EFFECTIVE RECURSION FORMULAE FOR COMPUTING INTERSECTION NUMBERS

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Abstract. This paper should be considered as a sequel to our previous paper on n-point functions of intersection numbers [5]. Here we prove several new effective recursion formulae for computing all intersection indices (integrals of ψ classes) on the moduli space of curves, inducting only on the genus. We also prove an improved recursion formula of n-point functions.

1. Introduction

We denote by $\overline{\mathcal{M}}_{g,n}$ the moduli space of stable n-pointed genus g complex algebraic curves. Let ψ_i be the first Chern class of the line bundle whose fiber over each pointed stable curve is the cotangent line at the i-th marked point.

We adopt Witten's notation in this paper,

$$\langle\tau_{d_1}\cdots\tau_{d_n}\rangle_g := \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1}\cdots\psi_n^{d_n}, \qquad \sum_{j=1}^n d_j = 3g+n-3.$$

Witten-Kontsevich theorem [9, 4] provides a recursive way to compute all these intersection numbers. However explicit and effective recursion formulae for computing intersection indices are still very rare and very welcome. We call the following generating function

$$F(x_1,\dots,x_n) = \sum_{g=0}^{\infty} \sum_{\sum d_j = 3g-3+n} \langle\tau_{d_1}\cdots\tau_{d_n}\rangle_g \prod_{j=1}^n x_j^{d_j}$$

the n-point function.

Our recursion formula [5] for normalized n-point function computes intersection indices recursively by decreasing the number of marked points. We will prove the following improved recursion formula of n-point functions in this paper.

Theorem 1.1. *For $n \geq 2$,*

$$F(x_1,\dots,x_n) = \sum_{r,s\geq 0} \frac{(2r+n-3)!!}{12^s(2r+2s+n-1)!!} S_r(x_1,\dots,x_n) \left(\sum_{j=1}^n x_j\right)^{3s},$$

where S_r is a homogeneous symmetric polynomial defined by

$$S_r(x_1,\dots,x_n) = \left(\frac{1}{2\sum_{j=1}^n x_j} \sum_{\underline{n}=I\coprod J} (\sum_{i\in I} x_i)^2 (\sum_{i\in J} x_i)^2 F(x_I)F(x_J)\right)_{3r+n-3}$$

$$= \frac{1}{2\sum_{j=1}^n x_j} \sum_{\underline{n}=I\coprod J} (\sum_{i\in I} x_i)^2 (\sum_{i\in J} x_i)^2 \sum_{r'=0}^{r} F_{r'}(x_I)F_{r-r'}(x_J),$$

where $I, J \neq \emptyset$, $\underline{n} = \{1, 2, \ldots, n\}$ and $F_g(x_I)$ denotes the degree $3g + |I| - 3$ homogeneous component of the $|I|$-point function $F(x_{k_1}, \ldots, x_{k_{|I|}})$, where $k_j \in I$.

It is natural to ask whether there exists a recursion formula which explicitly expresses intersection indices in terms of intersection indices with strictly lower genus. Motivated by Witten's KdV coefficient equation and our n-point function formula, we find such a recursion formula.

Theorem 1.2. *Let $d_j \geq 0$ and $\sum_{j=1}^n d_j = 3g + n - 3$. Then*

$$\begin{aligned}(2g+n-1)(2g+n-2)\langle\prod_{j=1}^n \tau_{d_j}\rangle_g \\ = \frac{2d_1+3}{12}\langle\tau_0^4\tau_{d_1+1}\prod_{j=2}^n \tau_{d_j}\rangle_{g-1} - \frac{2g+n-1}{6}\langle\tau_0^3\prod_{j=1}^n\tau_{d_j}\rangle_{g-1} \\ + \sum_{\{2,\ldots,n\}=I\coprod J}(2d_1+3)\langle\tau_{d_1+1}\tau_0^2\prod_{i\in I}\tau_{d_i}\rangle_{g'}\langle\tau_0^2\prod_{i\in J}\tau_{d_i}\rangle_{g-g'} \\ - \sum_{\{2,\ldots,n\}=I\coprod J}(2g+n-1)\langle\tau_{d_1}\tau_0\prod_{i\in I}\tau_{d_i}\rangle_{g'}\langle\tau_0^2\prod_{i\in J}\tau_{d_i}\rangle_{g-g'}.\end{aligned}$$

It's not difficult to see that when indices $d_j \geq 1$, all non-zero intesection indices on the right hands have genera strictly less than g.

The above recursion formula, together with the string and dilaton equations, provides an effective recursive algorithm for computing intersection indices on moduli spaces of curves by inducting solely on genus g. We have written a Maple program implementing the above recursion formula to compute intersection indices which is available at [10].

Besides our n-point function formula and the above new recursion formula, the only known effective formula for computing intersection indices is the following DVV formula [1, 2] (equivalent to Virasoro constraints)

$$\begin{aligned}\langle\tau_{k+1}\tau_{d_1}\cdots\tau_{d_n}\rangle_g = \frac{1}{(2k+3)!!}\Bigg[\sum_{j=1}^n\frac{(2k+2d_j+1)!!}{(2d_j-1)!!}\langle\tau_{d_1}\cdots\tau_{d_j+k}\cdots\tau_{d_n}\rangle_g \\ +\frac{1}{2}\sum_{r+s=k-1}(2r+1)!!(2s+1)!!\langle\tau_r\tau_s\tau_{d_1}\cdots\tau_{d_n}\rangle_{g-1} \\ +\frac{1}{2}\sum_{r+s=k-1}(2r+1)!!(2s+1)!!\sum_{\underline{n}=I\coprod J}\langle\tau_r\prod_{i\in I}\tau_{d_i}\rangle_{g'}\langle\tau_s\prod_{i\in J}\tau_{d_i}\rangle_{g-g'}\Bigg]\end{aligned}$$

which computes intersection indices by inducting on both the genus and the number of marked points. We know that Mirzakhani's recursion formula [7] of Weil-Petersson volumes is essentially equivalent to the DVV formula [6, 8].

We also found the following simple identity, which plays a key role in the discovery of Theorem 1.2.

Theorem 1.3. *Let $d_j \geq 0$ and $\sum_{j=1}^n d_j = 3g+n-2$. Then*

$$(2g+n-1)\langle\tau_0\prod_{j=1}^n\tau_{d_j}\rangle_g = \frac{1}{12}\langle\tau_0^4\prod_{j=1}^n\tau_{d_j}\rangle_{g-1} + \frac{1}{2}\sum_{\underline{n}=I\coprod J}\langle\tau_0^2\prod_{i\in I}\tau_{d_i}\rangle_{g'}\langle\tau_0^2\prod_{i\in J}\tau_{d_i}\rangle_{g-g'}.$$

In fact, Theorem 1.3 is also very suitable for computing intersection indices. Note that the non-zero intersection indices on the right hand side have strictly lower genus. We may then compute inductively on the maximum index, say d_1, and use the string equation

$$\langle\tau_{d_1}\prod_{j=2}^n\tau_{d_j}\rangle_g = \langle\tau_0\tau_{d_1+1}\prod_{j=2}^n\tau_{d_j}\rangle_g - \sum_{i=2}^n\langle\tau_{d_1+1}\tau_{d_i-1}\prod_{j\neq i,1}\tau_{d_j}\rangle_g.$$

Theorem 1.2 and 1.3 are proved by applying our n-point function formula and Witten's KdV coefficient equation.

Theorem 1.2 tells us that the intersection numbers on moduli spaces of curves are determined by intersection numbers on the boundaries. On the other hand, a theorem of Ionel [3] says when $g \geq 2$, any product of degree at least g of descendant or tautological classes vanishes when restricted to $\mathcal{M}_{g,n}$.

2. Proof of Theorems 1.1 ∼ 1.3

Consider the following "normalized" n-point function

$$G(x_1,\dots,x_n) = \exp\left(\frac{-\sum_{j=1}^n x_j^3}{24}\right)\cdot F(x_1,\dots,x_n).$$

In [5], we have proved the following recursion formula.

Theorem 2.1. [5] *For $n \geq 2$,*

$$G(x_1,\dots,x_n) = \sum_{r,s\geq 0}\frac{(2r+n-3)!!}{4^s(2r+2s+n-1)!!}P_r(x_1,\dots,x_n)\Delta(x_1,\dots,x_n)^s,$$

where P_r and Δ are homogeneous symmetric polynomials defined by

$$\Delta(x_1,\dots,x_n) = \frac{(\sum_{j=1}^n x_j)^3 - \sum_{j=1}^n x_j^3}{3},$$

$$P_r(x_1,\dots,x_n) = \left(\frac{1}{2\sum_{j=1}^n x_j}\sum_{\underline{n}=I\coprod J}(\sum_{i\in I}x_i)^2(\sum_{i\in J}x_i)^2 G(x_I)G(x_J)\right)_{3r+n-3}$$

$$= \frac{1}{2\sum_{j=1}^n x_j}\sum_{\substack{\underline{n}=I\coprod J\\ I,J\neq\emptyset}}(\sum_{i\in I}x_i)^2(\sum_{i\in J}x_i)^2\sum_{r'=0}^r G_{r'}(x_I)G_{r-r'}(x_J).$$

We also need the following lemma.

Lemma 2.2. [5] *Let P and Δ be as defined in Theorem 2.1. Then*

$$G_g(x_1,\dots,x_n) = \frac{1}{(2g+n-1)}P_g(x_1,\dots,x_n) + \frac{\Delta(x_1,\dots,x_n)}{4(2g+n-1)}G_{g-1}(x_1,\dots,x_n).$$

In terms of n-point functions, it's not difficult to see that Theorem 1.3 can be rephrased as the following proposition.

Proposition 2.3. *Let $F(x_1,\dots,x_n)$ be n-point functions. Then*

$$\sum_{g=0}^{\infty}(2g+n-1)\left(\sum_{i=1}^{n}x_i\right)F_g(x_1,\dots,x_n)=\frac{1}{12}\left(\sum_{i=1}^{n}x_i\right)^4F(x_1,\dots,x_n)$$
$$+\frac{1}{2}\sum_{\underline{n}=I\coprod J}\left(\sum_{i\in I}x_i\right)^2\left(\sum_{i\in J}x_i\right)^2F(x_I)F(x_J).$$

Proof. For convenience of notation, we define

$$H=\exp\left(\frac{\sum_{i=1}^{n}x_i^3}{24}\right),\qquad H^{-1}=\exp\left(\frac{-\sum_{i=1}^{n}x_i^3}{24}\right),$$

$$H_d=\frac{1}{d!}\left(\frac{\sum_{i=1}^{n}x_i^3}{24}\right)^d,\qquad H_d^{-1}=\frac{1}{d!}\left(\frac{-\sum_{i=1}^{n}x_i^3}{24}\right)^d.$$

Note that $\sum_{i=0}^{d}H_iH_{d-i}^{-1}=0$ if $d>0$. We have

$$\frac{H^{-1}\cdot RHS}{\sum_{i=1}^{n}x_i}=\sum_{g=0}^{\infty}\left(\frac{1}{12}\left(\sum_{i=1}^{n}x_i\right)^3G_{g-1}(x_1,\dots,x_n)+P_g(x_1,\dots,x_n)\right)$$
$$=\sum_{g=0}^{\infty}\left((2g+n-1)G_g(x_1,\dots,x_n)+\frac{1}{12}\left(\sum_{i=1}^{n}x_i^3\right)G_{g-1}(x_1,\dots,x_n)\right)$$

where we applied Lemma 2.2 in the second equation.

$$\frac{H^{-1}\cdot LHS}{\sum_{i=1}^{n}x_i}=\sum_{g=0}^{\infty}\sum_{a+b+c=g}(2a+2b+n-1)G_a(x_1,\dots,x_n)H_bH_c^{-1}$$
$$=\sum_{g=0}^{\infty}\sum_{a=0}^{g}(2a+n-1)G_a(x_1,\dots,x_n)\sum_{b+c=g-a}H_bH_c^{-1}$$
$$+\sum_{g=0}^{\infty}\sum_{a+b+c=g}G_a(x_1,\dots,x_n)2bH_bH_c^{-1}$$
$$=\sum_{g=0}^{\infty}(2g+n-1)G_g(x_1,\dots,x_n)+\sum_{g=0}^{\infty}\frac{1}{12}\left(\sum_{i=1}^{n}x_i^3\right)G_{g-1}(x_1,\dots,x_n).$$

So we conclude the proof of the proposition. ∎

The following corollary clearly imply Theorem 1.1.

Corollary 2.4.

$$F_g(x_1,\ldots,x_n) = \frac{1}{12(2g+n-1)}\left(\sum_{j=1}^n x_j\right)^3 F_{g-1}(x_1,\ldots,x_n)$$
$$+\frac{1}{(2g+n-1)}\left(\frac{1}{2(\sum_{j=1}^n x_j)}\sum_{\underline{n}=I\coprod J}\left(\sum_{i\in I} x_i\right)^2\left(\sum_{i\in J} x_i\right)^2 F(x_I)F(x_J)\right)_{3g+n-3}.$$

The following is a reformulation of Witten's KdV coefficient equation (see [5]).

Lemma 2.5. *We have*

$$\left(2y\frac{\partial}{\partial y}+1\right)\left(\left(y+\sum_{j=1}^n x_j\right)^2 F(y,x_1,\ldots,x_n)\right) =$$
$$\left(\frac{y}{4}\left(y+\sum_{j=1}^n x_j\right)^4 + y\left(y+\sum_{j=1}^n x_j\right)\right)F(y,x_1,\ldots,x_n)$$
$$+\,y\sum_{\substack{\underline{n}=I\coprod J\\ J\neq\emptyset}}\left(\left(y+\sum_{i\in I}x_i\right)\left(\sum_{i\in J}x_i\right)^3+2\left(y+\sum_{i\in I}x_i\right)^2\left(\sum_{i\in J}x_i\right)^2\right)F(y,x_I)F(x_J).$$

From Theorem 1.3, we can group the first and third terms on the right hand side of Theorem 1.2 and further simplify to the following recursion relation.

$$(2g+n-1)\langle\tau_r\prod_{j=1}^n\tau_{d_j}\rangle_g = (2r+3)\langle\tau_0\tau_{r+1}\prod_{j=1}^n\tau_{d_j}\rangle_g$$
$$-\frac{1}{6}\langle\tau_0^3\tau_r\prod_{j=1}^n\tau_{d_j}\rangle_{g-1} - \sum_{\underline{n}=I\coprod J}\langle\tau_0\tau_r\prod_{i\in I}\tau_{d_i}\rangle_{g'}\langle\tau_0^2\prod_{i\in J}\tau_{d_i}\rangle_{g-g'}.$$

So we need only prove the following equivalent statement of Theorem 1.2.

Proposition 2.6. *We have*

$$y\sum_{g=0}^{\infty}(2g+n-1)F_g(y,x_1,\ldots,x_n) = 2y\frac{\partial}{\partial y}\left(\left(\sum_{j=1}^n y+x_j\right)F(y,x_1,\ldots,x_n)\right)$$
$$+\left(\left(y+\sum_{j=1}^n x_j\right)-\frac{y}{6}\left(y+\sum_{j=1}^n x_j\right)^3\right)F(y,x_1,\ldots,x_n)$$
$$-\,y\sum_{\substack{\underline{n}=I\coprod J\\ J\neq\emptyset}}\left(y+\sum_{i\in I}x_i\right)\left(\sum_{i\in J}x_i\right)^2 F(y,x_I)F(x_J).$$

Proof. From Lemma 2.5, it's not difficult to get the following equation for the part of differentiation with respect to y.

$$2y\left(y+\sum_{j=1}^{n}x_j\right)\frac{\partial}{\partial y}\left(\left(y+\sum_{j=1}^{n}x_j\right)F(y,x_1,\dots,x_n)\right)$$
$$=\left(\frac{y}{4}\left(y+\sum_{j=1}^{n}x_j\right)^4-y\left(y+\sum_{j=1}^{n}x_j\right)-\left(y+\sum_{j=1}^{n}x_j\right)^2\right)F(y,x_1,\dots,x_n)$$
$$+y\sum_{\substack{\underline{n}=I\coprod J\\ J\neq\emptyset}}\left(\left(y+\sum_{i\in I}x_i\right)\left(\sum_{i\in J}x_i\right)^3+2\left(y+\sum_{i\in I}x_i\right)^2\left(\sum_{i\in J}x_i\right)^2\right)F(y,x_I)F(x_J).$$

Multiply each side of the equation in Proposition 2.6 by $y+\sum_{j=1}^n x_j$ and substitute the differential part using the above equation, we get

$$y\sum_{g=0}^{\infty}(2g+n-1)\left(y+\sum_{i=1}^{n}x_i\right)F_g(y,x_1,\dots,x_n)$$
$$=\left(\frac{y}{12}\left(y+\sum_{j=1}^{n}x_j\right)^4-y\left(y+\sum_{j=1}^{n}x_j\right)\right)F(y,x_1,\dots,x_n)$$
$$+y\sum_{\substack{\underline{n}=I\coprod J\\ J\neq\emptyset}}\left(y+\sum_{i\in I}x_i\right)^2\left(\sum_{i\in J}x_i\right)^2F(y,x_I)F(x_J).$$

Add to each side with the term

$$y\left(y+\sum_{j=1}^{n}x_j\right)F(y,x_1,\dots,x_n),$$

we get the equation of Proposition 2.3. So we conclude the proof of Theorem 1.2. ☐

3. More recursion formulae of intersection numbers

It's easy to see that Theorem 1.1 implies the following identity.

Proposition 3.1. *Let $n\geq 2$, $d_j\geq 0$ and $\sum_{j=1}^n d_j=3g+n-2$. Then*

$$(2g+n-1)!!\langle\tau_0\prod_{j=1}^{n}\tau_{d_j}\rangle_g$$
$$=\sum_{r+s=g}\frac{(2r+n-3)!!}{12^s}\sum_{a+b=3s}\binom{3s}{a}\sum_{\substack{\underline{n}=I\coprod J\\ I,J\neq\emptyset}}\langle\tau_0^{a+2}\prod_{i\in I}\tau_{d_i}\rangle_{r'}\langle\tau_0^{b+2}\prod_{i\in J}\tau_{d_i}\rangle_{r-r'}.$$

The following identity can be proved by similar methods of the last section.

Proposition 3.2. *Let* $d_j \geq 0$*,* $d_1 \geq 1$ *and* $\sum_{j=1}^n d_j = 3g+n-3$*. Then*

$$\begin{aligned}
&(2d_1+1)\langle\tau_1^2\prod_{j=1}^n\tau_{d_j}\rangle_g \\
&\quad= \frac{(2d_1+3)(2d_1+1)}{12}\langle\tau_{d_1+1}\tau_0^4\prod_{j=2}^n\tau_{d_j}\rangle_{g-1} - \frac{1}{2}\langle\tau_{d_1-1}\tau_0^2\tau_1^2\prod_{j=2}^n\tau_{d_j}\rangle_{g-1} \\
&\quad+ (2d_1+3)(2d_1+1)\sum_{\{2,\ldots,n\}=I\coprod J}\langle\tau_{d_1+1}\tau_0^2\prod_{i\in I}\tau_{d_i}\rangle_{g'}\langle\tau_0^2\prod_{i\in J}\tau_{d_i}\rangle_{g-g'} \\
&\quad- (2g+n-1)\sum_{\{2,\ldots,n\}=I\coprod J}\Bigg(\langle\tau_{d_1-1}\tau_1\prod_{i\in I}\tau_{d_i}\rangle_{g'}\langle\tau_0^2\prod_{i\in J}\tau_{d_i}\rangle_{g-g'} \\
&\qquad\qquad +2\langle\tau_{d_1-1}\tau_0\prod_{i\in I}\tau_{d_i}\rangle_{g'}\langle\tau_0\tau_1\prod_{i\in J}\tau_{d_i}\rangle_{g-g'}\Bigg).
\end{aligned}$$

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