# THE EXTENSION FOR MEAN CURVATURE FLOW WITH FINITE INTEGRAL CURVATURE IN RIEMANNIAN MANIFOLDS *

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**Abstract**

We investigate the integral conditions to extend the mean curvature flow in a Riemannian manifold. We prove that the mean curvature flow solution with finite total mean curvature on a finite time interval $[0, T)$ can be extended over time $T$. Moreover, we show that the condition is optimal in some sense.

## 1 Introduction

Let $(M, g)$ be a compact $n$-dimensional manifold without boundary, and let $F_t : M^n \to N^{n+1}$ be a one-parameter family of smooth hypersurfaces immersed in a Riemannian manifold $(N, h)$. We say that $M_t = F_t(M)$ is a solution of the mean curvature flow if $F_t$ satisfies

$$\begin{cases} \frac{\partial}{\partial t}F(x,t) &= -H(x,t)\nu(x,t) \\ F(x,0) &= F_0(x), \end{cases}$$

where $F(x, t) = F_t(x)$, $H(x, t)$ is the mean curvature, $\nu(x, t)$ is the unit outward normal vector, and $F_0$ is some given initial hypersurface.

When the ambient space is the Euclidean space $\mathbb{R}^{n+1}$, G. Huisken [4] showed that the solution of the mean curvature flow converges to a round point in a finite time for convex initial hypersurface. He also proved that if the second fundamental form is uniformly bounded, then the mean curvature flow can be extended over time. If the ambient space is a Riemannian manifold, G. Huisken [5] proved the similar convergence theorem for certain initial compact hypersurface and gave an sufficient condition to assure the extension over time for mean curvature flow. Distinct from the above pointwise conditions, in our previous work [12] we investigated the integral conditions to extend the mean curvature flow on closed hypersurfaces in $\mathbb{R}^{n+1}$, which

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is optimal in some sense. Almost at the same time, N. Le and N. Šešum [7] studied the same question independently with a different method.

In this paper, we study the mean curvature flow of hypersurfaces in a Riemannian manifold with bounded geometry, which generalizes our results in [12]. We recall that a Riemannian manifold is said to have bounded geometry if (i): the sectional curvature is bounded; (ii): the first covariant derivative of the curvature tensor is bounded; (iii): the injective radius is bounded from below by a positive constant. In this paper we always assume that the ambient space $N^{n+1}$ is a complete Riemannian manifold with bounded geometry. We prove that when the space-time integration of the mean curvature is finite and the second fundamental form is bounded from below, the mean curvature flow can be extended.

**Theorem 1.1.** *Let $F_t : M^n \longrightarrow N^{n+1}$ $(n \geq 3)$ be a solution of the mean curvature flow of closed hypersurfaces on a finite time interval $[0, T)$. If*
(1) *there is a positive constant $C$ such that $h_{ij} \geq -C$ for $(x, t) \in M \times [0, T)$,*
(2) $||H||_{\alpha, M\times[0,T)} = \left(\int_0^T \int_M |H|^\alpha d\mu dt\right)^{\frac{1}{\alpha}} < +\infty$ *for some $\alpha \geq n+2$,*
*then this flow can be extended over time $T$.*

Suppose that the sectional curvature $K_N$ of $N^{n+1}$ satisfies

$$-K_1 \leq K_N \leq K_2,$$

where $K_1$ and $K_2$ are nonnegative constants. We also will prove the following theorem.

**Theorem 1.2.** *Let $F_t : M^n \longrightarrow N^{n+1}$ $(n \geq 3)$ be a solution of the mean curvature flow of closed hypersurfaces on a finite time interval $[0, T)$. If*
(1) $H^2 > n^2 K_1$ *at $t = 0$,*
(2) $||H||_{\alpha, M\times[0,T)} = \left(\int_0^T \int_M |H|^\alpha d\mu dt\right)^{\frac{1}{\alpha}} < +\infty$ *for some $\alpha \geq n+2$,*
*then this flow can be extended over time $T$.*

The following example shows that the condition $\alpha \geq n+2$ in Theorem 1.1 and 1.2 is optimal when the ambient space is a complete simply connected space form.

**Example.** (i) For the case where $N^{n+1} = \mathbb{R}^{n+1}$, set $\mathbb{S}^n = \{x \in \mathbb{R}^{n+1} : \sum_{i=1}^n x_i^2 = 1\}$. Let $F$ be the standard isometric embedding of $\mathbb{S}^n$ into $\mathbb{R}^{n+1}$. It is clear that $F(t) = \sqrt{1-2nt}F$ is the solution to the mean curvature flow, where $T = \frac{1}{2n}$ is the maximal existence time. By a simple computation, we have $g_{ij}(t) = (1-2nt)g_{ij}$, $H(t) = \frac{n}{\sqrt{1-2nt}}$ and $h_{ij}(t) > 0$. Hence

$$\begin{aligned} ||H||_{\alpha, M\times[0,T)} &= \left(\int_0^T \int_M |H|^\alpha d\mu dt\right)^{\frac{1}{\alpha}} \\ &= C_1 \left(\int_0^T (T-t)^{\frac{n-\alpha}{2}} dt\right)^{\frac{1}{\alpha}}, \end{aligned}$$

where $C_1$ is a positive constant. It follows that

$$||H||_{\alpha,M\times[0,T)}\begin{cases} =\infty, & for \quad \alpha\geq n+2,\\ <\infty, & for \quad \alpha<n+2.\end{cases}$$

This implies that the condition $\alpha\geq n+2$ in Theorem 1.1 and 1.2 is optimal when $N^{n+1}=\mathbb{R}^{n+1}$.

(ii) Let $\mathbb{F}^{n+1}(c)$ be a complete simply connected space form with constant curvature $c$. We consider the case $N^{n+1}=\mathbb{F}^{n+1}(c)$, where $c=\pm1$, that is, $N^{n+1}=\mathbb{S}^{n+1}$ or $\mathbb{H}^{n+1}$. Let $M=\mathbb{S}^n(r_0)$ be a total umbilical sphere of radius $r_0$ in $N^{n+1}$ with constant mean curvature $H_0$ satisfying $H_0>0$ when $c=1$, and $H_0^2>n^2$ when $c=-1$. Put $d=\frac{H_0^2}{H_0^2+n^2c}$. Let $\mathbb{S}^n(r(t))$ be a sphere with radius $r(t)=\frac{\sqrt{H_0^2+n^2c}}{\sqrt{H^2(t)+n^2c}}r_0$, where $H(t)=\sqrt{\frac{n^2cde^{2nct}}{1-de^{2nct}}}$. Then $\mathbb{S}^n(r(t))$ is a family of total umbilical spheres with constant mean curvature $H(t)$, which satisfies the mean curvature flow with initial value $M=\mathbb{S}^n(r_0)\subset N^{n+1}$. It is clear that the maximal existence time is $T=-\frac{\ln d}{2n}$ when $c=1$, and $T=\frac{\ln d}{2n}$ when $c=-1$, the second fundamental form $h_{ij}$ satisfies $h_{ij}>0$, and the volume of $\mathbb{S}^n(r(t))$ is $V(t)=\left(\frac{H_0^2+n^2c}{H^2(t)+n^2c}\right)^{\frac{n}{2}}V_0$, where $V_0$ is the volume of $\mathbb{S}^n(r_0)$. Hence

$$\begin{aligned} ||H||_{\alpha,M\times[0,T)} &= \left(\int_0^T\int_M|H|^\alpha d\mu dt\right)^{\frac{1}{\alpha}}\\ &= \left(\int_0^T H^\alpha(t)V(t)dt\right)^{\frac{1}{\alpha}}\\ &= C_2\left(\int_0^T(n^{2-n}de^{2nct})^{\frac{\alpha}{2}}\left(\frac{1-de^{2nct}}{c}\right)^{\frac{n-\alpha}{2}}dt\right)^{\frac{1}{\alpha}}, \end{aligned}$$

where $C_2$ is a positive constant. Since $(n^{2-n}de^{2nct})^{\frac{\alpha}{2}}$ has positive upper and lower bounds because of the finiteness of $T$, and the integral

$$\int_0^T\left(\frac{1-de^{2nct}}{c}\right)^{\frac{n-\alpha}{2}}dt\begin{cases} =\infty, & for \quad \alpha\geq n+2,\\ <\infty, & for \quad \alpha<n+2,\end{cases}$$

it follows that

$$||H||_{\alpha,M\times[0,T)}\begin{cases} =\infty, & for \quad \alpha\geq n+2,\\ <\infty, & for \quad \alpha<n+2.\end{cases}$$

This implies that the condition $\alpha\geq n+2$ in Theorem 1.1 and 1.2 is optimal when $N^{n+1}=\mathbb{S}^{n+1}$ or $\mathbb{H}^{n+1}$.

## 2 Preliminaries

Let $F_t: M^n\to N^{n+1}$ be a one-parameter family of smooth hypersurfaces immersed in a Riemannian manifold $N$. Denote by $g=\{g_{ij}\}$ and $A=\{h_{ij}\}$ the induced metric and the second fundamental form of $M$ respectively, and $H$ is the mean curvature of $M$, which is the trace of $A$. We put $\bar{\nabla}$ and $\bar{R}ic$ be the connection and the Ricci tensor of $N$, and $R_{ABCD}$, $A$, $B$, $C$, $D = 0, 1, \cdots, n$, be components of the curvature tensor of $N$ with respect to some local coordinates such that $e_0 = \nu$.

Firstly, we recall some evolution equations (see [2], [5] or [15]).

**Lemma 2.1.** *Along the mean curvature flow in Riemannian manifolds, we have the following evolution equations*

$$\begin{aligned}
\frac{\partial}{\partial t} g_{ij} &= -2Hh_{ij}, \\
\frac{\partial |A|^2}{\partial t} &= \triangle |A|^2 - 2|\nabla A|^2 + 2|A|^2(|A|^2 + \bar{R}ic(\nu, \nu)) \\
&\quad -4(h^{ij}h^m_j \bar{R}_{mli}{}^{\ l} - h^{ij}h^{lm}\bar{R}_{milj}) - 2h^{ij}(\bar{\nabla}_j \bar{R}_{0li}{}^{\ l} + \bar{\nabla}_l \bar{R}_{0ij}{}^{\ l}), \\
\frac{\partial}{\partial t} H &= \triangle H + H(|A|^2 + \bar{R}ic(\nu, \nu)).
\end{aligned}$$

We denote by $\omega_n$ the volume of the unit ball in $\mathbb{R}^n$. The following Sobolev inequality can be found in [6].

**Lemma 2.2.** *Let $M^n \subset N^{n+p}$ be an $n(\geq 2)$-dimensional closed submanifold in a Riemannian manifold $N^{n+p}$ with codimension $p \geq 1$. Denote by $i_N$ the positive lower bound of the injective radius of $N$ restricted on $M$. Assume $K_N \leq b^2$ and let $h$ be a non-negative $C^1$ function on $M$. Then*

$$\left( \int_M h^{\frac{n}{n-1}} d\mu \right)^{\frac{n-1}{n}} \leq C(n, \alpha) \int_M \Big[ |\nabla h| + h|H| \Big] d\mu,$$

*provided*

$$b^2 (1-\alpha)^{-\frac{2}{n}} (\omega_n^{-1} Vol(supp\ h))^{\frac{2}{n}} \leq 1 \text{ and } 2\rho_0 \leq i_N,$$

*where*

$$\rho_0 = \begin{cases} b^{-1} \sin^{-1} b(1-\alpha)^{-\frac{1}{n}} (\omega_n^{-1} Vol(supp\ h))^{\frac{1}{n}} & for\ b\ real, \\ (1-\alpha)^{-\frac{1}{n}} (\omega_n^{-1} Vol(supp\ h))^{\frac{1}{n}} & for\ b\ imaginary. \end{cases}$$

*Here $\alpha$ is a free parameter, $0 < \alpha < 1$, and*

$$C(n, \alpha) = \frac{1}{2}\pi \cdot 2^{n-2} \alpha^{-1} (1-\alpha)^{-\frac{1}{n}} \frac{n}{n-1} \omega_n^{-\frac{1}{n}}.$$

*For $b$ imaginary, we may omit the factor $\frac{1}{2}\pi$ in the definition of $C(n, \alpha)$.*

The following lemma gives a proper form of the Sobolev inequality, which can be found in [10]. Here we outline the proof.

**Lemma 2.3.** *Let $M^n \subset N^{n+p}$ be a $n(\geq 3)$-dimensional closed submanifold in a Riemannian manifold $N^{n+p}$ with codimension $p \geq 1$. Denote by $i_N$ the positive* *lower bound of the injective radius of $N$ restricted on $M$. Assume $K_N \leq K_2$, where $K_2$ is a non-negative constant and let $f$ be a non-negative $C^1$ function on $M$ satisfying*

$$K_2(n+1)^{\frac{2}{n}}(\omega_n^{-1}Vol(supp\ f))^{\frac{2}{n}} \leq 1, \tag{1}$$

$$2K_2^{-\frac{1}{2}}\sin^{-1} K_2^{\frac{1}{2}}(n+1)^{\frac{1}{n}}(\omega_n^{-1}Vol(supp\ f))^{\frac{1}{n}} \leq i_N. \tag{2}$$

*Then*

$$\| \nabla f \|_2^2 \geq \frac{(n-2)^2}{4(n-1)^2(1+t)}\left[\frac{1}{C^2(n)} \| f \|^2_{\frac{2n}{n-2}} - H_0^2\left(1+\frac{1}{t}\right) \| f \|_2^2\right],$$

*where $H_0 = \max_{x\in M} H$ and $C(n) = C(n, \frac{n}{n+1})$*

*Proof.* For all $g \in C^1(M)$, $g \geq 0$ satisfying (1) and (2), Lemma 2.2 implies

$$\| g \|_{\frac{n}{n-1}} \leq C(n)\int_M (| \nabla g | + Hg)d\mu. \tag{3}$$

Substituting $g = f^{\frac{2(n-1)}{n-2}}$ into (3) gives

$$\left(\int_M f^{\frac{2n}{n-2}}d\mu\right)^{\frac{n-1}{n}} \leq \frac{2(n-1)}{n-2}C(n)\int_M f^{\frac{n}{n-2}} | \nabla f | \,d\mu + C(n)\int_M Hf^{\frac{2(n-1)}{n-2}}d\mu.$$

By Hölder's inequality, we get

$$\| f \|_{\frac{2n}{n-2}} \leq C(n)\left[\frac{2(n-1)}{n-2} \| \nabla f \|_2 + H_0 \| f \|_2\right].$$

This implies

$$\| f \|^2_{\frac{2n}{n-2}} \leq C^2(n)\left[\frac{4(n-1)^2(1+t)}{(n-2)^2} \| \nabla f \|_2^2 + H_0^2\left(1+\frac{1}{t}\right) \| f \|_2^2\right],$$

which is desired.

## 3 An estimate of the mean curvature by its $L^{n+2}$-norm

In this section, we prove the following theorem, which plays an important role in the proof of Theorem 1.1.

**Theorem 3.1.** *Suppose that $F_t : M^n \longrightarrow N^{n+1}$ $(n \geq 3)$ is a mean curvature flow for $t \in [0, T_0]$, and the second fundamental form $A$ is uniformly bounded on time interval $[0, T_0]$. Then*

$$\max_{(x,t)\in M\times[\frac{T_0}{2},T_0]} H^2(x,t) \leq C_3\left(\int_0^{T_0}\int_{M_t} |H|^{n+2}d\mu dt\right)^{\frac{2}{n+2}},$$

*where $C_3$ is a constant depending only on $n$, $T_0$, $\sup_{(x,t)\in M\times[0,T_0]}|A|$, $K_1$, $K_2$ and the injectivity radius lower bound $i_N>0$ of $N$.*

*Proof.* The evolution equation of $H^2$ is

$$\frac{\partial}{\partial t}H^2=\triangle H^2-2\mid\nabla H\mid^2+2H^2\mid A\mid^2+2H^2\bar{R}ic(\nu,\nu).$$

Since $\mid A\mid$ is bounded, we obtain

$$\frac{\partial}{\partial t}H^2\leq\triangle H^2+\beta H^2, \tag{4}$$

where $\beta$ is a positive constant depending only on $n$, $\sup_{(x,t)\in M\times[0,T_0]}|A|$ and $K_2$. For $0<R<R'<\infty$ and $x\in M$, we set

$$\eta=\begin{cases}1 & x\in B_{g(0)}(x,R),\\ \eta\in[0,1]\ and\ |\nabla\eta|_{g(0)}\leq\frac{1}{R'-R} & x\in B_{g(0)}(x,R')\setminus B_{g(0)}(x,R),\\ 0 & x\in M\setminus B_{g(0)}(x,R').\end{cases}$$

Since *supp* $\eta\subseteq B_{g(0)}(x,R')$, $\eta$ satisfies (1) and (2) with respect to $g(0)$ for $R'$ sufficiently small. On the other hand, the area of some fixed subset in $M$ is non-increasing along the mean curvature flow, hence $\eta$ satisfies (1) and (2) with respect to each $g(t)$ for $t\in[0,T_0]$.

Fix $R'>0$ sufficiently small, for any point $x\in M_t$, we denote by $B(R')=B_{g(0)}(x,R')$ the geodesic ball with radius $R'$ centered at $x$ with respect to the metric $g(0)$. Putting $f=|H|^2$, then for any $p\geq 2$, the inequality (4) implies

$$\frac{1}{p}\frac{\partial}{\partial t}\int_{B(R')}f^p\eta^2\leq\int_{B(R')}\eta^2f^{p-1}\triangle f+\int_{B(R')}\beta f^p\eta^2+\frac{1}{p}\int_{B(R')}f^p\eta^2\frac{\partial}{\partial t}d\upsilon_t.$$

Integrating by parts yields

$$\begin{aligned}\int_{B(R')}\eta^2f^{p-1}\triangle f &= -\frac{4(p-1)}{p^2}\int_{B(R')}\mid\nabla(f^{\frac{p}{2}}\eta)\mid^2+\frac{4}{p^2}\int_{B(R')}\mid\nabla\eta\mid^2 f^p\\ &\quad+\frac{4(p-2)}{p^2}\int_{B(R')}\nabla(f^{\frac{p}{2}}\eta)f^{\frac{p}{2}}\nabla\eta\\ &\leq -\frac{2}{p}\int_{B(R')}\mid\nabla(f^{\frac{p}{2}}\eta)\mid^2+\frac{2}{p}\int_{B(R')}\mid\nabla\eta\mid^2 f^p.\end{aligned}$$

Thus

$$\begin{aligned}\frac{1}{p}\frac{\partial}{\partial t}\int_{B(R')}f^p\eta^2 &\leq -\frac{2}{p}\parallel\nabla(f^{\frac{p}{2}}\eta)\parallel_2^2+\beta\parallel f^{\frac{p}{2}}\eta\parallel_2^2\\ &\quad+\frac{2}{p}\int_{B(R')}\mid\nabla\eta\mid^2 f^p+\frac{1}{p}\int_{B(R')}f^p\eta^2\frac{\partial}{\partial t}d\upsilon_t\\ &\leq -\frac{2}{p}\parallel\nabla(f^{\frac{p}{2}}\eta)\parallel_2^2+\beta\parallel f^{\frac{p}{2}}\eta\parallel_2^2+\frac{2}{p}\int_{B(R')}\mid\nabla\eta\mid^2 f^p.\end{aligned}$$

Hence

$$\frac{\partial}{\partial t}\int_{B(R')} f^p\eta^2 + \int_{B(R')} \mid \nabla(f^{\frac{p}{2}}\eta) \mid^2 \leq 2\int_{B(R')} \mid \nabla\eta \mid^2 f^p + \beta p\int_{B(R')} f^p\eta^2. \qquad (5)$$

For any $0 < \tau < \tau' < T_0$, define a function $\psi$ on $[0, T_0]$ by

$$\psi(t) = \begin{cases} 0 & 0 \leq t \leq \tau, \\ \frac{t-\tau}{\tau'-\tau} & \tau \leq t \leq \tau', \\ 1 & \tau' \leq t \leq T_0. \end{cases}$$

Multiplying (5) by $\psi(t)$ gives

$$\begin{aligned} & \frac{\partial}{\partial t}\Big(\psi\int_{B(R')} f^p\eta^2\Big) + \psi\int_{B(R')} \mid \nabla(f^{\frac{p}{2}}\eta) \mid^2 \\ \leq\ & 2\psi\int_{B(R')} \mid \nabla\eta \mid^2 f^p + (\beta p\psi + \psi')\int_{B(R')} f^p\eta^2. \end{aligned} \qquad (6)$$

By integrating (6) on $[\tau, t]$ we obtain

$$\begin{aligned} & \int_{B(R')} f^p\eta^2 + \int_{\tau'}^{t}\int_{B(R')} \mid \nabla(f^{\frac{p}{2}}\eta) \mid^2 \\ \leq\ & 2\int_{\tau}^{T_0}\int_{B(R')} \mid \nabla\eta \mid^2 f^p + \Big(\beta p + \frac{1}{\tau'-\tau}\Big)\int_{\tau}^{T_0}\int_{B(R')} f^p\eta^2. \end{aligned}$$

For $R'$ sufficiently small, the following Sobolev inequality holds:

$$\begin{aligned} \Big(\int_{B(R')} f^{\frac{pn}{n-2}}\eta^{\frac{2n}{n-2}}\Big)^{\frac{n-2}{n}} &= \| f^{\frac{p}{2}}\eta \|^2_{\frac{2n}{n-2}} \\ &\leq \frac{4(n-1)^2(1+s)C^2(n)}{(n-2)^2} \| \nabla(f^{\frac{p}{2}}\eta) \|_2^2 \\ &\quad + H_0^2 C^2(n)\Big(1+\frac{1}{s}\Big) \| f^{\frac{p}{2}}\eta \|_2^2 . \end{aligned}$$

Hence

$$\begin{aligned} & \int_{\tau'}^{T_0}\int_{B(R')} f^{p(1+\frac{2}{n})}\eta^{2+\frac{1}{n}} \\ \leq\ & \int_{\tau'}^{T_0}\Big(\int_{B(R')} f^p\eta^2\Big)^{\frac{2}{n}}\Big(\int_{B(R')} f^{\frac{np}{n-2}}\eta^{\frac{2n}{n-2}}\Big)^{\frac{n-2}{n}} \\ \leq\ & \max_{t\in[\tau', T_0]}\Big(\int_{B(R')} f^p\eta^2\Big)^{\frac{2}{n}} \times \int_{\tau}^{T_0}\Big[\frac{4(n-1)^2(1+s)C^2(n)}{(n-2)^2} \| \nabla(f^{\frac{p}{2}}\eta) \|_2^2 \\ & + H_0^2 C^2(n)\Big(1+\frac{1}{s}\Big) \| f^{\frac{p}{2}}\eta \|_2^2 \Big] \\ \leq\ & C_4 \max_{t\in[\tau', T_0]}\Big(\int_{B(R')} f^p\eta^2\Big)^{\frac{2}{n}} \times \int_{\tau}^{T_0}\Big[ \| \nabla(f^{\frac{p}{2}}\eta) \|_2^2 + \| f^{\frac{p}{2}}\eta \|_2^2 \Big] \end{aligned}$$

$$\leq \; C_4 \left[ 2\int_{\tau}^{T_0}\int_{B(R')} \mid \nabla\eta \mid^2 f^p + \Big(\beta p + \frac{1}{\tau' - \tau}\Big) \int_{\tau}^{T_0}\int_{B(R')} f^p\eta^2 \right]^{1+\frac{2}{n}},$$

where we put $s = 1$ and $C_4$ is a constant depending on $n$ and $\sup_{(x,t)\in M\times[0,T_0]} |A|$.

Note that $|\nabla\eta|_{g(t)} \leq |\nabla\eta|^2_{g(0)} e^{lt}$, where $l = \max_{0\leq t\leq T_0} ||\frac{\partial g}{\partial t}||_{g(t)}$. Thus

$$\int_{\tau}^{T_0}\int_{B(R')} \mid \nabla\eta \mid^2 f^p \leq \int_{\tau}^{T_0}\int_{B(R')} \Big( \mid \nabla\eta \mid_{g(0)} e^{\frac{1}{2}lt}\Big)^2 f^p \leq \frac{e^{C_5 T_0}}{(R' - R)^2}\int_{\tau}^{T_0}\int_{B(R')} f^p,$$

where $C_5$ is a constant depending on $n$ and $\sup_{(x,t)\in M\times[0,T_0]} |A|^2$. Then it follows that

$$\begin{aligned} \int_{\tau}^{T_0}\int_{B(R)} f^{p(1+\frac{2}{n})} d\mu_t dt \;\; &\leq \;\; C_4 \left(\beta p + \frac{1}{\tau' - \tau} + \frac{2e^{C_5 T_0}}{(R' - R)^2}\right)^{1+\frac{2}{n}} \\ &\quad \times \left(\int_{\tau}^{T_0}\int_{B(R')} f^p d\mu_t dt\right)^{1+\frac{2}{n}}. \end{aligned} \tag{7}$$

Putting $L(p, t, R) = \int_t^{T_0}\int_{B(R)} f^p$, we obtain from (7)

$$L\Big(p\Big(1 + \frac{2}{n}\Big), \tau', R\Big) \leq C_4 \left(\beta p + \frac{1}{\tau' - \tau} + \frac{2e^{C_5 T_0}}{(R' - R)^2}\right)^{1+\frac{2}{n}} L(p, \tau, R')^{1+\frac{2}{n}}. \tag{8}$$

We set $\mu = 1 + \frac{2}{n}$, $p_k = \frac{n+2}{2}\mu^k$, $\tau_k = \Big(1 - \frac{1}{\mu^{k+1}}\Big)t$ and $R_k = \frac{R}{2}\Big(1 + \frac{1}{\mu^{k/2}}\Big)$, where $k = 0, 1, 2, \cdots$. Then it follows from (8) that

$$\begin{aligned} L(p_{k+1}, \tau_{k+1}, R_{k+1})^{\frac{1}{p_{k+1}}} \quad &\leq \quad C_4^{\frac{1}{p_{k+1}}} \Big[\frac{(n+2)\beta}{2} + \frac{\mu^2}{\mu - 1}\cdot\frac{1}{t} + \frac{4e^{C_5 T_0}}{R'^2}\cdot\frac{\mu}{(\sqrt{\mu} - 1)^2}\Big]^{\frac{1}{p_k}} \\ &\quad \times\mu^{\frac{k}{p_k}} L(p_k, \tau_k, R_k)^{\frac{1}{p_k}}. \end{aligned}$$

Hence for any $m \geq 1$,

$$\begin{aligned} & L(p_{m+1}, \tau_{m+1}, R_{m+1})^{\frac{1}{p_{m+1}}} \\ \leq \;\; & C_4^{\sum_{k=0}^m \frac{1}{p_{k+1}}} \Big[\frac{(n+2)\beta}{2} + \frac{\mu^2}{\mu - 1}\cdot\frac{1}{t} + \frac{4e^{C_5 T_0}}{R'^2}\cdot\frac{\mu}{(\sqrt{\mu} - 1)^2}\Big]^{\sum_{k=0}^m \frac{1}{p_k}} \\ & \times\mu^{\sum_{k=0}^m \frac{k}{p_k}} L(p_0, \tau_0, R_0)^{\frac{1}{p_0}}. \end{aligned}$$

As $m \to +\infty$, we conclude

$$f(x, t) \leq C_6^{\frac{n}{n+2}}\Big(C_6 + \frac{1}{t} + \frac{e^{C_5 T_0}}{R'^2}\Big)\Big(1 + \frac{2}{n}\Big)^{\frac{n}{2}} \left(\int_0^{T_0}\int_{M_t} f^{\frac{n+2}{2}}\right)^{\frac{2}{n+2}}, \tag{9}$$

where $C_6$ is a positive constant depending on $n$, $\sup_{M\times[0,T]} |A|$, $K_1$ and $K_2$.

Note that we choose $R'$ sufficient small such that

$$K_2(n+1)^{\frac{2}{n}}(\omega_n^{-1}Vol_{g(0)}(B(R'))^{\frac{2}{n}} \leq 1 \tag{10}$$

and

$$2K_2^{-\frac{1}{2}}\sin^{-1}K_2^{\frac{1}{2}}(n+1)^{\frac{1}{n}}(\omega_n^{-1}Vol_{g(0)}(B(R'))^{\frac{1}{n}} \leq i_N. \tag{11}$$

For $g(0)$, there is a non-positive constant $K$ depending on $n$, $\max_{x\in M_0}|A|$, $K_1$ and $K_2$ such that the sectional curvature of $M_0$ is bounded from below by $K$. By the Bishop-Gromov volume comparison theorem, we have

$$Vol_{g(0)}(B(R')) \leq Vol_K(B(R')),$$

where $Vol_K(B(R'))$ is the volume of the ball with radius $R'$ in the $n$-dimensional complete simply connected space form with constant curvature $K$. Let $R'$ be the largest number satisfying

$$K_2(n+1)^{\frac{2}{n}}(\omega_n^{-1}Vol_K(B(R'))^{\frac{2}{n}} \leq 1$$

and

$$2K_2^{-\frac{1}{2}}\sin^{-1}K_2^{\frac{1}{2}}(n+1)^{\frac{1}{n}}(\omega_n^{-1}Vol_K(B(R'))^{\frac{1}{n}} \leq i_N.$$

Then $R'$ only depends on $n$, $K_1$, $K_2$, $i_N$ and $\sup_{(x,t)\in M\times[0,T_0]}|A|$, and $Vol_{g(0)}(B(R'))$ satisfies (10) and (11). This implies that

$$\max_{(x,t)\in M\times[\frac{T_0}{2},T_0]} H^2(x,t) \leq C_3\left(\int_0^{T_0}\int_{M_t}|H|^{n+2}d\mu dt\right)^{\frac{2}{n+2}},$$

where $C_3$ is a constant depending on $n$, $T_0$, $\sup_{(x,t)\in M\times[0,T_0]}|A|$, $K_1$, $K_2$ and $i_N$, which is desired.

## 4 Mean curvature flow with finite total mean curvature

In this section we combine the above results to prove Theorem 1.1.

*Proof of Theorem 1.1.* It is sufficient to prove the theorem for $\alpha = n+2$ since by Hölder's inequality, $||H||_{\alpha,M\times[0,T)} < \infty$ implies $||H||_{n+2,M\times[0,T)} < \infty$ if $\alpha > n+2$. We argue by contradiction.

Suppose that the solution to the mean curvature flow can't be extended over $T$, then $|A|$ becomes unbounded as $t \to T$. Since $h_{ij} \geq -C$, we get $\sum_{i,j}(h_{ij}+C)^2 \leq C_7[tr(h_{ij}+C)]^2$, where $C_7$ is a constant depending only on $n$. Since $|A|^2$ is unbounded, we have $\sum_{i,j}(h_{ij}+C)^2$ is unbounded. This together with

$$[tr(h_{ij}+C)]^2 = (H+nC)^2 = H^2 + 2nCH + n^2C^2$$

implies that $H^2$ is unbounded. Namely,

$$\sup_{(x,t)\in M\times[0,T)} H^2(x,t)=\infty.$$

Choose an increasing time sequence $t^{(i)}$, $i=1,2,\cdots$, such that $\lim_{i\to\infty} t^{(i)}=T$. We take a sequence of points $x^{(i)}\in M$ satisfying

$$H^2(x^{(i)},t^{(i)})=\max_{(x,t)\in M\times[0,t^{(i)}]} H^2(x,t).$$

Then $\lim_{i\to\infty} H^2(x^{(i)},t^{(i)})=\infty$.

Putting $Q^{(i)}=H^2(x^{(i)},t^{(i)})$, we have $\lim_{i\to\infty} Q^{(i)}=\infty$. This together with $\lim_{i\to\infty} t^{(i)}=T>0$ implies that there exists a positive integer $i_0$ such that $Q^{(i)}t^{(i)}\geq 1$ and $Q^{(i)}\geq 1$ for $i\geq i_0$. For $i\geq i_0$ and $t\in[0,1]$, we consider the rescaled flows

$$F^{(i)}(t)=F\left(\frac{t-1}{Q^{(i)}}+t^{(i)}\right):(M,g^{(i)}(t))\longrightarrow(N,Q^{(i)}h).$$

Let $H_{(i)}$ and $g^{(i)}(t)=F^{(i)}(t)^*(Q^{(i)}h)$ be the mean curvature of $F^{(i)}(t)$ and the induced metric on $M$ induced by $F^{(i)}(t)$ respectively. Then $F^{(i)}(t):M\to\mathbb{R}^{n+1}$ is still a solution to the mean curvature flow on $t\in[0,1]$. Since $F_t$ satisfies $h_{ij}\geq -C$ for $(x,t)\in M\times[0,T)$, we have

$$H^2_{(i)}(x,t)\leq 1 \quad on \quad M\times[0,1],$$

$$h^{(i)}_{jk}\geq -\frac{C}{\sqrt{Q^{(i)}}} \quad on \quad M\times[0,1], \tag{12}$$

where $A^{(i)}=h^{(i)}_{jk}$ is the second fundamental form of $F^{(i)}(t)$. The inequality in (12) gives $h^{(i)}_{jk}+\frac{C}{\sqrt{Q^{(i)}}}\geq 0$. Hence

$$h^{(i)}_{jk}+\frac{C}{\sqrt{Q^{(i)}}}\leq tr\left(h^{(i)}_{jk}+\frac{C}{\sqrt{Q^{(i)}}}\right)\leq H_{(i)}+\frac{nC}{\sqrt{Q^{(i)}}},$$

which implies that $h^{(i)}_{jk}\leq H_{(i)}+\frac{(n-1)C}{\sqrt{Q^{(i)}}}$. Since $Q^{(i)}\geq 1$ for $i\geq i_0$, it follows that $|A^{(i)}|\leq C_8$, where $C_8$ is a constant independent of $i$ for $i\geq i_0$.

We consider the sequence $(M,g^{(i)}(t),x^{(i)})$, $t\in[0,1]$. It follows from [3] that there is a subsequence of $(M^{(i)},g^{(i)}(t),x^{(i)})$ converges to a Riemannian manifold $(\widetilde{M},\widetilde{g}(t),\widetilde{x})$, and the corresponding subsequence of immersions $F^{(i)}(t)$ converges to an immersion $\widetilde{F}(t):\widetilde{M}\to\mathbb{R}^{n+1}$ $t\in[0,1]$.

Since $(N,h)$ has bounded geometry and $Q^{(i)}\geq 1$ for $i\geq i_0$, $(N,Q^{(i)}h)$ also has bounded geometry with the same bounding constants as $(N,h)$ for each $i\geq i_0$. It follows from Theorem 3.1 that

$$\max_{(x,t)\in M^{(i)}\times[\frac{1}{2},1]} H^2_{(i)}(x,t)\leq C_9\left(\int_0^1\int_{M_t}|H_{(i)}|^{n+2}d\mu_{g^{(i)}(t)}dt\right)^{\frac{2}{n+2}},$$

where $C_9$ is a constant independent of $i$ for $i \geq i_0$. Hence

$$\begin{aligned}\max_{(x,t)\in\widetilde{M}\times[\frac{1}{2},1]} \widetilde{H}^2(x,t) &\leq \lim_{i\to\infty} C_9 \left(\int_0^1 \int_{M_t} |H_{(i)}|^{n+2} d\mu_{g^{(i)}(t)} dt\right)^{\frac{2}{n+2}} \\ &\leq \lim_{i\to\infty} C_9 \left(\int_{t^{(i)}}^{t^{(i)}+(Q^{(i)})^{-1}} \int_{M_t} |H|^{n+2} d\mu dt\right)^{\frac{2}{n+2}}. \quad (13)\end{aligned}$$

The equality in (13) holds because $\int_0^T \int_M H^{n+2} d\mu dt < +\infty$ and $\lim_{i\to\infty}(Q^{(i)})^{-1} = 0$. On the other hand, according to the choice of the points, we have

$$\widetilde{H}^2(\widetilde{x},1) = \lim_{i\to\infty} H^2_{(i)}(x^{(i)},1) = 1.$$

This is a contradiction. We complete the proof of Theorem 1.1.

We can prove Theorem 1.2 by a similar method.

*Proof of Theorem 1.2.* Since $H^2 > n^2 K_1$ is a strict inequality and $M$ is compact, there is a positive constant $\varepsilon$ such that $H^2 \geq n^2 K_1 + \varepsilon$. From Lemma 3.1 we know that $H^2 \geq n^2 K_1 + \varepsilon$ is preserved along the flow. Moreover, $H^2 > n^2 K_1$ at $t = 0$ implies $|A|^2 \leq C_{10} H^2$, where $C_{10}$ is a constant. We put $f_0 = \frac{|A|^2}{H^2}$, then we can obtain the evolution of $f_0$ by Lemma 5.2 in [5]:

$$\begin{aligned}\frac{\partial}{\partial t} f_0 &= \triangle f_0 + \frac{2}{H}\langle \nabla_l H, \nabla_l f_0\rangle - \frac{2}{H^4}|\nabla_i H h_{kl} - \nabla_i h_{kl} H|^2 \\ &\quad -\frac{1}{H^2}[4(h^{ij}h_{jl}\bar{R}^{l\ \ m}_{mi} - h^{ij}h^{lm}\bar{R}_{iljm}) + h^{ij}(\bar{\nabla}_j \bar{R}_{0li}{}^{l} + \bar{\nabla}_l \bar{R}_{0ij}{}^{l})].\end{aligned}$$

This implies that

$$\frac{\partial}{\partial t} f_0 \leq \triangle f_0 + C_{11} f_0 + C_{12},$$

where $C_{11}$ and $C_{12}$ are constants independent of $t$. By the maximum principle and the finiteness of $T$, there exists some positive constant $C_{13}$ independent of $t$ such that $|A|^2 \leq C_{13} H^2$ for $t \in [0,T)$.

We only need to prove the theorem for $\alpha = n+2$, and we still argue by contradiction. If the solution to the mean curvature flow can't be extended over time $T$, then $|A|^2$ becomes unbounded as $t \to T$, and $|A|^2 \leq C_{13} H^2$ implies that $H^2$ also becomes unbounded. Let $(x^{(i)}, t^{(i)})$, $Q^{(i)}$, $F^{(i)}(t)$, $g^{(i)}(t)$ and $(\widetilde{M}, \tilde{g}(t), \tilde{x})$ be the same as we denoted in the proof of Theorem 1.1, $A^{(i)}$ and $H_{(i)}$ be the second fundamental form and mean curvature of the immersion $F^{(i)}(t)$ respectively. Then $|A^{(i)}|^2 \leq C_{13}|H_{(i)}|^2$ for $(x,t) \in M \times [0,1]$, which implies that $A^{(i)}$ is bounded by a constant independent of $i$, for $t \in [0,1]$. Then by the conclusion of Theorem 3.1, we have

$$\max_{(x,t)\in M^{(i)}\times[\frac{1}{2},1]} H^2_{(i)}(x,t) \leq C_{14}\left(\int_0^1 \int_{M_t} |H_{(i)}|^{n+2} d\mu_{g^{(i)}(t)} dt\right)^{\frac{2}{n+2}},$$

where $C_{14}$ is a constant independent of $i$. Then by a similar process to the proof of Theorem 1.1, we can get a contradiction to complete the proof.

**Remark 4.1.** By a similar argument, we can extend the mean curvature flow in the case where $(M, g)$ is a complete non-compact Riemannian manifold. But in that case, the condition $\alpha \geq n+2$ have to be changed to $\alpha = n+2$, since the Hölder's inequality doesn't hold.

**Remark 4.2.** In [14], we have investigated the integral conditions to extend mean curvature flow where $M^n$ is a submanifold in $N^{n+p}$ with codimension $p \geq 2$.

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