DESCENDENT INTEGRALS AND TAUTOLOGICAL RINGS OF MODULI SPACES OF CURVES

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Dedicated to Professor Shing-Tung Yau on the occasion of his 60th birthday

Abstract. The main objective of this paper is to give a summary of our recent work on recursion formulae for intersection numbers on moduli spaces of curves and their applications. We also present a conjectural relation between tautological rings and the mock theta function.

Contents

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1. Introduction

The moduli space of curves is a very important subject of study in algebraic geometry and mathematical physics. Mathematicians realized long ago that the study of a single curve is not sufficient even to understand the curve itself; the most fascinating aspect of the story is that properties of curves can be better understood when comparing general members in a natural family of curves in question.

The study of moduli spaces of curves started with Riemann. Although its construction was consolidated only in the late 1960's in the hands of Mumford, Deligne, Gieseker, etc., the moduli space of curves was already implicit in the work of the classical Italian school of algebraic geometry. It has never lost its value since its creation; we see much beautiful mathematics flourishing around it in the past years.

In this paper, we will only touch on one aspect of the moduli space of curves, namely the intersection theory. There is a large amount of literature on the intersection theory of moduli spaces of curves, a nice account can be found in Vakil's survey paper [74].

Since Mumford's pioneering work [68], we know that certain geometrically natural cohomology classes on moduli spaces of curves are of primary interest.

A ground-breaking achievement in the early 1990's is the celebrated Witten conjecture, giving a surprising connection of intersection theory on moduli spaces of curves to the realm of integrable systems. Shortly afterwards, Kontsevich gave a remarkable proof by expressing integrals of ψ classes as a new type of matrix integrals. Moreover, the Witten-Kontsevich theorem provides an effective recursive way to compute integrals of ψ classes, or descendent integrals. On the other hand, Gromov-Witten theory, also dubbed modern enumerative geometry, revolutionized the study of Hodge integrals on moduli spaces of curves, namely integrals of mixed ψ and λ classes.

Another important development in the 1990's is Faber's remarkable conjectures on the tautological rings of moduli spaces of curves.

In the meantime, string duality has produced many conjectures about the moduli spaces of stable curves and stable maps, such as the mirror formula and the Mariño-Vafa formula. The interactions of string theory and moduli spaces has been one of the most exciting research fields in mathematics for the past several years. For more discussion, see the survey article [55].

In this paper, we first review the Witten-Kontsevich theorem, which is the starting point of many results discussed here. After that, we will give a survey of our work on explicit effective recursion formulae for computing descendent integrals, higher Weil-Petersson volumes of moduli spaces of curves and Witten's r-spin intersection numbers. Most of these formulae provide feasible algorithms for implementation on computers.

We also describe our proof of the Faber intersection number conjecture using a recursive formula of n-point functions and discuss possible relations between tautological rings and mock theta functions.

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2. Intersection numbers and the Witten-Kontsevich theorem

Denote by $\overline{\mathcal{M}}_{g,n}$ the moduli space of stable n-pointed genus g complex algebraic curves. We have the morphism that forgets the last marked point

$$\pi : \overline{\mathcal{M}}_{g,n+1} \longrightarrow \overline{\mathcal{M}}_{g,n}.$$

Denote by $\sigma_1, \dots, \sigma_n$ the canonical sections of π. Let ω_π be the relative dualizing sheaf; we have the following tautological classes on moduli spaces of curves.

$$\begin{aligned}
\psi_i &= c_1(\sigma_i^*(\omega_\pi)) \\
\kappa_i &= \pi_*(\psi_{n+1}^{i+1}) \\
\lambda_k &= c_k(\mathbb{E}), \quad 1 \le k \le g,
\end{aligned}$$

where $\mathbb{E} = \pi_*(\omega_\pi)$ is the Hodge bundle.

Intuitively, ψ_i is the first Chern class of the line bundle corresponding to the cotangent space of the universal curve at the i-th marked point and the fiber of $\mathbb{E}$ is the space of holomorphic one-forms on the algebraic curve.

We use Witten's notation

$$\langle \tau_{d_1} \cdots \tau_{d_n} \kappa_{a_1} \cdots \kappa_{a_m} \mid \lambda_1^{k_1} \cdots \lambda_g^{k_g} \rangle := \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \cdots \psi_n^{d_n} \kappa_{a_1} \cdots \kappa_{a_m} \lambda_1^{k_1} \cdots \lambda_g^{k_g}.$$

These intersection numbers are called the Hodge integrals. They are rational numbers because the moduli spaces of curves are orbifolds (with quotient singularities) except in genus zero. Their degrees should add up to $\dim \overline{\mathcal{M}}_{g,n} = 3g - 3 + n$.

Intersection numbers of pure ψ classes $\langle \tau_{d_1} \cdots \tau_{d_n} \rangle$ are often called descendent integrals. Intersection numbers of pure κ classes $\langle \kappa_{a_1} \cdots \kappa_{a_m} \rangle$ are called higher Weil-Petersson volumes.

The classes κ_i were first introduced by Mumford [68] on $\overline{\mathcal{M}}_g$; their generalization to $\overline{\mathcal{M}}_{g,n}$ here is due to Arbarello-Cornalba [2].

Mumford's 1983 paper [68] initiated the systematic study of the intersection theory of the moduli space of curves. In particular, Mumford computed the Chow ring of $\overline{\mathcal{M}}_2$,

$$A^*(\overline{\mathcal{M}}_2) \cong [\lambda, \delta]/(5\lambda^3 - \lambda^2\delta, 110\lambda^2 - 21\lambda\delta + \delta^2),$$

where $\lambda = \lambda_1$ and $\delta = \delta_0 + \delta_1$ is the full boundary divisor.

The Chow rings of $\overline{\mathcal{M}}_g$ for $g \le 5$ have been studied by Faber [13] and Izadi [35].

2.1. **Witten-Kontsevich theorem.** In 1990, Witten [75] made a striking conjecture (first proved by Kontsevich [42]) that the generating function of ψ class intersection numbers is governed by the KdV hierarchy. Now Witten's conjecture has many different proofs [8, 36, 37, 39, 65, 70].

Roughly speaking, Witten's motivation comes from the two seemingly unrelated mathematical models that describe the physical theory of two-dimensional gravity. One is the counting of triangulations of surfaces, which is related to matrix models and the other is the intersection theory of $\overline{\mathcal{M}}_{g,n}$. The partition function of the first model is known to obey the KdV hierarchy.

Kontsevich's remarkable proof uses a combinatorial description of moduli spaces and Feynman diagram techniques. A very readable exposition can be found in [50].

Witten's conjecture revolutionized the intersection theory of moduli spaces of curves and motivated a surge of subsequent developments: Gromov-Witten theory, Faber's conjecture [15] and the Virasoro conjecture of Eguchi-Hori-Xiong-Katz [10]. Below we follow Witten's nice exposition [75].

The KdV hierarchy is the following hierarchy of differential equations for $n \geq 1$,

$$\frac{\partial U}{\partial t_n} = \frac{\partial R_{n+1}}{\partial t_0}, \tag{1}$$

where R_n are Gelfand-Dikii differential polynomials in $U, \partial U/\partial t_0, \partial^2 U/\partial t_0^2, \ldots$, defined recursively by

$$R_1 = U, \qquad \frac{\partial R_{n+1}}{\partial t_0} = \frac{1}{2n+1}\left(\frac{\partial U}{\partial t_0}R_n + 2U\frac{\partial R_n}{\partial t_0} + \frac{1}{4}\frac{\partial^3 R_n}{\partial t_0^3}\right). \tag{2}$$

It is easy to see that

$$R_2 = \frac{1}{2}U^2 + \frac{1}{12}\frac{\partial^2 U}{\partial t_0^2},$$

$$R_3 = \frac{1}{6}U^3 + \frac{U}{12}\frac{\partial^3 U}{\partial t_0^3} + \frac{1}{24}(\frac{\partial U}{\partial t_0})^2 + \frac{1}{240}\frac{\partial^4 U}{\partial t_0^4},$$

$$\vdots$$

The Witten-Kontsevich theorem states that the generating function

$$F(t_0, t_1, \ldots) = \sum_g \sum_{\mathbf{n}} \langle \prod_{i=0}^{\infty} \tau_i^{n_i} \rangle_g \prod_{i=0}^{\infty} \frac{t_i^{n_i}}{n_i!} \tag{3}$$

is a τ-function for the KdV hierarchy, i.e. $U = \partial^2 F/\partial t_0^2$ obeys all equations in the KdV hierarchy. The first equation in the KdV hierarchy is the classical KdV equation

$$\frac{\partial U}{\partial t_1} = U\frac{\partial U}{\partial t_0} + \frac{1}{12}\frac{\partial^3 U}{\partial t_0^3}. \tag{4}$$

In addition, F obeys the string equation

$$\frac{\partial F}{\partial t_0} = \frac{t_0^2}{2} + \sum_{i=0}^{\infty} t_{i+1}\frac{\partial F}{\partial t_i} \tag{5}$$

and the dilaton equation

$$\frac{\partial F}{\partial t_1} = \frac{1}{24} + \sum_{i=0}^{\infty} \frac{2i+1}{3} t_i \frac{\partial F}{\partial t_i}. \tag{6}$$

The string and dilaton equations can be proved directly in algebraic geometry, e.g. see [62]. Witten introduced the following notation for the derivatives of F,

$$\langle\langle \tau_{d_1} \cdots \tau_{d_n} \rangle\rangle := \frac{\partial^n F}{\partial t_{d_1} \cdots \partial t_{d_n}}.$$

The left-hand side of (1) is the same as

$$\frac{\partial}{\partial t_0}\langle\langle\tau_n\tau_0\rangle\rangle.$$

By the string equation, it is clear that upon integrating both sides of (1) once in t_0 and using the recursion relation (2), we get

$$\langle\langle\tau_n\tau_0\tau_0\rangle\rangle = \frac{1}{2n+1}\langle\langle\tau_{n-1}\tau_0\rangle\rangle\langle\langle\tau_0^3\rangle\rangle + 2\langle\langle\tau_{n-1}\tau_0^2\rangle\rangle\langle\langle\tau_0^2\rangle\rangle + \frac{1}{4}\langle\langle\tau_{n-1}\tau_0^4\rangle\rangle. \tag{7}$$

The above discussion may be summarized as the following:

Proposition 2.1. *Let F be the generating function* (3) *and $U = \partial^2 F/\partial t_0^2$. Then we have the following equivalent statements of the Witten-Kontsevich theorem:*

i) *U satisfies the KdV hierarchy* (1) *and the string equation* (5)*;*
ii) *U satisfies the first KdV equation* (4)*, the string equation and the dilaton equation;*
iii) *F satisfies the recursion formula* (7) *and the string equation.*

Moreover, the Witten-Kontsevich theorem uniquely determines F.

Proof. The only nontrivial part is that (ii) implies (i), which is proved in [59] (Corollary 2.4). □

2.2. **Virasoro constraints.** The Witten-Kontsevich theorem has an important reformulation due to Dijkgraaf, Verlinde, and Verlinde [9], in terms of the Virasoro constraints.

Define a family of differential operators L_k for $k \geq -1$ by

$$\begin{aligned} L_k = &-\frac{1}{2}(2k+3)!!\frac{\partial}{\partial t_{k+1}} + \frac{1}{2}\sum_{j=0}^{\infty}\frac{(2(j+k)+1)!!}{(2j-1)!!}t_j\frac{\partial}{\partial t_{j+k}} \\ &+ \frac{1}{4}\sum_{d_1+d_2=k-1}(2d_1+1)!!(2d_2+1)!!\frac{\partial^2}{\partial t_{d_1}\partial t_{d_2}} + \frac{\delta_{k,-1}t_0^2}{4} + \frac{\delta_{k,0}}{48}, \end{aligned} \tag{8}$$

It is straightforward to verify that these operators satisfy the Virasoro relations

$$[L_n, L_m] = (n-m)V_{n+m}.$$

Dijkgraaf, Verlinde, and Verlinde [9] have proved that the KdV form of Witten's conjecture is equivalent to the following Virasoro constraints. An elegant exposition of the proof can be found in [25].

Proposition 2.2. *(DVV formula) Let F be the generating function of descendent integrals defined in* (3)*. We have $L_k(\exp F) = 0$ for $k \geq -1$. More explicitly,*

$$\begin{aligned} \langle\tau_{k+1}\tau_{d_1}\cdots\tau_{d_n}\rangle_g = \frac{1}{(2k+3)!!}\Bigg[&\sum_{j=1}^{n}\frac{(2k+2d_j+1)!!}{(2d_j-1)!!}\langle\tau_{d_1}\cdots\tau_{d_j+k}\cdots\tau_{d_n}\rangle_g \\ &+ \frac{1}{2}\sum_{r+s=k-1}(2r+1)!!(2s+1)!!\langle\tau_r\tau_s\tau_{d_1}\cdots\tau_{d_n}\rangle_{g-1} \\ &+\frac{1}{2}\sum_{r+s=k-1}(2r+1)!!(2s+1)!!\sum_{\underline{n}=I\coprod J}\langle\tau_r\prod_{i\in I}\tau_{d_i}\rangle_{g'}\langle\tau_s\prod_{i\in J}\tau_{d_i}\rangle_{g-g'}\Bigg]. \end{aligned} \tag{9}$$

3. The n-point function

Definition 3.1. In [15], the following generating function,

$$F(x_1,\cdots,x_n)=\sum_{g=0}^{\infty}\sum_{\sum d_i=3g-3+n}\langle\tau_{d_1}\cdots\tau_{d_n}\rangle_g\prod_{i=1}^{n}x_i^{d_i}$$

is called the n-point function.

The coefficients of n-point functions encode all information of intersection numbers of ψ classes. Okounkov [69] obtained a beautiful expression of the n-point functions using n-dimensional error-function-type integrals. However, it is very difficult to extract coefficients from Okounkov's analytic formula.

Brézin and Hikami [4] apply correlation functions of GUE ensemble to uncover explicit formulae of n-point functions.

3.1. **A recursive formula of n-point functions.** Consider the following "normalized" n-point function

$$G(x_1,\dots,x_n)=\exp\left(\frac{-\sum_{j=1}^{n}x_j^3}{24}\right)F(x_1,\dots,x_n).$$

The one-point function $G(x)=\frac{1}{x^2}$ is due to Witten; we have also Dijkgraaf's two-point function

$$G(x,y)=\frac{1}{x+y}\sum_{k\geq 0}\frac{k!}{(2k+1)!}\left(\frac{1}{2}xy(x+y)\right)^k$$

and Zagier's three-point function [78] which we learned from Prof. C. Faber,

$$G(x,y,z)=\sum_{r,s\geq 0}\frac{r!S_r(x,y,z)}{4^r(2r+1)!!\cdot 2}\cdot\frac{\Delta^s}{8^s(r+s+1)!},$$

where $S_r(x,y,z)$ and Δ are the homogeneous symmetric polynomials defined by

$$S_r(x,y,z)=\frac{(xy)^r(x+y)^{r+1}+(yz)^r(y+z)^{r+1}+(zx)^r(z+x)^{r+1}}{x+y+z}\in\mathbb{Z}[x,y,z],$$
$$\Delta(x,y,z)=(x+y)(y+z)(z+x)=\frac{(x+y+z)^3}{3}-\frac{x^3+y^3+z^3}{3}.$$

The two- and three-point functions were found in the early 1990's. These explicit form of two- and three-point functions played a crucial role in Faber's pioneering work on tautological rings [15]. Since then it has been a prominent open problem to find explicit formulae of n-point functions, closed or recursive. Faber's work [15] indicated clearly that this is probably the first step toward a proof of his intersection number conjecture.

By solving the differential equation coming from Witten's KdV coefficient equation (7), we get a recursive formula for normalized n-point functions, generalizing Dijkgraaf and Zagier's formulae.

Theorem 3.2. [59] *For $n\geq 2$,*

$$G(x_1,\dots,x_n)=\sum_{r,s\geq 0}\frac{(2r+n-3)!!}{4^s(2r+2s+n-1)!!}P_r(x_1,\dots,x_n)\Delta(x_1,\dots,x_n)^s,$$

where P_r and Δ are homogeneous symmetric polynomials defined by

$$\Delta(x_1,\dots,x_n) = \frac{(\sum_{j=1}^n x_j)^3 - \sum_{j=1}^n x_j^3}{3},$$

$$\begin{aligned} P_r(x_1,\dots,x_n) &= \left(\frac{1}{2\sum_{j=1}^n x_j}\sum_{\underline{n}=I\coprod J}(\sum_{i\in I}x_i)^2(\sum_{i\in J}x_i)^2 G(x_I)G(x_J)\right)_{3r+n-3} \\ &= \frac{1}{2\sum_{j=1}^n x_j}\sum_{\underline{n}=I\coprod J}(\sum_{i\in I}x_i)^2(\sum_{i\in J}x_i)^2\sum_{r'=0}^{r} G_{r'}(x_I)G_{r-r'}(x_J), \end{aligned}$$

where $I, J \neq \emptyset$, $\underline{n} = \{1,2,\dots,n\}$ and $G_g(x_I)$ denotes the degree $3g+|I|-3$ homogeneous component of the normalized $|I|$-point function $G(x_{k_1},\dots,x_{k_{|I|}})$, where $k_j \in I$.

Dijkgraaf and Zagier's formulae gave us much inspiration in writing down the general pattern of the above recursive formula. On the other hand, it took us great effort to get the correct coefficients on the right hand side.

Proposition 3.3. [59] *The recursion relation in Theorem 3.2 is equivalent to either one of the following statements.*

i) *The normalized n-point functions satisfy the following recursion relation*

$$G_g(x_1,\dots,x_n) = \frac{1}{(2g+n-1)}P_g(x_1,\dots,x_n) + \frac{\Delta(x_1,\dots,x_n)}{4(2g+n-1)}G_{g-1}(x_1,\dots,x_n).$$

ii) *The n-point functions $F_g(x_1,\dots,x_n)$ satisfy the following recursion relation*

$$\begin{aligned} (2g+n-1)\left(\sum_{i=1}^n x_i\right)F_g(x_1,\dots,x_n) &= \frac{1}{12}\left(\sum_{i=1}^n x_i\right)^4 F_{g-1}(x_1,\dots,x_n) \\ &\quad + \frac{1}{2}\sum_{g'=0}^{g}\sum_{\underline{n}=I\coprod J}\left(\sum_{i\in I}x_i\right)^2\left(\sum_{i\in J}x_i\right)^2 F_{g'}(x_I)F_{g-g'}(x_J). \end{aligned}$$

After we used Theorem 3.2 to prove Proposition 3.3(ii), we realized that the latter has already been embodied in the Witten-Kontsevich theorem. Integrating the KdV equation (4) once in t_0, we get

$$\frac{\partial^2 F}{\partial t_1 \partial t_0} = \frac{1}{2}\left(\frac{\partial F}{\partial t_0^2}\right)^2 + \frac{1}{12}\frac{\partial^4 F}{\partial t_0^4} + H(t_1,t_2,\dots), \tag{10}$$

where H does not depend on t_0. In fact, we can prove $H=0$ using the string equation and the KdV equation (4); details can be found in Section 2 of [59]. So equation (10) can be rewritten as

$$\langle\langle\tau_0\tau_1\rangle\rangle = \frac{1}{12}\langle\langle\tau_0^4\rangle\rangle + \frac{1}{2}\langle\langle\tau_0^2\rangle\rangle\langle\langle\tau_0^2\rangle\rangle.$$

A minute's thought will convince you that this is just the identity in Proposition 3.3(ii) after we apply the string equation (5) and the dilaton equation (6).

The normalized n-point function G has some nice vanishing properties not possessed by the original n-point function F.

Theorem 3.4. [59] *Let $\mathcal{C}\left(\prod_{j=1}^n x_j^{d_j}, p(x_1,\dots,x_n)\right)$ denote the coefficient of $\prod_{j=1}^n x_j^{d_j}$ in a polynomial or formal power series $p(x_1,\dots,x_n)$.*

i) *Let* $k > 2g-2+n$, $d_j \geq 0$ *and* $\sum_{j=1}^n d_j = 3g-2+n-k$. *Then*

$$\mathcal{C}\left(z^k \prod_{j=1}^n x_j^{d_j}, G_g(z, x_1, \ldots, x_n)\right) = 0.$$

ii) *Let* $d_j \geq 0$, $\sum_{j=1}^n d_j = g$ *and* $a = \#\{j \mid d_j = 0\}$. *Then*

$$\mathcal{C}\left(z^{2g-2+n} \prod_{j=1}^n x_j^{d_j}, G_g(z, x_1, \ldots, x_n)\right) = \frac{1}{4^g \prod_{j=1}^n (2d_j+1)!!}.$$

iii) *Let* $d_j \geq 0$, $\sum_{j=1}^n d_j = g+1$, $a = \#\{j \mid d_j = 0\}$ *and* $b = \#\{j \mid d_j = 1\}$. *Then*

$$\mathcal{C}\left(z^{2g-3+n} \prod_{j=1}^n x_j^{d_j}, G_g(z, x_1, \ldots, x_n)\right) = \frac{2g^2 + (2n-1)g + \frac{n^2-n}{2} - 3 + \frac{5a-a^2}{2}}{4^g \prod_{j=1}^n (2d_j+1)!!}.$$

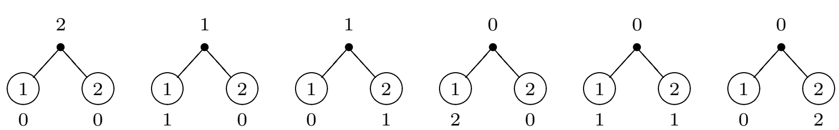
As an important application of the n-point functions, we proved some new identities of descendent integrals, which led to a proof of the Faber intersection number conjecture (see Section 6). Some of these results (e.g. Theorem 3.4) have also found applications in Zhou's important work [79] on Hurwitz-Hodge integrals.

Next we give an interesting combinatorial interpretation of n-point functions in terms of summation over binary trees.

Recall that a binary tree T is a tree such that each node $v \in V(T)$ either has no children ($v \in L(T)$ is a leaf) or has two children ($v \notin L(T)$).

Let T be a binary tree. Let $n = |L(T)|$ be the number of leaves. We assign an integer $g(v) \geq 0$ to each node $v \in V(T)$ and label the n leaves with distinct values $\ell(v) \in \{1, \ldots, n\}$. Then we call such T a "weighted marked binary tree" (abbreviated "WMB tree") and call $g(T) = \sum_{v \in V(T)} g(v)$ the total weight of T.

Here are all the WMB trees with $(g, n) = (2, 2)$:



Proposition 3.5. [59] *Denote by WMB(g,n) the set of isomorphism classes of all WMB trees with total weight g and n leaves. Then*

$$12^g \left(\prod_{j=1}^n x_j\right) \cdot (x_1 + \cdots + x_n)^2 F_g(x_1, \ldots, x_n)$$

$$= \sum_{T \in \mathrm{WMB}(g,n)} \prod_{v \in V(T)} \frac{\left(|L(v)| - 3 + \sum\limits_{\substack{w \in D(v) \\ w \neq v}} 2g(w)\right)!!}{\left(|L(v)| - 1 + \sum\limits_{w \in D(v)} 2g(w)\right)!!} \left(\sum_{w \in L(v)} x_{\ell(w)}\right)^{3g(v)+1},$$

where $D(v) \subset V(T)$ *is the set of all descendants of* v *and* $L(v) = D(v) \cap L(T)$.

The idea of proof of the above formula is simple. We apply Proposition 3.3(ii) repeatedly, until the right hand side contains only one-point functions. Note that partitions of indices are in one-to-one correspondence with binary trees.

3.2. **An effective recursion formulae of descendent integrals.** In [59] (Proposition 5.1), we proved a recursion formula which explicitly expresses any descendent integral in terms of those with strictly lower genus.

Proposition 3.6. *Let* $d_j \geq 0$ *and* $\sum_{j=1}^n d_j = 3g+n-3$. *Then*

$$
\begin{aligned}
&(2g+n-1)(2g+n-2)\langle\prod_{j=1}^{n}\tau_{d_j}\rangle_g\\
&\qquad=\frac{2d_1+3}{12}\langle\tau_0^4\tau_{d_1+1}\prod_{j=2}^{n}\tau_{d_j}\rangle_{g-1}-\frac{2g+n-1}{6}\langle\tau_0^3\prod_{j=1}^{n}\tau_{d_j}\rangle_{g-1}\\
&\qquad+\sum_{\{2,\ldots,n\}=I\coprod J}(2d_1+3)\langle\tau_{d_1+1}\tau_0^2\prod_{i\in I}\tau_{d_i}\rangle_{g'}\langle\tau_0^2\prod_{i\in J}\tau_{d_i}\rangle_{g-g'}\\
&\qquad\qquad-\sum_{\{2,\ldots,n\}=I\coprod J}(2g+n-1)\langle\tau_{d_1}\tau_0\prod_{i\in I}\tau_{d_i}\rangle_{g'}\langle\tau_0^2\prod_{i\in J}\tau_{d_i}\rangle_{g-g'}.
\end{aligned}
$$

By the string equation, we may assume all indices $d_j \geq 1$; then every non-zero descendent integral on the right hand side has genus strictly less than g. So everything reduces to the following well-known identity of genus zero intersection numbers:

$$\langle\tau_{d_1}\cdots\tau_{d_n}\rangle_0=\binom{n-3}{d_1,\ldots,d_n}.$$

4. Hodge integrals

In this section, we study the integrals of products of ψ and λ classes on $\overline{\mathcal{M}}_{g,n}$. Hodge integrals arise naturally in the localization computation of Gromov-Witten theory [43, 31], since the Atiyah-Bott localization formula expresses $\mathbb{C}^*$-equivariant classes over $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r,d)$ in terms of summation over the fixed point loci consisting of products of moduli spaces of marked curves, whose summands are just the products of Hodge integrals.

4.1. **Faber's algorithm.** Faber's algorithm [14] reduces the calculation of general Hodge integrals to those with pure ψ classes.

Step 1: Eliminating κ classes

The fact that intersection numbers involving both κ classes and ψ classes can be reduced to intersection numbers involving only ψ classes was already known to Witten [75], and has been developed by Arbarello-Cornalba [2], Faber [14] and Kaufmann-Manin-Zagier [38] into a beautiful combinatorial formalism.

First we fix notation as in [38]. Consider the semigroup N^∞ of sequences $\mathbf{m}=(m_1,m_2,\dots)$ where m_i are nonnegative integers and $m_i=0$ for sufficiently large i.

Let $\mathbf{m},\mathbf{a_1},\ldots,\mathbf{a_n}\in N^\infty$, $\mathbf{m}=\sum_{i=1}^n\mathbf{a_i}$.

$$|\mathbf{m}|:=\sum_{i\geq1}im_i,\quad ||\mathbf{m}||:=\sum_{i\geq1}m_i,\quad \binom{\mathbf{m}}{\mathbf{a_1},\ldots,\mathbf{a_n}}:=\prod_{i\geq1}\binom{m_i}{a_1(i),\ldots,a_n(i)}.$$

We have the following formula [38] to remove κ classes.

$$\langle\prod_{j=1}^{n}\tau_{d_j}\kappa(\mathbf{m})\Psi\rangle_g=\sum_{k=0}^{||\mathbf{m}||}\frac{(-1)^{||\mathbf{m}||-k}}{k!}\sum_{\substack{\mathbf{m}=\mathbf{m_1}+\cdots+\mathbf{m_k}\\ \mathbf{m_i}\neq\mathbf{0}}}\binom{\mathbf{m}}{\mathbf{m_1},\dots,\mathbf{m_k}}\langle\prod_{j=1}^{n}\tau_{d_j}\prod_{j=1}^{k}\tau_{|\mathbf{m_j}|+1}\Psi\rangle_g, \tag{11}$$

where $\kappa(\mathbf{m})\triangleq\prod_{i\geq1}\kappa_i^{m(i)}$ and Ψ is the pull-back of classes from $\overline{\mathcal{M}}_g$ under forgetful morphisms, such as $\mathrm{ch}(\mathbb{E})$ and λ classes.

Step 2: Substituting λ classes by Chern characters

There is a universal formula to express λ classes in terms of $\mathrm{ch}(\mathbb{E})$,

$$\lambda_j=\sum_{\mu\vdash j}(-1)^{j-\ell(\mu)}\prod_{r\geq1}\frac{((r-1)!)^{m_r}}{m_r!}\mathrm{ch}_\mu(\mathbb{E}),\quad j\geq1,$$

where the sum ranges over all partitions μ of j, $\ell(\mu)$ is the length of μ and m_r is the number of r in μ, and $\mathrm{ch}_\mu(\mathbb{E})=\mathrm{ch}_{\mu_1}(\mathbb{E})\cdots\mathrm{ch}_{\mu_\ell}(\mathbb{E})$. Since $\mathrm{ch}_{2k}(\mathbb{E})=0$ when $k>0$, we may consider only partitions into odd numbers.

Step 3: Applying Mumford's formula

So we arrive at the following integrals with only $\mathrm{ch}(\mathbb{E})$ and ψ classes,

$$\int_{\overline{\mathcal{M}}_{g,n}}\psi_1^{d_1}\cdots\psi_n^{d_n}\mathrm{ch}_{2k_1-1}(\mathbb{E})\cdots\mathrm{ch}_{2k_\ell-1}(\mathbb{E}).$$

We will apply Mumford's formula [68]

$$\mathrm{ch}_{2k-1}(\mathbb{E})=\frac{B_{2k}}{(2k)!}\left[\kappa_{2k-1}-\sum_{i=1}^{n}\psi_i^{2k-1}+\frac{1}{2}\sum_{\xi\in\Delta}p_{\xi*}\left(\sum_{i=0}^{2k-2}\psi_1^i(-\psi_2)^{2k-2-i}\right)\right], \tag{12}$$

where Δ is the set of boundary divisors and B_{2g} is the $2g$-th Bernoulli number.

From [41], we know the pull-back behavior of Hodge bundles under the two natural boundary gluing morphisms.

$$p:\overline{\mathcal{M}}_{g_1,n_1+1}\times\overline{\mathcal{M}}_{g_2,n_2+1}\to\overline{\mathcal{M}}_{g_1+g_2,n_1+n_2},\quad p^*(\mathbb{E}')=\mathbb{E}_1\oplus\mathbb{E}_2.$$

$$p:\overline{\mathcal{M}}_{g-1,n+2}\to\overline{\mathcal{M}}_{g,n},\quad p^*(\mathbb{E}')=\mathbb{E}\oplus\mathbb{C}.$$

Where $\mathbb{E}'$ is the Hodge bundle of the target moduli spaces and $\mathbb{C}$ is the trivial line bundle.

Since $\mathrm{ch}(\mathbb{E}_1\oplus\mathbb{E}_2)=\mathrm{ch}(\mathbb{E}_1)+\mathrm{ch}(\mathbb{E}_2)$, we have

$$\begin{aligned}&\langle\prod_{i=1}^{n}\tau_{d_i}\prod_{i=1}^{\ell}\mathrm{ch}_{2k_i-1}(\mathbb{E})\rangle_g\\ &\quad=\frac{B_{2k_1}}{(2k_1)!!}\Bigg(\langle\tau_{2k_1}\prod_{i=1}^{n}\tau_{d_i}\prod_{i=2}^{\ell}\mathrm{ch}_{2k_i-1}(\mathbb{E})\rangle_g+\sum_{j=1}^{n}\langle\tau_{d_j+2k_1-1}\prod_{i\neq j}\tau_{d_i}\prod_{i=2}^{\ell}\mathrm{ch}_{2k_i-1}(\mathbb{E})\rangle_g\\ &\qquad\qquad+\frac{1}{2}\sum_{j=0}^{2k_1-2}(-1)^j\langle\tau_j\tau_{2k_1-2-j}\prod_{i=1}^{n}\tau_{d_i}\prod_{i=2}^{\ell}\mathrm{ch}_{2k_i-1}(\mathbb{E})\rangle_{g-1}\end{aligned}$$

$$+\frac{1}{2}\sum_{\substack{I\coprod J=\underline{n}\\ I'\coprod J'=\{2,\ldots,\ell\}}}\sum_{j=0}^{2k_1-2}(-1)^j\langle\tau_j\prod_{i\in I}\tau_{d_i}\prod_{i\in I'}\mathrm{ch}_{2k_i-1}(\mathbb{E})\rangle_{g'}\langle\tau_{2k_1-2-j}\prod_{i\in J}\tau_{d_i}\prod_{i\in J'}\mathrm{ch}_{2k_i-1}(\mathbb{E})\rangle_{g-g'}\Bigg)$$

So we can reduce the integral to pure ψ classes by induction on the number of $\mathrm{ch}(\mathbb{E})$. In fact, Faber's algorithm [14] computes more general intersection numbers, which may contain boundary divisors.

4.2. **Hodge integral formulae.** There are very few closed formulae for Hodge integrals. For example, Getzler and Pandharipande [26] showed that the degree zero Virasoro conjecture for $\mathbb{P}^1$, $\mathbb{P}^2$ and $\mathbb{P}^3$ implies respectively the following three Hodge integral formulae:

$$\langle\tau_{d_1}\cdots\tau_{d_n}\mid\lambda_g\rangle_g=\binom{2g+n-3}{d_1,\ldots,d_n}\frac{2^{2g-1}-1}{2^{2g-1}}\frac{|B_{2g}|}{(2g)!},\tag{13}$$

$$\langle\tau_{d_1}\cdots\tau_{d_n}\mid\lambda_g\lambda_{g-1}\rangle_g=\frac{(2g-3+n)!|B_{2g}|}{2^{2g-1}(2g)!\prod_{j=1}^n(2d_j-1)!!},\tag{14}$$

$$\langle\lambda_{g-1}^3\rangle_g=\frac{1}{(2g-2)!}\frac{|B_{2g-2}|}{2g-2}\frac{|B_{2g}|}{2g}.\tag{15}$$

We will see in Section 6 that (14) is an equivalent formulation of the Faber intersection number conjecture. The λ_g theorem (13) was first proved by Faber and Pandharipande [17]. Goulden, Jackson and Vakil [29] have a short proof using the ELSV formula. The identity (15) is proved in [16].

Liu, Liu and Zhou [54] gave a unified proof of (13),(15) as a consequence of the Mariño-Vafa formula.

The following Hodge integral identity is proved in our paper [58].

Proposition 4.1. *Let $g\geq 2$, $d_j\geq 1$ and $\sum_{j=1}^n(d_j-1)=g$. Then*

$$\begin{aligned}
&-\frac{(2g-2)!}{|B_{2g-2}|}\int_{\overline{\mathcal{M}}_{g,n}}\psi_1^{d_1}\cdots\psi_n^{d_n}\mathrm{ch}_{2g-3}(\mathbb{E})\\
&\qquad=\frac{2g-2}{|B_{2g-2}|}\left(\int_{\overline{\mathcal{M}}_{g,n}}\psi_1^{d_1}\cdots\psi_n^{d_n}\lambda_{g-1}\lambda_{g-2}-3\int_{\overline{\mathcal{M}}_{g,n}}\psi_1^{d_1}\cdots\psi_n^{d_n}\lambda_{g-3}\lambda_g\right)\\
&\qquad=\frac{1}{2}\sum_{j=0}^{2g-4}(-1)^j\langle\tau_{2g-4-j}\tau_j\tau_{d_1}\cdots\tau_{d_n}\rangle_{g-1}+\frac{(2g-3+n)!}{2^{2g+1}(2g-3)!}\cdot\frac{1}{\prod_{j=1}^n(2d_j-1)!!}.
\end{aligned}$$

In fact, from Mumford's formula (12), the above Hodge integral is equivalent to the following identity:

$$\begin{aligned}
\frac{(2g-3+n)!}{2^{2g+1}(2g-3)!\prod_{j=1}^n(2d_j-1)!!}&=\langle\tau_{2g-2}\prod_{j=1}^n\tau_{d_j}\rangle_g-\sum_{j=1}^n\langle\tau_{d_j+2g-3}\prod_{i\neq j}\tau_{d_i}\rangle_g\\
&\quad+\frac{1}{2}\sum_{\underline{n}=I\coprod J}\sum_{j=0}^{2g-4}(-1)^j\langle\tau_j\prod_{i\in I}\tau_{d_i}\rangle_{g'}\langle\tau_{2g-4-j}\prod_{i\in J}\tau_{d_i}\rangle_{g-g'},
\end{aligned}$$

which can be proved by packing the right-hand side as coefficients of n-point functions; see discussions in Section 6.

The remarkable ELSV formula of Ekedahl, Lando, Shapiro, and Vainshtein [12] relates single Hurwitz numbers to intersection theory on the moduli space of curves.

Theorem 4.2. *(ELSV formula) Let $n = l(\mu)$ and $r = 2g-2+|\mu|+n$. Then*

$$H_{g,\mu} = r!\prod_{i=1}^{n}\left(\frac{\mu_i^{\mu_i}}{\mu_i!}\right)\int_{\overline{\mathcal{M}}_{g,n}}\frac{1-\lambda_1+\cdots+(-1)^g\lambda_g}{(1-\mu_1\psi_1)\cdots(1-\mu_n\psi_n)}, \tag{16}$$

The ELSV formula was originally proved by studying the degree of the Lyashko-Looijenga mapping, which can be expressed in terms of the top Segre class of the completed Hurwitz space, regarded as a cone over $\overline{\mathcal{M}}_{g,n}$. The ELSV formula is more succinctly recovered using virtual localization on moduli spaces of relative stable morphisms [32, 53]. It can also be derived as a limit of the Mariño-Vafa formula [54, 55].

A key step in Kazarian-Lando's proof [37] of the Witten-Kontsevich theorem is to invert ELSV and eliminate λ classes. So assertions about descendent integrals may be proved by studying Hurwitz numbers.

Proposition 4.3. *(Kazarian-Lando)* [37] *Let $\sum_{i=1}^{n} d_i = 3g-3+n$. Then*

$$\langle\tau_{d_1}\cdots\tau_{d_n}\rangle_g = \sum_{\mu_1=1}^{d_1+1}\cdots\sum_{\mu_n=1}^{d_n+1}\left(\frac{|\mathrm{Aut}(\mu)|}{(2g-2+|\mu|+n)!}\prod_{i=1}^{n}\frac{(-1)^{d_i+1-\mu_i}}{(d_i+1-\mu_i)!\mu_i^{\mu_i-1}}\right)H_{g,\mu}.$$

In [28], Goulden, Jackson and Vakil proposed a conjectural ELSV-type formula expressing one-part double Hurwitz numbers in terms of intersection theory on some compactified universal Picard variety. Recently, D. Zvonkine has proposed a conjectural ELSV-type formula on moduli spaces of r-spin curves.

5. Higher Weil-Petersson volumes

For $\mathbf{b}\in N^{\infty}$, we denote by $V_{g,n}(\mathbf{b})$ the higher Weil-Petersson volume

$$V_{g,n}(\mathbf{b}) := \langle\tau_0^n\kappa(\mathbf{b})\rangle_g = \int_{\overline{\mathcal{M}}_{g,n}}\kappa(\mathbf{b}).$$

Also we write $V_g(\mathbf{b})$ instead of $V_{g,0}(\mathbf{b})$. If $\mathbf{b} = (3g-3+n, 0, 0, \dots)$, we get the classical Weil-Petersson volumes.

Higher Weil-Petersson volumes were extensively studied in the paper [38]. In particular, they obtained a closed formula for $V_{0,n}(\mathbf{b})$.

5.1. **Generalization of Mirzakhani's recursion formula.** In a series of innovative papers [64, 65], Mirzakhani utilizes hyperbolic geometry to obtain a beautiful recursion formula of the Weil-Petersson volumes of the moduli spaces of bordered Riemann surfaces. By taking derivatives in Mirzakhani's recursion, Mulase and Safnuk [67] obtained a differential form of Mirzakhani's recursion formula involving integrals of κ_1 and ψ classes on moduli spaces of curves, which is immediately seen to imply the DVV formula (9).

Wolpert's formula [77] tells us that

$$\kappa_1 = \frac{1}{2\pi^2}\omega_{WP},$$

where ω_{WP} is the Weil-Petersson Kähler form. However, Wolpert's formula has no counterpart for higher degree κ classes.

In the papers [56, 57], we have proved that the Mulase-Safnuk form of Mirzakhani's recursion formula is in fact equivalent to the Witten-Kontsevich theorem. The proof can be generalized to prove an analogue of the Mulase-Safnuk form of Mirzakhani's recursion containing arbitrary higher degree κ classes.

Theorem 5.1. [56] *Let $\mathbf{b}\in N^\infty$ and $d_j\geq 0$. Then*

$$\begin{aligned}(17)\quad &(2d_1+1)!!\langle\kappa(\mathbf{b})\tau_{d_1}\cdots\tau_{d_n}\rangle_g\\ &=\sum_{j=2}^n\sum_{\mathbf{L}+\mathbf{L}'=\mathbf{b}}\alpha_{\mathbf{L}}\binom{\mathbf{b}}{\mathbf{L}}\frac{(2(|\mathbf{L}|+d_1+d_j)-1)!!}{(2d_j-1)!!}\langle\kappa(\mathbf{L}')\tau_{|\mathbf{L}|+d_1+d_j-1}\prod_{i\neq 1,j}\tau_{d_i}\rangle_g\\ &+\frac{1}{2}\sum_{\mathbf{L}+\mathbf{L}'=\mathbf{b}}\sum_{r+s=|\mathbf{L}|+d_1-2}\alpha_{\mathbf{L}}\binom{\mathbf{b}}{\mathbf{L}}(2r+1)!!(2s+1)!!\langle\kappa(\mathbf{L}')\tau_r\tau_s\prod_{i=2}^n\tau_{d_i}\rangle_{g-1}\\ &+\frac{1}{2}\sum_{\substack{\mathbf{L}+\mathbf{e}+\mathbf{f}=\mathbf{b}\\ I\coprod J=\{2,\dots,n\}}}\sum_{r+s=|\mathbf{L}|+d_1-2}\alpha_{\mathbf{L}}\binom{\mathbf{b}}{\mathbf{L},\mathbf{e},\mathbf{f}}(2r+1)!!(2s+1)!!\\ &\qquad\times\langle\kappa(\mathbf{e})\tau_r\prod_{i\in I}\tau_{d_i}\rangle_{g'}\langle\kappa(\mathbf{f})\tau_s\prod_{i\in J}\tau_{d_i}\rangle_{g-g'},\end{aligned}$$

where the constants $\alpha_{\mathbf{L}}$ are determined recursively from the following formula

$$\sum_{\mathbf{L}+\mathbf{L}'=\mathbf{b}}\frac{(-1)^{||\mathbf{L}||}\alpha_{\mathbf{L}}}{\mathbf{L}!\mathbf{L}'!(2|\mathbf{L}'|+1)!!}=0,\qquad \mathbf{b}\neq 0,$$

namely

$$\alpha_{\mathbf{b}}=\mathbf{b}!\sum_{\substack{\mathbf{L}+\mathbf{L}'=\mathbf{b}\\ \mathbf{L}'\neq\mathbf{0}}}\frac{(-1)^{||\mathbf{L}'||-1}\alpha_{\mathbf{L}}}{\mathbf{L}!\mathbf{L}'!(2|\mathbf{L}'|+1)!!},\qquad \mathbf{b}\neq 0,$$

with the initial value $\alpha_{\mathbf{0}}=1$.

Denoting $\alpha(\ell,0,0,\dots)$ by α_ℓ, we recover Mirzakhani's recursion formula with

$$\alpha_\ell=l!\beta_\ell=(-1)^{\ell-1}(2^{2\ell}-2)\frac{B_{2\ell}}{(2\ell-1)!!}.$$

We also have

$$\alpha(\boldsymbol{\delta}_\ell)=\frac{1}{(2\ell+1)!!},$$

where $\boldsymbol{\delta}_\ell$ denotes the sequence with 1 at the ℓ-th place and zeros elsewhere.

Note that Theorems 5.1 holds only for $n\geq 1$. If $n=0$, i.e. for higher Weil-Petersson volumes of $\overline{\mathcal{M}}_g$, we may apply the following formula first, which is a special case of Proposition 3.1 of the paper [56].

$$(18)\qquad \langle\kappa(\mathbf{b})\rangle_g=\frac{1}{2g-2}\sum_{\mathbf{L}+\mathbf{L}'=\mathbf{b}}(-1)^{||\mathbf{L}||}\binom{\mathbf{b}}{\mathbf{L}}\langle\tau_{|\mathbf{L}|+1}\kappa(\mathbf{L}')\rangle_g.$$

So we can use Theorems 5.1 to compute any intersection numbers of ψ and κ classes recursively with the three initial values

$$\langle\tau_0\kappa_1\rangle_1=\frac{1}{24},\qquad\langle\tau_0^3\rangle_0=1,\qquad\langle\tau_1\rangle_1=\frac{1}{24}.$$

The idea to look at the reciprocal of $\alpha_{\mathbf{L}}$ was inspired by the work of Mulase and Safnuk [67], where they considered the reciprocal of α_ℓ.

For the proof of Theorem 5.1, we first transfer the inversion of constants $\alpha_{\mathbf{L}}$ to the left hand side of (17), then we carry out the computation by applying the formula (11) and conclude the result from the DVV formula (9).

5.2. **Recursion formulae of higher Weil-Petersson volumes.** Although there are recursion formulae for higher Weil-Petersson volumes in genus zero [38, 80], it seems difficult to generalize the methods of these papers to deduce explicit recursion formulae between $V_{g,n}(\mathbf{b})$ valid in all genera. We remark that Zograf's elegant algorithm [81] for computing Weil-Petersson volumes seems also not easy to generalize to higher degree κ classes.

The following recursion formulae are proved in [56].

Proposition 5.2. [56] *Let* $\mathbf{b} \in N^\infty$ *and* $n \geq 1$. *Then*

$$\begin{aligned}\big(2g-1+||\mathbf{b}||\big)V_{g,n}(\mathbf{b}) = \frac{1}{12}V_{g-1,n+3}(\mathbf{b}) - \sum_{\substack{\mathbf{L}+\mathbf{L}'=\mathbf{b}\\ ||\mathbf{L}'||\geq 2}} \binom{\mathbf{b}}{\mathbf{L}} V_{g,n}(\mathbf{L}+\boldsymbol{\delta}_{|\mathbf{L}'|})\\ + \frac{1}{2}\sum_{\substack{\mathbf{L}+\mathbf{L}'=\mathbf{b}\\ \mathbf{L}\neq\mathbf{0},\mathbf{L}'\neq\mathbf{0}}}\sum_{r+s=n-1}\binom{\mathbf{b}}{\mathbf{L}}\binom{n-1}{r}V_{g',r+2}(\mathbf{L})V_{g-g',s+2}(\mathbf{L}').\end{aligned}$$

Proposition 5.2 is an effective formula for computing higher Weil-Petersson volumes recursively by induction on g and $||\mathbf{b}||$, with initial values

$$V_{0,3}(0) = 1 \quad \text{and} \quad V_{0,n}(\boldsymbol{\delta}_{n-3}) = 1,\ n \geq 4.$$

Proposition 5.3. [56] *Let* $g \geq 2$ *and* $\mathbf{b} \in N^\infty$. *Then*

$$\begin{aligned}\big((2g-1)(2g-2)+(4g-3)||\mathbf{b}||+||\mathbf{b}||^2\big)V_g(\mathbf{b}) = 5\sum_{\mathbf{L}+\mathbf{L}'=\mathbf{b}}\binom{\mathbf{b}}{\mathbf{L}}V_{g,1}(\mathbf{L}+\boldsymbol{\delta}_{|\mathbf{L}'|+1})\\ -\frac{1}{6}\sum_{\mathbf{L}+\mathbf{L}'=\mathbf{b}}\binom{\mathbf{b}}{\mathbf{L}}V_{g-1,3}(\mathbf{L}+\boldsymbol{\delta}_{|\mathbf{L}'|}) - \sum_{\mathbf{L}+\mathbf{e}+\mathbf{f}=\mathbf{b}}\binom{\mathbf{b}}{\mathbf{L},\mathbf{e},\mathbf{f}}V_{g',1}(\mathbf{e}+\boldsymbol{\delta}_{|\mathbf{L}|})V_{g-g',2}(\kappa(\mathbf{f}))\\ -(2g-1+||\mathbf{b}||)\sum_{\substack{\mathbf{L}+\mathbf{L}'=\mathbf{b}\\ ||\mathbf{L}'||\geq 2}}\binom{\mathbf{b}}{\mathbf{L}}V_g(\mathbf{L}+\boldsymbol{\delta}_{|\mathbf{L}'|})\\ -\sum_{\substack{\mathbf{L}+\mathbf{L}'=\mathbf{b}\\ ||\mathbf{L}'||\geq 2}}\binom{\mathbf{b}}{\mathbf{L}}\sum_{\mathbf{e}+\mathbf{f}=\mathbf{L}+\boldsymbol{\delta}_{|\mathbf{L}'|}}\binom{\mathbf{L}+\boldsymbol{\delta}_{|\mathbf{L}'|}}{\mathbf{e}}V_g(\mathbf{e}+\boldsymbol{\delta}_{|\mathbf{f}|}).\end{aligned}$$

By induction on $||\mathbf{b}||$, Proposition 5.3 reduces the computation of $V_g(\mathbf{b})$ to the cases of $V_{g,n}(\mathbf{b})$ for $n \geq 1$, which have been computed by Proposition 5.2. Therefore Propositions 5.2 and 5.3 completely determine higher Weil-Petersson volumes of moduli spaces of curves.

The virtue of the above recursion formulae is that they do not involve ψ classes. So if one is only interested in computing higher Weil-Petersson volumes, the above recursions are more efficient both in speed and space, especially when we utilize "option remember" in Maple procedures.

On the other hand, we know that intersection numbers of mixed ψ and κ classes can be expressed by integrals of pure κ classes using the following well-known formula repeatedly.

We give a proof here, since the same argument is also used in the proof of Propositions 5.2 and 5.3.

Proposition 5.4. *Let $d_n \geq 1$. Then*

$$\langle \tau_{d_1} \cdots \tau_{d_n} \kappa(\mathbf{b}) \rangle_g = \sum_{\mathbf{L}+\mathbf{L}'=\mathbf{b}} \binom{\mathbf{b}}{\mathbf{L}} \langle \tau_{d_1} \cdots \tau_{d_{n-1}} \kappa(\mathbf{L}') \kappa_{|\mathbf{L}|+d_n-1} \rangle_g.$$

Proof. We need some results from [2]. Let $\pi_n : \overline{\mathcal{M}}_{g,n} \longrightarrow \overline{\mathcal{M}}_{g,n-1}$ be the morphism that forgets the last marked point. Then

i) $\pi_{n*}(\psi_1^{d_1} \cdots \psi_{n-1}^{d_{n-1}} \psi_n^{d_n}) = \psi_1^{d_1} \cdots \psi_{n-1}^{d_{n-1}} \kappa_{d_n-1}$ for $d_n \geq 1$,
ii) $\kappa_a = \pi_n^*(\kappa_a) + \psi_n^a$ on $\overline{\mathcal{M}}_{g,n}$.

So if $d_n > 0$, by the projection formula we have

$$\begin{aligned} \langle \tau_{d_1} \cdots \tau_{d_n} \kappa(\mathbf{b}) \rangle_g &= \int_{\overline{\mathcal{M}}_{g,n-1}} \pi_{n*} \left(\psi_1^{d_1} \cdots \psi_n^{d_n} \prod_{i\geq 1} (\pi_n^* \kappa_i + \psi_n^i)^{b(i)} \right) \\ &= \sum_{\mathbf{L}+\mathbf{L}'=\mathbf{b}} \binom{\mathbf{b}}{\mathbf{L}} \langle \tau_{d_1} \cdots \tau_{d_{n-1}} \kappa(\mathbf{L}') \kappa_{|\mathbf{L}|+d_n-1} \rangle_g \end{aligned}$$

□

6. Faber's conjecture on tautological rings

Denote by $\mathcal{M}_g$ the moduli space of Riemann surfaces of genus $g \geq 2$. The tautological ring $\mathcal{R}^*(\mathcal{M}_g)$ is defined to be the $\mathbb{Q}$-subalgebra of the Chow ring $\mathcal{A}^*(\mathcal{M}_g)$ generated by the tautological classes κ_i and λ_i.

Proposition 6.1. *$\mathcal{R}^*(\mathcal{M}_g)$ has the following properties:*

i) (Mumford [68]) $\mathcal{R}^*(\mathcal{M}_g)$ *is in fact generated by the $g-2$ classes $\kappa_1, \ldots, \kappa_{g-2}$;*
ii) (Looijenga [51]) $\mathcal{R}^j(\mathcal{M}_g) = 0$ *for $j > g-2$ and* $\dim \mathcal{R}^{g-2}(\mathcal{M}_g) \leq 1$ (*Faber* [15] *showed that actually* $\dim \mathcal{R}^{g-2}(\mathcal{M}_g) = 1$).

Around 1993, Faber [14] proposed a series of remarkable conjectures about the structure of the tautological ring $\mathcal{R}^*(\mathcal{M}_g)$. Faber's conjectures have aroused a lot of interest and motivated tremendous progress toward understanding the topology of the moduli space of curves.

Faber's conjecture is mentioned as a fundamental question in monographs such as [33] (pp. 68-70) and [22] (pp. 148-155).

Roughly speaking, Faber's conjecture asserts that "$\mathcal{R}^*(\mathcal{M}_g)$ behaves like the cohomology ring of a $(g-2)$-dimensional complex projective manifold." We now state it precisely.

i) (Perfect pairing conjecture) When an isomorphism $\mathcal{R}^{g-2}(\mathcal{M}_g) \cong \mathbb{Q}$ is fixed, the following natural pairing is perfect

$$\mathcal{R}^k(\mathcal{M}_g) \times \mathcal{R}^{g-2-k}(\mathcal{M}_g) \longrightarrow \mathcal{R}^{g-2}(\mathcal{M}_g) = \mathbb{Q}; \tag{19}$$

Faber's perfect pairing conjecture is still open to this day.

ii) The $[g/3]$ classes $\kappa_1, \ldots, \kappa_{[g/3]}$ generate the ring, with no relations in degrees $\leq [g/3]$;

The part (ii) of Faber's conjecture has been proved by Morita [66] and Ionel [34].

Another important part of Faber's conjecture is the intersection number conjecture, which we will discuss in some detail.

6.1. **The Faber intersection number conjecture.** Faber predicted the top intersections as the following relations in $\mathcal{R}^{g-2}(\mathcal{M}_g)$,

$$\pi_*(\psi_1^{d_1+1}\cdots\psi_n^{d_n+1}) = \frac{(2g-3+n)!(2g-1)!!}{(2g-1)!\prod_{j=1}^n(2d_j+1)!!}\kappa_{g-2}, \quad \text{for } \sum_{j=1}^n d_j = g-2, \tag{20}$$

where $\pi : \overline{\mathcal{M}}_{g,n} \longrightarrow \overline{\mathcal{M}}_g$ is the forgetful morphism.

Thus the Faber intersection number conjecture determines the ring structure of $\mathcal{R}^*(\mathcal{M}_g)$ if Faber's perfect pairing conjecture is true.

Since $\lambda_g\lambda_{g-1}$ vanishes on the boundary of $\overline{\mathcal{M}}_g$, the Faber intersection number conjecture is equivalent to

$$\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1}\cdots\psi_n^{d_n}\lambda_g\lambda_{g-1} = \frac{(2g-3+n)!|B_{2g}|}{2^{2g-1}(2g)!\prod_{j=1}^n(2d_j-1)!!}. \tag{21}$$

By Mumford's formula [68] for the Chern character of the Hodge bundle, the above identity is equivalent to

$$\begin{aligned}\frac{(2g-3+n)!}{2^{2g-1}(2g-1)!\prod_{j=1}^n(2d_j-1)!!} =& \langle\tau_{2g}\prod_{j=1}^n\tau_{d_j}\rangle_g - \sum_{j=1}^n\langle\tau_{d_j+2g-1}\prod_{i\neq j}\tau_{d_i}\rangle_g\\ &+\frac{1}{2}\sum_{j=0}^{2g-2}(-1)^j\langle\tau_{2g-2-j}\tau_j\prod_{i=1}^n\tau_{d_i}\rangle_{g-1}\\ &+\frac{1}{2}\sum_{\underline{n}=I\coprod J}\sum_{j=0}^{2g-2}(-1)^j\langle\tau_j\prod_{i\in I}\tau_{d_i}\rangle_{g'}\langle\tau_{2g-2-j}\prod_{i\in J}\tau_{d_i}\rangle_{g-g'},\end{aligned} \tag{22}$$

where $d_j \geq 1$, $\sum_{j=1}^n d_j = g+n-2$.

The identity (21) was shown to follow from the degree 0 Virasoro conjecture for $\mathbb{P}^2$ by Getzler and Pandharipande [26]. Givental [27] has announced a proof of the Virasoro conjecture for $\mathbb{P}^n$. Y.-P. Lee and R. Pandharipande are writing a book [52] giving details. Recently Teleman [73] announced a proof of the Virasoro conjecture for all manifolds with semi-simple quantum cohomology. His argument depends crucially on the Mumford conjecture about the stable rational cohomology rings of the moduli spaces proved by Madsen and Weiss.

However, the Virasoro conjecture is a huge machinery and conceals the combinatorial structure of intersection numbers. The proof of the Mumford conjecture is also highly nontrivial. So a more direct proof of the Faber intersection number conjecture is very much desired.

Goulden, Jackson and Vakil [30] recently give an enlightening proof of the identity (20) for up to three points. Their remarkable proof relied on relative virtual localization in Gromov-Witten theory and some tour de force combinatorial computations.

6.2. **Relations with n-point functions.** Now we describe our approach to proving the identity (22); the details are in [58].

Since the one- and two-point functions in genus 0 are

$$F_0(x) = \frac{1}{x^2}, \qquad F_0(x,y) = \frac{1}{x+y} = \sum_{k=0}^\infty(-1)^k\frac{x^k}{y^{k+1}},$$

it is consistent to define the virtual intersection numbers

$$\langle\tau_{-2}\rangle_0 = 1, \qquad \langle\tau_k\tau_{-1-k}\rangle_0 = (-1)^k,\ k\geq 0.$$

For $a,b\in\mathbb{Z}$, we introduce the following notation:

$$L_g^{a,b}(y,x_1\dots,x_n) \triangleq \sum_{g'=0}^{g}\sum_{\underline{n}=I\coprod J}(y+\sum_{i\in I}x_i)^a(-y+\sum_{i\in J}x_i)^b F_{g'}(y,x_I)F_{g-g'}(-y,x_J),$$

We regard $L_g^{a,b}(y,x_1\dots,x_n)$ as a formal series in $\mathbb{Q}[x_1,\dots,x_n][[y,y^{-1}]]$ with $\deg y<\infty$.

$$\begin{aligned}\frac{1}{2}\sum_{\underline{n}=I\coprod J}\sum_{j=0}^{2g-2}(-1)^j\langle\tau_j\prod_{i\in I}\tau_{d_i}\rangle_{g'}\langle\tau_{2g-2-j}\prod_{i\in J}\tau_{d_i}\rangle_{g-g'}+\langle\prod_{j=1}^{n}\tau_{d_j}\tau_{2g}\rangle_g-\sum_{j=1}^{n}\langle\tau_{d_j+2g-1}\prod_{i\neq j}\tau_{d_i}\rangle_g\\
=\frac{1}{2}\sum_{\underline{n}=I\coprod J}\sum_{j\in\mathbb{Z}}(-1)^j\langle\tau_j\prod_{i\in I}\tau_{d_i}\rangle_{g'}\langle\tau_{2g-2-j}\prod_{i\in J}\tau_{d_i}\rangle_{g-g'}\\
=\left[\sum_{g'=0}^{g}\sum_{\underline{n}=I\coprod J}F_{g'}(y,x_I)F_{g-g'}(-y,x_J)\right]_{y^{2g-2}\prod_{i=1}^n x_i^{d_i}}\\
=\left[L_g^{0,0}(y,x_1,\dots,x_n)\right]_{y^{2g-2}\prod_{i=1}^n x_i^{d_i}}.\end{aligned}$$

The right-hand side of (22) may be written as the coefficients of the n-point functions.

$$[F_{g-1}(y,-y,x_1,\dots,x_n)]_{y^{2g-2}}+\left[L_g^{0,0}(y,x_1,\dots,x_n)\right]_{y^{2g-2}\prod_{i=1}^n x_i^{d_i}}. \tag{23}$$

So in order to prove the Faber intersection number conjecture, it is sufficient to prove the following results.

Proposition 6.2. *We have*

i)

$$\left[L_g^{0,0}(y,x_1\dots,x_n)\right]_{y^{2g-2}}=0;$$

ii) *For $d_j\geq 1$ and $\sum_{j=1}^n d_j=g+n$,*

$$\left[L_g^{2,2}(y,x_1\dots,x_n)\right]_{y^{2g}\prod_{j=1}^n x_j^{d_j}}=\frac{(2g+n+1)!}{4^g(2g+1)!\prod_{j=1}^n(2d_j-1)!!}.$$

Next we apply Proposition 3.3(ii) to prove a strengthened version of the above proposition inductively.

7. Dimension of tautological rings

If $\mathbf{m}\in N^\infty$ and $|\mathbf{m}|=g-2$, then, from Faber's intersection number identity (20) and equation (11), we have in $\mathcal{R}^{g-2}(\mathcal{M}_g)$

$$\kappa(\mathbf{m})=\sum_{r=1}^{||\mathbf{m}||}\frac{(-1)^{||\mathbf{m}||-r}}{r!}\sum_{\substack{\mathbf{m}=\mathbf{m_1}+\dots+\mathbf{m_r}\\ \mathbf{m_i}\neq\mathbf{0}}}\binom{\mathbf{m}}{\mathbf{m_1},\dots,\mathbf{m_r}}\frac{(2g-3+r)!\kappa_{g-2}}{(2g-2)!!\prod_{j=1}^r(2|\mathbf{m}_j|+1)!!}.$$

Let $0 \leq k \leq g-2$ and denote by $p(k)$ the partition number of k. Define a matrix V_g^k of size $p(k) \times p(g-2-k)$ with entries

$$(V_g^k)_{\mathbf{L},\mathbf{L}'} = \sum_{r=1}^{||\mathbf{L}+\mathbf{L}'||} \frac{(-1)^{||\mathbf{L}+\mathbf{L}'||-r}}{r!} \sum_{\substack{\mathbf{L}+\mathbf{L}'=\mathbf{m_1}+\cdots+\mathbf{m_r}\\ \mathbf{m_i}\neq\mathbf{0}}} \binom{\mathbf{L}+\mathbf{L}'}{\mathbf{m_1},\ldots,\mathbf{m_r}} \frac{(2g-3+r)!}{\prod_{j=1}^r (2|\mathbf{m}_j|+1)!!},$$

where $\mathbf{L},\mathbf{L}' \in N^\infty$ and $|\mathbf{L}| = k$, $|\mathbf{L}'| = g-2-k$.

We call V_g^k the Faber intersection matrix. Instead of using the above closed formula directly, we [60] have some recursive ways to compute entries of V_g^k. As a result, we have computed V_g^k for all $g \leq 36$. We are interested in the rank of the Faber intersection matrix, so we introduce the notation

$$R_g^k := \operatorname{rank} V_g^k, \qquad R_g := \sum_{k=0}^{g-2} R_g^k.$$

Obviously we have $R_g^k = R_g^{g-2-k}$, $0 \leq k \leq g-2$. If Faber's perfect pairing conjecture (19) is true, then we have for $0 \leq k \leq g-2$,

$$R_g^k = \dim(R^k(\mathcal{M}_g)),$$
$$R_g = \dim \mathcal{R}^*(\mathcal{M}_g).$$

Since Faber has verified his conjecture for all $g \leq 23$, the above relations hold at least when $g \leq 23$. Like the importance of cohomology, the dimensions of tautological rings are important invariants of moduli spaces of curves.

7.1. **Ramanujan's mock theta functions.** First we recall the standard notation of basic hypergeometric series.

$$(a)_n = (a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1}), \qquad n \geq 1$$
$$(a)_0 = 1, \qquad (a)_\infty = (a;q)_\infty = \prod_{n=0}^\infty (1-aq^n).$$

Recall Ramanujan's third order mock theta function $\omega(q)$

$$\begin{aligned}(24)\quad \omega(q) = \sum_{n=0}^\infty \omega(n) q^n &:= \sum_{n=0}^\infty \frac{q^{2n^2+2n}}{\prod_{j=0}^n (1-q^{2j+1})^2}\\ &= 1+2q+3q^2+4q^3+6q^4+8q^5+10q^6+14q^7+18q^8+22q^9\\ &\quad +29q^{10}+36q^{11}+44q^{12}+56q^{13}+68q^{14}+82q^{15}+\cdots\end{aligned}$$

In 1966, Andrews proved asymptotic formulae for $\omega(n)$ and another third order mock theta function $f(q)$,

$$\begin{aligned}(25)\quad f(q) = \sum_{n=0}^\infty f(n) q^n &:= \sum_{n=0}^\infty \frac{q^{n^2}}{\prod_{j=1}^n (1+q^j)^2}\\ &= 1+q-2q^2+3q^3-3q^4+3q^5-5q^6+7q^7-6q^8+6q^9+\cdots\end{aligned}$$

In 2003, Andrews [1] improved his asymptotic formulae for $f(n)$ to a conjectural exact formula. Around the same time, Zwegers [82] found a relationship between mock theta functions and vector-valued modular forms. Andrews' conjectural exact formula for $f(n)$ was proved by Bringmann and Ono [7] using Zwegers' results, along with the theory of Maass forms and Poincaré series. Recently, Garthwaite [21] proved an analogue of Andrews' exact formula for $\omega(n)$ following the method of Bringmann and Ono.

We first introduce some notation. Let $e(x) := e^{2\pi i x}$. If $k \geq 1$ and n are integers, define

$$A_k(n) := \frac{1}{2}\sqrt{\frac{k}{12}} \sum_{\substack{x \pmod{24k} \\ x^2 \equiv -24n+1 \pmod{24k}}} \chi_{12}(x) \cdot e\left(\frac{x}{12k}\right), \tag{26}$$

where the sum runs over the residue classes modulo $24k$, and where we use the Kronecker symbol

$$\chi_{12}(x) := \left(\frac{12}{x}\right) = \begin{cases} 0, & x \text{ is not coprime to } 6, \\ 1, & x \equiv \pm 1 \mod 12, \\ -1, & x \equiv \pm 5 \mod 12. \end{cases}$$

Let $I_{1/2}$ be the I-Bessel function

$$I_{1/2}(z) = \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} \sinh z.$$

Now we can state Garthwaite's formula [21]:

$$\omega(n) = \frac{\pi}{2\sqrt{2}}(3n+2)^{-1/4} \sum_{k=1}^{\infty} \frac{(-1)^{k-1} A_{2k-1}\left(nk - \frac{3k(k-1)}{2}\right)}{2k-1} \cdot I_{1/2}\left(\frac{\pi\sqrt{3n+2}}{6k-3}\right). \tag{27}$$

Actually this series approaches the exact value very rapidly and can be effectively used to compute exact values of $\omega(n)$.

Lemma 7.1. *When n is odd, $\omega(n)$ is even.*

Proof. First we note that

$$\frac{1}{(1-q^{2j+1})^2} = \sum_{k=0}^{\infty} (k+1) q^{(2j+1)k}.$$

The lemma follows easily from (24). □

Compared with its definition (24), $\omega(q)$ has a simpler expression. The following lemma is well-known to experts, but we include a proof here for the reader's convenience.

Lemma 7.2.

$$\sum_{n=0}^{\infty} \omega(n) q^n = \sum_{n=0}^{\infty} \frac{q^n}{\prod_{j=0}^{n}(1-q^{2j+1})}.$$

Proof. Consider the following q-series,

$$F(t) = \sum_{n=0}^{\infty} \frac{t^n}{(tq)_n},$$

where $|q| < 1$. Since $(tq)_n = (1 - tq^n)(tq)_{n-1}$, we have

$$(1-t)F(t) = 1 + \sum_{n=1}^{\infty} \left(\frac{1}{(tq)_n} - \frac{1}{(tq)_{n-1}}\right) t^n$$

$$
\begin{aligned}
&= 1 + \sum_{n=1}^{\infty} \left(\frac{1}{(tq)_n} - \frac{(1 - tq^n)}{(tq)_n} \right) t^n \\
&= 1 + t \sum_{n=1}^{\infty} \frac{(tq)^n}{(tq)_n} \\
&= 1 + \frac{t^2 q}{1 - tq} F(tq).
\end{aligned}
$$

In the last equation, we used $(tq)_n = (1 - tq)(tq^2)_{n-1}$.

Let $R(t) = (1 - t)F(t)$. Then

$$R(t) = 1 + \frac{t^2 q}{(1 - tq)(1 - tq)} R(tq)$$

By iteration, we have

$$R(t) = 1 + \sum_{n=0}^{r-1} \frac{t^{2n+2} q^{(n+1)^2}}{(tq)_{n+1}^2} + \prod_{n=0}^{r} \frac{t^2 q^{2n+1}}{(1 - tq^{n+1})^2} R(tq^{r+1})$$

Letting $r \to \infty$, we get

$$(1 - t)F(t) = \sum_{n=0}^{\infty} \frac{t^{2n} q^{n^2}}{(tq)_n^2}$$

Substituting q by q^2, t by q, we get the desired equation,

$$\sum_{n=0}^{\infty} \frac{q^n}{(1-q)(1-q^3)\cdots(1-q^{2n+1})} = \sum_{n=0}^{\infty} \frac{q^{2n^2+2n}}{(1-q)^2(1-q^3)^2\cdots(1-q^{2n+1})^2}$$

$\square$

7.2. **Asymptotics of tautological dimensions.** With the help of the website "The On-Line Encyclopedia of Integer Sequences", we discovered the surprising coincidence that $R_g = \omega(g-2)$ for $2 \le g \le 17$. However, when $g > 17$, this is no longer true. Let us use the notation $\omega_g := \omega(g-2)$.

g	18	19	20	21	22	23	24	25	26	27	28	29	30
ω_g	101	122	146	176	210	248	296	350	410	484	566	660	772
R_g	102	122	146	178	211	250	300	352	415	492	574	670	788

In general, we have the following conjecture.

Conjecture 7.3. *For all $g \ge 2$, we have*

$$R_g \ge \omega_g \tag{28}$$

and there exists some constant $C > 0$, such that

$$\lim_{g \to \infty} \frac{\omega_g}{R_g} = C.$$

Faber's computation reveals that there should exist a uniquely determined integer sequence $a(n)$ with $a(n) = 0$ for $n \le 0$, such that

$$\dim \mathcal{R}^k(\mathcal{M}_g) = \begin{cases} p(k) - a(3k - g), & 0 \le k \le \frac{g-2}{2}, \\ \dim \mathcal{R}^{g-2-k}(\mathcal{M}_g), & \frac{g-2}{2} < k \le g - 2. \end{cases}$$

Thanks to Faber's verification, we know that $R_g^k = \dim \mathcal{R}^{g-2-k}(\mathcal{M}_g)$ for at least $g \leq 23$. Faber computed $a(n)$ for $n \leq 10$, we extend this to $n \leq 15$. Of course here we need to assume that Faber's perfect pairing conjecture continues to hold in higher genera.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$a(n)$	1	1	2	3	5	6	10	13	18	24	33	41	56	71	91

Motivated by Faber's calculation, we extracted two integer sequences from the mock theta function $\omega(q)$, which may be of independent interest.

Proposition 7.4. *There exist two integer sequences $p_\omega(n), a_\omega(n)$ with $a_\omega(n) = 0$ for $n \leq 0$ such that for all $g \geq 2$, if we define*

$$\omega_g^k := \begin{cases} p_\omega(k) - a_\omega(3k-g)), & 0 \leq k \leq \frac{g-2}{2}, \\ \omega_g^{g-2-k}, & \frac{g-2}{2} < k \leq g-2, \end{cases}$$

then $\omega_g = \sum_{k=0}^{g-2} \omega_g^k$. Moreover, $p_\omega(n)$ and $a_\omega(n)$ are uniquely determined.

Proof. Consider ω_{2m}^{m-1} and ω_{2m+1}^{m-1}; then we have

$$\begin{aligned} p_\omega(m-1) - a_\omega(m-3) = \omega_{2m}^{m-1} &= \omega_{2m} - 2\sum_{i=0}^{m-2} \omega_{2m}^i \\ &= \omega_{2m} - 2\sum_{i=0}^{m-2} (p_\omega(i) - a_\omega(3i-2m)), \end{aligned}$$

$$\begin{aligned} p_\omega(m-1) - a_\omega(m-4) = \omega_{2m+1}^{m-1} &= \frac{\omega_{2m+1} - 2\sum_{i=0}^{m-2} \omega_{2m+1}^i}{2} \\ &= \frac{\omega_{2m+1}}{2} - \sum_{i=0}^{m-2} (p_\omega(i) - a_\omega(3i-2m-1)). \end{aligned}$$

It is not difficult to see that p_ω and a_ω are uniquely determined recursively by the above two identities. The integrality of $\omega_g^k, p_\omega, a_\omega$ is guaranteed by Lemma 7.1 that ω_{2m+1} is even. $\square$

Let $p(n)$ denote the partition number of n. We may compare $p_\omega(n)$ and $p(n)$ in the following tables.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
$p(n)$	1	1	2	3	5	7	11	15	22	30	42	56	77	101	135	176	231	297
$p_\omega(n)$	1	1	2	3	5	7	11	15	22	30	41	56	75	100	132	172	225	289
$a_\omega(n)$	0	1	1	2	3	5	7	10	13	18	25	34	44	58	74	97	125	160

Comparing the first few values of $a_\omega(n)$ and $a(n)$ leads us to guess that $a_\omega(n) \geq a(n)$ may always hold.

n	50	100	300	500	800
$p_\omega(n)$	191817	176074114	$\approx 8 \times 10^{15}$	$\approx 2 \times 10^{21}$	$\approx 5 \times 10^{27}$
$p_\omega(n)/p(n)$	0.9392	0.9239	0.9074	0.9021	0.8983

Table 1. Rank of Faber's intersection matrix

g	R_g^0	R_g^1	R_g^2	R_g^3	R_g^4	R_g^5	R_g^6	R_g^7	R_g^8	R_g^9	R_g^{10}
2	1										
3	1	1									
4	1	1	1								
5	1	1	1	1							
6	1	1	2	1	1						
7	1	1	2	2	1	1					
8	1	1	2	2	2	1	1				
9	1	1	2	3	3	2	1	1			
10	1	1	2	3	4	3	2	1	1		
11	1	1	2	3	4	4	3	2	1	1	
12	1	1	2	3	5	5	5	3	2	1	1
13	1	1	2	3	5	6	6	5	3	2	1
14	1	1	2	3	5	6	8	6	5	3	2
15	1	1	2	3	5	7	9	9	7	5	3
16	1	1	2	3	5	7	10	10	10	7	5
17	1	1	2	3	5	7	10	12	12	10	7
18	1	1	2	3	5	7	11	13	**16**	13	11
19	1	1	2	3	5	7	11	14	17	17	14
20	1	1	2	3	5	7	11	14	19	20	19
21	1	1	2	3	5	7	11	15	20	**24**	**24**
22	1	1	2	3	5	7	11	15	21	25	**29**
23	1	1	2	3	5	7	11	15	21	27	**32**

Table 2. Decomposition of ω_g

g	ω_g^0	ω_g^1	ω_g^2	ω_g^3	ω_g^4	ω_g^5	ω_g^6	ω_g^7	ω_g^8	ω_g^9	ω_g^{10}
18	1	1	2	3	5	7	11	13	**15**	13	11
19	1	1	2	3	5	7	11	14	17	17	14
20	1	1	2	3	5	7	11	14	19	20	19
21	1	1	2	3	5	7	11	15	20	**23**	**23**
22	1	1	2	3	5	7	11	15	21	25	**28**
23	1	1	2	3	5	7	11	15	21	27	**31**

Conjecture 7.5. *For all $n \geq 0$, we have*

$$p(n) \geq p_\omega(n), \qquad a(n) \leq a_\omega(n) \tag{29}$$

and there exists some constant $C > 0$, such that

$$\lim_{n\to\infty} \frac{p_\omega(n)}{p(n)} = C.$$

Note that the two inequalities (29) together imply (28). We may also strengthen the inequality (28) in Conjecture 7.3 as follows.

Conjecture 7.6. *Let $g \geq 2$ and $0 \leq k \leq g-2$. Then we have*

$$R_g^k \geq \omega_g^k.$$

We have checked Conjecture 7.5 when $g \leq 36$. Values of R_g^k and ω_g^k for $g \leq 23$ are listed in Table 1 and Table 2 respectively; note that $R_g^k = \omega_g^k$ when $g \leq 17$. We use bold numbers whenever the corresponding values are different.

8. Gromov-Witten invariants

Gromov-Witten theory is a generalization of the intersection theory of moduli spaces of curves. In fact, the intersection theory of $\overline{\mathcal{M}}_{g,n}$ corresponds to the Gromov-Witten theory of a point. A very readable exposition of Gromov-Witten invariants can be found in [25].

Let X be a smooth projective variety and $\overline{\mathcal{M}}_{g,n}(X,\beta)$ denote the moduli stack of stable maps of genus g and degree $\beta \in H_2(X,\mathbb{Z})$ with n marked points. There are several canonical morphisms:

i) Let ev be the evaluation maps at the marked points:

$$\begin{gathered} ev : \overline{\mathcal{M}}_{g,n}(X,\beta) \to X^n \\ (f : C \to X, x_1, \dots, x_n) \mapsto \big(f(x_1), \dots, f(x_n)\big) \in X^n. \end{gathered} \tag{30}$$

ii) Let st be the forgetful map to the domain curve followed by stabilization:

$$st : \overline{\mathcal{M}}_{g,n+1}(X,\beta) \to \overline{\mathcal{M}}_{g,n}. \tag{31}$$

iii) Let π be the map of forgetting the last marked point x_{n+1} and stabilizing the resulting domain curve:

$$\pi : \overline{\mathcal{M}}_{g,n+1}(X,\beta) \to \overline{\mathcal{M}}_{g,n}(X,\beta) \tag{32}$$

The forgetful morphism π has n canonical sections

$$\sigma_i : \overline{\mathcal{M}}_{g,n}(X,\beta) \to \overline{\mathcal{M}}_{g,n+1}(X,\beta),$$

corresponding to the n marked points. Let

$$\omega = \omega_{\overline{\mathcal{M}}_{g,n+1}(X,\beta)/\overline{\mathcal{M}}_{g,n}(X,\beta)}$$

be the relative dualizing sheaf and Ψ_i the cohomology class $c_1(\sigma_i^*\omega)$.

If $\gamma_1, \dots, \gamma_n \in H^*(X,\mathbb{Q})$, the Gromov-Witten invariants are defined by

$$\langle \tau_{d_1}(\gamma_1) \dots \tau_{d_n}(\gamma_n) \rangle_{g,\beta}^X = \int_{[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{\mathrm{virt}}} \Psi_1^{d_1} \cdots \Psi_n^{d_n} \cup \mathrm{ev}^*(\gamma_1 \boxtimes \cdots \boxtimes \gamma_n).$$

We also denote the insertion $\tau_k(\gamma_a)$ by $\tau_{k,a}$.

Given a basis $\{\gamma_a\}$ for $H^*(X,\mathbb{Q})$, we may use $g_{ab} = \int_X \gamma_a \cup \gamma_b$ and its inverse g^{ab} to lower and raise indices. We denote $\gamma^a = g^{ab}\gamma_b$ and apply the Einstein summation convention.

The genus g Gromov-Witten potential of X is defined by

$$\langle\langle \tau_{d_1}(\gamma_1)\cdots\tau_{d_n}(\gamma_n)\rangle\rangle_g = \sum_\beta \left\langle \tau_{d_1}(\gamma_1)\cdots\tau_{d_n}(\gamma_n)\exp\left(\sum_{m,a} t^a_m \tau_m(\gamma_a)\right)\right\rangle^X_{g,\beta} q^\beta.$$

8.1. **Universal equations of Gromov-Witten invariants.** There are some universal equations satisfied by Gromov-Witten invariants. Universal means that they do not depend on the target manifolds.

We may pull back tautological relations on $\overline{\mathcal{M}}_{g,n}$ via the map st in (31) to get universal equations for Gromov-Witten invariants by the splitting axiom and cotangent line comparison equations [44].

From the simple fact that the three boundary divisors of $\overline{\mathcal{M}}_{0,4} \cong \mathbb{P}^1$ are equal, we get the WDVV equation:

$$\langle\langle \tau_{k_1,a_1}\tau_{k_2,a_2}\gamma_\alpha\rangle\rangle_0 \langle\langle \gamma^\alpha \tau_{k_3,a_3}\tau_{k_4,a_4}\rangle\rangle_0 = \langle\langle \tau_{k_1,a_1}\tau_{k_3,a_3}\gamma_\alpha\rangle\rangle_0 \langle\langle \gamma^\alpha \tau_{k_2,a_2}\tau_{k_4,a_4}\rangle\rangle_0, \tag{33}$$

which is the associativity condition of the quantum cohomology ring.

On $\overline{\mathcal{M}}_{1,1}$, we have $\psi_1 = \frac{1}{12}\delta$, which implies the genus one topological recursion relation

$$\langle\langle \tau_k(x)\rangle\rangle_1 = \langle\langle \tau_{k-1}(x)\gamma_\alpha\rangle\rangle_0 \langle\langle \gamma^\alpha\rangle\rangle_1 + \frac{1}{24}\langle\langle \tau_{k-1}(x)\gamma_\alpha\gamma^\alpha\rangle\rangle_0.$$

Other known topological recursion relations in genus $g \leq 3$ are given in [3, 6, 23, 24, 40].

Xiaobo Liu introduced the "T operator" to facilitate the transformation from topological relations on moduli spaces of curves to universal equations of Gromov-Witten invariants by the splitting axiom. The interested reader may consult [47] for a discussion on relations among known universal equations.

Using the WDVV equation, Kontsevich derived a recursion formula for the number N_d of degree d plane rational curves passing through $3d-1$ general points, illustrating the power of Gromov-Witten theory in classical enumerative geomtry.

Inspired by Givental's axiomatic Gromov-Witten theory, Y.-P. Lee [45, 46] had proposed an algorithm that, conjecturally, computes all tautological equations on $\overline{\mathcal{M}}_{g,n}$ using only linear algebra. Faber, Shadrin and Zvonkine [18] proved that Y.-P. Lee's algorithm is correct if and only if the Gorenstein conjecture on the tautological cohomology ring of $\overline{\mathcal{M}}_{g,n}$ is true.

There are also universal equations which do not come from the tautological ring of moduli space of curves. For example, we have the so-called string equation, dilaton equation and divisor equation, respectively, in the following.

$$\langle \tau_{0,0}\tau_{k_1,a_1}\cdots\tau_{k_n,a_n}\rangle^X_{g,\beta} = \sum_{i=1}^n \langle \tau_{k_1,a_1}\cdots\tau_{k_i-1,a_i}\cdots\tau_{k_n,a_n}\rangle^X_{g,\beta} \tag{34}$$

$$\langle \tau_{1,0}\tau_{k_1,a_1}\cdots\tau_{k_n,a_n}\rangle^X_{g,\beta} = (2g-2+n)\langle \tau_{k_1,\alpha_1}\cdots\tau_{k_n,\alpha_n}\rangle^X_{g,\beta} \tag{35}$$

$$\begin{aligned}\langle \tau_0(\omega)\tau_{k_1,a_1}\cdots\tau_{k_n,a_n}\rangle^X_{g,\beta} =& \big(\omega\cap\beta\big)\langle \tau_{k_1,a_1}\cdots\tau_{k_n,a_n}\rangle^X_{g,\beta} \\ &+ \sum_{i=1}^n \langle \tau_{k_1,a_1}\cdots\tau_{k_i-1}(\omega\cup\gamma_a)\cdots\tau_{k_n,a_n}\rangle^X_{g,\beta},\end{aligned} \tag{36}$$

where $\omega \in H^2(X,\mathbb{Q})$.

8.2. **Some vanishing identities.** We adopt Gathmann's convention [20] in the following to simplify notation, namely we define

$$\langle \tau_{-2}(pt) \rangle_{0,0}^X = 1,$$

$$\langle \tau_m(\gamma_1)\tau_{-1-m}(\gamma_2) \rangle_{0,0}^X = (-1)^{\max(m,-1-m)} \int_X \gamma_1 \cdot \gamma_2, \quad m \in \mathbb{Z}.$$

All other Gromov-Witten invariants that contain a negative power of a cotangent line are defined to be zero.

Motivated by our work on intersection numbers on moduli spaces of curves, we [58] conjectured the following universal equations for Gromov-Witten invariants valid in all genera.

Conjecture 8.1. [58] *Let $x_i, y_i \in H^*(X)$ and $k \geq 2g - 3 + r + s$. Then*

$$\sum_{g'=0}^{g} \sum_{j \in \mathbb{Z}} (-1)^j \langle\langle \tau_j(\gamma_a) \prod_{i=1}^{r} \tau_{p_i}(x_i) \rangle\rangle_{g'} \langle\langle \tau_{k-j}(\gamma^a) \prod_{i=1}^{s} \tau_{q_i}(y_i) \rangle\rangle_{g-g'} = 0.$$

Note that j runs over all integers.

Conjecture 8.2. [58] *Let $k > g$. Then*

$$\sum_{j=0}^{2k} (-1)^j \langle\langle \tau_j(T_a)\tau_{2k-j}(T^a) \rangle\rangle_g^X = 0. \tag{37}$$

We also have

$$\frac{1}{2} \sum_{j=0}^{2g-2} (-1)^j \langle\langle \tau_j(T_a)\tau_{2g-2-j}(T^a) \rangle\rangle_{g-1} = \frac{(2g)!}{B_{2g}} \langle\langle \mathrm{ch}_{2g-1}(\mathbb{E}) \rangle\rangle_g. \tag{38}$$

Note that by the Chern character formula of Faber and Pandharipande [16], we have the equivalence

$$\text{Conjecture 8.1 } (r = s = 0) \iff \text{Conjecture 8.2}$$

When X is a point, the above conjectures have been proved in [58] using the recursive formula of n-point functions. Recently, X. Liu and Pandharipande [55, 49] give a complete proof of the above conjectures. Their proof uses virtual localization to get topological recursion relations (TRR) expressing ψ_1^{2g+r} in terms of boundary classes in $A^{2g+r}(\overline{\mathcal{M}}_{g,1})$, which are then pulled back to get the universal equations of Conjecture 8.1.

As we have seen, it is relatively straightforward to go from TRR to universal equations for Gromov-Witten invariants. This is not always easy when an identity of descendent integrals is not a TRR. An example is the Virasoro conjecture [10] for Gromov-Witten invariants, which is a generalization of the DVV formula (9) and was not discovered until 6 years later. In the same line, one would hope to find matrix models or corresponding integrable hierarchies for a general target X. Besides the point case, we know that the Gromov-Witten potential of $X = \mathbb{CP}^1$ is governed by the Toda hierarchy [11, 71].

9. Witten's r-spin numbers

In this section, we present an algorithm for computing Witten's r-spin intersection numbers. First we recall Witten's definition of r-spin numbers [76].

Let Σ be a Riemann surface of genus g with marked points $x_1, x_2, \ldots, x_s$. Fix an integer $r \geq 2$. Label each marked point x_i by an integer m_i, $0 \leq m_i \leq r - 1$. Consider the line bundle $\mathcal{S} = K \otimes (\otimes_{i=1}^{s} \mathcal{O}(x_i)^{-m_i})$ over Σ, where K denotes the canonical line bundle. If $2g-2-\sum_{i=1}^{s} m_i$ is divisible by r, then there are r^{2g} isomorphism classes of line bundles $\mathcal{T}$ such that $\mathcal{T}^{\otimes r} \cong \mathcal{S}$. The choice of an isomorphism class of $\mathcal{T}$ determines a cover $\mathcal{M}_{g,s}^{1/r}$ of $\mathcal{M}_{g,s}$. The compactification of $\mathcal{M}_{g,s}^{1/r}$ is denoted by $\overline{\mathcal{M}}_{g,s}^{1/r}$.

Let $\mathcal{V}$ be a vector bundle over $\overline{\mathcal{M}}_{g,s}^{1/r}$ whose fiber is the dual space to $H^1(\Sigma, \mathcal{T})$. The top Chern class $c_D(\mathcal{V})$ of this bundle has degree $D = (g-1)(r-2)/r + \sum_{i=1}^{s} m_i/r$.

We associate with each marked point x_i an integer $n_i \geq 0$. Witten's r-spin intersection numbers are defined by

$$\langle \tau_{n_1,m_1} \cdots \tau_{n_s,m_s} \rangle_g = \frac{1}{r^g} \int_{\overline{\mathcal{M}}_{g,s}^{1/r}} \prod_{i=1}^{s} \psi(x_i)^{n_i} \cdot c_D(\mathcal{V}),$$

which is non-zero only if

$$(r+1)(2g-2) + rs = r\sum_{j=1}^{s} n_j + \sum_{j=1}^{s} m_j. \tag{39}$$

Consider the differential operator

$$Q = D^r + \sum_{i=0}^{r-2} \gamma_i(x) D^i, \qquad \text{where } D = \frac{\sqrt{-1}}{\sqrt{r}} \frac{\partial}{\partial x}.$$

There exists a unique pseudo-differential operator $Q^{1/r} = D + \sum_{i>0} w_{-i} D^{-i}$.

The Gelfand–Dikii equations read

$$i\frac{\partial Q}{\partial t_{n,m}} = [Q_+^{n+(m+1)/r}, Q] \cdot \frac{c_{n,m}}{\sqrt{r}}, \tag{40}$$

where

$$c_{n,m} = \frac{(-1)^n r^{n+1}}{(m+1)(r+m+1)\cdots(nr+m+1)}.$$

9.1. **Generalized Witten's conjecture.** Consider the formal series F in variables $t_{n,m}$, $n \geq 0$ and $0 \leq m \leq r-1$,

$$F(t_{0,0}, t_{0,1}, \dots) = \sum_{d_{n,m}} \langle \prod_{n,m} \tau_{n,m}^{d_{n,m}} \rangle \prod_{n,m} \frac{t_{n,m}^{d_{n,m}}}{d_{n,m}!}.$$

The conjecture of Witten is that this F is the string solution of the r-Gelfand–Dikii hierarchy, namely

$$\frac{\partial F}{\partial t_{0,0}} = \frac{1}{2} \sum_{i,j=0}^{r-2} \delta_{i+j,r-2} t_{0,i} t_{0,j} + \sum_{n=0}^{\infty} \sum_{m=0}^{r-2} t_{n+1,m} \frac{\partial F}{\partial t_{n,m}}, \tag{41}$$

$$\frac{\partial^2 F}{\partial t_{0,0} \partial t_{n,m}} = -c_{n,m} \operatorname{res}(Q^{n+\frac{m+1}{r}}), \tag{42}$$

where Q satisfies the Gelfand-Dikii equations and $t_{0,0}$ is identified with x.

When $r = 2$, the above assertion is just the Witten-Kontsevich theorem. Witten's r-spin conjecture for any $r \geq 2$ has been proved recently by Faber, Shadrin and Zvonkine [18]. In fact, Witten's r-spin theory corresponds to A_{r-1} singularity. Fan, Javis and Ruan [19] have developed a Gromov-Witten type quantum theory for all non-degenerate quasi-homogeneous singularity and proved the more general ADE-integrable hierarchy conjecture of Witten.

9.2. **An algorithm for computing Witten's r-spin numbers.** We proved a structure theorem about formal pseudo-differential operators in a forthcoming paper [61]. If combined with the generalized Witten conjecture, it can be used to derive the following effective recursion formulae for computing Witten's r-spin numbers.

Theorem 9.1. [61] *For fixed $r \geq 2$, we have*

$$\langle\langle \tau_{1,0}\tau_{0,0}\rangle\rangle_g = \frac{1}{2}\langle\langle \tau_{0,0}\tau_{0,m'}\rangle\rangle_{g'}\eta^{m'm''}\langle\langle \tau_{0,m''}\tau_{0,0}\rangle\rangle_{g-g'} + \mathrm{Low}(r),$$

where $\mathrm{Low}(r)$ *is an explicit sum of products of* $\langle\langle \cdots \rangle\rangle$ *with genera strictly lower than* g.

In particular, when $r = 2$, we have

$$\langle\langle \tau_{1,0}\tau_{0,0}\rangle\rangle_g = \frac{1}{2}\langle\langle \tau_{0,0}^2\rangle\rangle_{g'}\langle\langle \tau_{0,0}^2\rangle\rangle_{g-g'} + \frac{1}{12}\langle\langle \tau_{0,0}^4\rangle\rangle_{g-1},$$

when $r = 3$, we have

$$\langle\langle \tau_{1,0}\tau_{0,0}\rangle\rangle_g = \langle\langle \tau_{0,0}\tau_{0,1}\rangle\rangle_{g'}\langle\langle \tau_{0,0}^2\rangle\rangle_{g-g'} + \frac{1}{6}\langle\langle \tau_{0,0}^3\tau_{0,1}\rangle\rangle_{g-1},$$

when $r = 4$, we have

$$\begin{aligned}\langle\langle \tau_{1,0}\tau_{0,0}\rangle\rangle_g &= \langle\langle \tau_{0,0}\tau_{0,2}\rangle\rangle_{g'}\langle\langle \tau_{0,0}^2\rangle\rangle_{g-g'} + \frac{1}{2}\langle\langle \tau_{0,0}\tau_{0,1}\rangle\rangle_{g'}\langle\langle \tau_{0,1}\tau_{0,0}\rangle\rangle_{g-g'}\\ &\quad + \frac{1}{4}\langle\langle \tau_{0,0}^3\tau_{0,2}\rangle\rangle_{g-1} + \frac{1}{48}\langle\langle \tau_{0,0}^2\rangle\rangle_{g'}\langle\langle \tau_{0,0}^4\rangle\rangle_{g-1-g'} + \frac{1}{32}\langle\langle \tau_{0,0}^3\rangle\rangle_{g'}\langle\langle \tau_{0,0}^3\rangle\rangle_{g-1-g'}\\ &\quad + \frac{1}{480}\langle\langle \tau_{0,0}^6\rangle\rangle_{g-2}.\end{aligned}$$

Now we describe how to use the above Theorem to compute intersection numbers. It consists of three steps.

i) When $g = 0$, these intersection numbers can be computed by the WDVV equations. An algorithm has been given by Witten [76].

ii) Let $g \geq 1$. For an intersection number containing a puncture operator $\langle \tau_{0,0}\tau_{n_1,m_1}\cdots\tau_{n_s,m_s}\rangle_g$, we have from Theorem 1.1 and the dilaton equation

$$\begin{aligned}&(2g-1+s-a)\langle \tau_{0,0}\tau_{n_1,m_1}\cdots\tau_{n_s,m_s}\rangle_g\\ &\qquad = \tilde{\sum_{\underline{s}=I\coprod J}}\langle \tau_{0,0}\tau_{0,m'}\prod_{i\in I}\tau_{n_i,m_i}\rangle_{g'}\eta^{m'm''}\langle \tau_{0,m''}\tau_{0,0}\prod_{i\in J}\tau_{n_i,m_i}\rangle_{g-g'} + \mathrm{Low}(r)\end{aligned}$$

where $a = \#\{i \mid n_i = 0\}$. Note that in the summation

$$\tilde{\sum_{\underline{s}=I\coprod J}}$$

we rule out the cases $I = \{i_1\}$ and $n_{i_1} = 0$ or $J = \{i_1\}$ and $n_{i_1} = 0$. Then the right-hand side follows by induction on genera or number of marked points.

iii) For any intersection number $\langle \tau_{n_1,m_1}\cdots\tau_{n_s,m_s}\rangle_g$ with $n_1 \geq n_2 \geq \cdots \geq n_s$, we apply the string equation first

$$\langle \tau_{n_1,m_1}\cdots\tau_{n_s,m_s}\rangle_g = \langle \tau_{0,0}\tau_{n_1+1,m_1}\cdots\tau_{n_s,m_s}\rangle_g - \sum_{j=2}^{s}\langle \tau_{n_1+1,m_1}\tau_{n_j-1,m_j}\prod_{i\neq 1,j}\tau_{n_i,m_i}\rangle_g$$

The first term on the right-hand side follows from step (ii) and the second term follows by induction on the maximum descendent index.

We have written a Maple program according to the above algorithm. Some r-spin numbers when $r = 3$ and 4 are listed below. These values agree with previous results in [5, 40, 72].

Table 3. Witten's r-spin numbers ($r = 3$)

$\langle\tau_{1,0}\rangle_1$	$\frac{1}{12}$	$\langle\tau_{1,1}\tau_{3,1}\rangle_2$	$\frac{11}{4320}$	$\langle\tau_{0,1}\tau_{0,1}\tau_{2,1}\rangle_1$	$\frac{1}{36}$
$\langle\tau_{6,1}\rangle_3$	$\frac{1}{31104}$	$\langle\tau_{2,1}\tau_{2,1}\rangle_2$	$\frac{17}{4320}$	$\langle\tau_{0,1}\tau_{1,1}\tau_{1,1}\rangle_1$	$\frac{1}{36}$
$\langle\tau_{9,0}\rangle_4$	$\frac{1}{746496}$	$\langle\tau_{0,1}\tau_{7,0}\rangle_3$	$\frac{1}{15552}$	$\langle\tau_{0,1}\tau_{0,1}\tau_{5,0}\rangle_2$	$\frac{1}{432}$
$\langle\tau_{14,1}\rangle_6$	$\frac{1}{4837294080}$	$\langle\tau_{1,1}\tau_{6,0}\rangle_3$	$\frac{19}{77760}$	$\langle\tau_{0,1}\tau_{1,1}\tau_{4,0}\rangle_2$	$\frac{13}{2160}$
$\langle\tau_{17,0}\rangle_7$	$\frac{1}{162533081088}$	$\langle\tau_{2,0}\tau_{5,1}\rangle_3$	$\frac{103}{217728}$	$\langle\tau_{0,1}\tau_{2,0}\tau_{3,1}\rangle_2$	$\frac{1}{108}$
$\langle\tau_{22,1}\rangle_9$	$\frac{1}{1805510340771840}$	$\langle\tau_{2,1}\tau_{5,0}\rangle_3$	$\frac{47}{77760}$	$\langle\tau_{0,1}\tau_{2,1}\tau_{3,0}\rangle_2$	$\frac{23}{2160}$
$\langle\tau_{25,0}\rangle_{10}$	$\frac{1}{75831434312417280}$	$\langle\tau_{3,0}\tau_{4,1}\rangle_3$	$\frac{443}{544320}$	$\langle\tau_{1,1}\tau_{1,1}\tau_{3,0}\rangle_2$	$\frac{29}{2160}$
$\langle\tau_{30,1}\rangle_{12}$	$\frac{1}{123548906006608084992 0}$	$\langle\tau_{3,1}\tau_{4,0}\rangle_3$	$\frac{67}{77760}$	$\langle\tau_{1,1}\tau_{2,0}\tau_{2,1}\rangle_2$	$\frac{19}{1080}$

Table 4. Witten's r-spin numbers ($r = 4$)

$\langle\tau_{1,0}\rangle_1$	$\frac{1}{8}$	$\langle\tau_{0,2}\tau_{1,2}\rangle_1$	$\frac{1}{96}$	$\langle\tau_{0,1}\tau_{0,1}\tau_{2,2}\rangle_1$	$\frac{1}{32}$
$\langle\tau_{3,2}\rangle_2$	$\frac{3}{2560}$	$\langle\tau_{0,1}\tau_{4,1}\rangle_2$	$\frac{1}{320}$	$\langle\tau_{0,1}\tau_{0,2}\tau_{2,1}\rangle_1$	$\frac{1}{24}$
$\langle\tau_{6,0}\rangle_3$	$\frac{3}{20480}$	$\langle\tau_{0,2}\tau_{4,0}\rangle_2$	$\frac{19}{7680}$	$\langle\tau_{0,1}\tau_{1,1}\tau_{1,2}\rangle_1$	$\frac{1}{24}$
$\langle\tau_{8,2}\rangle_4$	$\frac{77}{39321600}$	$\langle\tau_{1,1}\tau_{3,1}\rangle_2$	$\frac{7}{960}$	$\langle\tau_{0,2}\tau_{0,2}\tau_{2,0}\rangle_1$	$\frac{1}{48}$
$\langle\tau_{11,0}\rangle_5$	$\frac{19}{104857600}$	$\langle\tau_{1,2}\tau_{3,0}\rangle_2$	$\frac{41}{7680}$	$\langle\tau_{0,1}\tau_{0,1}\tau_{5,0}\rangle_2$	$\frac{13}{2560}$
$\langle\tau_{13,2}\rangle_6$	$\frac{59}{33554432000}$	$\langle\tau_{2,0}\tau_{2,2}\rangle_2$	$\frac{49}{7680}$	$\langle\tau_{0,1}\tau_{1,1}\tau_{4,0}\rangle_2$	$\frac{1}{64}$
$\langle\tau_{16,0}\rangle_7$	$\frac{39}{268435456000}$	$\langle\tau_{2,1}\tau_{2,1}\rangle_2$	$\frac{11}{960}$	$\langle\tau_{0,1}\tau_{2,1}\tau_{3,0}\rangle_2$	$\frac{9}{320}$
$\langle\tau_{18,2}\rangle_8$	$\frac{9367}{7215545057280000}$	$\langle\tau_{2,0}\tau_{5,0}\rangle_3$	$\frac{43}{20480}$	$\langle\tau_{1,1}\tau_{1,1}\tau_{3,0}\rangle_2$	$\frac{7}{192}$
$\langle\tau_{21,0}\rangle_9$	$\frac{2363}{24739011624960000}$	$\langle\tau_{3,0}\tau_{4,0}\rangle_3$	$\frac{7}{2048}$	$\langle\tau_{1,1}\tau_{2,0}\tau_{2,1}\rangle_2$	$\frac{1}{20}$
$\langle\tau_{23,2}\rangle_{10}$	$\frac{23567}{30786325577728000000}$	$\langle\tau_{1,2}\tau_{5,2}\rangle_3$	$\frac{311}{1720320}$	$\langle\tau_{0,2}\tau_{2,2}\tau_{2,2}\rangle_2$	$\frac{11}{3072}$
$\langle\tau_{26,0}\rangle_{11}$	$\frac{5443}{1055531162664960000000}$	$\langle\tau_{2,2}\tau_{4,2}\rangle_3$	$\frac{67}{172032}$	$\langle\tau_{1,2}\tau_{1,2}\tau_{2,2}\rangle_2$	$\frac{7}{1536}$

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