aligned}$$

In other words, we have leg 4 in all possible positions inserted into string $(5,6,...,n)$ and sum up s_{41} and all s_{4t} with number t at the right hand side of leg 4. For expression simplicity we re-mark the legs as $(2,4,3,5,6,\ldots,n,1) \to (1,2,3,4,5,\ldots,n-1,n)$, then we have

$$\sum_{i=3}^{n}\left[\mathcal{A}(1,3,\ldots,i-1,2,i,\ldots,n-1,n)\sum_{j=i}^{n}s_{2j}\right] \equiv I_n = 0 \ , \qquad (6.15)$$

which means that we have leg 2 in all possible positions inserted into string $(3,4,\ldots,n-1)$ and sum up s_{2j} with number j at the right hand side of leg 2. Note that $\sum_{j=1}^{n}s_{2j}=0$, i.e., $\sum_{j=i}^{n}s_{2j} = -\sum_{j=1}^{i-1}s_{2j}$, thus we have a dual form of BCJ relations

$$\sum_{i=3}^{n}\left[\mathcal{A}(1,3,\ldots,i-1,2,i,\ldots,n-1,n)\sum_{j=1}^{i-1}s_{2j}\right] = 0 \ , \qquad (6.16)$$

with number j at the left hand side of leg 2.

Since the lower point BCJ relations can be checked directly, we focus on the n-point relations and assume that BCJ relations are true for super-amplitudes lower than n-point. We use BCFW recursion of

SYM theory to expand all super-amplitudes in (6.15) under $(1,n)$-shifting, such as

$$\begin{aligned}
&\mathcal{A}(1,3,\ldots,i-1,2,i,\ldots,n-1,n) \\
&=\sum_{k=3}^{i-2}\int d^4\eta_{\hat{P}_k}\mathcal{A}(\hat{1},3,\ldots,k,\hat{P}_k;\eta_{\hat{P}_k})\frac{1}{P_k^2}\mathcal{A}(-\hat{P}_k,k+1,\ldots,i-1,2,i,\ldots,n-1,\hat{n};\eta_{\hat{P}_k}) \\
&\quad+\int d^4\eta_{\hat{P}_{i-1}}\mathcal{A}(\hat{1},3,\ldots,i-1,\hat{P}_{i-1};\eta_{\hat{P}_{i-1}})\frac{1}{P_{i-1}^2}\mathcal{A}(-\hat{P}_{i-1},2,i,\ldots,n-1,\hat{n};\eta_{\hat{P}_{i-1}})\Big|_{i\neq 3} \\
&\quad+\int d^4\eta_{\hat{P}_2}\mathcal{A}(\hat{1},3,\ldots,i-1,2,\hat{P}_2;\eta_{\hat{P}_2})\frac{1}{P_2^2}\mathcal{A}(-\hat{P}_2,i,\ldots,n-1,\hat{n};\eta_{\hat{P}_2})\Big|_{i\neq n} \\
&+\sum_{k=i}^{n-2}\int d^4\eta_{\hat{P}_k}\mathcal{A}(\hat{1},3,\ldots,i-1,2,i,\ldots,k,\hat{P}_k;\eta_{\hat{P}_k})\frac{1}{P_k^2}\mathcal{A}(-\hat{P}_k,k+1,\ldots,n-1,\hat{n};\eta_{\hat{P}_k}) \\
&\equiv A_i+B_i+C_i+D_i\ . \qquad (6.17)
\end{aligned}$$

A_i is terms split at the left hand side of leg $(i-1)$ and D_i is terms at the right hand side of leg i. B_i and C_i are special terms which do not exist when i takes the boundary value $i=3$ or n. In later discussion we denote $A=\sum_{i=3}^{n}(A_i\sum_{j=i}^{n}s_{2j})$, which is the corresponding contribution of all the super-amplitudes, so for B,C,D. The left hand side of (6.15) after BCFW expansion is $I_n=A+B+C+D$. Firstly we consider the term A, which is

$$\sum_{i=3}^{n}\sum_{k=3}^{i-2}\left[\int d^4\eta_{\hat{P}_k}\mathcal{A}(\hat{1},3,\ldots,k,\hat{P}_k;\eta_{P_k})\frac{1}{P_k^2}\mathcal{A}(-\hat{P}_k,k+1,\ldots,i-1,2,i,\ldots,n-1,\hat{n};\eta_{P_k})\sum_{j=i}^{n}s_{2j}\right]\ , \qquad (6.18)$$

where propagator is distinguished by index k and index i denotes distinct super-amplitude. In order to group terms of same channel into one super-space integration $\eta_{\hat{P}_k}$, we should exchange the order of summation from $\sum_{i=3}^{n}\sum_{k=3}^{i-2}$ to $\sum_{k=3}^{n-2}\sum_{i=k+2}^{n}$, i.e.,

$$\begin{aligned}
\sum_{k=3}^{n-2}\quad&\int d^4\eta_{\hat{P}_k}\Big\{\mathcal{A}(\hat{1},3,\ldots,k,\hat{P}_k;\eta_{\hat{P}_k})\frac{1}{P_k^2}\times \\
&\Big[\sum_{i=k+2}^{n}\Big(\mathcal{A}(-\hat{P}_k,k+1,\ldots,i-1,2,i,\ldots,n-1,\hat{n};\eta_{\hat{P}_k})\sum_{j=i}^{n}s_{2j}\Big)\Big]\Big\}\ , \qquad (6.19)
\end{aligned}$$

where propagator P_k is related by index k, so is the shifted Grassmann variable $\eta_{\hat{P}_k}$. All z in super-amplitudes and kinematic factors s_{2j} should be replaced by z_P, which are fixed by on-shell equation $\hat{P}^2(z)=0$. Now let us consider the term in square brackets. It is noticed that in the right hand side sub-super-amplitudes every momentum is on-shell and the shifted momenta become $\hat{P}_k=\hat{P}_k(z_{P_k})$ and $\hat{n}=\hat{n}(z_{P_k})$, but $\sum_{j=i}^{n}s_{2j}$ is evaluated with un-shifted momenta. In order to use lower point BCJ relations we should rewrite the factor $\sum_{j=i}^{n} s_{2j}$ as $\sum_{j=i}^{\hat{n}} s_{2j}(z_{P_k}) + (s_{2n} - s_{2\hat{n}}(z_{P_k}))$. Note that all z in these factors have been replaced by z_{P_k}, thus the term in square brackets is split into two parts,

$$\begin{aligned}
&\sum_{i=k+2}^{n} \left(\mathcal{A}(-\hat{P}_k, k+1, \ldots, i-1, 2, i, \ldots, n-1, \hat{n}; \eta_{\hat{P}_k}) \sum_{j=i}^{n} s_{2j} \right) \\
&= \sum_{i=k+2}^{n} \left(\mathcal{A}(-\hat{P}_k, k+1, \ldots, i-1, 2, i, \ldots, n-1, \hat{n}; \eta_{\hat{P}_k}) \sum_{j=i}^{\hat{n}} s_{2j} \right) \\
&\quad + (s_{2n} - s_{2\hat{n}}) \sum_{i=k+2}^{n} \mathcal{A}(-\hat{P}_k, k+1, \ldots, i-1, 2, i, \ldots, n-1, \hat{n}; \eta_{\hat{P}_k}) \ .
\end{aligned} \tag{6.20}$$

Using $(n-k+1)$-point BCJ relation the first term becomes

$$-\mathcal{A}(-\hat{P}_k, 2, k+1, \ldots, n-1, \hat{n}; \eta_{\hat{P}_k}) \sum_{j=k+1}^{\hat{n}} s_{2j} \ ,$$

and using $(n-k+1)$-point $U(1)$-decoupling relation the second term turns to

$$(s_{2\hat{n}} - s_{2n}) \mathcal{A}(-\hat{P}_k, k+1, \ldots, n-1, \hat{n}, 2; \eta_{\hat{P}_k}) \ .$$

After pulling them back into term A and changing index k to $i-1$, we combine them with term B. Many terms naturally cancel and we have the result of $A+B$,

$$\sum_{i=4}^{n} \left[\int d^4\eta_{\hat{P}_{i-1}} \mathcal{A}(\hat{1}, 3, \ldots, i-1, \hat{P}_{i-1}; \eta_{\hat{P}_{i-1}}) \frac{1}{P_{i-1}^2} \mathcal{A}(-\hat{P}_{i-1}, i, \ldots, n-1, \hat{n}, 2; \eta_{\hat{P}_{i-1}})(s_{2\hat{n}}(z_{P_{i-1}}) - s_{2n}) \right] \ . \tag{6.21}$$

The other two terms in (6.17), C and D, are disposed in almost the same manner, while we change the kinematic factors $\sum_{j=i}^{\hat{n}} s_{2j}$ to $-\sum_{j=\hat{1}}^{i-1} s_{2j}$ in order to use the dual form of BCJ relations. In order to use BCJ relations of lower point, we should rewrite $\sum_{j=1}^{i-1} s_{2j}$ as $\sum_{j=\hat{1}}^{i-1} s_{2j} + (s_{21} - s_{2\hat{1}})$. Notice the important property $\hat{P}_1(z) + \hat{P}_n(z) = P_1 + P_n$ of BCFW recursion relation, we have $s_{2\hat{n}} - s_{2n} = -(s_{2\hat{1}} - s_{21})$. With this in hand we can easily repeat calculation of $C+D$, which gives the result

$$\sum_{i=3}^{n-1} \left[\int d^4\eta_{\hat{P}_{i-1}} \mathcal{A}(2, \hat{1}, 3, \ldots, i-1, \hat{P}_{i-1}; \eta_{\hat{P}_{i-1}}) \frac{1}{P_{i-1}^2} \mathcal{A}(-\hat{P}_{i-1}, i, \ldots, n-1, \hat{n}; \eta_{\hat{P}_{i-1}})(s_{2\hat{n}}(z_{P_{i-1}}) - s_{2n}) \right] \ . \tag{6.22}$$

Experience in doing BCFW recursion relation indicates that $I_n = A + B + C + D$ is related to the supersymmetric BCFW recursion relation of $\mathcal{A}(2, 1, 3, \ldots, n)$ under $(1, n)$-shifting. In fact let us consider integration

$$\oint \frac{dz}{z} \mathcal{A}(2, \hat{1}, 3, \ldots, n-1, \hat{n}) \times s_{2\hat{n}}(z) \ . \tag{6.23}$$

Residue at pole $z = 0$ gives

$$\mathcal{A}(2, 1, 3, \ldots, n)s_{2n} \ ,$$

and sum of poles at shifted super-amplitude gives

$$-\sum_{i=4}^{n}\left[\int d^4\eta_{\hat{P}_{i-1}}\mathcal{A}(\hat{1}, 3, \ldots, i-1, \hat{P}_{i-1}; \eta_{\hat{P}_{i-1}})\frac{1}{P^2_{i-1}}\mathcal{A}(-\hat{P}_{i-1}, i, \ldots, n-1, \hat{n}, 2; \eta_{\hat{P}_{i-1}})s_{2\hat{n}}(z_{P_{i-1}})\right] \ ,$$

and

$$-\sum_{i=3}^{n-1}\left[\int d^4\eta_{\hat{P}_{i-1}}\mathcal{A}(2, \hat{1}, 3, \ldots, i-1, \hat{P}_{i-1}; \eta_{\hat{P}_{i-1}})\frac{1}{P^2_{i-1}}\mathcal{A}(-\hat{P}_{i-1}, i, \ldots, n-1, \hat{n}; \eta_{\hat{P}_{i-1}})s_{2\hat{n}}(z_{P_{i-1}})\right] \ .$$

Adding these terms together we reproduce $-I_n$. Since the shifted momenta $(1, n)$ are not adjacent, the super-amplitude behaves as $1/z^2$ when z approaches to infinity[6], so that the integration (6.23) equals to zero. Thus we have $I_n = 0$, which finishes the proof.

7. Conclusion

It is interesting to see that in $\mathcal{N} = 4$ SYM theory there are same relations of amplitudes as pure gluon amplitudes[9, 10, 12], and one can prove them using BCFW recursion relation of $\mathcal{N} = 4$ SYM theory, like the pure gluon case[17]. This verifies the conjecture that matter amplitudes also obey the similar BCJ relations[20]. Deduced from relations between amplitudes of $\mathcal{N} = 4$ SYM theory and amplitudes of $\mathcal{N} = 8$ super-gravity, it is challenging and interesting to think whether one can use supersymmetric BCFW recursion relation to guess or prove relations of super-gravity amplitudes. Since BCFW recursion relation naturally groups many terms into different channels, we wander if it could shed some light on computing gravity amplitudes through square relations of BCJ.

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