

COMPUTING TOP INTERSECTIONS IN THE TAUTOLOGICAL RING OF \mathcal{M}_g

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ABSTRACT. We derive effective recursion formulae of top intersections in the tautological ring $R^*(\mathcal{M}_g)$ of the moduli space of curves of genus $g \geq 2$. As an application, we prove a convolution-type tautological relation in $R^{g-2}(\mathcal{M}_g)$.

1. INTRODUCTION

We denote by $\overline{\mathcal{M}}_{g,n}$ the moduli space of stable n -pointed genus g complex algebraic curves. We have the morphism that forgets the last marked point,

$$\pi_{n+1} : \overline{\mathcal{M}}_{g,n+1} \longrightarrow \overline{\mathcal{M}}_{g,n}.$$

Denote by $\sigma_1, \dots, \sigma_n$ the canonical sections of π , and by D_1, \dots, D_n the corresponding divisors in $\overline{\mathcal{M}}_{g,n+1}$. Let ω_π be the relative dualizing sheaf. We have the following tautological classes on moduli spaces of curves.

$$\begin{aligned} \psi_i &= c_1(\sigma_i^*(\omega_\pi)), \\ \kappa_i &= \pi_* \left(c_1 \left(\omega_\pi \left(\sum D_i \right) \right)^{i+1} \right), \\ \lambda_l &= c_l(\pi_*(\omega_\pi)), \quad 1 \leq l \leq g. \end{aligned}$$

The definition of κ classes on $\overline{\mathcal{M}}_{g,n}$ is due to Arbarello-Cornalba [1], generalizing Mumford-Morita-Miller classes

$$\kappa_i = \pi_* \left(c_1(\omega_\pi)^{i+1} \right) \in \mathcal{A}^i(\mathcal{M}_g),$$

where $\mathcal{A}^*(\mathcal{M}_g)$ is the rational Chow ring of \mathcal{M}_g .

The tautological ring $R^*(\mathcal{M}_g)$ is defined to be the subalgebra of $\mathcal{A}^*(\mathcal{M}_g)$ generated by the tautological classes κ_i .

We use Witten's notation to denote intersection numbers:

$$\langle \tau_{d_1} \cdots \tau_{d_n} \kappa_{a_1} \cdots \kappa_{a_m} \mid \lambda_1^{k_1} \cdots \lambda_g^{k_g} \rangle := \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \cdots \psi_n^{d_n} \kappa_{a_1} \cdots \kappa_{a_m} \lambda_1^{k_1} \cdots \lambda_g^{k_g}.$$

These are rational numbers and are called the Hodge integrals, which can be computed by Faber's algorithm [7] based on Mumford's formula for Chern characters of Hodge bundles and the celebrated Witten-Kontsevich theorem [22, 14].

1.1. Faber's conjecture. Around 1993, Faber [6] proposed a series of remarkable conjectures about the structure of $R^*(\mathcal{M}_g)$:

i) For $0 \leq k \leq g-2$, the natural product

$$R^k(\mathcal{M}_g) \times R^{g-2-k}(\mathcal{M}_g) \rightarrow R^{g-2}(\mathcal{M}_g) \cong \mathbb{Q}$$

is a perfect pairing.

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- ii) The $[g/3]$ classes $\kappa_1, \dots, \kappa_{[g/3]}$ generate the ring $R^*(\mathcal{M}_g)$, with no relations in degrees $\leq [g/3]$.
- iii) Let $\sum_{j=1}^n d_j = g - 2$ and $d_j \geq 0$. Then

$$(1) \quad \sum_{\sigma \in S_n} \kappa_\sigma = \frac{(2g - 3 + n)!}{(2g - 2)!! \prod_{j=1}^n (2d_j + 1)!!} \kappa_{g-2},$$

where κ_σ is defined as follows: write the permutation σ as a product of $\nu(\sigma)$ disjoint cycles $\sigma = \beta_1 \cdots \beta_{\nu(\sigma)}$, where we think of the symmetric group S_n as acting on the n -tuple (d_1, \dots, d_n) . Denote by $|\beta|$ the sum of the elements of a cycle β . Then $\kappa_\sigma = \kappa_{|\beta_1|} \kappa_{|\beta_2|} \cdots \kappa_{|\beta_{\nu(\sigma)}|}$.

It is a theorem of Looijenga [18] that

$$\dim R^k(\mathcal{M}_g) = 0, \quad k > g - 2,$$

$$\dim R^{g-2}(\mathcal{M}_g) \leq 1.$$

Faber proved that actually $\dim R^{g-2}(\mathcal{M}_g) = 1$.

Part (i) of Faber's conjecture is also called Faber's perfect pairing conjecture, which is still open. Faber has verified $g \leq 23$.

Part (ii) has been proved independently by Morita [19] and Ionel [11] by very different methods. As pointed out by Faber [6], Harer's stability result implies that there is no relation in degrees $\leq [g/3]$.

Part (iii) is known as the Faber intersection number conjecture and is equivalent to

$$(2) \quad \langle \tau_{d_1+1} \cdots \tau_{d_n+1} \mid \lambda_g \lambda_{g-1} \rangle_g = \frac{(2g - 3 + n)!}{(2g - 2)!! \prod_{j=1}^n (2d_j + 1)!!} \langle \kappa_{g-2} \mid \lambda_g \lambda_{g-1} \rangle_g.$$

A short and direct proof of the Faber intersection number conjecture can be found in [16]. Also see [8, 10] and the most recently [3] for different approaches to the problem.

In fact, Faber [6] further proposed that the tautological ring $R^*(\mathcal{M}_g)$ behaves like the algebraic cohomology ring of a nonsingular projective variety of dimension $g - 2$, i.e. it satisfies the Hard Lefschetz and Hodge Positivity properties with respect to κ_1 .

Faber's intersection number conjecture determines the top intersections in $R^{g-2}(\mathcal{M}_g)$. If we assume Faber's perfect pairing conjecture, then the ring structure of $\mathcal{R}^*(\mathcal{M}_g)$ is also determined.

A central theme in Faber's conjecture is to explicitly describe relations in the tautological rings. However, it is a highly nontrivial task to identify tautological relations in $R^*(\mathcal{M}_g)$ when g becomes larger. In this paper, we will consider only tautological relations of top degree in $R^{g-2}(\mathcal{M}_g)$. Already known examples include Faber-Zagier's formula [6]

$$(3) \quad \kappa_1^{g-2} = \frac{1}{g-1} 2^{2g-5} ((g-2)!)^2 \kappa_{g-2}.$$

and Pandharipande's formula [21]

$$(4) \quad \sum_{i=0}^{g-2} (-1)^i \lambda_i \kappa_{g-2-i} = \frac{2^{g-1}}{g!} \kappa_{g-2}.$$

in $R^{g-2}(\mathcal{M}_g)$, $g \geq 2$.

Now we describe the main results of this paper. We will prove two effective recursive formulae of different flavors for computing top intersections in $R^{g-2}(\mathcal{M}_g)$ (see Theorems 3.6 and 3.8). For example, Theorem 3.8 can be equivalently stated as the following:

Theorem 1.1. *Let $g \geq 3$ and $|\mathbf{m}| = g - 2$. Then the following relation*

$$(5) \quad \kappa(\mathbf{m}) = \frac{1}{(|\mathbf{m}|| - 1)} \sum_{\substack{\mathbf{L} + \mathbf{L}' = \mathbf{m} \\ \|\mathbf{L}\| \geq 2}} A_{g, \mathbf{L}} \binom{\mathbf{m}}{\mathbf{L}} \kappa(\mathbf{L}' + \delta_{|\mathbf{L}|})$$

holds in $R^{g-2}(\mathcal{M}_g)$, where $A_{g, \mathbf{L}} = \mathbf{L}! D_{g, \mathbf{L}}$ are some explicitly known constants.

Note that in the right-hand side of equation (5), $\|\mathbf{L}' + \delta_{|\mathbf{L}|}\| < \|\mathbf{m}\|$, so equation (5) is indeed an effective recursion relation. Our strategy of proof is to exploit the method used in [15]. From an algorithmic point of view, recursion formulae are often more effective than closed formulae, since previously computed values can be reused in recursive computations.

Our recursion formulae will be used to compute Faber's intersection matrix, whose rank is equal to the dimension of $R^*(\mathcal{M}_g)$ by Faber's perfect pairing conjecture. On the other hand, the cohomological dimension $H^*(\mathcal{M}_g)$ is also an outstanding open problem [2].

One objective of this paper is to gain a better understanding of Faber-Zagier's formula. In Section 4, we prove an interesting Bernoulli number identity equivalent to Faber-Zagier's formula.

In the final section, we prove the following convolution-type tautological relation:

Theorem 1.2. *Let $g \geq 3$. We have the following relation in $R^{g-2}(\mathcal{M}_g)$,*

$$(6) \quad \sum_{i=0}^{g-2} D_{g, g-2-i} \frac{\kappa_1^i \kappa_{g-2-i}}{i!} = 0,$$

where $D_{g, k}$ are given by

$$(7) \quad D_{g, k} = \frac{3}{2(g-2)} \cdot \frac{1}{k!} + \frac{2g-1}{2(g-2)} \sum_{j=0}^k \frac{(2j+1)(-1)^{j+1} 2^{2j} B_{2j}}{j!(k-j)!}.$$

where B_{2j} is the $2j$ -th Bernoulli number. We have $D_{g, 0} = -1$, $D_{g, 1} = \frac{g+1}{g-2}$, $D_{g, 2} = \frac{17g-4}{6(g-2)}$.

We don't know whether the expression of $D_{g, k}$ can be simplified.

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2. THE FABER INTERSECTION MATRIX

First we fix notation. Consider the semigroup N^∞ of sequences $\mathbf{m} = (m(1), m(2), \dots)$ where $m(i)$ are nonnegative integers and $m(i) = 0$ for sufficiently large i . We sometimes also use $(1^{m(1)} 2^{m(2)} \dots)$ to denote \mathbf{m} .

Let $\mathbf{m}, \mathbf{a}_1, \dots, \mathbf{a}_n \in N^\infty$, $\mathbf{m} = \sum_{i=1}^n \mathbf{a}_i$.

$$|\mathbf{m}| := \sum_{i \geq 1} im(i) \quad \|\mathbf{m}\| := \sum_{i \geq 1} m(i) \quad \binom{\mathbf{m}}{\mathbf{a}_1, \dots, \mathbf{a}_n} := \prod_{i \geq 1} \binom{m(i)}{a_1(i), \dots, a_n(i)}.$$

Let $\mathbf{m} \in N^\infty$, we denote a formal monomial of κ classes by

$$\kappa(\mathbf{m}) := \prod_{i \geq 1} \kappa_i^{m(i)}.$$

If $|\mathbf{m}| = g - 2$, then from Faber's intersection number identity (2) and the formula expressing ψ classes by κ classes [13], we have the following relation in $R^{g-2}(\mathcal{M}_g)$,

$$(8) \quad \kappa(\mathbf{m}) = \text{Fab}_g(\mathbf{m})\kappa_{g-2},$$

where the proportional constant $\text{Fab}_g(\mathbf{m})$ is given by

$$\text{Fab}_g(\mathbf{m}) = \sum_{r=1}^{|\mathbf{m}|} \frac{(-1)^{|\mathbf{m}|-r}}{r!} \sum_{\substack{\mathbf{m}=\mathbf{m}_1+\dots+\mathbf{m}_r \\ \mathbf{m}_1 \neq \mathbf{0}}} \binom{\mathbf{m}}{\mathbf{m}_1, \dots, \mathbf{m}_r} \frac{(2g-3+r)!}{(2g-2)!! \prod_{j=1}^r (2|\mathbf{m}_j|+1)!!}.$$

Let $g \geq 2$ and $0 \leq k \leq g - 2$. Denote by $p(n)$ the number of partitions of n . Define a matrix V_g^k of size $p(k) \times p(g-2-k)$ with entries

$$(9) \quad (V_g^k)_{\mathbf{L}, \mathbf{L}'} = \text{Fab}_g(\mathbf{L} + \mathbf{L}'),$$

where $\mathbf{L}, \mathbf{L}' \in N^\infty$ and $|\mathbf{L}| = k$, $|\mathbf{L}'| = g - 2 - k$.

We call V_g^k the Faber intersection matrix. If Faber's perfect pairing conjecture is true, then we have

$$(10) \quad \text{rank } V_g^k = \dim R^k(\mathcal{M}_g), \quad 0 \leq k \leq g - 2.$$

Faber has verified his conjecture for all $g \leq 23$, so the above relation holds for at least $g \leq 23$. Thus, we may get useful information of $R^*(\mathcal{M}_g)$ from the Faber intersection matrix.

In the next section, we will develop a recursive method for computing entries of V_g^k . As a result, we have computed V_g^k for all $g \leq 36$. These data is used to check a conjectural relationship between $\text{rank } V_g^k$ and Ramanujan's mock theta function (see Section 7 of [17]).

The recursive algorithm did not reduce computational complexity in theory. But the recursively reusable data rendered the computation more efficient.

If we calculate V_g^k by equation (9) on a computer, the required CPU time is proportional to the number of vector partitions.

Definition 2.1. For $\mathbf{m} \in N^\infty$, denote by $P(\mathbf{m})$ the number of distinct representations of \mathbf{m} as an unordered sum in N^∞ ,

$$\mathbf{m} = \mathbf{m}_1 + \mathbf{m}_2 + \dots + \mathbf{m}_r,$$

where $\mathbf{m}_i \neq \mathbf{0}$. We call such $P(\mathbf{m})$ the vector partition number.

We make the convention that $P(\mathbf{0}) = 1$. There is a simple recursion formula for computing $P(\mathbf{m})$, due to Cheema and Motzkin [5].

Lemma 2.2. (Cheema-Motzkin) For $\mathbf{m} = (m_1, m_2, \dots, m_s)$ with $m_1 \neq 0$, we have

$$(11) \quad m_1 P(\mathbf{m}) = \sum_{a_i \geq 0} \left(P(m_1 - a_1, \dots, m_s - a_s) \sum_{\substack{k > 0 \\ k | \gcd(a_1, \dots, a_s)}} \frac{a_1}{k} \right).$$

Proof. Let $\mathbf{x} = (x_1, x_2, \dots)$ be a sequence of formal variables and $\mathbf{x}^{\mathbf{m}} = x_1^{m(1)} x_2^{m(2)} \dots$ for $\mathbf{m} \in N^\infty$. We have

$$\sum_{\mathbf{m} \in N^\infty} P(\mathbf{m}) \mathbf{x}^{\mathbf{m}} = \prod_{\substack{\mathbf{a} \in N^\infty \\ \mathbf{a} \neq \mathbf{0}}} (1 - \mathbf{x}^{\mathbf{a}})^{-1}.$$

Hence

$$\log \sum_{\mathbf{m} \in N^\infty} P(\mathbf{m}) \mathbf{x}^{\mathbf{m}} = - \sum_{\substack{\mathbf{a} \in N^\infty \\ \mathbf{a} \neq \mathbf{0}}} \log(1 - \mathbf{x}^{\mathbf{a}}) = \sum_{\substack{\mathbf{a} \in N^\infty \\ \mathbf{a} \neq \mathbf{0}}} \sum_{k=1}^{\infty} \frac{\mathbf{x}^{k\mathbf{a}}}{k}$$

Differentiating with respect to x_1 , we get

$$\sum_{\mathbf{m} \in N^\infty} m_1 P(\mathbf{m}) \mathbf{x}^{\mathbf{m}} = \sum_{\substack{\mathbf{m} \in N^\infty \\ \mathbf{a} \neq \mathbf{0}}} P(\mathbf{m}) \mathbf{x}^{\mathbf{m}} \sum_{\substack{\mathbf{a} \in N^\infty \\ \mathbf{a} \neq \mathbf{0}}} \sum_{k=1}^{\infty} a_1 \mathbf{x}^{k\mathbf{a}}.$$

Equating coefficients of $\mathbf{x}^{\mathbf{m}}$, we get the desired identity (11). \square

For a given genus g , the complexity of computing V_g^k for all $0 \leq k \leq g-2$ is measured by the following quantity

$$D(g-2) := \sum_{|\mathbf{m}|=g-2} P(\mathbf{m}),$$

Some values of $D(n)$ are listed as follows.

n	0	1	2	3	4	5	10	20	30	40
$D(n)$	1	1	3	6	14	27	817	318106	71832114	11668071461

Proposition 2.3. $D(n)$ has a simple generating function

$$\sum_{n=0}^{\infty} D(n) x^n = \prod_{n=1}^{\infty} \frac{1}{(1-x^n)^{p(n)}}.$$

Proof. We have

$$\sum_{\mathbf{m}} P(\mathbf{m}) \mathbf{x}^{\mathbf{m}} = \prod_{\mathbf{m} \neq \mathbf{0}} \frac{1}{(1 - \mathbf{x}^{\mathbf{m}})}.$$

Substitute x_i by x^i , we get the desired result. \square

$D(n)$ can also be regarded as the number of double partitions of n . The following asymptotic formula is proved by Kaneiwa [12]

$$D(n) \sim e^{\frac{\pi^2 n}{6 \log n}},$$

which gives the computational complexity of the Faber intersection matrix.

Let $\mathcal{I}(\mathcal{M}_g)$ be the ideal of polynomial relations of κ classes in \mathcal{M}_g and let $\mathcal{I}^k(\mathcal{M}_g)$ be the group of relations in degree k . We have

$$\dim R^k(\mathcal{M}_g) + \dim \mathcal{I}^k(\mathcal{M}_g) = p(k).$$

Let $s \geq 0$ and $g = 3k - s$. Faber [6] pointed out that when $k \geq s + 2$ (i.e. $2k \leq g - 2$), $\dim \mathcal{I}^k(\mathcal{M}_g)$ depends only on s . Denoting this number by $a(s)$, Faber has [6] computed the first ten values. Our calculation of the rank of V_g^k for $g \leq 36$ extends Faber's table of the function a .

s	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$a(s)$	1	1	2	3	5	6	10	13	18	24	33	41	56	71	91

In [6], Faber says ‘‘Zagier and I have a favourite guess of this function a , but there are many functions with ten prescribed values.’’

We have also tried to guess the function a , but failed. For example, define

$$f(s) = \sum_{0 \leq r \leq \lfloor s/3 \rfloor} p(s+1-3r) - p(s-3r).$$

We have $f(s) = a(s)$, $s \leq 10$, but $f(11) = 34 \neq a(11)$.

3. COMPUTING TOP INTERSECTIONS IN $R^{g-2}(\mathcal{M}_g)$

Let $d_j \geq 1$ and $\sum_j d_j + |\mathbf{m}| = g - 2 + n$. We define the following quantities

$$(12) \quad F_{g,n}(\mathbf{m}) := \frac{(2g-2)!! \prod_{j=1}^n (2d_j - 1)!!}{(2g+n-3)! \mathbf{m}!} \cdot \frac{\langle \prod_{j=1}^n \tau_{d_j} \kappa(\mathbf{m}) \mid \lambda_g \lambda_{g-1} \rangle_g}{\langle \kappa_{g-2} \mid \lambda_g \lambda_{g-1} \rangle_g}.$$

Proposition 3.1. *The above definition of $F_{g,n}(\mathbf{m})$ is independent of d_j and $F_{g,n}(\mathbf{0}) = 1$.*

Proof. From the formula expressing ψ by κ in [13], we have

$$\begin{aligned} & \left\langle \prod_{j=1}^n \tau_{d_j} \kappa(\mathbf{m}) \mid \lambda_g \lambda_{g-1} \right\rangle_g \\ &= \sum_{r=0}^{|\mathbf{m}|} \frac{(-1)^{|\mathbf{m}|-r}}{r!} \sum_{\substack{\mathbf{m}=\mathbf{m}_1+\dots+\mathbf{m}_r \\ \mathbf{m}_i \neq \mathbf{0}}} \binom{\mathbf{m}}{\mathbf{m}_1, \dots, \mathbf{m}_r} \left\langle \prod_{j=1}^n \tau_{d_j} \prod_{j=1}^r \tau_{|\mathbf{m}_j|+1} \mid \lambda_g \lambda_{g-1} \right\rangle_g \end{aligned}$$

So the proposition follows directly from Faber's intersection number identity (2). \square

We will also write $F_g(\mathbf{m})$ instead of $F_{g,0}(\mathbf{m})$. We are particularly interested in $F_g(\mathbf{m})$ when $|\mathbf{m}| = g - 2$, since they determine relations in $R^{g-2}(\mathcal{M}_g)$

$$(13) \quad \kappa(\mathbf{m}) = \frac{(2g-3)!! \mathbf{m}! F_g(\mathbf{m})}{2g-2} \kappa_{g-2}.$$

It is important to notice that we may extend $F_g(\mathbf{m})$ to be defined for all $\mathbf{m} \in N^\infty$ using the following Lemma 3.3 and Theorem 3.6.

We define constants $\beta_{\mathbf{L}}$ and $\gamma_{\mathbf{L}}$ by

$$\sum_{\mathbf{L}+\mathbf{L}'=\mathbf{b}} \frac{(-1)^{|\mathbf{L}|} \beta_{\mathbf{L}}}{\mathbf{L}'! (2|\mathbf{L}'|+1)!!} = 0, \quad \mathbf{b} \neq \mathbf{0},$$

$$\sum_{\mathbf{L}+\mathbf{L}'=\mathbf{b}} \frac{(-1)^{|\mathbf{L}|} \gamma_{\mathbf{L}}}{\mathbf{L}'! (2|\mathbf{L}'|-1)!!} = 0, \quad \mathbf{b} \neq \mathbf{0},$$

with the initial values $\beta_{\mathbf{0}} = \gamma_{\mathbf{0}} = 1$. We also denote their reciprocals by

$$\gamma_{\mathbf{L}}^{-1} := \frac{(-1)^{|\mathbf{L}|}}{\mathbf{L}'! (2|\mathbf{L}'|-1)!!}, \quad \beta_{\mathbf{L}}^{-1} := \frac{(-1)^{|\mathbf{L}|}}{\mathbf{L}'! (2|\mathbf{L}'|+1)!!}.$$

Lemma 3.2. *Let $n \geq 0$ and $|\mathbf{m}| \leq g - 2$. Then*

$$(14) \quad (2g+n-1)F_{g,n+2}(\mathbf{m}) = \sum_{\mathbf{L}+\mathbf{L}'=\mathbf{m}} (2g+n-1-2|\mathbf{L}'|) \gamma_{\mathbf{L}} F_{g,n+1}(\mathbf{L}').$$

Proof. We recall the following recursive formula from Theorem 5.3 of [15]

$$\begin{aligned} & \langle \tau_d \tau_{d_0} \prod_{j=1}^n \tau_{d_j} \kappa(\mathbf{m}) \mid \lambda_g \lambda_{g-1} \rangle_g \\ &= \sum_{\mathbf{L} + \mathbf{L}' = \mathbf{m}} \frac{\gamma_{\mathbf{L}} \mathbf{m}!}{\mathbf{L}'!} \frac{(2d + 2d_0 + 2|\mathbf{L}| - 1)!!}{(2d - 1)!!(2d_0 - 1)!!} \langle \tau_{d_0 + d + |\mathbf{L}| - 1} \prod_{j=1}^n \tau_{d_j} \kappa(\mathbf{L}') \mid \lambda_g \lambda_{g-1} \rangle_g \\ &+ \sum_{\mathbf{L} + \mathbf{L}' = \mathbf{m}} \sum_{j=1}^n \frac{\gamma_{\mathbf{L}} \mathbf{m}!}{\mathbf{L}'!} \frac{(2d + 2d_j + 2|\mathbf{L}| - 3)!!}{(2d - 1)!!(2d_j - 3)!!} \langle \tau_{d_0} \tau_{d_j + d + |\mathbf{L}| - 1} \prod_{i \neq j} \tau_{d_i} \kappa(\mathbf{L}') \mid \lambda_g \lambda_{g-1} \rangle_g. \end{aligned}$$

When $\mathbf{m} = 0$, it was obtained by Getzler and Pandharipande [8] from degree 0 Virasoro constraints for \mathbb{P}^2 .

So we have

$$\begin{aligned} (2g + n - 1)F_{g,n+2}(\mathbf{m}) &= \sum_{\mathbf{L} + \mathbf{L}' = \mathbf{m}} (2d + 2d_0 + 2|\mathbf{L}| - 1) \gamma_{\mathbf{L}} F_{g,n+1}(\mathbf{L}') \\ &+ \sum_{\mathbf{L} + \mathbf{L}' = \mathbf{m}} \sum_{j=1}^n (2d_j - 1) \gamma_{\mathbf{L}} F_{g,n+1}(\mathbf{L}') \\ &= \sum_{\mathbf{L} + \mathbf{L}' = \mathbf{m}} (2g + n - 1 - 2|\mathbf{L}'|) \gamma_{\mathbf{L}} F_{g,n+1}(\mathbf{L}'). \end{aligned}$$

In the last equation, we used $d + d_0 + \sum_{j=1}^n d_j = g + n - |\mathbf{m}|$. \square

Lemma 3.2 can also be proved using the following identity instead

$$\sum_{\mathbf{L} + \mathbf{L}' = \mathbf{b}} (-1)^{|\mathbf{L}|} \binom{\mathbf{b}}{\mathbf{L}} \langle \tau_{|\mathbf{L}|} \prod_{j=1}^n \tau_{d_j} \kappa(\mathbf{L}') \mid \lambda_g \lambda_{g-1} \rangle_g = \sum_{j=1}^n \langle \tau_{d_j - 1} \prod_{i \neq j} \tau_{d_i} \kappa(\mathbf{b}) \mid \lambda_g \lambda_{g-1} \rangle_g,$$

which can be proved by the same argument of Proposition 3.1 of [15].

Lemma 3.3. *Let $n \geq 0$ and $|\mathbf{m}| \leq g - 2$. Then*

$$(15) \quad F_{g,n}(\mathbf{m}) = \sum_{\mathbf{L} + \mathbf{L}' = \mathbf{m}} \beta_{\mathbf{L}}^{-1} F_{g,n+1}(\mathbf{L}').$$

Proof. We use the following identity

$$(2g - 2 + n) \langle \prod_{j=1}^n \tau_{d_j} \kappa(\mathbf{m}) \mid \lambda_g \lambda_{g-1} \rangle_g = \sum_{\mathbf{L} + \mathbf{L}' = \mathbf{m}} (-1)^{|\mathbf{L}|} \binom{\mathbf{m}}{\mathbf{L}} \langle \tau_{|\mathbf{L}| + 1} \prod_{j=1}^n \tau_{d_j} \kappa(\mathbf{L}') \mid \lambda_g \lambda_{g-1} \rangle_g,$$

which can be proved by the same argument of Proposition 3.1 of [15]. By a direct calculation as Lemma 3.2, we get the desired result. \square

Lemma 3.4. *Let $n \geq 0$ and $|\mathbf{m}| \leq g - 2$. Then*

$$(16) \quad 2|\mathbf{m}| F_{g,n+1}(\mathbf{m}) = \sum_{\substack{\mathbf{e} + \mathbf{f} + \mathbf{L} = \mathbf{m} \\ \mathbf{L} \neq \mathbf{m}}} \beta_{\mathbf{e}}^{-1} \gamma_{\mathbf{f}} (2g + n - 1 - 2|\mathbf{L}|) F_{g,n+1}(\mathbf{L}).$$

Proof. The result follows by applying Lemma 3.2 to the right hand side of equation (15) in Lemma 3.3. We have

$$\begin{aligned}
(2g+n-1)F_{g,n+1}(\mathbf{m}) &= \sum_{\mathbf{e}+\mathbf{b}=\mathbf{m}} \beta_{\mathbf{e}}^{-1}(2g+n-1)F_{g,n+2}(\mathbf{b}) \\
&= \sum_{\mathbf{e}+\mathbf{b}=\mathbf{m}} \beta_{\mathbf{e}}^{-1} \sum_{\mathbf{f}+\mathbf{L}=\mathbf{b}} \gamma_{\mathbf{f}}(2g+n-1-2|\mathbf{L}|)F_{g,n+1}(\mathbf{L}) \\
&= \sum_{\mathbf{e}+\mathbf{f}+\mathbf{L}=\mathbf{m}} \beta_{\mathbf{e}}^{-1} \gamma_{\mathbf{f}}(2g+n-1-2|\mathbf{L}|)F_{g,n+1}(\mathbf{L}).
\end{aligned}$$

So we get the desired identity. \square

Proposition 3.5. *Let $n \geq 0$ and $|\mathbf{m}| \leq g-2$. Then*

$$(17) \quad 2|\mathbf{m}|F_{g,n}(\mathbf{m}) = (2g+n-2) \sum_{\substack{\mathbf{L}+\mathbf{L}'=\mathbf{m} \\ \mathbf{L}' \neq \mathbf{0}}} C_{\mathbf{L}}F_{g,n}(\mathbf{L}'),$$

where $C_{\mathbf{0}} = -1$ and

$$C_{\mathbf{L}} = \sum_{\mathbf{e}+\mathbf{f}=\mathbf{L}} 2|\mathbf{e}|\beta_{\mathbf{e}}\beta_{\mathbf{f}}^{-1} = - \sum_{\mathbf{e}+\mathbf{f}=\mathbf{L}} \gamma_{\mathbf{e}}^{-1}\beta_{\mathbf{f}}, \quad \mathbf{L} \neq \mathbf{0}.$$

Proof. From Lemma 3.3 and Lemma 3.4, we have

$$\begin{aligned}
2|\mathbf{m}| \sum_{\mathbf{L}+\mathbf{L}'=\mathbf{m}} \beta_{\mathbf{L}}F_{g,n}(\mathbf{L}') &= 2|\mathbf{m}|F_{g,n+1}(\mathbf{m}) \\
&= \sum_{\substack{\mathbf{e}+\mathbf{f}+\mathbf{L}=\mathbf{m} \\ \mathbf{L} \neq \mathbf{m}}} \beta_{\mathbf{e}}^{-1} \gamma_{\mathbf{f}}(2g+n-1-2|\mathbf{L}|) \sum_{\mathbf{L}'+\mathbf{L}''=\mathbf{L}} \beta_{\mathbf{L}'}F_{g,n}(\mathbf{L}'') \\
&= \sum_{\mathbf{e}+\mathbf{f}+\mathbf{L}=\mathbf{m}} \beta_{\mathbf{e}}^{-1} \gamma_{\mathbf{f}}(2g+n-1-2|\mathbf{L}|) \sum_{\mathbf{L}'+\mathbf{L}''=\mathbf{L}} \beta_{\mathbf{L}'}F_{g,n}(\mathbf{L}'') - (2g+n-1-2|\mathbf{m}|) \sum_{\mathbf{L}+\mathbf{L}'=\mathbf{m}} \beta_{\mathbf{L}}F_{g,n}(\mathbf{L}').
\end{aligned}$$

Subtract $2|\mathbf{m}| \sum_{\mathbf{L}+\mathbf{L}'=\mathbf{m}} \beta_{\mathbf{L}}F_{g,n}(\mathbf{L}')$ from each side, we have

$$\begin{aligned}
(18) \quad 0 &= (2g+n-1) \sum_{\mathbf{f}+\mathbf{L}''=\mathbf{m}} \gamma_{\mathbf{f}}F_{g,n}(\mathbf{L}'') - 2 \sum_{\mathbf{e}+\mathbf{f}+\mathbf{L}'+\mathbf{L}''=\mathbf{m}} |\mathbf{L}''|\beta_{\mathbf{e}}^{-1} \gamma_{\mathbf{f}}\beta_{\mathbf{L}'}F_{g,n}(\mathbf{L}'') \\
&\quad - 2 \sum_{\mathbf{e}+\mathbf{f}+\mathbf{L}'+\mathbf{L}''=\mathbf{m}} |\mathbf{L}'|\beta_{\mathbf{e}}^{-1} \gamma_{\mathbf{f}}\beta_{\mathbf{L}'}F_{g,n}(\mathbf{L}'') - (2g+n-1) \sum_{\mathbf{L}+\mathbf{L}'=\mathbf{m}} \beta_{\mathbf{L}}F_{g,n}(\mathbf{L}').
\end{aligned}$$

Now we simplify the third term in (18)

$$\begin{aligned}
- \sum_{\mathbf{e}+\mathbf{f}+\mathbf{L}'+\mathbf{L}''=\mathbf{m}} 2|\mathbf{L}'|\beta_{\mathbf{e}}^{-1} \gamma_{\mathbf{f}}\beta_{\mathbf{L}'}F_{g,n}(\mathbf{L}'') &= \sum_{\mathbf{e}+\mathbf{f}+\mathbf{L}'+\mathbf{L}''=\mathbf{m}} 2|\mathbf{e}|\beta_{\mathbf{e}}\gamma_{\mathbf{f}}\beta_{\mathbf{L}'}F_{g,n}(\mathbf{L}'') \\
&= \sum_{\mathbf{e}+\mathbf{f}+\mathbf{L}'+\mathbf{L}''=\mathbf{m}} (2|\mathbf{e}|+1)\beta_{\mathbf{e}}^{-1} \gamma_{\mathbf{f}}\beta_{\mathbf{L}'}F_{g,n}(\mathbf{L}'') - \sum_{\mathbf{e}+\mathbf{f}+\mathbf{L}'+\mathbf{L}''=\mathbf{m}} \beta_{\mathbf{e}}^{-1} \gamma_{\mathbf{f}}\beta_{\mathbf{L}'}F_{g,n}(\mathbf{L}'') \\
&= \sum_{\mathbf{e}+\mathbf{f}+\mathbf{L}'+\mathbf{L}''=\mathbf{m}} \gamma_{\mathbf{e}}^{-1} \gamma_{\mathbf{f}}\beta_{\mathbf{L}'}F_{g,n}(\mathbf{L}'') - \sum_{\mathbf{L}+\mathbf{L}'=\mathbf{m}} \gamma_{\mathbf{L}}F_{g,n}(\mathbf{L}') \\
&= \sum_{\mathbf{L}+\mathbf{L}'=\mathbf{m}} \beta_{\mathbf{L}}F_{g,n}(\mathbf{L}') - \sum_{\mathbf{L}+\mathbf{L}'=\mathbf{m}} \gamma_{\mathbf{L}}F_{g,n}(\mathbf{L}').
\end{aligned}$$

Substitute into (18), we have

$$0 = (2g + n - 1) \sum_{\mathbf{L} + \mathbf{L}' = \mathbf{m}} \gamma_{\mathbf{L}} F_{g,n}(\mathbf{L}') - 2 \sum_{\mathbf{L} + \mathbf{L}' = \mathbf{m}} |\mathbf{L}'| \gamma_{\mathbf{L}} F_{g,n}(\mathbf{L}'') \\ + \sum_{\mathbf{L} + \mathbf{L}' = \mathbf{m}} \beta_{\mathbf{L}} F_{g,n}(\mathbf{L}') - \sum_{\mathbf{L} + \mathbf{L}' = \mathbf{m}} \gamma_{\mathbf{L}} F_{g,n}(\mathbf{L}') - (2g + n - 1) \sum_{\mathbf{L} + \mathbf{L}' = \mathbf{m}} \beta_{\mathbf{L}} F_{g,n}(\mathbf{L}').$$

So we get

$$2 \sum_{\mathbf{L} + \mathbf{L}' = \mathbf{m}} |\mathbf{L}'| \gamma_{\mathbf{L}} F_{g,n}(\mathbf{L}') = (2g + n - 2) \sum_{\mathbf{L} + \mathbf{L}' = \mathbf{m}} \gamma_{\mathbf{L}} F_{g,n}(\mathbf{L}') - (2g + n - 2) \sum_{\mathbf{L} + \mathbf{L}' = \mathbf{m}} \beta_{\mathbf{L}} F_{g,n}(\mathbf{L}').$$

Convoluting both sides by $\gamma_{\mathbf{L}}^{-1}$, we have

$$2|\mathbf{m}|F_{g,n}(\mathbf{m}) = (2g + n - 2)F_{g,n}(\mathbf{m}) - (2g + n - 2) \sum_{\mathbf{L} + \mathbf{L}' = \mathbf{m}} \left(\sum_{\mathbf{e} + \mathbf{f} = \mathbf{L}} \gamma_{\mathbf{e}}^{-1} \beta_{\mathbf{f}} \right) F_{g,n}(\mathbf{L}') \\ = -(2g + n - 2) \sum_{\substack{\mathbf{L} + \mathbf{L}' = \mathbf{m} \\ \mathbf{L} \neq \mathbf{0}}} \left(\sum_{\mathbf{e} + \mathbf{f} = \mathbf{L}} \gamma_{\mathbf{e}}^{-1} \beta_{\mathbf{f}} \right) F_{g,n}(\mathbf{L}') \\ = -(2g + n - 2) \sum_{\substack{\mathbf{L} + \mathbf{L}' = \mathbf{m} \\ \mathbf{L} \neq \mathbf{0}}} \left(\sum_{\mathbf{e} + \mathbf{f} = \mathbf{L}} (2|\mathbf{e}| + 1) \beta_{\mathbf{e}}^{-1} \beta_{\mathbf{f}} \right) F_{g,n}(\mathbf{L}') \\ = (2g + n - 2) \sum_{\substack{\mathbf{L} + \mathbf{L}' = \mathbf{m} \\ \mathbf{L} \neq \mathbf{0}}} \left(\sum_{\mathbf{e} + \mathbf{f} = \mathbf{L}} 2|\mathbf{e}| \beta_{\mathbf{e}} \beta_{\mathbf{f}}^{-1} \right) F_{g,n}(\mathbf{L}').$$

□

Our first main result follows as an immediately corollary of the above proposition.

Theorem 3.6. *Let $|\mathbf{m}| \leq g - 2$. Then*

$$(19) \quad |\mathbf{m}|F_g(\mathbf{m}) = (g - 1) \sum_{\substack{\mathbf{L} + \mathbf{L}' = \mathbf{m} \\ \mathbf{L} \neq \mathbf{0}}} C_{\mathbf{L}} F_g(\mathbf{L}').$$

The constant $C_{\mathbf{L}}$ is defined in Proposition 3.5. In particular,

$$F_g(\delta_k) = \frac{2g - 2}{(2k + 1)!!}, \quad k \geq 1.$$

δ_k denotes the sequence with 1 at the k -th place and zeros elsewhere.

Proof. For the last assertion, just note that

$$C_{\delta_k} = 2k \cdot \beta_{\delta_k} = \frac{2k}{(2k + 1)!!}.$$

□

Theorem 3.6 gives an efficient way to compute constants $F_g(\mathbf{m})$ when $|\mathbf{m}| = g - 2$, hence tautological relations in $\mathcal{R}^{g-2}(\mathcal{M}_g)$ through equation (13). The identity (19) may be expanded to get a closed formula of $F_g(\mathbf{m})$.

Corollary 3.7. *For $\mathbf{m} \neq \mathbf{0}$, we have*

$$F_g(\mathbf{m}) = \sum_{k=1}^{|\mathbf{m}|} (g-1)^k \sum_{\substack{\mathbf{m}=\mathbf{m}_1+\dots+\mathbf{m}_k \\ \mathbf{m}_i \neq \mathbf{0}}} \frac{\prod_{j=1}^k C_{\mathbf{m}_j}}{\prod_{j=1}^k |\mathbf{m}_1 + \dots + \mathbf{m}_j|}.$$

Now we come to our second main result, in contrast with Theorem 3.6, it is a formula that gives recursive relations only among those $F_g(\mathbf{m})$ with $|\mathbf{m}| = g-2$.

Theorem 3.8. *Let $g > 2$ and $|\mathbf{m}| = g-2$. Then we have*

$$(20) \quad (|\mathbf{m}| - 1)F_g(\mathbf{m}) = \sum_{\substack{\mathbf{L}+\mathbf{L}'=\mathbf{m} \\ \|\mathbf{L}'\| \geq 2}} D_{g,\mathbf{L}'} \frac{(\mathbf{L} + \boldsymbol{\delta}_{g-2-|\mathbf{L}|})!}{\mathbf{L}!} F_g(\mathbf{L} + \boldsymbol{\delta}_{g-2-|\mathbf{L}|}),$$

where the constant $D_{g,\mathbf{L}'}$ is given by

$$D_{g,\mathbf{L}'} = -\frac{1}{\mathbf{L}'!} + \frac{2g-1}{2(g-2)} \sum_{\substack{\mathbf{L}_1+\mathbf{L}_2=\mathbf{L}' \\ \|\mathbf{L}_1\| \geq 1}} C_{\mathbf{L}_1} \frac{(1+2|\mathbf{L}_1|)!!}{\mathbf{L}_2!}.$$

The constant $C_{\mathbf{L}}$ is defined in Proposition 3.5.

Proof. By the projection formula and $\kappa_0 = 2g-2$, we have

$$(21) \quad \begin{aligned} \langle \tau_1 \kappa(\mathbf{m}) \mid \lambda_g \lambda_{g-1} \rangle_g &= \int_{\mathcal{M}_g} \pi_*(\psi \cdot \prod_{i \geq 1} (\pi^* \kappa_i + \psi^i)^{m_i}) \lambda_g \lambda_{g-1} \\ &= \sum_{\mathbf{L}+\mathbf{L}'=\mathbf{m}} \binom{\mathbf{m}}{\mathbf{L}} \int_{\mathcal{M}_g} \kappa(\mathbf{L}) \kappa_{|\mathbf{L}'|} \lambda_g \lambda_{g-1} \\ &= (2g-2) \int_{\mathcal{M}_g} \kappa(\mathbf{m}) \lambda_g \lambda_{g-1} + \sum_{\substack{\mathbf{L}+\mathbf{L}'=\mathbf{m} \\ \mathbf{L}' \neq \mathbf{0}}} \binom{\mathbf{m}}{\mathbf{L}} \int_{\mathcal{M}_g} \kappa(\mathbf{L}) \kappa_{|\mathbf{L}'|} \lambda_g \lambda_{g-1}. \end{aligned}$$

From equation (12), the above equation becomes

$$(22) \quad \begin{aligned} F_{g,1}(\mathbf{m}) &= F_g(\mathbf{m}) + \frac{1}{2g-2} \sum_{\substack{\mathbf{L}+\mathbf{L}'=\mathbf{m} \\ \mathbf{L}' \neq \mathbf{0}}} \frac{1}{\mathbf{L}'!} F_g(\mathbf{L} + \boldsymbol{\delta}_{|\mathbf{L}'|}) (\mathbf{L} + \boldsymbol{\delta}_{|\mathbf{L}'|})! \\ &= \left(1 + \frac{|\mathbf{m}|}{2g-2}\right) F_g(\mathbf{m}) + \frac{1}{2g-2} \sum_{\substack{\mathbf{L}+\mathbf{L}'=\mathbf{m} \\ \|\mathbf{L}'\| \geq 2}} \frac{1}{\mathbf{L}'!} F_g(\mathbf{L} + \boldsymbol{\delta}_{|\mathbf{L}'|}) (\mathbf{L} + \boldsymbol{\delta}_{|\mathbf{L}'|})! \end{aligned}$$

Take $n = 1$ in Proposition 3.5, we have

$$2|\mathbf{m}|F_{g,1}(\mathbf{m}) = (2g-1) \sum_{\substack{\mathbf{L}+\mathbf{L}'=\mathbf{m} \\ \mathbf{L}' \neq \mathbf{0}}} C_{\mathbf{L}} F_{g,1}(\mathbf{L}')$$

Substitute equation (22) into both sides of the above identity and adjust the indices, we get

$$\begin{aligned}
 & (g-2)(\|\mathbf{m}\| - 1)F_g(\mathbf{m}) \\
 &= \frac{2g-1}{2} \sum_{\substack{\mathbf{L}+\mathbf{L}'=\mathbf{m} \\ \|\mathbf{L}'\|\geq 2}} \sum_{\substack{\mathbf{L}_1+\mathbf{L}_2=\mathbf{L}' \\ \|\mathbf{L}_1\|\geq 1}} C_{\mathbf{L}_1} \frac{(1+2\|\mathbf{L}_1\|)!!}{\mathbf{L}_1!\mathbf{L}_2!} F_g(\mathbf{L} + \delta_{g-2-|\mathbf{L}|})(\mathbf{L} + \delta_{g-2-|\mathbf{L}|})! \\
 & \quad - (g-2) \sum_{\substack{\mathbf{L}+\mathbf{L}'=\mathbf{m} \\ \|\mathbf{L}'\|\geq 2}} \frac{1}{\mathbf{L}'!\mathbf{L}'!} F_g(\mathbf{L} + \delta_{g-2-|\mathbf{L}|})(\mathbf{L} + \delta_{g-2-|\mathbf{L}|})!,
 \end{aligned}$$

which is just our desired identity. \square

We know $F_g(\delta_{g-2}) = \frac{2g-2}{(2g-3)!!}$. So Theorem 3.8 gives a recursive formula to compute $F(\mathbf{m})$ when $|\mathbf{m}| = g-2$ by induction on $\|\mathbf{m}\|$. We have written a maple program to implement Theorems 3.6 and 3.8, both of them give the correct results.

Remark 3.9. Here we comment on the signs of the constants that we met in this section. Although we are not able to give a proof for now, numerical evidence strongly suggests that $\beta_{\mathbf{L}} > 0, \gamma_{\mathbf{L}} > 0$ for all $L \in N^\infty$ and $C_{\mathbf{L}} > 0, D_{g,\mathbf{L}} > 0$ for all $\mathbf{L} \neq 0$ and $g \geq 3$. When $\mathbf{L} = (1^k)$, this is easy to verify (see the next section). One may wonder at the seemly exceptional initial values $C_0 = D_0 = -1$; in fact, their negativity is essential for the tautological relation in Theorem 1.2.

4. IDENTITIES OF BERNOULLI NUMBERS

Let $k \geq 0$. Denote by C_k the constants $C_{(k,0,0,\dots)}$ defined in Proposition 17. The same convention apply for $\beta_k, \beta_k^{-1}, \gamma_k, \gamma_k^{-1}, D_{g,k}$. Recall that we have (see [15])

$$(23) \quad \beta_k = \frac{(-1)^k (2 - 2^{2k}) B_{2k}}{k!(2k-1)!!},$$

$$(24) \quad \beta_k^{-1} = \frac{(-1)^k}{k!(2k+1)!!},$$

$$(25) \quad \gamma_k = \frac{E_{2k}}{(2k-1)!!},$$

$$(26) \quad \gamma_k^{-1} = \frac{(-1)^k}{k!(2k-1)!!},$$

where B_k, E_k are the Bernoulli and Euler numbers respectively

$$\frac{t}{e^t - 1} = \sum_{m=0}^{\infty} B_m \frac{t^m}{m!},$$

$$\sec t = \sum_{m=0}^{\infty} E_m \frac{t^m}{m!}.$$

We have $B_1 = -\frac{1}{2}, B_{2k+1} = 0$ for $k \geq 1$ and $E_{2k+1} = 0$ for $k \geq 0$.

We first prove a simple closed formula for C_k .

Lemma 4.1. *Let $k \geq 0$. Then*

$$C_k = - \sum_{j=0}^k \gamma_j^{-1} \beta_{k-j} = \frac{(-1)^{k+1} 2^{2k} B_{2k}}{k!(2k-1)!}.$$

Proof. From (23) and (26), we have

$$\begin{aligned} - \sum_{j=0}^k \gamma_j^{-1} \beta_{k-j} &= \sum_{j=0}^k \frac{(-1)^{k-j}}{(k-j)!(2k-2j-1)!} \cdot \frac{(-1)^{j+1} (2-2^{2j}) B_{2j}}{j!(2j-1)!} \\ &= \frac{(-1)^{k+1} (2-2^{2k}) B_{2k}}{k!(2k-1)!} + \sum_{j=0}^{k-1} \frac{(-1)^{k-j}}{(k-j)!(2k-2j-1)!} \cdot \frac{(-1)^{j+1} (2-2^{2j}) B_{2j}}{j!(2j-1)!}. \end{aligned}$$

So we need to prove

$$2(2^{2k}-1)(2k+1)B_{2k} = \sum_{j=0}^{k-1} \binom{2k+1}{2j} (2-2^{2j})(2k-2j+1)B_{2j},$$

which is easily seen to follow from the following lemma. □

Lemma 4.2. *Let $n \geq 2$. Then*

$$2(2^n-1)(n+1)B_n = \sum_{j=0}^{n-1} \binom{n+1}{j} (2-2^j)(n-j+1)B_j.$$

Proof. Multiplying $\frac{1}{(n+1)!}$ on both sides and taking generating functions, we have

$$\begin{aligned} A := \sum_{n=0}^{\infty} \frac{2(2^n-1)B_n}{n!} t^n &= \frac{-2t}{\sinh t} + 2 \sum_{n=0}^{\infty} \frac{B_n}{n!} t^n \\ &= \frac{-2t}{\sinh t} + t \coth(t/2) - t \end{aligned}$$

and

$$\begin{aligned} B &:= \sum_{n=0}^{\infty} \sum_{j=0}^{n-1} \frac{2-2^j}{j!(n+1-j)!} (n-j+1)B_j t^n \\ &= \sum_{n=0}^{\infty} \sum_{j=0}^n \frac{2-2^j}{j!(n-j)!} B_j t^n - \sum_{n=0}^{\infty} \frac{2-2^n}{n!} B_n t^n \\ &= \frac{te^t}{\sinh t} - \frac{t}{\sinh t}. \end{aligned}$$

Since the lemma only cares about coefficients of t^n with $n \geq 2$, it is sufficient to prove that

$$A + 2t = B.$$

Multiply $(e^{2t}-1)/t$ on each side, it is easy to check that both sides equal $2e^{2t}-2e^t$. □

We denote $F_g(k, 0, 0, \dots)$ by $F_g(k)$.

Lemma 4.3. *Let $k \geq 1$. Then*

$$(27) \quad F_g(k) = 2^{2k} \sum_{n=1}^k (g-1)^n \sum_{\substack{a_1+\dots+a_n=k \\ a_i>0}} \prod_{j=1}^n \frac{|B_{2a_j}|}{a_j!(2a_j-1)!! \cdot |a_1+\dots+a_j|}.$$

Proof. This follows directly from Corollary 3.7 and Lemma 4.1. Also note that $|B_{2m}| = (-1)^{m+1} B_{2m}$, $m > 0$. \square

Lemma 4.4. *Let $k \geq 0$. Then*

$$F_0(k) = \frac{(-1)^k 2^{2k}}{(k+1)!(2k+1)!!}.$$

Proof. By Lemma 4.1 and Theorem 3.6, we need to prove that for $k \geq 0$,

$$\begin{aligned} \frac{(-1)^k k \cdot 2^{2k}}{(k+1)!(2k+1)!!} &= \sum_{j=1}^k \frac{(-1)^{j+1} 2^{2j} B_{2j}}{(2j-1)!! j!} \cdot \frac{(-1)^{k-j} 2^{2(k-j)}}{(k-j+1)!(2k-2j+1)!!} \\ &= \frac{(-1)^{k+1} 2^{2k}}{(k+1)!(2k+1)!!} + \sum_{j=0}^k \frac{(-1)^{j+1} 2^{2j} B_{2j}}{(2j-1)!! j!} \cdot \frac{(-1)^{k-j} 2^{2(k-j)}}{(k-j+1)!(2k-2j+1)!!} \end{aligned}$$

It is not difficult to simplify the above equation to

$$\sum_{j=0}^k \binom{2k+2}{2j} B_{2j} = k+1.$$

We have

$$\begin{aligned} \sum_{j=0}^k \binom{2k+2}{2j} B_{2j} &= \sum_{j=0}^{2k+1} \binom{2k+2}{j} B_j - \binom{2k+2}{1} B_1 \\ &= k+1, \end{aligned}$$

where we used the well-known formula

$$\sum_{j=0}^m \binom{m+1}{j} B_j = 0, \quad m > 0.$$

This completes the proof. \square

The following Faber-Zagier's formula [6] may be proved using the Faber intersection number conjecture and the Cauchy residue formula [23].

Proposition 4.5. (Faber-Zagier) *Let $g \geq 2$. Then*

$$(28) \quad F_g(g-2) = \frac{2^{2g-4}(g-2)!}{(2g-3)!!}.$$

Proof. By (8) and (13), it is easy to see that Faber-Zagier's formula is equivalent to the following combinatorial lemma. \square

The proof of the following Lemma 4.6 is due to Jian Zhou [23].

Lemma 4.6. *Let $g \geq 1$. Then*

$$\sum_{k=1}^g \left(\frac{(-1)^k}{k!} (2g+1+k) \sum_{\substack{g=m_1+\dots+m_k \\ m_i > 0}} \binom{2g+k}{2m_1+1, \dots, 2m_k+1} \right) = (-1)^g 2^{2g} (g!)^2$$

Proof. We will use Cauchy's residue formula

$$\begin{aligned} & \sum_{k=1}^g \frac{(-1)^k}{k!} (2g+1+k) \sum_{\substack{m_1+\dots+m_k=g \\ m_i > 0}} \binom{2g+k}{2m_1+1, \dots, 2m_k+1} \\ &= \sum_{k=1}^g (-1)^k \prod_{j=1}^{2g+1} (k+j) \sum_{\substack{m_1+\dots+m_k=g \\ m_i > 0}} \frac{1}{(2m_1+1)!} \cdots \frac{1}{(2m_k+1)!} \\ &= \sum_{k=1}^g (-1)^k \prod_{j=1}^{2g+1} (k+j) \operatorname{Res}_{z=0} \frac{1}{z^{2g+1+k}} f(z)^k \\ &= \operatorname{Res}_{z=0} \left(\sum_{k=1}^g (-1)^k \prod_{j=1}^{2g+1} (k+j) \frac{1}{z^{2g+1+k}} f(z)^k \right), \end{aligned}$$

where

$$f(z) = \sum_{m>0} \frac{z^{2m+1}}{(2m+1)!}.$$

To take the summation over k , we notice that

$$\prod_{j=1}^{2g+1} (k+j) \frac{1}{z^{2g+1+k}} = -z \partial_w^{2g+1} |_{w=z} w^{-k-1},$$

hence we can proceed as follows:

$$\begin{aligned} & \operatorname{Res}_{z=0} \sum_{k=1}^g (-1)^k \prod_{j=1}^{2g+1} (k+j) \frac{1}{z^{2g+1+k}} f(z)^k \\ &= - \operatorname{Res}_{z=0} \sum_{k=1}^g (-1)^k z \partial_w^{2g+1} |_{w=z} w^{-k-1} f(z)^k \\ &= - \operatorname{Res}_{z=0} z \partial_w^{2g+1} |_{w=z} \frac{-w^{-2} f(z)}{1+w^{-1} f(z)} \\ &= - \operatorname{Res}_{z=0} z \partial_w^{2g+1} |_{w=z} \left(\frac{1}{w+f(z)} - \frac{1}{w} \right) \\ &= (2g+1)! \operatorname{Res}_{z=0} \frac{z}{(z+f(z))^{2g+2}} \\ &= (2g+1)! \operatorname{Res}_{z=0} \frac{z}{\sinh^{2g+2} z} \\ &= (2g+1)! \frac{1}{2\pi i} \oint_z \frac{z}{\sinh^{2g+2} z} dz. \end{aligned}$$

To evaluate the contour integral, we make the following change of variable. Take $u = \sinh z$, $z = \operatorname{arcsinh} u$, so that $dz = (1 + u^2)^{-1/2} du$, therefore,

$$\frac{1}{2\pi i} \oint_z \frac{z}{\sinh^{2g+2} z} dz = \frac{1}{2\pi i} \oint_u \frac{1}{u^{2g+2}} \operatorname{arcsinh} u \cdot (1 + u^2)^{-1/2} du$$

However, if we write

$$\operatorname{arcsinh} u \cdot (1 + u^2)^{-1/2} = \sum_{g=0}^{\infty} a_g u^{2g+1},$$

and differentiate both sides

$$\begin{aligned} \sum_{g=0}^{\infty} (2g+1) a_g u^{2g} &= -u(1+u^2)^{-3/2} \operatorname{arcsinh} u + (1+u^2)^{-1} \\ &= (1+u^2)^{-1} \left(1 - \sum_{g=0}^{\infty} a_g u^{2g+2}\right), \end{aligned}$$

We get $a_0 = 1$ and

$$a_{g+1} = -\frac{2g+2}{2g+3} a_g, \quad g \geq 0.$$

From the recursion relation one gets the following unique solution:

$$a_g = -\frac{(-1)^g}{(2g+1)!} 2^{2g} (g!)^2.$$

This completes the proof. \square

By Lemma 4.3, we may rewrite Faber-Zagier's formula (28) as the following identity of Bernoulli numbers.

Proposition 4.7. *Let $g \geq 1$. Then*

$$(29) \quad \sum_{n=1}^g (g+1)^n \sum_{\substack{a_1+\dots+a_n=g \\ a_i>0}} \prod_{j=1}^n \frac{|B_{2a_j}|}{a_j!(2a_j-1)!! \cdot |a_1+\dots+a_j|} = \frac{g!}{(2g+1)!!}.$$

It shall be interesting to find a direct combinatorial proof of (29). However, we are not able to find a general explicit expression for $F_g(k)$ and it seems not easy to prove (29) using Theorem 3.6.

For comparison, we note that Lemma 4.3 and Lemma 4.4 give the following identity

$$(30) \quad \sum_{n=1}^k \sum_{\substack{a_1+\dots+a_n=k \\ a_i>0}} \prod_{j=1}^n \frac{B_{2a_j}}{a_j!(2a_j-1)!! \cdot |a_1+\dots+a_j|} = \frac{1}{(k+1)!(2k+1)!!}.$$

5. PROOF OF THEOREM 1.2

First note that the explicit value of $D_{g,k}$ in (7) follows from Lemma 4.1. Now Theorem 1.2 is an easy consequence of Theorem 3.8 and Faber-Zagier's formula (3),

$$\begin{aligned} \sum_{i=0}^{g-4} D_{g,g-2-i} \frac{\kappa_1^i \kappa_{g-2-i}}{i!} &= \frac{(2g-3)!!}{2g-2} \sum_{i=0}^{g-4} D_{g,g-2-i} F_g(1^i, g-2-i) \\ &= \frac{(2g-3)!!}{2g-2} (g-3) F_g(1^{g-2}) \\ &= \frac{g-3}{(g-2)!} \kappa_1^{g-2}. \end{aligned}$$

On the other hand, from $D_{g,0} = -1$, $D_{g,1} = \frac{g+1}{g-2}$, we have

$$\begin{aligned} D_{g,0} \frac{\kappa_1^{g-2} \kappa_0}{(g-2)!} + D_{g,1} \frac{\kappa_1^{g-2}}{(g-3)!} &= -\frac{2g-2}{(g-2)!} \kappa_1^{g-2} + \frac{g+1}{(g-2)!} \kappa_1^{g-2} \\ &= \frac{3-g}{g-1} \kappa_1^{g-2}. \end{aligned}$$

Adding up the above two sets of equations, we complete the proof of Theorem 1.2.

Proposition 5.1. *Let $g \geq 3$. If we assume that Faber's perfect pairing conjecture is true in codimension one, namely the natural product*

$$R^1(\mathcal{M}_g) \times R^{g-3}(\mathcal{M}_g) \rightarrow R^{g-2}(\mathcal{M}_g) \cong \mathbb{Q}$$

is nondegenerate, then the following relation

$$(31) \quad D_{g,g-2} \frac{g-1}{2^{2g-5}((g-2)!)^2} \kappa_1^{g-3} + \sum_{i=0}^{g-3} D_{g,g-3-i} \frac{\kappa_1^i \kappa_{g-3-i}}{(i+1)!} = 0$$

holds in $R^{g-3}(\mathcal{M}_g)$.

Proof. By Theorem 1.2 and Faber-Zagier's formula (3), we have

$$\begin{aligned} \sum_{i=0}^{g-2} D_{g,g-2-i} \frac{\kappa_1^i \kappa_{g-2-i}}{i!} &= D_{g,g-2} \kappa_{g-2} + \sum_{i=1}^{g-2} D_{g,g-2-i} \frac{\kappa_1^i \kappa_{g-2-i}}{i!} \\ &= \kappa_1 \cdot \left(D_{g,g-2} \frac{g-1}{2^{2g-5}((g-2)!)^2} \kappa_1^{g-3} + \sum_{i=1}^{g-2} D_{g,g-2-i} \frac{\kappa_1^{i-1} \kappa_{g-2-i}}{i!} \right) = 0. \end{aligned}$$

So the relation (31) follows from Faber's perfect pairing conjecture in codimension one and a substitution of i by $i+1$. \square

Example 5.2. Take $g = 6$ in (31), we get

$$\begin{aligned} &D_{6,4} \frac{5}{2^7(4!)^2} \kappa_1^3 + D_{6,0} \frac{10}{4!} \kappa_1^3 + D_{6,1} \frac{1}{3!} \kappa_1^3 + D_{6,2} \frac{1}{2!} \kappa_1 \kappa_2 + D_{6,3} \kappa_3 \\ &= -\frac{5161}{41472} \kappa_1^3 + \frac{49}{24} \kappa_1 \kappa_2 + \frac{395}{72} \kappa_3 \end{aligned}$$

Substitute into the above identity the following relations in $R^*(\mathcal{M}_6)$ computed by Faber [6],

$$\kappa_3 = \frac{5}{2304} \kappa_1^3, \quad \kappa_1 \kappa_2 = \frac{127}{2304} \kappa_1^3,$$

we then checked that the tautological relation (31) holds when $g = 6$.

Remark 5.3. Counterexamples of analogues of Faber’s perfect pairing conjecture on partially compactified moduli spaces of curves have recently been found by Cavalieri and Yang [4], but we still have reasons to believe that Faber’s original perfect pairing conjecture for $R^*(\mathcal{M}_g)$ is true, in view of the beautiful combinatorial structures of $R^*(\mathcal{M}_g)$ even in the top degree. On the other hand, finding explicit tautological relations in lower degrees is a much harder but rewarding problem. In this regard, we refer the interested readers to [6, 11].

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