Two-point boundary value problems and exact controllability for several kinds of linear and nonlinear wave equations

De-Xing Kong* and Qing-You Sun†

Abstract: In this paper we introduce some new concepts for second-order hyperbolic equations: two-point boundary value problem, global exact controllability and exact controllability. For several kinds of important linear and nonlinear wave equations arising from physics and geometry, we prove the existence of smooth solutions of the two-point boundary value problems and show the global exact controllability of these wave equations. In particular, we investigate the two-point boundary value problem for one-dimensional wave equation defined on a closed curve and prove the existence of smooth solution which implies the exact controllability of this kind of wave equation. Furthermore, based on this, we study the two-point boundary value problems for the wave equation defined on a strip with Dirichlet or Neumann boundary conditions and show that the equation still possesses the exact controllability in these cases. Finally, as an application, we introduce the hyperbolic curvature flow and obtain a result analogous to the well-known theorem of Gage and Hamilton for the curvature flow of plane curves.

Key words and phrases: wave equation, two-point boundary value problem, global exact controllability, exact controllability, Dirichlet boundary condition, Neumann boundary condition, hyperbolic curvature flow, Gage-Hamilton's theorem.

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*Department of Mathematics, Zhejiang University, Hangzhou 310027, China.
†Center of Mathematical Sciences, Zhejiang University, Hangzhou 310027, China.

1 Introduction

Consider the following seconder-order hyperbolic partial differential equation

$$\mathscr{P}(t, x, u, Du, D^2u) = 0, \tag{1.1}$$

where t is the time variable, $x = (x_1, \cdots x_n)$ stand for the spacial variables, $u = u(t, x)$ is the unknown function, $\mathscr{P}$ is a given smooth function of the independent of variables $t, x_1, \cdots, x_n$, the unknown function u, the first-order partial derivatives $Du = (u_t, u_{x_1}, \cdots, u_{x_n})$ and the second-order partial derivatives $D^2u = (u_{tt}, u_{tx_1}, \cdots)$. Since we only consider the hyperbolic case, the initial data associated with the equation (1.1) read

$$t = 0: \quad u = u_0(x), \quad u_t = u_1(x), \tag{1.2}$$

where $u_0(x)$ and $u_1(x)$ are two given functions which stand for the initial position and the initial velocity, respectively. The equation (1.1) and the initial data (1.2) constitute the famous Cauchy problem. Another important problem related to (1.1) is the following so-called *two-point boundary value problem* for the equation (1.1):

Two-point Boundary Value Problem (TBVP): *Given two suitable smooth functions $u_0(x), u_T(x)$ and a positive constant T, can we find a C^2-smooth function $u = u(t, x)$ defined on the strip $[0, T] \times \mathbb{R}^n$ such that the function $u = u(t, x)$ satisfies the equation* (1.1) *on the domain $[0, T] \times \mathbb{R}^n$, the initial condition*

$$u(0, x) = u_0(x), \quad \forall\, x \in \mathbb{R}^n \tag{1.3}$$

and the terminal condition

$$u(T, x) = u_T(x), \quad \forall\, x \in \mathbb{R}^n? \tag{1.4}$$

Another Statement of TBVP: *Given two suitable smooth functions $u_0(x), u_T(x)$ and a positive constant T, can we find an initial velocity $u_t(0, x) = u_1(x)$ such that the Cauchy problem* (1.1)-(1.2) *has a solution $u = u(t, x) \in C^2([0, T] \times \mathbb{R}^n)$ which satisfies the terminal condition* (1.4)*?*

The system (1.1) together with (1.3)-(1.4) can be viewed as a distributed parameter control system when the initial velocity function $u_1(x)$ is considered as a control input.

We now give the following definition.

Definition 1.1. *For any given* $T > 0$*, if the TBVP* (1.1), (1.3)-(1.4) *has a* C^2 *solution on the strip* $[0, T] \times \mathbb{R}^n$*, then the equation* (1.1) *is called to possess the global exact controllability; If the TBVP* (1.1), (1.3)-(1.4) *admits a* C^2 *solution on the strip* $[0, T] \times \mathbb{R}^n$ *for some given* $T > 0$*, then we say that the equation* (1.1) *possesses the exact controllability.*

Remark 1.1. *For the system of seconder-order hyperbolic partial differential equations, we have similar concepts and definitions; For higher-order hyperbolic partial differential equations, we have a similar discussion.*

The wave equations play an important role in both theoretical and applied fields, they include two classes: linear wave equations and nonlinear wave equations. The classical wave equation is an important second-order linear partial differential equation of waves, such as sound waves, light waves and water waves. It arises in fields such as acoustics, electromagnetics, fluid dynamics, and general relativity. Historically, the problem of a vibrating string such as that of a musical instrument was studied by Jean le Rond d'Alembert, Leonhard Euler, Daniel Bernoulli, and Joseph-Louis Lagrange.

The wave equation is the prototypical example of a hyperbolic partial differential equation. In its simplest form, the wave equation refers to a unknown scalar function $y = y(t, x)$ which satisfies

$$y_{tt} - c^2 \Delta y = 0, \tag{1.5}$$

where $\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$ denotes the Laplacian and c is a fixed constant which stands for the propagation speed of the wave. One of aims of the present is to investigate the TBVP for (1.5), more precisely we study the following TBVP:

TBVP for (1.5)**:** *Given two functions* $f(x), g(x) \in C^{[n/2]+2}(\mathbb{R}^n)$ *and a positive constant* T*, can we find a* C^2*-smooth function* $y = y(t, x)$ *defined on the strip* $[0, T] \times \mathbb{R}^n$ *such that the function* $y = y(t, x)$ *satisfies the equation* (1.5) *on the domain* $[0, T] \times \mathbb{R}^n$*, the initial condition*

$$y(0, x) = f(x), \quad \forall\, x \in \mathbb{R}^n \tag{1.6}$$

and the terminal condition

$$y(T, x) = g(x), \quad \forall\, x \in \mathbb{R}^n? \tag{1.7}$$

Another Statement: *Given two functions $f(x), g(x) \in C^{[n/2]+2}(\mathbb{R}^n)$ and a positive constant T, can we find an initial velocity $v = v(x)$ such that the Cauchy problem for the wave equation* (1.5) *with the initial data*

$$t = 0: \quad y = f(x), \quad y_t = v(x) \tag{1.8}$$

has a solution $y = (t, x) \in C^2([0, T] \times \mathbb{R}^n)$ which satisfies the terminal condition (1.7)*?*

In this paper we shall show the global exact controllability of the equation (1.5) and some nonlinear wave equations arising from geometry and the theory of relativistic strings.

It is well-known that there are many deep and beautiful results on the TBVP for ordinary differential equations, however, according to the authors' knowledge, few of results on the TBVP for hyperbolic equations, even for (linear or nonlinear) wave equations have been known. Therefore, we can say that the result presented in this paper is the first result on this research topic.

On the other hand, the study on *boundary* control problems for hyperbolic systems was initiated by D.L. Russell in the 1960s. In [15], using the characteristic method, he showed that a class of $n \times n$ first order linear hyperbolic systems is exactly boundary controllable. This work led to an intensive investigation of controllability and stabilization of linear hyperbolic systems for more than 30 years. The literature pertaining to this study is now absolutely enormous, we refer to two excellent review papers Russell [16] and Lions [14]. However, while it may be fair to say that the study of boundary control of linear hyperbolic systems is now nearly complete, the study of nonlinear hyperbolic systems is still vastly open. Up to now, some results on the exact boundary controllability of (abstract) semilinear wave equations have been obtained (see [1], [2], [4], [12]-[13], [19]-[21] and references cited therein). As for boundary control of quasilinear hyperbolic systems, there have been few results so far. Motivated by Ruessell's work, Cirinà [3] studied boundary control of general quasilinear hyperbolic systems. Using a different approach from that of Russell, he proved that the system is *locally* exactly boundary controllable in the sense that the C^1 norms of both initial and terminal states are required to be small.

All the above results are obtained under the assumption that the initial and terminal states are smooth, and they are discussed in the framework of classical solutions. For the global exact controllability of system (1.1) in the case that the initial and terminal states maybe contain discontinuity points of the first kind, up to now only a few of results have been known. In Kong [7], the author investigated the global exact boundary controllability of 2×2 quasilinear hyperbolic system of conservation laws with linearly degenerate characteristics and proved that the system with nonlinear boundary conditions is globally exactly boundary controllable in the class of piecewise C^1 functions. Later, by a new constructive method, Kong and Yao reproved the global exact boundary controllability of a class of quasilinear hyperbolic systems of conservation laws with linearly degenerate characteristics, shown that the system with nonlinear boundary conditions is globally exactly boundary controllable in the class of piecewise C^1 functions, in particular, gave the optimal control time of the system (see [9]).

Here we would like to remark that the TBVP and the boundary control problems are essentially different two kinds of problems. Both of them play an important role in both theoretical and applied aspects.

We now state the first result in this paper.

Theorem 1.1. *The TBVP* (1.5)-(1.7) *admits a C^2-smooth solution $y = y(t,x)$ defined on the strip $[0,T] \times \mathbb{R}^n$.*

Theorem 1.1 implies that the wave equation (1.5) possesses the global exact controllability.

Remark 1.2. *The solution of the TBVP of* (1.5)-(1.7) *does not possess the uniqueness (see Remarks 2.2 and 3.2 for the details).*

Remark 1.3. *Theorem 1.1 can be generalized to the case of inhomogeneous wave equations, i.e.,*

$$y_{tt} - c^2 \Delta y = F(t,x), \tag{1.9}$$

where $F(t,x)$ is a given function which stands for the source term of the system.

Remark 1.4. *We have similar results for some nonlinear wave equations including a wave map equation arising from geometry and the equations for the motion of relativistic strings in the Minkowski space-time $\mathbb{R}^{1+n}$ (see Section 7 for the details). These nonlinear wave equations play an important role in both mathematics and physics.*

However, some problems arising from engineering, control theory etc. can be reduced to the TBVP for wave equations defined on a closed curve, say, a circle. The typical example is the vibration of a closed elastic string. In this case, since the wave equation is defined on a closed curve and then the solution must possess the periodicity, the method used in the proof of Theorem 1.1 will no longer work. It needs some new ideas and new technologies to study such a kind of problems.

In this paper we also investigate this kind of problems mentioned above. For simplicity, we consider the following TBVP[1]

$$\begin{cases} y_{tt} - y_{\theta\theta} = 0, \quad \forall\, (t,\theta) \in [0,T] \times \mathbb{R}, \\ y(0,\theta) = f(\theta), \quad \forall\, \theta \in \mathbb{R}, \\ y(T,\theta) = g(\theta), \quad \forall\, \theta \in \mathbb{R}, \end{cases} \tag{1.10}$$

where $y = y(t,\theta)$ is the unknown function of the variables t and θ, T is a given positive constant, $f(\theta)$ and $g(\theta)$ are two given periodic C^3 functions of $\theta \in \mathbb{R}$, say, the period is L, in which L is a positive real number. The second result in the present paper is the following theorem.

Theorem 1.2. *The TBVP* (1.10) *admits a global L-periodic C^2 solution $y = y(t,\theta)$ defined on the domain $[0,T] \times \mathbb{R}$, provided that $\dfrac{T}{L}$ is a rational number with $\dfrac{2T}{L} \notin \mathbb{N}$.*

Remark 1.5. *Theorem 1.2 implies that the wave equation in* (1.10) *possesses the exact controllability. On the other hand, in general, the solution of the TBVP* (1.10) *is not unique (see the proof of Theorem 1.2 in Section 4 for the details).*

Remark 1.6. *In Theorem 1.2, if $\dfrac{2T}{L} \in \mathbb{N}$, then there exists a relationship between $f(\theta)$ and $g(\theta)$. This means that $f(\theta)$ and $g(\theta)$ can not be given arbitrarily. See Remark 4.2 for the details.*

As a consequence, we consider the following TBVP defined on a circle

$$\begin{cases} y_{tt} - y_{\theta\theta} = 0, \quad \forall\, (t,\theta) \in [0,T] \times \mathbb{S}^1, \\ y(0,\theta) = f(\theta), \quad \forall\, \theta \in \mathbb{S}^1, \\ y(T,\theta) = g(\theta), \quad \forall\, \theta \in \mathbb{S}^1, \end{cases} \tag{1.11}$$

where $\mathbb{S}^1$ stands for the unit circle, $f(\theta)$ and $g(\theta)$ are two given C^3 functions of $\theta \in \mathbb{S}^1$. By Theorem 1.2, we have

[1]In fact, by scaling, any wave equation $y_{tt} - c^2 y_{\theta\theta} = 0$ with the propagation speed c can be reduced to the wave equation in (1.10). Therefore, in this paper, without loss of generality, we only consider the wave equation with the propagation speed $c = 1$.

Theorem 1.3. *The TBVP* (1.11) *admits a global* C^2 *solution* $y = y(t, \theta)$ *defined on the cylinder* $[0, T] \times \mathbb{S}^1$*, provided that* $\dfrac{T}{2\pi}$ *is a rational number and* $\dfrac{T}{\pi} \notin \mathbb{N}$*.*

Remark 1.7. *Theorem 1.3 implies that the wave equation defined on a circle possesses the exact controllability.*

Remark 1.8. *For the* $(1 + n)$*-dimensional wave equation* (1.5)*, we have similar results, provided that the initial/terminal data* $f(x)$ *and* $g(x)$ *in* (1.6)-(1.7) *are* $C^{[n/2]+3}$ *smooth functions and are periodic in* $r = \sqrt{x_1^2 + \cdots + x_n^2}$*. In fact, in the present situation, using Theorem 1.2, by a way similar to the proof of Theorem 1.1 we can prove that the TBVP* (1.5)-(1.7) *admits a global* C^2 *solution* $y = y(t, x)$ *defined on the strip* $[0, T] \times \mathbb{R}^n$*, provided that* $\dfrac{T}{L}$ *is a rational number with* $\dfrac{2T}{L} \notin \mathbb{N}$*, where* L *is the period of* $f(x)$*,* $g(x)$ *with respect to* r*. In other words, in this case the equation* (1.5) *still possesses the exact controllability.*

Based on the results mentioned above, we further study the two-point boundary value problems for the wave equation defined on a strip with Dirichlet or Neumann boundary conditions and show that the equation still possesses the exact controllability in these cases. Finally, as an application of the results mentioned above, we introduce the hyperbolic curvature flow and obtain a result analogous to the well-known theorem of Gage and Hamilton [5] for the curvature flow of plane curves.

This paper is organized as follows. Section 2 is devoted to the study on the global exact controllability of one-dimensional wave equation. Based on Section 2, in Section 3 we investigate the global exact controllability of linear wave equations in several space variables and we prove Theorem 1.1. In Section 4, we give the proof of Theorem 1.2 and Theorem 1.3, respectively. As some applications of Theorem 1.2, in Sections 5 and 6 we study the two-point boundary value problems for the wave equation defined on a strip with (homogeneous or inhomogeneous) Dirichlet or Neumann boundary conditions and show that the equation still possesses the exact controllability in these cases. The global exact controllability of some nonlinear wave equations arising from geometry and physics has been investigated in Section 7. In Section 8 we introduce the hyperbolic curvature flow and prove a result analogous to the one shown by Gage and Hamilton in [5] for curvature flow of plane curves. In Section 9 we give a summary and some discussions and then present several open problems.

2 One-dimensional wave equation

This section concerns the global exact controllability of one-dimensional wave equation, which is a basis of the present paper.

Consider the following TBVP for one-dimensional wave equation

$$\begin{cases} y_{tt} - y_{xx} = 0, \\ y(0,x) = f(x), \\ y(T,x) = g(x), \end{cases} \tag{2.1}$$

where T is an arbitrary fixed positive constant, $f(x)$ and $g(x)$ are two given functions of $x \in \mathbb{R}$. We have the following theorem which is a special case of Theorem 1.1 but fundamental in this paper.

Theorem 2.1. *Suppose that $f(x)$ and $g(x)$ are two given C^2-smooth functions of $x \in \mathbb{R}$. Then the TBVP (2.1) admits a C^2-smooth solution $y = y(t,x)$ defined on the strip $[0,T] \times \mathbb{R}$.*

Proof. Introduce

$$\tilde{y}(t,x) = y(t,x) - \frac{f(x-t) + f(x+t)}{2}, \tag{2.2}$$

and

$$\tilde{f}(x) = g(x) - \frac{f(x-T) + f(x+T)}{2}. \tag{2.3}$$

Obviously, $\tilde{f}(x)$ is a C^2-smooth function of $x \in \mathbb{R}$. By means of $\tilde{y}(t,x)$ and $\tilde{f}(x)$, the TBVP (2.1) can be equivalently rewritten as

$$\begin{cases} \tilde{y}_{tt} - \tilde{y}_{xx} = 0, \\ \tilde{y}(0,x) = 0, \\ \tilde{y}(T,x) = \tilde{f}(x), \end{cases} \tag{2.4}$$

Therefore, in order to prove Theorem 2.1, it suffices to show that the TBVP (2.4) has a C^2-smooth solution $\tilde{y} = \tilde{y}(t,x)$ defined on the strip $[0,T] \times \mathbb{R}$.

By d'Alembert formula, the solution of the following Cauchy problem

$$\begin{cases} \tilde{y}_{tt} - \tilde{y}_{xx} = 0, \\ t = 0: \quad \tilde{y} = 0, \quad \tilde{y}_t = v(x) \end{cases} \tag{2.5}$$

reads

$$\tilde{y}(t,x) = \frac{1}{2}\int_{x-t}^{x+t} v(\tau)d\tau, \tag{2.6}$$

where $v(x)$ is a C^1-smooth function to be determined, which stands for the initial velocity.

We next show that there indeed exists an initial velocity $v(x)$ such that the solution $\tilde{y} = \tilde{y}(t,x)$ of the Cauchy problem (2.5), defined by (2.6), satisfies the terminal condition in the TBVP (2.4), i.e.,

$$\tilde{y}(T,x) = \tilde{f}(x). \tag{2.7}$$

To do so, we construct a C^1 function $u(x)$ on $[-T,T]$ which satisfies

$$\int_{-T}^{T} u(\tau)d\tau = 2\tilde{f}(0), \quad u(T) - u(-T) = 2\tilde{f}'(0) \quad \text{and} \quad u'_-(T) - u'_+(-T) = 2\tilde{f}''(0). \tag{2.8}$$

Define

$$v(x) = \begin{cases} u(x-2NT) + 2\displaystyle\sum_{i=1}^{N} \tilde{f}'\big(x-(2i-1)T\big), & \forall\, x \geq 0, \\ u(x+2NT) - 2\displaystyle\sum_{i=1}^{N} \tilde{f}'\big(x+(2i-1)T\big), & \forall\, x < 0, \end{cases} \tag{2.9}$$

where N is given by

$$N = \left[\frac{|x|+T}{2T}\right].$$

In (2.9), the terms including the summation disappear in the case of $N=0$.

We claim that the function $v(x)$ defined by (2.9) is C^1-smooth.

In what follows, we distinguish two cases to show this fact.

Case A: $x \geq 0$. In this case, for every $x \neq (2N-1)T \ (N \in \mathbb{N})$, by (2.9) it is easy to check that $v(x)$ is C^1-smooth at such a point x.

When $x = (2N-1)T$, it follows from (2.9) that

$$\lim_{\alpha\to x^+} v(\alpha) = v(x) = u(-T) + 2\sum_{i=0}^{N-1} \tilde{f}'(2iT) \tag{2.10}$$

and

$$\lim_{\alpha\to x^-} v(\alpha) = \lim_{\alpha\to x^-} u\big(\alpha-2(N-1)T\big) + 2\sum_{i=1}^{N-1} \lim_{\alpha\to x^-} \tilde{f}'\big(\alpha-(2i-1)T\big) = u(T) + 2\sum_{i=1}^{N-1} \tilde{f}'(2iT). \tag{2.11}$$

Combining (2.10) and (2.11) gives

$$v(x) - \lim_{\alpha\to x^-} v(\alpha) = u(-T) - u(T) + 2\tilde{f}'(0). \tag{2.12}$$

Noting the second equation in (2.8) yields the continuity of $v(x)$ for all $x \geq 0$.

On the other hand, it follows from (2.9) that

$$v'_-(x) = u'_-\big(x - 2(N-1)T\big) + 2\sum_{i=1}^{N-1} \tilde{f}''\big(x - (2i-1)T\big) = u'_-(T) + 2\sum_{i=1}^{N-1} \tilde{f}''\big((2N-2i)T\big) \tag{2.13}$$

and

$$v'_+(x) = u'_+(x - 2NT) + 2\sum_{i=1}^{N} \tilde{f}''\big(x - (2i-1)T\big) = u'_+(-T) + 2\sum_{i=1}^{N} \tilde{f}''\big((2N-2i)T\big). \tag{2.14}$$

Combining (2.13) and (2.14) gives

$$v'_+(x) - v'_-(x) = u'_+(-T) - u'_-(T) + 2\tilde{f}''(0). \tag{2.15}$$

Noting the third equation in (2.8), we have

$$v'_+(x) = v'_-(x). \tag{2.16}$$

Summarizing the above argument yields that the function $v(x)$ defined by (2.9) is C^1-smooth for all $x \geq 0$.

Case B: $x < 0$. Similarly, we can prove that the function $v(x)$ is C^1-smooth in the present situation.

Combining Cases A and B, we have shown that the function $v(x)$ defined by (2.9) is a C^1-smooth function of $x \in \mathbb{R}$, and then the solution $\tilde{y} = \tilde{y}(t,x)$, defined by (2.6), of the Cauchy problem (2.5) is C^2-smooth on the whole upper plane $\mathbb{R}^+ \times \mathbb{R}$.

We now claim that, for the initial velocity $v(x)$ defined by (2.9), the solution $\tilde{y} = \tilde{y}(t,x)$ defined by (2.6) satisfies the terminal condition (2.7).

In fact, we verify this statement by distinguishing the following two cases:

Case 1: $x \geq 0$. In the present situation, it holds that

$$x \leq 2NT + T \leq x + 2T.$$

Thus it follows from (2.9) that

$$\begin{aligned}
\frac{1}{2}\int_x^{x+2T} v(\tau)d\tau =\ & \frac{1}{2}\left[\int_x^{2NT+T} v(\tau)d\tau + \int_{2NT+T}^{x+2T} v(\tau)d\tau\right] \\
=\ & \int_x^{2NT+T}\left[\frac{1}{2}u(\tau-2NT) + \sum_{i=1}^{N}\tilde{f}'\big(\tau-(2i-1)T\big)\right]d\tau + \\
& \int_{2NT+T}^{x+2T}\left[\frac{1}{2}u\big(\tau-2(N+1)T\big) + \sum_{i=1}^{N+1}\tilde{f}'\big(\tau-(2i-1)T\big)\right]d\tau \\
=\ & \frac{1}{2}\left[\int_x^{2NT+T} u(\tau)d\tau + \int_{2NT+T}^{x+2T} u(\tau)d\tau\right] + \\
& \sum_{i=1}^{N}\Big[\tilde{f}\big((2N-2i+2)T\big) - \tilde{f}\big(x-(2i-1)T\big)\Big] + \\
& \sum_{i=1}^{N+1}\Big[\tilde{f}\big(x-(2i-3)T\big) - \tilde{f}\big((2N-2i+2)T\big)\Big] \\
=\ & \frac{1}{2}\left[\int_{x-2NT}^{T} u(\tau)d\tau + \int_{-T}^{x-2NT} u(\tau)d\tau\right] + \\
& \sum_{i=1}^{N}\Big[\tilde{f}\big(x-(2i-3)T\big) - \tilde{f}\big(x-(2i-1)T\big)\Big] + \\
& \tilde{f}\big(x-(2N-1)T\big) - \tilde{f}(0) \\
=\ & \frac{1}{2}\int_{-T}^{T} u(\tau)d\tau + \tilde{f}(x+T) - \tilde{f}(0). \qquad (2.17)
\end{aligned}$$

Noting the first condition of (2.8), we obtain

$$\frac{1}{2}\int_x^{x+2T} v(\tau)d\tau = \tilde{f}(x+T). \qquad (2.18)$$

This leads to

$$\tilde{y}(T,x) = \frac{1}{2}\int_{x-T}^{x+T} v(\tau)d\tau = \tilde{f}(x). \qquad (2.19)$$

This is the desired terminal condition (2.7) for the case of $x \geq 0$.

Case 2: $x < 0$. In a similar manner, we can prove the terminal condition (2.7) holds for all $x < 0$.

The above discussion shows that $\tilde{y} = \tilde{y}(t,x)$ defined by (2.6) is a C^2-smooth solution of the TBVP (2.4) defined on the strip $[0,T] \times \mathbb{R}$. This proves Theorem 2.1. □

Remark 2.1. *In order to illustrate that it is easy to construct the function u satisfying* (2.8)*, in this remark we present two examples. The first example reads*

$$u = \frac{\tilde{f}''(0)}{2T}x^2 + \frac{\tilde{f}'(0)}{T}x + \frac{\tilde{f}(0)}{T} - \frac{\tilde{f}''(0)T}{6}. \qquad (2.20)$$

It is easy to check that the function u defined by (2.20) *satisfies all conditions in* (2.8)*. The second example is*

$$u(x) = \begin{cases} \tilde{h} + \frac{h}{2} + \frac{h}{2}\sin\left(\frac{\pi}{T}x + \frac{\pi}{2}\right), & \forall\, x \in [-T, 0), \\ \tilde{h} + h + \frac{1}{T}\tilde{f}''(0)x^2, & \forall\, x \in [0, T], \end{cases} \tag{2.21}$$

where

$$h = 2\tilde{f}'(0) - T\tilde{f}''(0) \quad \textit{and} \quad \tilde{h} = \frac{\tilde{f}(0)}{T} + \left(\frac{7}{12}T\tilde{f}''(0) - \frac{3}{2}\tilde{f}'(0)\right).$$

It is easy to verify that the function u defined by (2.21) *also satisfies all conditions in* (2.8)*.*

Remark 2.2. *The solution of the TBVP* (2.4) *(equivalently,* (2.2)*) is not unique. In fact, by the definition of $u(x)$ we observe that such a function u is not unique. This results the non-uniqueness of the initial velocity $v(x)$ and then the solution $\tilde{y} = \tilde{y}(t, x)$. For example, choose a C^1-smooth function $v_1(x)$ satisfying*

$$\int_{x-T}^{x+T} v_1(\tau)d\tau = 0. \tag{2.22}$$

This implies that $v_1(x)$ is a $2T$-periodic function and satisfies

$$\int_{-T}^{T} v_1(\tau)d\tau = 0. \tag{2.23}$$

Obviously, the function

$$\tilde{y}(t, x) = \frac{1}{2}\int_{x-t}^{x+t} \big(v(\tau) + v_1(\tau)\big)d\tau \tag{2.24}$$

also gives a C^2-smooth solution of the TBVP (2.4)*.*

The idea to construct $v(x)$ (see (2.9)) essentially comes from the *characteristic-quadrilateral identity* given in [10]. In fact, using the characteristic-quadrilateral identity, we have

$$\tilde{y}(A) + \tilde{y}(D) = \tilde{y}(B_1) + \tilde{y}(C_1). \tag{2.25}$$

See Figure 1. By the the initial condition in the TBVP (2.4), we get

$$\tilde{y}(A) = \tilde{y}(B_1) + \tilde{y}(C_1). \tag{2.26}$$

Similarly, we obtain

$$\tilde{y}(B_1) = \tilde{y}(B_2) + \tilde{y}(C_2), \quad \cdots, \quad \tilde{y}(B_{N-1}) = \tilde{y}(B_N) + \tilde{y}(C_N). \tag{2.27}$$

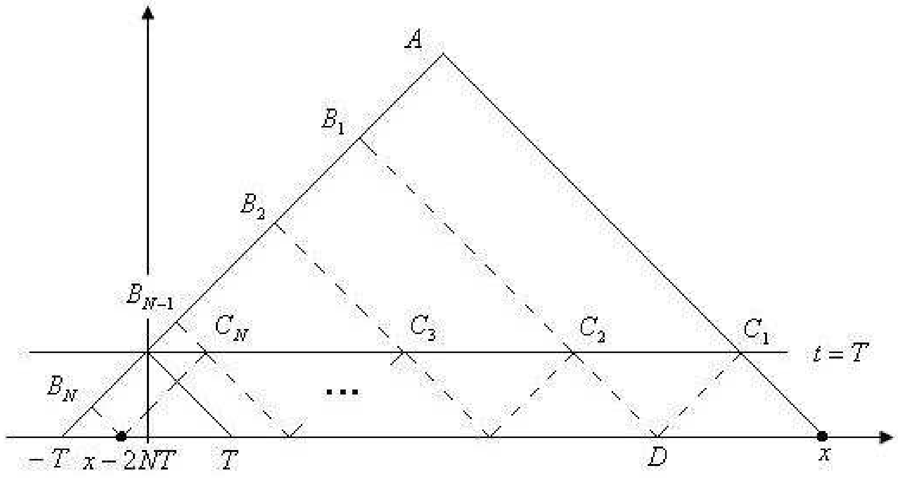

Figure 1: The method to construct the initial velocity $v(x)$ by characteristic-quadrilaterals

On the other hand, it follows from (2.6) that

$$\tilde{y}(A) = \frac{1}{2}\int_{-T}^{x} v(\tau)d\tau, \quad \tilde{y}(B_N) = \frac{1}{2}\int_{-T}^{x-2NT} v(\tau)d\tau. \tag{2.28}$$

Combining (2.26)-(2.28) gives the definition of $v(x)$ shown by (2.9).

Remark 2.3. *In fact, the existence of the C^2-solution of the TBVP* (2.4) *is equivalent to the existence of the C^1-solution of the following integral equation*

$$\int_{x-T}^{x+T} v(\tau)d\tau = \tilde{f}(x). \tag{2.29}$$

(2.29) *is a typical example of Volterra integral equations of the first kind. By the theory of Volterra integral equations, we can obtain the existence of the C^1-solution of the integral equation* (2.29). *However, by this theory we do not know how to construct the desired solution. But, in our theory we present a direct method to construct the desired solution of the TBVP* (2.1).

3 Wave equations in several space variables

In this section, we study the global exact controllability of wave equations in several space variables and prove Theorem 1.1.

Consider the TBVP (1.5)-(1.7), i.e., the following TBVP

$$\begin{cases} \Box\, y(t,x) = 0, \\ y(0,x) = f(x), \\ y(T,x) = g(x), \end{cases} \tag{3.1}$$

where $(t,x) \in \mathbb{R}^+ \times \mathbb{R}^n$, the symbol $\Box$ stands for the d'Alembert's operator, i.e.,

$$\Box = \partial_t^2 - \sum_{i=1}^{n} \partial_i^2 = \partial_t^2 - \Delta_x,$$

and $f(x), g(x) \in C^{[n/2]+2}(\mathbb{R}^n)$ are two given functions defined on $\mathbb{R}^n$. In (3.1), without loss of generality, we assume that the propagation speed c of wave is 1. Thus, in order to prove Theorem 1.1, it suffice to show the following theorem.

Theorem 3.1. *The TBVP* (3.1) *has a C^2-smooth solution $y = y(t,x)$ defined on the strip $[0,T] \times \mathbb{R}^n$.*

Remark 3.1. *Theorem 3.1 implies the global exact controllability of wave equations in several space variables. Similar result is true for the following inhomogeneous wave equations*

$$\Box\, y(t,x) = F(t,x).$$

We next give the proof of Theorem 3.1 (equivalently, Theorem 1.1).

Proof. We distinguish two cases to prove Theorem 3.1.

Case A: $n = 2k+1$ $(k \in \mathbb{N})$

Define the spherical mean of any given function $h(x)$ defined on $\mathbb{R}^n$:

$$\mathcal{A}_r h(x) = \frac{1}{\omega_{n-1}} \int_{S^{n-1}} h(x+ry)d\sigma(y), \tag{3.2}$$

where ω_{n-1} denotes the area of the unit sphere $S^{n-1} \subset \mathbb{R}^n$ and $d\sigma(y)$ stands for the Lebesgue measure on the unit sphere S^{n-1}. Let

$$w(t,r) = \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{k-1} \left(r^{2k-1}\mathcal{A}_r y(t,x)\right). \tag{3.3}$$

It is easy to check that the function $w(t,r)$ satisfies the following TBVP

$$\begin{cases} w_{tt} - w_{rr} = 0, \\ w(0,r) = \left(\dfrac{1}{r}\dfrac{\partial}{\partial r}\right)^{k-1} \left(r^{2k-1}\mathcal{A}_r f(x)\right), \\ w(T,r) = \left(\dfrac{1}{r}\dfrac{\partial}{\partial r}\right)^{k-1} \left(r^{2k-1}\mathcal{A}_r g(x)\right). \end{cases} \tag{3.4}$$

Thus, by Theorem 2.1, the TBVP (3.4) has a C^2-smooth solution defined on the domain $[0,T]\times\mathbb{R}$.

Denote

$$c_0=\prod_{i=1}^{k}(2i-1). \tag{3.5}$$

Then we have

$$y(t,x)=\lim_{r\to 0}\mathcal{A}_r y(t,x)=\lim_{r\to 0}\frac{1}{c_0 r}w(t,r). \tag{3.6}$$

It is easy to verify that the function $y=y(t,x)$ defined by (3.6) is a C^2-smooth solution of the TBVP (3.1) on the strip $[0,T]\times\mathbb{R}^n$.

Case B: $n=2k$ $(k\in\mathbb{N})$

By Hadamard's method of descent, we can also prove that the TBVP (3.1) has a C^2-smooth solution on the strip $[0,T]\times\mathbb{R}^n$. The main idea here is that if y solves a wave equation with n space variables, then it is also a solution of the corresponding wave equation with $n+1$ space variables, which happens to be independent of the last variable x_{n+1}. Here we omit the details. Thus the proof of Theorem 3.1 is completed. $\square$

Remark 3.2. *Noting Remark 2.2, we know that the solution of the TBVP* (3.1) *does not possesses the uniqueness.*

4 Wave equation defined on a closed curve

This section concerns the exact controllability of the wave equation defined on a circle. In other words, in this section we prove Theorem 1.2 and then Theorem 1.3.

To do so, we need the following Lemma (see Kong [8]).

Lemma 4.1. *Suppose that $F(x)$ is a L-periodic C^2 function of $x\in\mathbb{R}$, and its derivative of third order, i.e., $F'''(x)$, is piecewise smooth. Suppose furthermore that the Fourier series of $F(x)$ is given by*

$$F(x)=\frac{1}{2}A_0+\sum_{k=1}^{\infty}A_k\cos\left(\frac{2k\pi}{L}x\right)+\sum_{k=1}^{\infty}B_k\sin\left(\frac{2k\pi}{L}x\right), \tag{4.1}$$

where

$$A_0=\frac{2}{L}\int_0^L f(x)dx,\quad A_k=\frac{2}{L}\int_0^L f(x)\cos\left(\frac{2k\pi}{L}x\right)dx,\quad B_k=\frac{2}{L}\int_0^L f(x)\sin\left(\frac{2k\pi}{L}x\right)dx.$$

Then the series $\sum_{k=1}^{\infty}|k^2A_k|$ and $\sum_{k=1}^{\infty}|k^2B_k|$ are convergent.

Proof. It follows from (4.1) that

$$\begin{cases} \sum_{k=1}^{\infty} B_k \sin\left(\frac{2k\pi}{L}x\right) = \frac{1}{2}\big(F(x) - F(-x)\big) \triangleq G(x), \\ \frac{1}{2}A_0 + \sum_{k=1}^{\infty} A_k \cos\left(\frac{2k\pi}{L}x\right) = \frac{1}{2}\big(F(x) + F(-x)\big) \triangleq H(x). \end{cases} \tag{4.2}$$

Since $G(x)$ is a L-periodic odd function, its derivative of third order, i.e., $G'''(x)$, is a L-periodic piecewise continuous even function. So the form of the Fourier series of $G'''(x)$ should be

$$G'''(x) = \frac{1}{2}B_0^{(3)} + \sum_{k=1}^{\infty} B_k^{(3)} \cos\left(\frac{2k\pi}{L}x\right), \tag{4.3}$$

where $B_0^{(3)}$ and $B_k^{(3)}$ $(k = 1, 2, \cdots)$ stand for the coefficients of the Fourier series. By Parseval inequality, we obtain

$$\frac{1}{2}\left(B_0^{(3)}\right)^2 + \sum_{k=1}^{\infty}\left(B_k^{(3)}\right)^2 = \frac{2}{L}\int_0^L \left[G'''(x)\right]^2 dx < \infty. \tag{4.4}$$

On the other hand,

$$\begin{aligned} B_k^{(3)} &= \frac{2}{L}\int_0^L G'''(x)\cos\left(\frac{2k\pi}{L}x\right)dx \\ &= \frac{2}{L}\left[G''(x)\cos\left(\frac{2k\pi}{L}x\right)\right]\Big|_0^L + \frac{2k\pi}{L}\frac{2}{L}\int_0^L G''(x)\sin\left(\frac{2k\pi}{L}x\right)dx \\ &= \frac{2k\pi}{L}\frac{2}{L}\left[G'(x)\sin\left(\frac{2k\pi}{L}x\right)\right]\Big|_0^L - \left(\frac{2k\pi}{L}\right)^2\frac{2}{L}\int_0^L G'(x)\cos\left(\frac{2k\pi}{L}x\right)dx \\ &= -\left(\frac{2k\pi}{L}\right)^2\frac{2}{L}\left[G(x)\cos\left(\frac{2k\pi}{L}x\right)\right]\Big|_0^L - \left(\frac{2k\pi}{L}\right)^3\frac{2}{L}\int_0^L G(x)\sin\left(\frac{2k\pi}{L}x\right)dx \\ &= -\left(\frac{2k\pi}{L}\right)^3 B_k. \end{aligned} \tag{4.5}$$

Here we have made use of the fact that $G(x)$ and $G''(x)$ are L-periodic odd functions. So

$$\sum_{k=1}^{\infty}\left(k^3 B_k\right)^2 = \left(\frac{L}{2\pi}\right)^6 \sum_{k=1}^{\infty}\left(B_k^{(3)}\right)^2 < \infty. \tag{4.6}$$

Then by Cauchy inequality, it yields

$$\sum_{k=1}^{\infty}\left|k^2 B_k\right| \le \sqrt{\sum_{k=1}^{\infty}(k^3 B_k)^2 \cdot \sum_{k=1}^{\infty}\frac{1}{k^2}} < \infty. \tag{4.7}$$

Similarly, we can show that $\sum_{k=1}^{\infty}\left|k^2 A_k\right|$ is convergent.

Thus, the proof of Lemma 4.1 is completed. □

Remark 4.1. *Obviously, it follows from Lemma 4.1 that the series*

$$\sum_{k=1}^{\infty}|A_k|,\quad \sum_{k=1}^{\infty}|B_k|,\quad \sum_{k=1}^{\infty}|kA_k|,\quad \sum_{k=1}^{\infty}|kB_k|$$

are convergent.

We now prove Theorem 1.2.

Proof. Suppose that the Fourier series of the solution $y = y(t, \theta)$ to (1.10) reads

$$y(t,\theta) = \frac{1}{2}a_0(t) + \sum_{k=1}^{\infty} a_k(t)\cos\left(\frac{2k\pi}{L}\theta\right) + \sum_{k=1}^{\infty} b_k(t)\sin\left(\frac{2k\pi}{L}\theta\right), \tag{4.8}$$

where $a_0(t),\ a_k(t),\ b_k(t)$ stand for the coefficients of the Fourier series. Then by the first equation in (1.10), we obtain

$$\begin{cases} a_0''(t) = 0, \\ a_k''(t) + \left(\dfrac{2k\pi}{L}\right)^2 a_k(t) = 0, \\ b_k''(t) + \left(\dfrac{2k\pi}{L}\right)^2 b_k(t) = 0. \end{cases} \tag{4.9}$$

In particular,

$$\begin{cases} f(\theta) = \dfrac{1}{2}a_0(0) + \displaystyle\sum_{k=1}^{\infty} a_k(0)\cos\left(\frac{2k\pi}{L}\theta\right) + \sum_{k=1}^{\infty} b_k(0)\sin\left(\frac{2k\pi}{L}\theta\right), \\ g(\theta) = \dfrac{1}{2}a_0(T) + \displaystyle\sum_{k=1}^{\infty} a_k(T)\cos\left(\frac{2k\pi}{L}\theta\right) + \sum_{k=1}^{\infty} b_k(T)\sin\left(\frac{2k\pi}{L}\theta\right), \end{cases} \tag{4.10}$$

in which

$$\begin{cases} a_0(0) = \dfrac{2}{L}\displaystyle\int_0^L f(x)dx, \\ a_0(T) = \dfrac{2}{L}\displaystyle\int_0^L g(x)dx, \end{cases} \tag{4.11}$$

$$\begin{cases} a_k(0) = \dfrac{2}{L}\displaystyle\int_0^L f(x)\cos\left(\frac{2k\pi}{L}x\right)dx, \\ a_k(T) = \dfrac{2}{L}\displaystyle\int_0^L g(x)\cos\left(\frac{2k\pi}{L}x\right)dx, \end{cases} \tag{4.12}$$

and

$$\begin{cases} b_k(0) = \dfrac{2}{L}\displaystyle\int_0^L f(x)\sin\left(\frac{2k\pi}{L}x\right)dx, \\ b_k(T) = \dfrac{2}{L}\displaystyle\int_0^L g(x)\sin\left(\frac{2k\pi}{L}x\right)dx. \end{cases} \tag{4.13}$$

It follows from (4.9) that

$$\begin{cases} a_0(t) = \alpha_0 + \beta_0 t, \\ a_k(t) = \alpha_k \cos\left(\dfrac{2k\pi}{L}t\right) + \beta_k \sin\left(\dfrac{2k\pi}{L}t\right), \\ b_k(t) = \bar{\alpha}_k \cos\left(\dfrac{2k\pi}{L}t\right) + \bar{\beta}_k \sin\left(\dfrac{2k\pi}{L}t\right), \end{cases} \tag{4.14}$$

where $\alpha_0,\ \beta_0,\ \alpha_k,\ \beta_k,\ \bar{\alpha}_k$ and $\bar{\beta}_k$ are some constants. Comparing (4.11)-(4.13) with (4.14) gives

$$\begin{cases} \alpha_0 = \dfrac{2}{L}\displaystyle\int_0^L f(x)dx, \\ \alpha_0 + \beta_0 T = \dfrac{2}{L}\displaystyle\int_0^L g(x)dx, \end{cases} \tag{4.15}$$

$$\begin{cases} \alpha_k = \dfrac{2}{L}\displaystyle\int_0^L f(x)\cos\left(\dfrac{2k\pi}{L}x\right)dx, \\ \alpha_k \cos\left(\dfrac{2k\pi}{L}T\right) + \beta_k \sin\left(\dfrac{2k\pi}{L}T\right) = \dfrac{2}{L}\displaystyle\int_0^L g(x)\cos\left(\dfrac{2k\pi}{L}x\right)dx \end{cases} \tag{4.16}$$

and

$$\begin{cases} \bar{\alpha}_k = \dfrac{2}{L}\displaystyle\int_0^L f(x)\sin\left(\dfrac{2k\pi}{L}x\right)dx, \\ \bar{\alpha}_k \cos\left(\dfrac{2k\pi}{L}T\right) + \bar{\beta}_k \sin\left(\dfrac{2k\pi}{L}T\right) = \dfrac{2}{L}\displaystyle\int_0^L g(x)\sin\left(\dfrac{2k\pi}{L}x\right)dx. \end{cases} \tag{4.17}$$

That is,

$$\begin{cases} \alpha_0 = \dfrac{2}{L}\displaystyle\int_0^L f(x)dx, \\ \beta_0 = \dfrac{2}{LT}\displaystyle\int_0^L [g(x) - f(x)]dx, \end{cases} \tag{4.18}$$

$$\begin{cases} \alpha_k = \dfrac{2}{L}\displaystyle\int_0^L f(x)\cos\left(\dfrac{2k\pi}{L}x\right)dx, \\ \beta_k = \begin{cases} 0, \quad \text{if } \dfrac{2kT}{L} \in \mathbb{N}, \\ \dfrac{2}{L\sin\left(\frac{2k\pi}{L}T\right)}\displaystyle\int_0^L \left[g(x) - f(x)\cos\left(\dfrac{2k\pi}{L}T\right)\right]\cos\left(\dfrac{2k\pi}{L}x\right)dx, \quad \text{if } \dfrac{2kT}{L} \notin \mathbb{N} \end{cases} \end{cases} \tag{4.19}$$

and

$$\begin{cases} \bar{\alpha}_k = \dfrac{2}{L}\displaystyle\int_0^L f(x)\sin\left(\dfrac{2k\pi}{L}x\right)dx, \\ \bar{\beta}_k = \begin{cases} 0, \quad \text{if } \dfrac{2kT}{L} \in \mathbb{N}, \\ \dfrac{2}{L\sin\left(\frac{2k\pi}{L}T\right)}\displaystyle\int_0^L \left[g(x) - f(x)\cos\left(\dfrac{2k\pi}{L}T\right)\right]\sin\left(\dfrac{2k\pi}{L}x\right)dx, \quad \text{if } \dfrac{2kT}{L} \notin \mathbb{N}. \end{cases} \end{cases} \tag{4.20}$$

Here we would like to point out that, when $\frac{2kT}{L} \in \mathbb{N}$, i.e., $\sin\left(\frac{2k\pi}{L}T\right) = 0$, in (4.19)-(4.20) we take the corresponding coefficients β_k and $\bar{\beta}_k$ as zero. In fact, we may also take β_k and $\bar{\beta}_k$ as some series satisfying

$$\sum_{k=1}^{\infty} |k^2\beta_k| < \infty, \quad \sum_{k=1}^{\infty} |k^2\bar{\beta}_k| < \infty. \tag{4.21}$$

The different choice of β_k and $\bar{\beta}_k$ can give the different solution. This implies that the solution of the TBVP under consideration is not unique.

Since $\frac{T}{L}$ is a rational number and $\frac{2T}{L} \notin \mathbb{N}$, $\frac{2T}{L}$ can be expressed as a fraction $\frac{p}{q}$, where p and q are two irreducible integers and q is not less than 2, i.e., $q \geq 2$. By the property of sinusoid, we have

$$\left|\sin\left(\frac{2k\pi}{L}T\right)\right| = \left|\sin\left(\frac{kp\pi}{q}\right)\right| \geq \left|\sin\left(\frac{\pi}{q}\right)\right| \triangleq C_s, \quad \text{if } k \text{ is not the multiple of } q. \tag{4.22}$$

So for the given T and L, C_s is a constant. Then it follows from (4.14), (4.19), (4.21) and Remark 4.1 that

$$\begin{aligned}\sum_{k=1}^{\infty} |a_k(t)| &\leq \sum_{k=1}^{\infty} (|\alpha_k| + |\beta_k|) \\ &\leq \left(1 + \frac{1}{C_s}\right) \sum_{k=1}^{\infty} \left|\frac{2}{L}\int_0^L f(x)\cos\left(\frac{2k\pi}{L}x\right)dx\right| + \frac{1}{C_s}\sum_{k=1}^{\infty}\left|\frac{2}{L}\int_0^L g(x)\cos\left(\frac{2k\pi}{L}x\right)dx\right| \\ &< \infty. \end{aligned} \tag{4.23}$$

Similarly, we obtain from (4.14), (4.20), (4.21) and Remark 4.1 that

$$\sum_{k=1}^{\infty} |b_k(t)| < \infty. \tag{4.24}$$

Combining (4.23), (4.24) and (4.15) gives the convergence of the Fourier series (4.8) of the solution $y = y(t, \theta)$ immediately.

Moreover, by Lemma 4.1, Remark 4.1 and (4.21) we obtain

$$\sum_{k=1}^{\infty} |k\alpha_k| < \infty, \quad \sum_{k=1}^{\infty} |k\beta_k| < \infty, \quad \sum_{k=1}^{\infty} |k\bar{\alpha}_k| < \infty, \quad \sum_{k=1}^{\infty} \left|k\bar{\beta}_k\right| < \infty \tag{4.25}$$

and

$$\sum_{k=1}^{\infty} \left|k^2\alpha_k\right| < \infty, \quad \sum_{k=1}^{\infty} \left|k^2\beta_k\right| < \infty, \quad \sum_{k=1}^{\infty} \left|k^2\bar{\alpha}_k\right| < \infty, \quad \sum_{k=1}^{\infty} \left|k^2\bar{\beta}_k\right| < \infty. \tag{4.26}$$

Using (4.25) and (4.26), by a similar argument as used above, we can prove

$$\sum_{k=1}^{\infty} |ka_k(t)| < \infty, \quad \sum_{k=1}^{\infty} |kb_k(t)| < \infty, \quad \sum_{k=1}^{\infty} \left|a_k'(t)\right| < \infty, \quad \sum_{k=1}^{\infty} \left|b_k'(t)\right| < \infty \tag{4.27}$$

and

$$\begin{cases} \sum_{k=1}^{\infty} \left|k^2 a_k(t)\right| < \infty, \quad \sum_{k=1}^{\infty} \left|k^2 b_k(t)\right| < \infty, \quad \sum_{k=1}^{\infty} \left|k a_k'(t)\right| < \infty, \quad \sum_{k=1}^{\infty} \left|k b_k'(t)\right| < \infty, \\ \sum_{k=1}^{\infty} \left|a_k''(t)\right| < \infty, \quad \sum_{k=1}^{\infty} \left|b_k''(t)\right| < \infty. \end{cases} \tag{4.28}$$

Obviously, (4.27) and (4.28) imply that the Fourier series of the first-order derivatives of y (i.e., y_t and y_θ) and second-order derivatives of y (i.e., $y_{tt},\ y_{t\theta}$ and $y_{\theta\theta}$) are also convergent, respectively.

Thus, the proof of Theorem 1.2 is completed. □

Remark 4.2. *By* (4.10), (4.14), (4.18), *the first equation in* (4.19) *and the first equation in* (4.20), *if* $\frac{2T}{L} \in \mathbb{N}$, *then* $g(\theta)$ *satisfies*

$$\begin{aligned} g(\theta) &= \frac{1}{2}(\alpha_0 + \beta_0 T) + \sum_{k=1}^{\infty} \alpha_k \cos\left(\frac{2k\pi T}{L}\right) \cos\left(\frac{2k\pi}{L}\theta\right) + \sum_{k=1}^{\infty} \bar{\alpha}_k \cos\left(\frac{2k\pi T}{L}\right) \sin\left(\frac{2k\pi}{L}\theta\right) \\ &= \frac{1}{L}\int_0^L g(x)dx + \sum_{k=1}^{\infty} \frac{2}{L}\int_0^L f(x)\cos\left(\frac{2k\pi}{L}x\right)dx \cos\left(\frac{2k\pi}{L}\theta\right) \cos\left(\frac{2k\pi T}{L}\right) + \\ &\qquad \sum_{k=1}^{\infty} \frac{2}{L}\int_0^L f(x)\sin\left(\frac{2k\pi}{L}x\right)dx \sin\left(\frac{2k\pi}{L}\theta\right) \cos\left(\frac{2k\pi T}{L}\right). \end{aligned} \tag{4.29}$$

Then in this case, for any given $f(\theta)$, *the terminal value* $g(\theta)$ *can not be arbitrary. In particular, if* $\frac{T}{L} \in \mathbb{N}$, *then it should hold that*

$$g(\theta) = \frac{1}{L}\int_0^L [g(x) - f(x)]dx + f(\theta). \tag{4.30}$$

Remark 4.3. *If* $\frac{T}{L}$ *is a irrational number, then* (4.22) *is incorrect, and then we can not get the convergence in* (4.23), *etc.*

Remark 4.4. *Theorem 1.2 shows that, for some* T *satisfied the assumptions in Theorem 1.2, there exist some initial velocity* $v(\theta)$ *such that the Cauchy problem for the wave equation in* (1.10) *with the initial data*

$$t = 0: \quad y = f(\theta), \quad y_t = v(\theta) \tag{4.31}$$

has a solution $y = y(t,\theta) \in C^2([0,T] \times \mathbb{R})$ *which satisfies the terminal condition in* (1.10), *i.e., the third equation in* (1.10).

Theorem 1.3 comes from Theorem 1.2 directly.

5 Wave equation with homogeneous Dirichlet or Neumann boundary conditions

In this section, we investigate the TBVP for the wave equation defined on the domain $[0,T]\times[0,L]$ with homogeneous Dirichlet boundary conditions and homogeneous Neumann boundary conditions, respectively, where $T,\ L>0$ are two given real numbers.

More precisely, we consider the following problem for the wave equation with the homogeneous Dirichlet boundary conditions

$$\begin{cases} y_{tt}-y_{xx}=0, \\ y(0,x)=f(x), \\ y(T,x)=g(x), \\ y(t,0)=y(t,L)=0, \end{cases} \tag{5.1}$$

and the problem for the wave equation with the homogeneous Neumann boundary conditions

$$\begin{cases} y_{tt}-y_{xx}=0, \\ y(0,x)=f(x), \\ y(T,x)=g(x), \\ y_x(t,0)=y_x(t,L)=0, \end{cases} \tag{5.2}$$

where $y=y(t,x)$ is the unknown function of $(t,x)\in[0,T]\times[0,L]$, $f(x)$ and $g(x)$ are two given C^3 functions of $x\in[0,L]$. Moreover, f and g satisfy the compatibility conditions

$$f(x)\Big|_{x=L}^{x=0}=g(x)\Big|_{x=L}^{x=0}=0,\quad f''(x)\Big|_{x=L}^{x=0}=g''(x)\Big|_{x=L}^{x=0}=0,\quad \text{for the Dirichlet conditions;} \tag{5.3}$$

$$f'(x)\Big|_{x=L}^{x=0}=g'(x)\Big|_{x=L}^{x=0}=0,\quad \text{for the Neumann conditions.} \tag{5.4}$$

We have the following theorem which can be viewed as consequences of Theorem 1.2.

Theorem 5.1. *(A) Suppose that the compatibility conditions in* (5.3) *are satisfied,* $\dfrac{T}{L}$ *is a rational number and* $\dfrac{T}{L}\notin\mathbb{N}$. *Then the problem* (5.1) *admits a* C^2 *solution* $y=y(t,x)$ *on the domain* $[0,T]\times[0,L]$.

(B) Suppose that the compatibility conditions in (5.4) *are satisfied,* $\dfrac{T}{L}$ *is a rational number and* $\dfrac{T}{L} \notin \mathbb{N}$*. Then the problem* (5.2) *admits a* C^2 *solution* $y = y(t,x)$ *on the domain* $[0,T] \times [0,L]$*.*

Proof. For the case of the Dirichlet boundary conditions, we extend any C^2 solution $y = y(t,x)$ to the domain $[0,T] \times [-L,L]$ by

$$y(t,x) = -y(t,-x), \quad \text{for } x \in [-L,0], \tag{5.5}$$

and then extend $y = y(t,x)$ to be $2L$-periodic. See [10] and [11]. One easily checks that if the given initial/terminal data have the form in (5.1), the extended initial/terminal data are $2L$-periodic and given by (see [10]-[11])

$$\tilde{f}(x) \triangleq y(0,x) = \begin{cases} -f(-x), & \text{for } x \in [-L,0], \\ f(x), & \text{for } x \in [0,L], \end{cases} \tag{5.6}$$

$$\tilde{g}(x) \triangleq y(T,x) = \begin{cases} -g(-x), & \text{for } x \in [-L,0], \\ g(x), & \text{for } x \in [0,L], \end{cases} \tag{5.7}$$

respectively. Thus, we obtain an extended TBVP for the wave equation defined on the strip $[0,T] \times \mathbb{R}$. When the compatibility conditions in (5.3) are satisfied, this extended $y = y(t,x)$ is a C^2 solution of the extended TBVP with the extended initial/terminal data $\left(\tilde{f}(x), \tilde{g}(x)\right)$. Therefore, we may make use of Theorem 1.2 and obtain the solution, denoted by $\tilde{y} = \tilde{y}(t,x)$, to the extended TBVP defined on the strip $[0,T] \times \mathbb{R}$. Obviously, $\tilde{y} = \tilde{y}(t,x)$ is a $2L$-periodic odd C^2 function. So the Dirichlet boundary conditions are satisfied naturally. Let $y = y(t,x)$ be the restriction of $\tilde{y} = \tilde{y}(t,x)$ on the region $[0,T] \times [0,L]$. It is easy to see that $y = y(t,x)$ is the C^2 solution to (5.1).

Similarly, for the case of Neumann boundary conditions, we can extend $y = y(t,x)$ by

$$y(t,x) = y(t,-x), \quad \text{for } x \in [-L,0]. \tag{5.8}$$

See [10]-[11]. Then, $y = y(t,x)$ can be extended to be a classical $2L$-periodic C^2 solution of the extended TBVP for the wave equation with $2L$-periodic initial/terminal data given by

$$\tilde{f}(x) \triangleq y(0,x) = \begin{cases} f(-x), & \text{for } x \in [-L,0], \\ f(x), & \text{for } x \in [0,L], \end{cases} \tag{5.9}$$

$$\tilde{g}(x) \triangleq y(T,x) = \begin{cases} g(-x), & \text{for } x \in [-L,0], \\ g(x), & \text{for } x \in [0,L], \end{cases} \tag{5.10}$$

respectively. When the compatibility conditions in (5.4) are satisfied, by a similar argument as used above, we can prove the part (B) in Theorem 5.1.

Thus, the proof of Theorem 5.1 is completed. □

Remark 5.1. *Theorem 5.1 shows that the wave equation defined on the domain* $[0,T] \times [0,L]$ *still possesses the exact controllability in the cases of homogeneous Dirichlet boundary conditions and homogeneous Neumann boundary conditions, provided that the corresponding compatibility conditions are satisfied.*

6 Wave equation with inhomogeneous Dirichlet or Neumann boundary conditions

This section is devoted to the exact controllability of the wave equation defined on the domain $[0,T] \times [0,L]$ with inhomogeneous Dirichlet boundary conditions and inhomogeneous Neumann boundary conditions, respectively, where $T,\ L$ are two given positive real numbers.

More precisely, we consider the following TBVP for the wave equation with the inhomogeneous Dirichlet boundary conditions

$$\begin{cases} y_{tt} - y_{xx} = 0, \\ y(0,x) = f(x), \\ y(T,x) = g(x), \\ y(t,0) = h(t), \\ y(t,L) = l(t), \end{cases} \tag{6.1}$$

and the TBVP for the wave equation with the inhomogeneous Neumann boundary conditions

$$
\begin{cases}
y_{tt} - y_{xx} = 0, \\
y(0, x) = f(x), \\
y(T, x) = g(x), \\
y_x(t, 0) = H(t), \\
y_x(t, L) = K(t),
\end{cases}
\tag{6.2}
$$

where $y = y(t, x)$ is the unknown function of $(t, x) \in [0, T] \times [0, L]$, $f(x), g(x)$ are two given C^3 functions of $x \in [0, L]$ which stands for the initial data and terminal data, respectively, $h(t)$, $l(t)$ are two given C^3 functions of $t \in [0, T]$, and $H(t)$, $K(t)$ are two given C^2 functions of $t \in [0, T]$. Moreover, we assume that f, g, h, l, H and K satisfy the following compatibility conditions

$$
\begin{cases}
f(0) = h(0), \quad f''(0) = h''(0), \\
f(L) = l(0), \quad f''(L) = l''(0), \\
g(0) = h(T), \quad g''(0) = h''(T), \\
g(L) = l(T), \quad g''(L) = l''(T),
\end{cases}
\quad \text{for the Dirichlet conditions;}
\tag{6.3}
$$

$$
\begin{cases}
f'(0) = H(0), \\
f'(L) = K(0), \\
g'(0) = H(T), \\
g'(L) = K(T),
\end{cases}
\quad \text{for the Neumann conditions.}
\tag{6.4}
$$

We have

Theorem 6.1. *(A) Suppose that the compatibility conditions in* (6.3) *are satisfied,* $\dfrac{T}{L}$ *is a rational number and* $\dfrac{T}{L} \notin \mathbb{N}$. *Then the problem* (6.1) *admits a* C^2 *solution* $y = y(t, x)$ *on the domain* $[0, T] \times [0, L]$.

(B) Suppose that the compatibility conditions in (6.4) *are satisfied,* $\dfrac{T}{L}$ *is a rational number and* $T < L$. *Then the problem* (6.2) *admits a* C^2 *solution* $y = y(t, x)$ *on the domain* $[0, T] \times [0, L]$.

Proof. For the problem (6.1), noting that $\dfrac{T}{L} \notin \mathbb{N}$, without loss of generality, we may assume that $T < L$ (otherwise, we only need exchange the t-axes and the x-axes and then reduce the problem to the case of $T < L$). In this situation, we can extend $h(t)$ to the interval $[-T-L, T+L]$ by

$$h(t+L) + h(t-L) = 2l(t), \quad \text{for } t \in [0,T]. \tag{6.5}$$

Moreover we require that the extended $h(t)$ is a C^3 function on the interval $[-T-L, T+L]$.

Introduce

$$\tilde{y}(t,x) = y(t,x) - \frac{h(t+x)+h(t-x)}{2}. \tag{6.6}$$

Then by (6.5), the problem (6.1) becomes

$$\begin{cases} \tilde{y}_{tt} - \tilde{y}_{xx} = 0, \\ \tilde{f}(x) \triangleq \tilde{y}(0,x) = f(x) - \dfrac{h(x)+h(-x)}{2}, \\ \tilde{g}(x) \triangleq \tilde{y}(T,x) = g(x) - \dfrac{h(T+x)+h(T-x)}{2}, \\ \tilde{h}(t) \triangleq \tilde{y}(t,0) = 0, \\ \tilde{l}(t) \triangleq \tilde{y}(t,L) = 0. \end{cases} \tag{6.7}$$

And by (6.3), (6.5), the boundary data $\tilde{f}$, $\tilde{g}$, $\tilde{h}$ and $\tilde{l}$ satisfy the compatibility conditions

$$\begin{cases} \tilde{f}(0) = \tilde{f}(L) = 0, \quad \tilde{f}''(0) = \tilde{f}''(L) = 0, \\ \tilde{g}(0) = \tilde{g}(L) = 0, \quad \tilde{g}''(0) = \tilde{g}''(L) = 0. \end{cases} \tag{6.8}$$

Therefore, we can make use of Theorem 5.1 (A) and obtain the C^2 solution to the problem (6.7), and then the C^2 solution to (6.1). This proves the part (A) in Theorem 6.1.

Similarly, for the problem (6.2), noting $T < L$, we can extend $H(t)$ to the interval $[-L, T+L]$ by

$$H(t+L) + H(t-L) = 2K(t), \quad \text{for } t \in [0,T]. \tag{6.9}$$

As before, we require that the extended $H(t)$ is a C^2 function on $[-L, T+L]$.

Let

$$\tilde{y}(t,x) = y(t,x) - \frac{1}{2}\int_{t-x}^{t+x} H(\xi)d\xi. \tag{6.10}$$

Then by (6.9), the problem (6.2) becomes

$$\begin{cases} \tilde{y}_{tt} - \tilde{y}_{xx} = 0, \\ \tilde{f}(x) \triangleq \tilde{y}(0,x) = f(x) - \dfrac{1}{2}\displaystyle\int_{-x}^{x} H(\xi)d\xi, \\ \tilde{g}(x) \triangleq \tilde{y}(T,x) = g(x) - \dfrac{1}{2}\displaystyle\int_{T-x}^{T+x} H(\xi)d\xi, \\ \tilde{H}(t) \triangleq \tilde{y}_x(t,0) = 0, \\ \tilde{K}(t) \triangleq \tilde{y}_x(t,L) = 0. \end{cases} \tag{6.11}$$

And by (6.4), (6.9), the boundary data $\tilde{f}$, $\tilde{g}$, $\tilde{H}$ and $\tilde{K}$ satisfy the compatibility conditions

$$\begin{cases} \tilde{f}'(0) = \tilde{f}'(L) = 0, \\ \tilde{g}'(0) = \tilde{g}'(L) = 0. \end{cases} \tag{6.12}$$

Therefore, we can make use of Theorem 5.1 (B) and obtain the C^2 solution to the problem (6.11), and then the C^2 solution to (6.2). This proves the part (B) in Theorem 6.1.

Thus, the proof of Theorem 6.1 is completed. □

Remark 6.1. *Theorem 6.1 shows that the wave equation defined on the domain* $[0,T] \times [0,L]$ *still possesses the exact controllability even in the cases of inhomogeneous Dirichlet boundary conditions and inhomogeneous Neumann boundary conditions, provided that the corresponding compatibility conditions are satisfied.*

7 Some nonlinear wave equations

This section is devoted to the global exact controllability for some nonlinear wave equations including a wave map equation arising from geometry and the equations for the motion of relativistic strings in the Minkowski space-time $\mathbb{R}^{1+n}$.

7.1 A wave map equation

The theory of wave maps plays an important role in both mathematics and theoretical physics. The wave map equation is highly geometrical, and can be rewritten in many different ways. It is also related to the Einstein equations in general relativity. In this subsection, we consider the following TBVP for a kind of wave map equation[2]

$$\begin{cases} y_{tt} - y_{xx} = y_t^2 - y_x^2, \\ y(0, x) = f(x), \\ y(T, x) = g(x), \end{cases} \tag{7.1}$$

where T is a given positive real number, and $f(x), g(x)$ are two given C^2-smooth functions which stand for the initial and terminal states, respectively.

By making the following transformation on the unknown

$$z(t, x) = e^{-y(t,x)}, \tag{7.2}$$

the TBVP (7.1) can be rewritten as

$$\begin{cases} z_{tt} - z_{xx} = 0, \\ z(0, x) = e^{-f(x)} \triangleq \hat{f}(x), \\ z(T, x) = e^{-g(x)} \triangleq \hat{g}(x), \end{cases} \tag{7.3}$$

Obviously, the TBVP (7.1) has a C^2-smooth solution on the strip $[0, T] \times \mathbb{R}$ if and only if the TBVP (7.3) has a C^2-smooth positive solution on $[0, T] \times \mathbb{R}$. Therefore, in order to prove the existence of a C^2-smooth solution of the TBVP (7.1) on $[0, T] \times \mathbb{R}$, it suffices to show the existence of a C^2-smooth positive solution of the TBVP (7.3) on this domain. In fact, if so, $y(t, x) = -\ln z(t, x)$ is a C^2-smooth solution to the TBVP (7.1).

To do so, we suppose that

$$\inf f(x) > \sup g(x). \tag{7.4}$$

Similar to (2.2) and (2.3), we introduce

$$\tilde{z}(t, x) = z(t, x) - \frac{\hat{f}(x - t) + \hat{f}(x + t)}{2} \tag{7.5}$$

and

$$\begin{aligned} \tilde{f}(x) &= \hat{g}(x) - \frac{\hat{f}(x - T) + \hat{f}(x + T)}{2} \\ &= e^{-g(x)} - \frac{1}{2}\left[e^{-f(x-T)} + e^{-f(x+T)}\right] > 0. \end{aligned} \tag{7.6}$$

[2]In fact, the equation in (7.1) is nothing but the wave map from the Minkowski space $\mathbb{R}^{1+1}$ to the Riemmannian manifold $(\mathbb{R}, g)$, where $ds^2 = gdy^2 \triangleq e^{-2y}dy^2$.

In (7.6), we have made use of the assumption (7.4). Thus, the TBVP (7.3) can be rewritten as

$$\begin{cases} \tilde{z}_{tt} - \tilde{z}_{xx} = 0, \\ \tilde{z}(0, x) = 0, \\ \tilde{z}(T, x) = \tilde{f}(x). \end{cases} \tag{7.7}$$

As shown in Section 2, we may construct a non-negative C^1 function $u(x)$ defined on the interval $[-T, T]$ which satisfies the condition (2.8), and define an initial velocity $v(x)$ as shown in (2.9). By d'Alembert formula, the solution of the Cauchy problem for the wave equation in (7.7) with the initial data

$$\tilde{z}(0, x) = 0, \quad \tilde{z}_t(0, x) = v(x) \tag{7.8}$$

reads

$$\tilde{z}(t, x) = \frac{1}{2} \int_{x-t}^{x+t} v(\tau) d\tau. \tag{7.9}$$

It follows from (7.9), (7.5) and the fact that $\hat{f}(x) > 0$ that the TBVP (7.3) has a C^2-smooth positive solution $z = z(t, x)$, provided that $v(x) \geq 0$.

By the above argument, the key point to show the existence of a C^2-smooth positive solution of the TBVP (7.3) on $[0, T] \times \mathbb{R}$ is to construct a non-negative initial velocity $v(x)$. By (2.9), we have

Lemma 7.1. *Suppose that the function $\tilde{f}(x)$ defined by (7.6) satisfies*

$$\begin{cases} \displaystyle\sum_{i=1}^{N} \tilde{f}'\big(x - (2i-1)T\big) \geq 0, \quad \forall\, x > T, \\ \displaystyle\sum_{i=1}^{N} \tilde{f}'\big(x + (2i-1)T\big) \leq 0, \quad \forall\, x < -T. \end{cases} \tag{7.10}$$

Then the function $v(x)$ defined by (2.9) is non-negative.

Lemma 7.1 is obvious, here we omit its proof.

Summarizing the above argument yields the following theorem.

Theorem 7.1. *Suppose that $f(x)$ and $g(x)$ are two C^2-smooth functions and satisfy (7.4) and (7.10). Then the TBVP (7.1) admits a C^2-smooth solution $y = y(t, x)$ defined on the strip $[0, T] \times \mathbb{R}$.*

In order to understand the condition (7.10) clearly, we present the following two examples.

Example 7.1. *Choose the functions $f(x)$ and $g(x)$ such that the function $\tilde{f}(x)$ defined by* (7.6) *satisfies*

$$\begin{cases} \tilde{f}'(x) \geq 0, & \forall\, x > 0, \\ \tilde{f}'(x) \leq 0, & \forall\, x < 0. \end{cases} \tag{7.11}$$

In this case, it is easy to see that $\tilde{f}(x)$ satisfies the assumption (7.10)*, and then the TBVP* (7.1) *has a C^2-smooth solution on the domain $[0,T]\times\mathbb{R}$. For example, we may choose $f(x)$ and $g(x)$ satisfying*

$$\tilde{f}(x) = x^{2n} + c,$$

where $\tilde{f}(x)$ is defined by (7.6)*, c is a constant and n is a positive integer.*

Example 7.2. *Choose the functions $f(x)$ and $g(x)$ such that the function $\tilde{f}(x)$ defined by* (7.6) *satisfies*

$$\begin{cases} \tilde{f}'(x) \geq 0, & \forall\, x \in [0, 2T], \\ \tilde{f}'(x) = -\tilde{f}'(x+2T). \end{cases} \tag{7.12}$$

In the present situation, we have

$$\sum_{i=1}^{N} \tilde{f}'\big(x-(2i-1)T\big) \begin{cases} \geq 0, \; if \; N \; is \; odd, \\ = 0, \; if \; N \; is \; even, \end{cases} \qquad if \;\; x \geq 0 \tag{7.13}$$

and

$$\sum_{i=1}^{N} \tilde{f}'\big(x+(2i-1)T\big) \begin{cases} \leq 0, \; if \; N \; is \; odd, \\ = 0, \; if \; N \; is \; even, \end{cases} \qquad if \;\; x < 0. \tag{7.14}$$

This implies that $\tilde{f}(x)$ satisfies the assumption (7.10)*, and then the TBVP* (7.1) *has a C^2-smooth solution on the domain $[0,T]\times\mathbb{R}$. As an example, we may choose $f(x)$ and $g(x)$ satisfying*

$$\tilde{f}'(x) = \frac{\pi}{2T}\sin\frac{\pi x}{2T}, \quad i.e., \quad \tilde{f}(x) = -\cos\frac{\pi x}{2T} + c,$$

where $\tilde{f}(x)$ is defined by (7.6) *and c is a constant.*

7.2 The equations for the motion of relativistic strings in the Minkowski space-time $\mathbb{R}^{1+n}$

Let $X = (t, x_1, \cdots, x_n)$ be a position vector of a point in the $(1+n)$-dimensional Minkowski space $\mathbb{R}^{1+n}$. Consider the motion of a relativistic string and let $x_i = x_i(t,\theta)$ $(i = 1, \cdots, n)$

be the local equation of its world surface, where (t, θ) are the the surface parameters. The equations governing the motion of the string read (see [11])

$$|x_\theta|^2 x_{tt} - 2\langle x_t, x_\theta \rangle x_{t\theta} + (|x_t|^2 - 1) x_{\theta\theta} = 0. \tag{7.15}$$

The system (7.15) contains n nonlinear partial differential equations of second order. These equations also describe the extremal surfaces in the Minkowski space $\mathbb{R}^{1+n}$. Kong et al [11] considered the Cauchy problem for the equations (7.15) with the following initial data

$$t = 0: \quad x = p(\theta), \quad x_t = q(\theta), \tag{7.16}$$

where p is a given C^2 vector-valued function and q is a given C^1 vector-valued function. The Cauchy problem (7.15)-(7.16) describes the motion of a free relativistic string in the Minkowski space $\mathbb{R}^{1+n}$ with the initial position $p(\theta)$ and initial velocity (in the classical sense) $q(\theta)$. In particular, when $p(\theta)$ and $q(\theta)$ are periodic, the string under consideration is a closed one. It has shown that the global smooth solution of the Cauchy problem (7.15)-(7.16) exists and is unique (see [11]). On the other hand, it is well known that closed form representations of solutions for partial differential equations are very important and fundamental in both mathematical analysis and physical understanding. Unfortunately, nonlinear partial differential equations in general do not possess representations of solutions in closed form. Surprisingly, in the paper [10], we discovered a general solution formula in closed form for the nonlinear wave equations (7.15). By introducing a new concept of *generalized time-periodic function*, we proved that, if the initial data is periodic, then the smooth solution of the Cauchy problem (7.15)-(7.16) is *generalized time-periodic*, namely, the space-periodicity also implies the time-periodicity. This fact yields an interesting physical phenomenon: *the motion of closed strings is always generalized time-periodic.*

However, up to now there is not any result on the TBVP for the equations (7.15). In this subsection we investigate this problem and prove the global exact controllability of the equations (7.15).

By [6], under a very natural assumption which is a necessary and sufficient condition guaranteeing the motion is physical, there exists a globally diffeomorphic transformation of variables

$$\tau = t, \quad \vartheta = \vartheta(t, \theta) \tag{7.17}$$

such that the system (7.15) become

$$\tilde{x}_{\tau\tau} - \tilde{x}_{\vartheta\vartheta} = 0, \tag{7.18}$$

where

$$\tilde{x}(\tau,\vartheta) = x\big(t(\tau,\vartheta),\theta(\tau,\vartheta)\big), \tag{7.19}$$

in which $t = t(\tau,\vartheta),\ \theta = \theta(\tau,\vartheta)$ is the inverse of the transformation (7.17).

By Theorem 2.1, the system (7.18) possesses the global exact controllability. Noting the transformation (7.17) is globally diffeomorphic, we obtain the global exact controllability of the system (7.15).

In physics, the global exact controllability of the system (7.15) implies some interesting physical phenomena. For example, if we take the periodic initial data $p(\theta)$ which stands for a closed string, and a non-periodic terminal condition (e.g., $x(T,\theta) = \theta$) which denotes an open string, then the global exact controllability of the system (7.15) means that a closed string may become an open string. This fact implies that the topological structure of the string may change in its motion process.

8 Hyperbolic curvature flow and Gage-Hamilton's theorem

Let $\gamma(t)$ be a one parameter family of closed convex smooth curves in the plane. The position vector $X(t,s)$ parameterizes the curve and the curvature is $k(t,s)$. The hyperbolic curvature flow is described by the following wave equation

$$k_{tt} - k_{ss} = 0, \tag{8.1}$$

where the subscript ν denotes partial differentiation with respect ν.

Consider the TBVP for the wave equation (8.1) with the following initial condition and terminal condition

$$k(0,s) = f(s), \quad k(T,s) = k_0 > 0, \tag{8.2}$$

where $f(s)$ is a given non-negative periodic smooth function with the period L, T is a positive real number, and k_0 is a positive constant. We have

Theorem 8.1. *There exists a positive constant k_0 such that the TBVP (8.1)-(8.2) admits a non-negative smooth solution $k = k(t,s)$, provided that $\dfrac{T}{L}$ is a rational number with $\dfrac{2T}{L} \notin \mathbb{N}$.*

Remark 8.1. *By the basic theorem on plane curves, Theorem 8.1 implies that, under the hyperbolic curvature flow, any closed convex smooth curve in the plane can evolve into a* *circle in a finite time, and remains convex for sequent times. This is a result analogous to the well-know theorem of Gage and Hamilton for the mean curvature flow of plane curves.*

We now prove Theorem 8.1.

Proof. Consider the TBVP for the wave equation (8.1) with the following initial condition and terminal condition

$$k(0,s) = f(s), \quad k(T,s) = k_*, \tag{8.3}$$

where k_* is an arbitrary given positive constant. By Theorem 1.2, the TBVP (8.1), (8.3) has a global L-periodic solution $k = k(t,s)$. Define

$$v(s) = \partial_t k(0,s). \tag{8.4}$$

Clearly, $v(s)$ is the initial velocity corresponding to the solution $k = k(t,s)$. That is to say, the solution of the Cauchy problem for the wave equation (8.1) with the initial data $k(0,s) = f(s)$, $k_t(0,s) = v(s)$ is nothing but $k = k(t,s)$, moreover this solution satisfies the terminal data $k(T,s) = k_*$.

Notice that $v(s)$ is L-periodic. Let

$$M = \min_{s\in[0,L]} v(s). \tag{8.5}$$

If $M \geq 0$, then by D'Alembert formula

$$k(t,s) = \frac{f(s+t)+f(s-t)}{2} + \frac{1}{2}\int_{s-t}^{s+t} v(\tau)d\tau \geq 0, \quad \forall\ (t,s) \in [0,T]\times\mathbb{R}. \tag{8.6}$$

Taking $k_0 = k_*$, we finish the proof of Theorem 8.1.

Otherwise, it holds that $v(s) + |M| \geq 0$. Thus, it is easy to see that $\bar{k} \triangleq k(t,s) + |M|t$ is a global L-periodic smooth solution of the TBVP

$$\begin{cases} \bar{k}_{tt}(t,s) - \bar{k}_{ss}(t,x) = 0, \\ \bar{k}(0,s) = f(s), \\ \bar{k}(T,s) = k_* + |M|T. \end{cases} \tag{8.7}$$

Obviously, the initial velocity corresponding to the solution $\bar{k}(t,s)$ reads

$$\bar{v}(s) \triangleq \partial_t \bar{k}(0,s) = v(s) + |M| \geq 0. \tag{8.8}$$

This implies that the solution is non-negative, i.e., $\bar{k}(t,s) \geq 0$ for all $(t,s) \in [0,T]\times\mathbb{R}$. Taking $k_0 = k_* + |M|T$ gives the conclusion of Theorem 8.1. Thus, the proof of Theorem 8.1 is completed. $\square$

9 Summary and discussions

In the present paper we introduce three new concepts for second-order hyperbolic equations, they read two-point boundary value problem, global exact controllability and exact controllability, respectively. These second-order hyperbolic equations considered here include many important partial differential equations arising from both theoretical aspects and applied fields, e.g., mechanics (fluid mechanics and elasticity), physics, engineering, control theory and geometry, etc., the typical examples are wave equation, hyperbolic Monge-Ampère equation, wave map. For several kinds of important linear and nonlinear wave equations, in this paper we prove the existence of smooth solutions of the two-point boundary value problems and show the global exact controllability of these wave equations. In particular, we investigate the two-point boundary value problem for one-dimensional wave equation defined on a closed curve and prove the existence of smooth solution, this implies the exact controllability of this kind of wave equation. Furthermore, based on this, we study the two-point boundary value problems for the wave equation defined on a strip with Dirichlet or Neumann boundary conditions and show that the equation still possesses the exact controllability in these cases. Finally, as an application of Theorem 1.2, we introduce the hyperbolic curvature flow and prove a result analogous to the well-known theorem of Gage and Hamilton [5] for the curvature flow of plane curves. This result can be viewed as a simple application of hyperbolic partial differential equation to both geometry and topology.

Usually, "two-point boundary value problem" has another meaning: it applies to a boundary problem for a second order ODE in a bounded interval. The present paper generalizes this concept to the case of second-order hyperbolic partial differential equations. The two-point boundary value problem for partial differential equations is a new research topic. According to the authors' knowledge, up to now, few of results on the two-point boundary value problems for partial differential equations, in particular, hyperbolic partial differential equations (even for linear or nonlinear wave equations) have been known. Therefore, the present paper can be viewed as the first work in this new research topic.

It is well-known that, there are a lot of deep and beautiful results on the two-point boundary value problems for ordinary differential equations and for second-order differential inclusions. On the other hand, there are many important results on the *boundary* control problems for wave equations and hyperbolic systems. However, the two-point boundary value problems and the boundary control problems are essentially different two kinds of problems. Both of them play an important role in both theoretical and applied aspects.

The main aim of this paper is to introduce the concept "two-point boundary value problem" for second-order hyperbolic equations and to show the global exact controllability or exact controllability for several kinds of important linear and nonlinear wave equations. Although the results obtained in this paper are restrictive in some sense (they only apply to cases in which the solution to the Cauchy problem can be found explicitly: linear (classical) wave equations and their reformulations), these results shed further light on the study of this new research topic. In the future we will investigate the following open problems which seem to us more interesting and important: (i) what happens if we consider a general equation of the form

$$y_{tt} - y_{xx} = f(y)$$

with a regular and bounded function $f(y)$? (ii) what happens if we include regular coefficients and/or lower order terms? (iii) what happens if we consider a quasilinear wave equation of the form

$$y_{tt} - (\mathscr{C}(y_x))_x = 0$$

with a smooth and increasing function $\mathscr{C}(\nu)$, e.g., $\mathscr{C}(\nu) = \frac{\nu}{\sqrt{1+\nu^2}}$? (iv) what happens if we consider other models, particularly some nonlinear models arising from applied fields such as control theory, fluid dynamics, elasticity as well as engineering, etc.

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