The hyperbolic geometric flow on Riemann surfaces

De-Xing Kong[*], Kefeng Liu[†] and De-Liang Xu[‡]

Abstract

In this paper the authors study the hyperbolic geometric flow on Riemann surfaces. This new nonlinear geometric evolution equation was recently introduced by the first two authors motivated by Einstein equation and Hamilton's Ricci flow. We prove that, for any given initial metric on $\mathbb{R}^2$ in certain class of metrics, one can always choose suitable initial velocity symmetric tensor such that the solution exists for all time, and the scalar curvature corresponding to the solution metric g_{ij} keeps uniformly bounded for all time; moreover, if the initial velocity tensor is "large" enough, then the solution metric g_{ij} converges to the flat metric at an algebraic rate. If the initial velocity tensor does not satisfy the condition, then the solution blows up at a finite time, and the scalar curvature $R(t,x)$ goes to positive infinity as (t,x) tends to the blowup points, and a flow with surgery has to be considered. The authors attempt to show that, comparing to Ricci flow, the hyperbolic geometric flow has the following advantage: the surgery technique may be replaced by choosing suitable initial velocity tensor. Some geometric properties of hyperbolic geometric flow on general open and closed Riemann surfaces are also discussed.

Key words and phrases: hyperbolic geometric flow, Riemann surface, quasilinear hyperbolic system, global existence, blowup.

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[*]Center of Mathematical Sciences, Zhejiang University, Hangzhou 310027, China;
[†]Department of Mathematics, UCLA, CA 90095, USA;
[‡]Department of Mathematics, Shanghai Jiao Tong University, Shanghai 200240, China.

1 Introduction

Let $\mathscr{M}$ be an n-dimensional complete Riemannian manifold with Riemannian metric g_{ij}. The following general evolution equation for the metric g_{ij}

$$\frac{\partial^2 g_{ij}}{\partial t^2} + 2R_{ij} + \mathscr{F}_{ij}\left(g, \frac{\partial g}{\partial t}\right) = 0 \tag{HGF}$$

has been recently introduced by Kong and Liu [6] and named as *general version of hyperbolic geometric flow*, where $\mathscr{F}_{ij}$ are some given smooth functions of the Riemannian metric g and its first order derivative with respect to t. The most important three special cases are the so-called *standard hyperbolic geometric flow* or simply called *hyperbolic geometric flow*, the *Einstein's hyperbolic geometric flow* (see (1.1) and (1.12) below, respectively) and the *dissipative hyperbolic geometric flow* (see [5] or [2]). The present paper concerns the first two cases on Riemann surfaces.

In this paper we mainly study the evolution of a Riemannian metric g_{ij} on a Riemann surface $\mathscr{M}$ by its Ricci curvature tensor R_{ij} under the *hyperbolic geometric flow* equation

$$\frac{\partial^2 g_{ij}}{\partial t^2} = -2R_{ij}. \tag{1.1}$$

As the first step of our research on this topic, we are interested in the following initial metric on a surface of topological type $\mathbb{R}^2$

$$t = 0: \quad ds^2 = u_0(x)(dx^2 + dy^2), \tag{1.2}$$

where $u_0(x)$ is a smooth function with bounded C^2 norm and satisfies

$$0 < m \leqslant u_0(x) \leqslant M < \infty, \tag{1.3}$$

in which m, M are two positive constants. we shall prove the following result.

Theorem 1.1 *Given the initial metric (1.2) with (1.3), for any smooth function $u_1(x)$ satisfying*

(a) $u_1(x)$ has bounded C^1 norm;

(b) it holds that

$$u_1(x) \geqslant \frac{|u_0'(x)|}{\sqrt{u_0(x)}}, \quad \forall\, x \in \mathbb{R}, \tag{1.4}$$

the Cauchy problem

$$\begin{cases} \dfrac{\partial^2 g_{ij}}{\partial t^2} = -2R_{ij} \quad (i,j=1,2), \\ t = 0: \quad g_{ij} = u_0(x)\delta_{ij}, \quad \dfrac{\partial g_{ij}}{\partial t} = u_1(x)\delta_{ij} \quad (i,j=1,2) \end{cases} \tag{1.5}$$

has a unique smooth solution for all time $t \in \mathbb{R}$*, and the solution metric* g_{ij} *possesses the following form*

$$g_{ij} = u(t,x)\delta_{ij} \qquad (i,j = 1,2). \qquad \square \tag{1.6}$$

Theorem 1.1 will be proved in Section 2. This theorem gives a global existence result on smooth solutions of hyperbolic geometric flow. Based on Theorem 1.1 we can further prove the following theorem in Section 3.

Theorem 1.2 (I) *Under the assumptions mentioned in Theorem 1.1, the Cauchy problem (1.5) has a unique smooth solution with the form (1.6) for all time, moreover the scalar curvature* $R(t,x)$ *corresponding to the solution metric* g_{ij} *remains uniformly bounded, i.e.,*

$$|R(t,x)| \le k, \quad \forall\,(t,x) \in \mathbb{R}^+ \times \mathbb{R}, \tag{1.7}$$

where k *is a positive constant depending on* M*, the* C^2 *norm of* u_0 *and* C^1 *norm of* u_1*, but independent of* t *and* x*.*

(II) *Under the assumptions mentioned in Theorem 1.1, suppose furthermore that there exists a positive constant* ε *such that*

$$u_1(x) \geqslant \frac{|u_0'(x)|}{\sqrt{u_0(x)}} + \varepsilon, \quad \forall\, x \in \mathbb{R}, \tag{1.4a}$$

then the Cauchy problem (1.5) has a unique smooth solution with the form (1.6) for all time, moreover the solution metric g_{ij} *converges to one of flat curvature at an algebraic rate* $\dfrac{1}{(1+t)^\gamma}$*, i.e.,*

$$|R(t,x)| \le \frac{\tilde{k}}{(1+t)^\gamma}, \quad \forall\,(t,x) \in \mathbb{R}^+ \times \mathbb{R}, \tag{1.7a}$$

where $\gamma \in (1/2, 2]$ *and* $\tilde{k}$ *are two positive constants depending on* ε, M*, the* C^2 *norm of* u_0 *and* C^1 *norm of* u_1*.* $\square$

The condition (1.4) is a sufficient condition guaranteeing the global existence of smooth solution to the Cauchy problem (1.5). On the other hand, in some sense it is also a necessary condition, because we have the following theorem.

Theorem 1.3 *Suppose that* $u_0(x) \not\equiv 0$*, without loss of generality, we may assume that there exists a point* $x_0 \in \mathbb{R}$ *such that*

$$u_0'(x_0) < 0. \tag{1.8}$$

For the following initial velocity

$$u_1(x) \equiv \frac{u_0'(x)}{\sqrt{u_0(x)}}, \quad \forall\, x \in \mathbb{R}, \tag{1.9}$$

the Cauchy problem (1.5) has a unique smooth solution only in $[0, \tilde{T}_{\max}) \times \mathbb{R}$*, moreover there exists some point* $(\tilde{T}_{\max}, x_*)$ *such that the scalar curvature* $R(t,x)$ *satisfies*

$$R(t,x) \to \infty \qquad \text{as} \ \ (t,x) \nearrow (\tilde{T}_{\max}, x_*), \tag{1.10}$$

where

$$\tilde{T}_{\max} = -2 \left(\inf_{x \in \mathbb{R}} \left\{ u_0'(x) u_0^{-\frac{3}{2}}(x) \right\} \right)^{-1}. \qquad \square \tag{1.11}$$

Theorem 1.3 will be proved in Section 4.

Motivated by Theorems 1.1-1.2, we conjecture that any complete metric on a simply connected non-compact surface converges to a flat metric by choosing a suitable initial velocity tensor $\frac{\partial g_{ij}}{\partial t}(0,x)$. This should be true for higher dimensional manifolds of topological type $\mathbb{R}^n$ with suitable curvature assumption. We are now working on the general case in $\mathbb{R}^2$ other than (1.2)[1].

On the other hand, Theorem 1.3 shows that if we do not choose suitable initial velocity tensor $\frac{\partial g_{ij}}{\partial t}(0,x)$, the solution to the Cauchy problem (1.5) blows up in finite time, and the curvature tends to infinity when the points approach the blowup points. In this case, a flow with surgery has to be considered. In other words, this paper attempts to show that, by choosing a suitable initial velocity tensor, any complete metric on a surface of topological type $\mathbb{R}^2$ may flow to a flat surface under the hyperbolic geometric flow, otherwise the flow may blow up in finite time and the surgery technique has to be used. In some sense, we try to show the following interesting phenomenon: for the hyperbolic geometric flow, the surgery technique may be replaced by choosing suitable initial velocity tensor.

The following *Einstein's hyperbolic geometric flow* has also been introduced by Kong and Liu [6]

$$\frac{\partial^2 g_{ij}}{\partial t^2} + 2R_{ij} + \frac{1}{2} g^{pq} \frac{\partial g_{ij}}{\partial t} \frac{\partial g_{pq}}{\partial t} - g^{pq} \frac{\partial g_{ip}}{\partial t} \frac{\partial g_{qj}}{\partial t} = 0 \qquad (i,j = 1,2), \tag{1.12}$$

where g^{ij} is the inverse of g_{ij}. Noting the fact that the equation (1.12) is equivalent to (1.1) for the metric g_{ij} with the form (1.6), we know that all conclusions mentioned above hold for the Einstein's hyperbolic geometric flow (1.12).

[1]In fact, for the case of initial data $u_0 = u_0(ax+by), u_1 = u_1(ax+by)$, by the same way we can prove some results similar to Theorems 1.1-1.3, where a and b are two constants satisfying $a^2 + b^2 \neq 1$.

Here we would like to point out that, perhaps the method is more important than the results obtained in this paper. Our method may provide a new approach to the Penrose conjecture (see Penrose [7]) in general relativity and some of Yau's conjectures (e.g., problem 17 stated in Yau [8]) about noncompact complete manifolds with nonnegative curvature in differential geometry.

The paper is organized as follows. An interesting nonlinear partial differential equation related to the metric (1.6) is derived in Section 2. Based on this, we prove the global existence theorem on hyperbolic geometric flow, i.e., Theorem 1.1. The asymptotic behavior theorem, i.e., Theorem 1.2 is proved in Section 3, the proof depends on some new uniform *a priori* estimates on higher derivatives which are interesting in their own right. In section 4, we investigate the blowup phenomena and the formation of singularities in hyperbolic geometric flow. Section 5 is devoted to the discussion about the radial solutions to the hyperbolic geometric flow, i.e., the case $u = u(t, r)$ in which $r = \sqrt{x^2 + y^2}$. Section 6 concerns some geometric obstructions to the existence of smooth long-time solutions and periodic solutions of the hyperbolic geometric flow on general Riemann surfaces.

2 Global existence of hyperbolic geometric flow — Proof of Theorem 1.1

In our previous work [1], we have studied the flow of a metric by its Ricci curvature

$$\frac{\partial^2 g_{ij}}{\partial t^2} = -2R_{ij}$$

on n-dimensional manifolds, where $n \geq 5$. In some respects the higher dimensional cases are easier, due to the enough fast decay of the solution for the corresponding linear wave equations. For surfaces, some estimates on the curvature fails for that the solutions of the corresponding two-dimensional linear wave equations only possesses "slow" decay behavior. Therefore a new approach is needed.

On a surface, the hyperbolic geometric flow equation simplifies, because all of the information about curvature is contained in the scalar curvature function R. In our notation, $R = 2K$ where K is the Gauss curvature. The Ricci curvature is given by

$$R_{ij} = \frac{1}{2} R g_{ij}, \tag{2.1}$$

and the hyperbolic geometric flow equation simplifies the following equation for the special metric

$$\frac{\partial^2 g_{ij}}{\partial t^2} = -Rg_{ij}. \tag{2.2}$$

The metric for a surface can always be written (at least locally) in the following form

$$g_{ij} = u(t,x,y)\delta_{ij}, \tag{2.3}$$

where $u(t,x,y) > 0$. Therefore, we have

$$R = -\frac{\triangle \ln u}{u}. \tag{2.4}$$

Thus the equation (2.2) becomes

$$\frac{\partial^2 u}{\partial t^2} = \frac{\triangle \ln u}{u} \cdot u,$$

namely,

$$u_{tt} - \triangle \ln u = 0. \tag{2.5}$$

In order to prove Theorem 1.1, by the uniqueness of the smooth solution of nonlinear hyperbolic equations, it suffices to show that, for any given smooth function $u_1(x)$ satisfying the conditions (a)-(b), the following Cauchy problem

$$\begin{cases} u_{tt} - \triangle \ln u = 0, \\ t = 0: \quad u = u_0(x,y), \ u_t = u_1(x,y) \end{cases} \tag{2.6}$$

has a unique solution for all time; Moreover, the derivatives of the solution possess some algebraic decay estimates.

Notice that the initial data u_0 and u_1 only depend on the variable x and are independent of y. Therefore the Cauchy problem (2.6) may reduce to the following Cauchy problem for one-dimensional wave equation

$$\begin{cases} u_{tt} - (\ln u)_{xx} = 0, \\ t = 0: \quad u = u_0(x), \ u_t = u_1(x). \end{cases} \tag{2.7}$$

In what follows, we shall solve the Cauchy problem (2.7) and analyze its solution's decay behavior.

Denote

$$\phi = \ln u. \tag{2.8}$$

Then (2.5) reduces to

$$\phi_{tt} - e^{-\phi}\triangle\phi = -\phi_t^2. \tag{2.9}$$

In particular, the first equation in (2.7) becomes

$$\phi_{tt} - e^{-\phi}\phi_{xx} = -\phi_t^2. \tag{2.10}$$

Let

$$v = \phi_t, \quad w = \phi_x. \tag{2.11}$$

Then (2.10) can be rewritten as the following quasilinear system of first order

$$\begin{cases} \phi_t = v, \\ w_t - v_x = 0, \\ v_t - e^{-\phi} w_x = -v^2 \end{cases} \tag{2.12}$$

for smooth solutions.

Introduce

$$p = v + e^{-\frac{\phi}{2}} w, \quad q = v - e^{-\frac{\phi}{2}} w. \tag{2.13}$$

We have

Lemma 2.1 *p and q satisfy*

$$\begin{cases} p_t - \lambda p_x = -\dfrac{1}{4}\{p^2 + 3pq\}, \\ q_t + \lambda q_x = -\dfrac{1}{4}\{q^2 + 3pq\}, \end{cases} \tag{2.14}$$

where

$$\lambda = e^{-\frac{\phi}{2}}. \qquad \square \tag{2.15}$$

Proof. We now calculate

$$\begin{aligned} p_t - \lambda p_x &= \left(v + e^{-\frac{\phi}{2}} w\right)_t - e^{-\frac{\phi}{2}} \left(v + e^{-\frac{\phi}{2}} w\right)_x \\ &= v_t + e^{-\frac{\phi}{2}} w_t - e^{-\frac{\phi}{2}} v_x - e^{-\phi} w_x - \frac{1}{2} e^{-\frac{\phi}{2}} \phi_t w + \frac{1}{2} e^{-\phi} \phi_x w \\ &= -v^2 - \frac{1}{2} v w e^{-\frac{\phi}{2}} + \frac{1}{2} w^2 e^{-\phi} \\ &= -v^2 - \frac{1}{2} v \left(w e^{-\frac{\phi}{2}}\right) + \frac{1}{2} \left(w e^{-\frac{\phi}{2}}\right)^2 \\ &= -\left(\frac{p+q}{2}\right)^2 - \frac{1}{2} \frac{p+q}{2} \frac{p-q}{2} + \frac{1}{2} \left(\frac{p-q}{2}\right)^2 \\ &= -\frac{1}{4} \left(p^2 + 3pq\right). \end{aligned} \tag{2.16}$$

In (2.16) we have made use of (2.11), (2.12) and (2.13). In a similar way, we can prove the second equation in (2.14). Thus, the proof is finished. ■

By a direct calculation, from (2.14) we can obtain the following interesting lemma.

Lemma 2.2 *It holds that*

$$\begin{cases} p_t - (\lambda p)_x = -pq, \\ q_t + (\lambda q)_x = -pq. \end{cases} \qquad \Box \tag{2.17}$$

Noting (2.7)-(2.8), (2.11) and (2.13), we denote

$$p_0(x) \triangleq \frac{u_1(x)}{u_0(x)} + \frac{u_0'(x)}{u_0^{\frac{3}{2}}(x)}, \quad q_0(x) \triangleq \frac{u_1(x)}{u_0(x)} - \frac{u_0'(x)}{u_0^{\frac{3}{2}}(x)}. \tag{2.18}$$

It is easy to verify that the C^2 solution of the Cauchy problem (2.7) is equivalent to the C^1 solution of the Cauchy problem for the following quasilinear system of first order

$$\begin{cases} \phi_t = \dfrac{p+q}{2}, \\ p_t - \lambda p_x = -\dfrac{1}{4}(p^2 + 3pq), \\ q_t + \lambda q_x = -\dfrac{1}{4}(q^2 + 3pq) \end{cases} \tag{2.19}$$

with the initial data

$$t = 0: \quad \phi = \ln u_0(x), \ p = p_0(x), \ q = q_0(x), \tag{2.20}$$

where $\lambda = \lambda(\phi)$ is defined by (2.15), $p_0(x)$ and $q_0(x)$ are given by (2.18). Obviously, (2.19) is a strictly hyperbolic system with three distinguished characteristics

$$\lambda_1 = -\lambda, \quad \lambda_2 = 0, \quad \lambda_3 = \lambda. \tag{2.21}$$

In order to prove Theorem 1.1, it suffices to show the following theorem.

Theorem 2.1 *If $u_1(x)$ is a smooth function with bounded C^1 norm and satisfies*

$$u_1(x) \geqslant \frac{|u_0'(x)|}{\sqrt{u_0(x)}}, \quad \forall \ x \in \mathbb{R}, \tag{2.22}$$

then the Cauchy problem (2.19), (2.20) has a unique global smooth solution for all time $t \in \mathbb{R}$. $\Box$

Corollary 2.1 *Under the assumptions in Theorem 2.1, the Cauchy problem (2.7) has a unique global C^2 solution for all time $t \in \mathbb{R}$.* $\Box$

According to the existence and uniqueness theorem of smooth solution of hyperbolic systems of first order, there exists a locally smooth solution of the Cauchy problem (2.19)–(2.20). In order to prove Theorem 2.1, it suffices to establish the uniform *a priori* estimate on the C^1 norm of (ϕ, p, q) in the domain where the smooth solution of the Cauchy problem (2.19)–(2.20) exists. That is to say, we have to establish the uniform *a priori* estimates on the C^0 norm of (ϕ, p, q) and their derivatives of first order on the existence domain of smooth solution of the Cauchy problem (2.19)–(2.20). Noting (2.19), we see that the key point is to establish *a priori* estimate on the C^1 norm of p and q.

In order to prove Theorem 2.1, we need the following lemmas.

Lemma 2.3 *In the existence domain of the smooth solution of the Cauchy problem (2.19)–(2.20), it holds that*

$$0 \leqslant p(t,x) \leqslant \sup_{y\in\mathbb{R}} p_0(y) \tag{2.23}$$

and

$$0 \leqslant q(t,x) \leqslant \sup_{y\in\mathbb{R}} q_0(y). \qquad \square \tag{2.24}$$

Proof. In fact, passing through any point (t, x), we can draw two characteristics, defined by $\xi = \xi_\pm(\tau; t, x)$, which satisfy

$$\begin{cases} \frac{d\xi_\pm}{d\tau} = \pm\lambda(\tau; \xi_\pm(\tau; t, x)), \\ \xi_\pm(t; t, x) = x, \end{cases} \tag{2.25}$$

respectively. Noting the last two equations in (2.19), we observe that, along the characteristic $\xi = \xi_+(\tau; t, x)$, it holds that

$$p(t,x) = p_0(\xi_+(0;t,x)) \exp\left\{\int_0^t -\frac{1}{4}\left[p+3q\right](\tau;\xi_+(\tau;t,x))d\tau\right\}. \tag{2.26}$$

On the other hand, noting (2.18) and (2.20), we have

$$p_0(x) \geqslant 0, \quad \forall\, x \in \mathbb{R}, \tag{2.27}$$

and then by (2.26), we obtain

$$p(t,x) \geqslant 0$$

in the existence domain of the smooth solution of the Cauchy problem (2.19)-(2.20).

Similarly, we can prove

$$q(t,x) \geqslant 0. \tag{2.28}$$

On the other hand, noting (2.27), we obtain from (2.26) that, for any point (t, x) in the existence domain of the smooth solution

$$0 \leqslant p(t,x) \leqslant p_0(\xi_+(0;t,x)) \leqslant \sup_{y\in\mathbb{R}} p_0(y). \tag{2.29}$$

This is the desired inequality (2.23).

Similarly, we can prove (2.24). Thus, the proof is completed. ■

We next estimate p_x and q_x.

Let

$$r = p_x, \quad s = q_x. \tag{2.30}$$

Similar to Lemma 2.1, we have

Lemma 2.4 *r and s satisfy*

$$\begin{cases} r_t - \lambda r_x = -\frac{1}{4}\left[(2q+3p)r + 3ps\right], \\ s_t + \lambda s_x = -\frac{1}{4}\left[(2p+3q)s + 3qr\right]. \end{cases} \quad \square \tag{2.31}$$

Proof. By a direct calculation, we can easily prove (2.31). ■

Denote

$$r_0(x) \triangleq p_0'(x), \qquad s_0(x) \triangleq q_0'(x). \tag{2.32}$$

We have

Lemma 2.5 *In the existence domain of the smooth solution, it holds that*

$$|r(t,x)|, \quad |s(t,x)| \leqslant \max\left\{\sup_{y\in\mathbb{R}} |r_0(y)|, \quad \sup_{y\in\mathbb{R}} |s_0(y)|\right\}. \quad \square \tag{2.33}$$

Proof. Let

$$A = \frac{2q+3p}{4}, \quad B = \frac{3p}{4}, \quad \bar{A} = \frac{2p+3q}{4}, \quad \bar{B} = \frac{3q}{4}. \tag{2.34}$$

Then the system (2.31) can be rewritten as

$$\begin{cases} r_t - \lambda r_x = -Ar - Bs, \\ s_t + \lambda s_x = -\bar{A}s - \bar{B}r. \end{cases} \tag{2.35}$$

By Lemma 2.3, we have

$$A,\ \bar{A},\ B,\ \bar{B} \geqslant 0, \quad A \geqslant B, \quad \text{and} \quad \bar{A} \geqslant \bar{B}. \tag{2.36}$$

By the terminology in Kong [4], the system (2.35) (i.e., (2.31)) is *weakly dissipative*. Therefore, it follows from Theorem 2.3 in Kong [4] that

$$|r(t,x)|, \quad |s(t,x)| \leqslant \max\left\{\sup_{y\in I(t,x)} |r_0(y)|, \quad \sup_{y\in I(t,x)} |s_0(y)|\right\}, \tag{2.37}$$

where

$$I(t,x) = [\xi_-(0;t,x), \xi_+(0;t,x)]\,.$$

The desired estimate (2.33) comes from (2.37) directly. This proves Lemma 2.5. ■

We next estimate ϕ.

For any fixed point (t,x) in the existence domain of the smooth solution, it follows from the first equation in (2.19) that

$$\phi(t,x) = \phi(0,x) + \int_0^t \frac{p+q}{2}(\tau,x)d\tau. \tag{2.38}$$

Noting (2.20) and (2.23)-(2.24), we have

$$\ln u_0(x) \leqslant \phi(t,x) \leqslant \ln u_0(x) + \frac{1}{2}\left\{\sup_{y\in\mathbb{R}}|p_0(y)| + \sup_{y\in\mathbb{R}}|q_0(y)|\right\}t. \tag{2.39}$$

On the other hand, deriving the first equation in (2.19) with respect to x gives

$$(\phi_x)_t = \frac{1}{2}(p_x+q_x) = \frac{1}{2}(r+s), \tag{2.40}$$

that is,

$$\phi_x = \phi_x(0,x) + \frac{1}{2}\int_0^t (r+s)(\tau,x)d\tau. \tag{2.41}$$

Using (2.20) and noting (2.37), we have

$$|\phi_x(t,x)| \leqslant \frac{|u_0'(x)|}{u_0(x)} + \left[\sup_{y\in\mathbb{R}}|r_0(y)| + \sup_{y\in\mathbb{R}}|s_0(y)|\right]t. \tag{2.42}$$

Proof of Theorem 2.1. Notice that $u_0(x)$ is a smooth function with bounded C^2 norm and satisfies (1.7). On the other hand, notice that $u_1(x)$ is a smooth function with bounded C^1 norm and satisfies (2.22). Then it follows from (2.18) that

$$\begin{cases} 0 \leqslant p_0(x) \leqslant \sup_{x\in\mathbb{R}} p_0(x) \triangleq P_0 < \infty, \\ 0 \leqslant q_0(x) \leqslant \sup_{x\in\mathbb{R}} q_0(x) \triangleq Q_0 < \infty. \end{cases} \tag{2.43}$$

Thus, it follows from (2.39) that, in the existence of the smooth solution

$$\ln m \leqslant \phi(t,x) \leqslant \ln M + \frac{1}{2}(P_0+Q_0)t. \tag{2.44}$$

On the other hand, noting (2.15), we have

$$M^{-\frac{1}{2}}e^{-\frac{1}{4}(P_0+Q_0)t} \leqslant \lambda \leqslant m^{-\frac{1}{2}}. \tag{2.45}$$

From any interval $[a, b]$ in the x-axis, we introduce the following triangle domain, which is the strong determinate domain of the interval $[a, b]$

$$\triangle_{[a,b]} = \left\{(t,x)\,\middle|\, a + m^{-\frac{1}{2}}t \leqslant x \leqslant b + M^{-\frac{1}{2}}t\right\}. \tag{2.46}$$

It is easy to see that, in order to prove Theorem 2.1, it suffices to prove that, for arbitrary interval $[a, b]$ in the x-axis the Cauchy problem (2.19)-(2.20) has a unique smooth solution on $\triangle_{[a,b]}$.

Noting the estimates (2.23)-(2.24), (2.37), (2.39) and (2.42), for any point (t, x) in $\triangle_{[a,b]}$ we have following a priori estimates

$$0 \leqslant p(t,x) \leqslant \sup_{y\in[a,b]} p_0(y) \le P_0, \tag{2.47}$$

$$0 \leqslant q(t,x) \leqslant \sup_{y\in[a,b]} q_0(y) \le Q_0, \tag{2.48}$$

$$|r(t,x)|,\ |s(t,x)| \leqslant \max\left\{\sup_{y\in[a,b]} |r_0(y)|,\ \sup_{y\in[a,b]} |s_0(y)|\right\} < \infty, \tag{2.49}$$

$$\ln m \leqslant \phi(t,x) \leqslant \ln M + \frac{1}{2}\left\{\sup_{y\in[a,b]} |p_0(y)| + \sup_{y\in[a,b]} |q_0(y)|\right\} t \leqslant \ln M + \frac{1}{2}(P_0+Q_0)t_{[a,b]}, \tag{2.50}$$

$$|\phi_x(t,x)| \leqslant \frac{\sup_{x\in[a,b]} |u_0'(x)|}{m} + \left\{\sup_{y\in[a,b]} |r_0(y)| + \sup_{y\in[a,b]} |s_0(y)|\right\} t_{[a,b]}, \tag{2.51}$$

where

$$t_{[a,b]} = \frac{b-a}{m^{-\frac{1}{2}} - M^{-\frac{1}{2}}}.$$

In the estimates (2.47)-(2.51), we assume that the solution exists. The above *a priori* estimates (2.47)-(2.51) implies that the Cauchy problem (2.19)-(2.20) has a unique smooth solution on the whole triangle domain $\triangle_{[a,b]}$. This proves Theorem 2.1. ■

Theorem 1.1 follows from Theorem 2.1 immediately.

3 Asymptotic behavior — Proof of Theorem 1.2

In this section, we shall establish some uniform estimates on the global smooth solution (ϕ, p, q) of the Cauchy problem (2.19)-(2.20) as well as its derivatives r and s. Based on this, we can prove Theorem 1.2.

We first prove the part (I) of Theorem 1.2.

We now assume that (1.4) holds, i.e.,

$$u_1(x) \geqslant \frac{|u_0'(x)|}{\sqrt{u_0(x)}}, \quad \forall\, x \in \mathbb{R}. \tag{3.1}$$

Therefore, all the estimates mentioned in the previous section hold. Under the hypothesis (3.1), we next establish some decay estimates which play an important role in the proof of Theorem 1.2.

Noting the assumption (3.1) (i.e., (1.7)), we obtain from (2.18) that

$$p_0(x), \;\; q_0(x) \geqslant 0, \;\; \forall\, x \in \mathbb{R}. \tag{3.2}$$

In the present situation, similar to the argument in Section 2, we have

$$p(t,x) \geq 0, \;\; q(t,x) \geq 0, \quad \forall\, (t,x) \in \mathbb{R}^+ \times \mathbb{R}. \tag{3.3}$$

Thus, it follows from (2.19) that

$$p_t - \lambda p_x \leqslant -\frac{1}{4}p^2 \tag{3.4}$$

and

$$q_t + \lambda q_x \leqslant -\frac{1}{4}q^2. \tag{3.5}$$

For any fixed point $(0,\alpha)$ in the x-axis, along the characteristic $\xi = \xi_-(t;0,\alpha)$ it holds that

$$p(t,\xi_-(t;0,\alpha)) \leqslant \frac{p_0(\alpha)}{1+\frac{1}{4}p_0(\alpha)t} \leqslant P_0, \quad \forall\, t \geqslant 0. \tag{3.6}$$

This implies that, for any point $(t,x) \in \mathbb{R}^+ \times \mathbb{R}$, we have

$$p(t,x) \leqslant \frac{p_0(\xi_-(0;t,x))}{1+\frac{1}{4}p_0(\xi_-(0;t,x))t} \leqslant P_0, \tag{3.7}$$

namely,

$$p(t,x) \leqslant C_1, \quad \forall\, (t,x) \in \mathbb{R}^+ \times \mathbb{R}, \tag{3.8}$$

here and hereafter C_i $(i = 1, 2, \cdots)$ stand for the positive constants independent of t and x, but depending on ε, M and the C^2 norm of u_0 and C^1 norm of u_1.

Similarly, we can prove

$$q(t,x) \leqslant C_2, \quad \forall\, (t,x) \in \mathbb{R}^+ \times \mathbb{R}. \tag{3.9}$$

In what follows, by maximum principle for hyperbolic systems (see Theorem 2.1 in Kong [4]), we establish the decay estimates on r and s.

Noting (2.34), we have

$$A - B = \frac{1}{2}q \geq 0. \tag{3.10}$$

We now apply the maximum principle derived in Kong [4] to the system (2.35) and obtain

$$|r(t,x)| \leqslant \max\left\{\sup_{y\in I(t,x)}|r_0(y)|, \sup_{y\in I(t,x)}|s_0(y)|\right\}, \tag{3.11}$$

where $I(t,x)$ and $\xi_-(\tau;t,x)$ are defined as before. Let

$$N = \sup_{y\in\mathbb{R}}|r_0(y)| + \sup_{y\in\mathbb{R}}|s_0(y)|. \tag{3.12}$$

By (3.3), it follows from and (3.11) that

$$|r(t,x)| \leqslant N, \quad \forall\ (t,x) \in \mathbb{R}^+ \times \mathbb{R}. \tag{3.13}$$

Similarly, we have

$$|s(t,x)| \leqslant N, \quad \forall\ (t,x) \in \mathbb{R}^+ \times \mathbb{R}. \tag{3.14}$$

Next we estimate $\phi(t,x)$ and its derivatives. It follows from (2.39) that

$$\ln m \leqslant \phi(t,x) \leqslant \ln M + \frac{1}{2}(P_0 + Q_0)t, \tag{3.15}$$

as a consequence, by (2.8)

$$m \leqslant u(t,x) \leqslant M\exp\left\{\frac{1}{2}(P_0+Q_0)t\right\}. \tag{3.16}$$

To estimate ϕ_t and ϕ_x, we use (2.11) and (2.13). By (2.11) and (2.13), we have

$$\phi_t = \frac{1}{2}(p+q), \quad \phi_x = \frac{1}{2}e^{\frac{\phi}{2}}(p-q). \tag{3.17}$$

Thus, it follows from (3.8) and (3.9) that

$$|\phi_t(t,x)| \leqslant \frac{C_1+C_2}{2} \leqslant C_3, \tag{3.18}$$

$$|\phi_x(t,x)| \leqslant \frac{1}{2}\exp\left\{\frac{1}{2}\left[\ln M + \frac{1}{2}(P_0+Q_0)t\right]\right\}\frac{C_1+C_2}{2} \leqslant C_4\exp\{C_5 t\}. \tag{3.19}$$

We now estimate ϕ_{xx}. Noting (2.30) and (2.11), we obtain from (2.13) that

$$\begin{cases} r = p_x = v_x + e^{-\frac{\phi}{2}}\phi_{xx} - \dfrac{1}{2}e^{-\frac{\phi}{2}}\phi_x^2, \\ s = q_x = v_x - e^{-\frac{\phi}{2}}\phi_{xx} + \dfrac{1}{2}e^{-\frac{\phi}{2}}\phi_x^2. \end{cases} \tag{3.20}$$

It follows from (3.20) that

$$\phi_{xx} = \frac{1}{2}e^{\frac{\phi}{2}}\left\{(r-s) + e^{-\frac{\phi}{2}}\phi_x^2\right\} = \frac{1}{2}\left\{(r-s)e^{\frac{\phi}{2}} + \phi_x^2\right\}. \tag{3.21}$$

Thus, by (3.15), (3.13)–(3.14) and (3.19) we obtain

$$|\phi_{xx}(t,x)| \leqslant C_6 \exp\{C_7 t\}. \tag{3.22}$$

We finally estimate the scalar curvature R. By the definition, we have

$$R = \frac{(\ln u)_{xx}}{u}. \tag{3.23}$$

Noting (2.8), we obtain from (3.23) that

$$R(t,x) = \frac{\phi_{xx}}{e^{\phi}}. \tag{3.24}$$

Using (3.21), we have

$$R(t,x) = \frac{\frac{1}{2}\left\{(r-s)e^{\frac{\phi}{2}} + \phi_x^2\right\}}{e^{\phi}} = \frac{1}{2}\left\{(r-s)e^{-\frac{\phi}{2}} + \left(\phi_x e^{-\frac{\phi}{2}}\right)^2\right\}. \tag{3.25}$$

Thus, by (2.13) we get

$$R(t,x) = \frac{1}{2}\left\{(r-s)e^{-\frac{\phi}{2}} + \left(\frac{p-q}{2}\right)^2\right\}. \tag{3.26}$$

Then, using (3.13)–(3.15) and (3.8)–(3.9), we have

$$|R(t,x)| \leqslant \frac{1}{2}\left\{2N\cdot\exp\left\{-\frac{1}{2}\ln m\right\} + \frac{1}{4}(C_1+C_2)^2\right\} \leqslant C_8, \quad \forall\ (t,x)\in\mathbb{R}^+\times\mathbb{R}. \tag{3.27}$$

(3.27) implies that, for any fixed $(t,x)\in\mathbb{R}^2\times\mathbb{R}$ there exists a positive constant k such that

$$|R(t,x)| \leq k, \tag{3.28}$$

where k depends on m, M, the C^2 norm of u_0 and C^1 norm of u_1, but independent of t and x. This proves the part (I) of Theorem 1.2.

Remark 3.1 *From the above argument we observe that, in order to estimate the scalar curvature R, we need to estimate the quantities: $re^{-\frac{\phi}{2}}$ and $se^{-\frac{\phi}{2}}$. A more convenient way is as follows: let*

$$\tilde{r} = re^{-\frac{\phi}{2}}, \quad \tilde{s} = -se^{-\frac{\phi}{2}}, \tag{3.29}$$

then it follows from (2.31) that

$$\begin{cases} \tilde{r}_t - \lambda \tilde{r}_x = -\frac{3}{4}p(\tilde{r}-\tilde{s}), \\ \tilde{s}_t + \lambda \tilde{s}_x = -\frac{3}{4}q(\tilde{s}-\tilde{r}). \end{cases} \tag{3.30}$$

Noting (3.3) and using Remark 4 in Hong [3] or Theorem 2.4 in Kong [4], we have

$$\min\{\inf_{x\in\mathbb{R}} \tilde{r}(0,x), \inf_{x\in\mathbb{R}} \tilde{s}(0,x)\} \le \tilde{r}(t,x),\ \tilde{s}(t,x) \le \max\{\sup_{x\in\mathbb{R}} \tilde{r}(0,x), \sup_{x\in\mathbb{R}} \tilde{s}(0,x)\} \tag{3.31}$$

for any $(t,x) \in \mathbb{R}^+ \times \mathbb{R}$. *(3.31) gives the uniform estimates on the upper and lower bounds of the quantities* $re^{-\frac{\phi}{2}}$ *and* $se^{-\frac{\phi}{2}}$.

We next prove the part (Ⅱ) of Theorem 1.2.

In what follows, we assume that (1.4a) is satisfied, i.e., it holds that

$$u_1(x) \geqslant \frac{|u_0'(x)|}{\sqrt{u_0(x)}} + \varepsilon, \quad \forall\, x \in \mathbb{R}, \tag{1.4a}$$

where ε is an arbitrary positive constant. Obviously, under the assumption (1.4a), (2.22) is always true, and then all the estimates mentioned above hold. In fact, under the hypothesis (1.4a), we may furthermore establish some decay estimates which play an important role in the proof of the part (Ⅱ) of Theorem 1.2.

Noting the assumption (1.4a), we obtain from (2.18) that

$$p_0(x),\ \ q_0(x) \geqslant \frac{\varepsilon}{M} \triangleq M', \quad \forall\, x \in \mathbb{R}. \tag{3.32}$$

Introduce

$$\hat{p} = up, \quad \hat{q} = uq. \tag{3.33}$$

Then it follows from (2.8), (2.11) and (2.13) that

$$\hat{p} = u_t + \frac{1}{\sqrt{u}} u_x, \quad \hat{q} = u_t - \frac{1}{\sqrt{u}} u_x. \tag{3.34}$$

This gives

$$u(t,x) = u_0(x) + \int_0^t (\hat{p}+\hat{q})(\tau,x)d\tau. \tag{3.35}$$

On the other hand, similar to (2.14), we have

$$\begin{cases} \hat{p}_t - \frac{1}{\sqrt{u}}\hat{p}_x = \frac{1}{4u}\hat{p}(\hat{q}-\hat{p}), \\ \hat{q}_t + \frac{1}{\sqrt{u}}\hat{q}_x = \frac{1}{4u}\hat{q}(\hat{p}-\hat{q}). \end{cases} \tag{3.36}$$

Noting (3.3), (3.16) and (3.33), we observe that

$$\frac{1}{4u}\hat{p} \geq 0 \quad \frac{1}{4u}\hat{q} \geq 0, \quad \forall\, (t,x) \in \mathbb{R}^+ \times \mathbb{R}. \tag{3.37}$$

Then, by Remark 4 in Hong [3] or Theorem 2.4 in Kong [4], it follows from (3.36) that

$$\min\{\inf_{x\in\mathbb{R}} \hat{q}(0,x), \inf_{x\in\mathbb{R}} \hat{q}(0,x)\} \leq \hat{p}(t,x),\ \hat{q}(t,x) \leq \max\{\sup_{x\in\mathbb{R}} \hat{p}(0,x), \sup_{x\in\mathbb{R}} \hat{q}(0,x)\} \tag{3.38}$$

for any $(t,x) \in \mathbb{R}^+ \times \mathbb{R}$. Noting (1.3) and (3.32), we have

$$C_9 \leq \hat{p}(t,x),\ \ \hat{q}(t,x) \leq C_{10}, \quad \forall\, (t,x) \in \mathbb{R}^+ \times \mathbb{R}. \tag{3.39}$$

Thus, it follows from (3.35) that

$$C_{11}(1+t) \leq u(t,x) \leq C_{12}(1+t), \quad \forall\, (t,x) \in \mathbb{R}^+ \times \mathbb{R}. \tag{3.40}$$

Noting (3.33) gives

$$\frac{C_{13}}{1+t} \leq p(t,x),\ \ q(t,x) \leq \frac{C_{14}}{1+t}, \quad \forall\, (t,x) \in \mathbb{R}^+ \times \mathbb{R}, \tag{3.41}$$

where $C_{13} \leq C_9/C_{12}$ and $C_{14} \geq C_{10}/C_{11}$

We next establish some decay estimates on r and s.

Noting (2.34), we have

$$A - B = \frac{1}{2}q. \tag{3.42}$$

We now apply the maximum principle derived in Kong [4] to the system (2.35) and obtain

$$|r(t,x)| \leqslant \max\left\{\sup_{y\in I(t,x)} |r_0(y)|, \sup_{y\in I(t,x)} |s_0(y)|\right\} \exp\left\{-\int_0^t \frac{1}{2} q(\tau, \xi_-(\tau;t,x))d\tau\right\}, \tag{3.43}$$

where $I(t,x)$ and $\xi_-(\tau;t,x)$ are defined as in Section 2.

Let

$$N = \sup_{y\in\mathbb{R}} |r_0(y)| + \sup_{y\in\mathbb{R}} |s_0(y)|. \tag{3.44}$$

Noting (3.41), we obtain from (3.43) that

$$\begin{aligned} |r(t,x)| &\leqslant N \exp\left\{-\frac{C_{13}}{2}\int_0^t \frac{1}{1+\tau}d\tau\right\} \\ &\leqslant N \exp\left\{\ln(1+t)^{-C_{15}}\right\} \\ &= \frac{N}{(1+t)^{C_{15}}}. \end{aligned} \tag{3.45}$$

Similarly, we have

$$|s(t,x)| \leqslant \frac{N}{(1+t)^{C_{16}}}. \tag{3.46}$$

We now estimate ϕ_{xx}.

Noting (2.30) and (2.11), we obtain from (2.13) that

$$\begin{cases} r = p_x = v_x + e^{-\frac{\phi}{2}}\phi_{xx} - \dfrac{1}{2}e^{-\frac{\phi}{2}}\phi_x^2, \\ s = q_x = v_x - e^{-\frac{\phi}{2}}\phi_{xx} + \dfrac{1}{2}e^{-\frac{\phi}{2}}\phi_x^2. \end{cases} \tag{3.47}$$

It follows from (3.47) that

$$\phi_{xx} = \frac{1}{2}e^{\frac{\phi}{2}}\left\{(r-s) + e^{-\frac{\phi}{2}}\phi_x^2\right\} = \frac{1}{2}\left\{(r-s)e^{\frac{\phi}{2}} + \phi_x^2\right\}. \tag{3.48}$$

We finally estimate the scalar curvature R.

By the definition, we have

$$R = \frac{(\ln u)_{xx}}{u}. \tag{3.49}$$

Noting (2.8), we obtain from (3.49) that

$$R(t,x) = \frac{\phi_{xx}}{e^{\phi}}. \tag{3.50}$$

Using (3.48), we have

$$R(t,x) = \frac{\frac{1}{2}\left\{(r-s)e^{\frac{\phi}{2}} + \phi_x^2\right\}}{e^{\phi}} = \frac{1}{2}\left\{(r-s)e^{-\frac{\phi}{2}} + \left(\phi_x e^{-\frac{\phi}{2}}\right)^2\right\}. \tag{3.51}$$

Noting (2.8), (2.11) and (2.13), we get

$$R(t,x) = \frac{1}{2}\left\{\frac{r-s}{\sqrt{u}} + \left(\frac{p-q}{2}\right)^2\right\}. \tag{3.52}$$

Then, using (3.40)-(3.41) and (3.45)-(3.46), we have

$$\begin{aligned} |R(t,x)| &\leqslant \frac{1}{2}\left\{\frac{2N}{(1+t)^{\min\{C_{15},C_{16}\}}\sqrt{C_{11}(1+t)}} + \left(\frac{C_{14}-C_{13}}{2(1+t)}\right)^2\right\} \\ &\leqslant C_{17}\left\{\frac{1}{(1+t)^{\min\{C_{15},C_{16}\}+\frac{1}{2}}} + \frac{1}{(1+t)^2}\right\} \\ &\leqslant C_{18}\frac{1}{(1+t)^{\min\{2,\min\{C_{15},C_{16}\}+\frac{1}{2}\}}}, \qquad \forall\, (t,x) \in \mathbb{R}^+ \times \mathbb{R}. \end{aligned} \tag{3.53}$$

Taking $\tilde{k} \geq C_{18}$ and $\gamma = \min\{2, \min\{C_{15}, C_{16}\} + \frac{1}{2}\}$, we obtain from (3.53) that

$$|R(t,x)| \leq \frac{\tilde{k}}{(1+t)^{\gamma}}, \quad \forall\, (t,x) \in \mathbb{R}^+ \times \mathbb{R}. \tag{3.54}$$

This proves the part (II) of Theorem 1.2. Thus, the proof of Theorem 1.2 is completed.

4 Blowup phenomena and formation of singularities — Proof of Theorem 1.3

In this section we will investigate the blowup phenomena of hyperbolic geometric flow and the formation of singularities, provided that the assumption (1.7) is not satisfied.

Throughout this section, we assume that (1.2) holds. As in Theorem 1.3, we assume that there exists a point $x_0 \in \mathbb{R}$ such that

$$u_0'(x_0) < 0. \tag{4.1}$$

In this case, we choose

$$u_1(x) \equiv \frac{u_0'(x)}{\sqrt{u_0(x)}}, \quad \forall\, x \in \mathbb{R}. \tag{4.2}$$

In what follows, we shall prove

Theorem 4.1 *For the initial data $u_0(x)$ and $u_1(x)$ mentioned above, the Cauchy problem (1.5) has a unique smooth solution only in $[0, \tilde{T}_{\max}) \times \mathbb{R}$, where*

$$\tilde{T}_{\max} = -\frac{4}{\inf_{x\in\mathbb{R}}\{p_0(x)\}}, \tag{4.3}$$

where

$$p_0(x) = 2u_0'(x)u_0^{-\frac{3}{2}}(x).$$

Moreover, there exists some point $(\tilde{T}_{\max}, x_*)$ such that the scalar curvature $R(t,x)$ satisfies

$$R(t,x) \to \infty \qquad \text{as} \quad (t,x) \nearrow (\tilde{T}_{\max}, x_*). \qquad \square \tag{4.4}$$

Proof. As in Sections 2-3, it suffices to study the Cauchy problem (2.7), equivalently, the Cauchy problem

$$\begin{cases} \phi_t = \dfrac{p+q}{2}, \\ p_t - \lambda p_x = -\dfrac{1}{4}(p^2 + 3pq), \\ q_t + \lambda q_x = -\dfrac{1}{4}(q^2 + 3pq), \\ t = 0: \quad \phi = \ln u_0(x),\ p = p_0(x),\ q = 0, \end{cases} \tag{4.5}$$

where $p_0(x)$ is defined by (2.18).

Noting the last equation in (2.19), we observe that the above Cauchy problem becomes

$$\begin{cases} \phi_t = \dfrac{1}{2}p, \\ p_t - \lambda p_x = -\dfrac{1}{4}p^2, \end{cases} \tag{4.6}$$

$$t=0: \quad \phi=\ln u_0(x),\ p=p_0(x). \tag{4.7}$$

On the other hand, by (2.18) and (4.2), we have

$$p_0(x)=2\frac{u_0'(x)}{u_0^{\frac{3}{2}}(x)}. \tag{4.8}$$

Thus, passing through any fixed point $(0,\alpha)$ in the x-axis, we draw the characteristic $\xi=\xi_-(t;0,\alpha)$ which is defined by (2.25). Along the characteristic $\xi=\xi_-(t;0,\alpha)$, it follows from the second equation in (4.6) that

$$p\left(t,\xi_-(t;0,\alpha)\right)=\frac{p_0(\alpha)}{1+\frac{1}{4}p_0(\alpha)t}. \tag{4.9}$$

In particular,

$$p\left(t,\xi_-(t;0,x_0)\right)=\frac{p_0(x_0)}{1+\frac{1}{4}p_0(x_0)t} \tag{4.10}$$

for $t\in\left[0,-\dfrac{4}{p_0(x_0)}\right)$. Here we have made use of (4.1). By (4.1), we find $p_0(x_0)<0$, and then $-\dfrac{4}{p_0(x_0)}>0$. (4.10) implies that

$$p\left(t,\xi_-(t;0,x_0)\right)\searrow -\infty \quad \text{as} \quad t\nearrow -\frac{4}{p_0(x_0)}. \tag{4.11}$$

(4.11) shows that the smooth solution of the Cauchy problem (4.6)-(4.7) exists only in finite time. For any $\alpha\in\mathbb{R}$, (4.9) always holds. Combining this and (4.11) gives (4.3).

We next prove (4.4). For simplicity, we assume that

$$p_0(x_0)=\inf_{x\in\mathbb{R}}p_0(x)<0. \tag{4.12}$$

By (4.9), we have

$$|p(t,x)|=\frac{|p_0(x)|_{C^0}}{1-\frac{1}{4}|p_0(x_0)|t}, \quad \forall\,(t,x)\in[0,\widetilde{T}_{\max})\times\mathbb{R}. \tag{4.13}$$

On the other hand, it follows from the first equation in (4.6) that

$$\varphi(t,x)=\varphi_0(x)+\frac{1}{2}\int_0^t p(\tau,x)d\tau, \tag{4.14}$$

and then

$$\begin{aligned}|\varphi(t,x)|&\leqslant|\ln M|+\frac{1}{2}\int_0^t\frac{|p_0(x)|_{C^0}}{1-\frac{1}{4}|p_0(x_0)|\tau}d\tau\\&=|\ln M|+\frac{2|p_0(x)|_{C^0}}{|p_0(x_0)|}\ln\left(1-\frac{1}{4}|p_0(x_0)|t\right)^{-1}, \quad \forall\,(t,x)\in[0,\widetilde{T}_{\max})\times\mathbb{R}.\end{aligned} \tag{4.15}$$

We next estimate r and s. Noting the second equation in (2.30) and the fact $q \equiv 0$, we have

$$s \equiv 0, \quad \forall\,(t,x) \in [0,\widetilde{T}_{\max}) \times \mathbb{R}. \tag{4.16}$$

Thus the Cauchy problem (2.31)-(2.32) reduces to

$$\begin{cases} r_t - \lambda r_x = -\dfrac{3}{4} pr, \\ t = 0 : r = r_0(x) \triangleq p_0'(x). \end{cases} \tag{4.17}$$

Thus, along the characteristic $\xi = \xi_-(t;0,\alpha)$, it holds that

$$r = p_0'(\alpha) \exp\left\{ -\frac{3}{4} \int_0^t p(\tau, \xi_-(\tau;0,\alpha)) d\tau \right\}. \tag{4.18}$$

By (4.13), we have

$$|r(t,x)| \le |p_0'(x)|_{C^0} \exp\left\{ \frac{3|p_0(x)|_{C^0}}{|p_0(x_0)|} \ln\left(1 - \frac{1}{4}|p_0(x_0)|t\right)^{-1} \right\}, \quad \forall\,(t,x) \in [0,\widetilde{T}_{\max}) \times \mathbb{R}. \tag{4.19}$$

In particular,

$$r\,(t, \xi_-(t;0,x_0)) = p_0'(x_0) \exp\left\{ -\frac{3}{4} \int_0^t p(\tau, \xi_-(\tau;0,x_0)) d\tau \right\}, \quad \forall\, t < \widetilde{T}_{\max}. \tag{4.20}$$

Noting (4.12), we have

$$p_0'(x_0) = 0, \tag{4.21}$$

and then,

$$r\,(t, \xi_-(t;0,x_0)) \equiv 0, \quad \forall\, t \in [0,\widetilde{T}_{\max}). \tag{4.22}$$

We finally estimate R. By (3.26) and (4.22), we have

$$\begin{aligned} R\,(t, \xi_-(t;0,x_0)) &= \frac{1}{2}\left\{ r \exp\{-\frac{\varphi}{2}\} + \frac{1}{4}p^2 \right\} (t, \xi_-(t;0,x_0)) \\ &= \frac{1}{8} p^2\,(t, \xi_-(t;0,x_0)). \end{aligned} \tag{4.23}$$

Noting (4.10), we obtain

$$R\,(t, \xi_-(t;0,x_0)) = \frac{1}{8} \frac{p_0^2(x_0)}{\left(1 - \frac{1}{4}|p_0(x_0)|t\right)^2} \nearrow +\infty \quad \text{as} \quad t \nearrow \widetilde{T}_{\max}. \tag{4.24}$$

Denote

$$x_* = \xi_-(\widetilde{T}_{\max};0,x_0) \triangleq \lim_{t \to \widetilde{T}_{\max}} \xi_-(t;0,x_0). \tag{4.25}$$

It is easy to see that the desired (4.4) comes from (4.24) directly. This proves Theorem 4.1. ■

Theorem 4.1 is nothing but Theorem 1.3. Thus Theorem 1.3 has been proved.

Remark 4.1 *Under the assumptions of Theorem 4.1, the scalar curvature R goes to positive infinity at algebraic rate* $\frac{1}{(\widetilde{T}_{max}-t)^2}$ *as (t,x) tends to the blow up point $(\widetilde{T}_{max}, x_*)$.* □

5 Radial solutions to hyperbolic geometric flow

In this section, we shall investigate the radial hyperbolic geometric flow, i.e., we shall consider the radial solution $u = u(t,r)$ of the nonlinear wave equation (2.5). More precisely speaking, we consider the Cauchy problem

$$\begin{cases} u_{tt} - \triangle \ln u = 0, \\ t = 0: \quad u = u_0(r), \quad u_t = u_1(r), \end{cases} \tag{5.1}$$

where $u_0,\ u_1$ are smooth functions of r, in which

$$r = \sqrt{x^2+y^2} \geqslant 0. \tag{5.2}$$

In what follows, we will state and prove some formulas which are useful in the study of radial solutions to the hyperbolic geometric flow.

As in (2.8), let

$$\psi(t,r) = \ln u(t,r). \tag{5.3}$$

Then (2.5) becomes

$$\psi_{tt} - e^{-\psi}\left(\psi_{rr} + \frac{1}{r}\psi_r\right) + \psi_t^2 = 0. \tag{5.4}$$

Similar to (2.11), we denote

$$v = \psi_t, \quad w = \psi_r. \tag{5.5}$$

Then, the equation (5.4) can be equivalently rewritten as

$$\begin{cases} w_t - v_r = 0, \\ v_t - e^{-\psi}\left(w_r + \frac{1}{r}w\right) + v^2 = 0. \end{cases} \tag{5.6}$$

Introduce

$$\mu = v + e^{-\frac{\psi}{2}}w, \quad \nu = v - e^{-\frac{\psi}{2}}w, \tag{5.7}$$

we have

Lemma 5.1 *μ and ν satisfy the following equations*

$$\begin{cases} \mu_t - \chi\mu_r = -\left(\frac{\mu+\nu}{2}\right)^2 + \frac{1}{2}\left(\frac{\chi}{r} - \frac{\nu}{2}\right)(\mu-\nu), \\ \nu_t + \chi\nu_r = -\left(\frac{\mu+\nu}{2}\right)^2 + \frac{1}{2}\left(\frac{\chi}{r} + \frac{\mu}{2}\right)(\mu-\nu), \end{cases} \tag{5.8}$$

where

$$\chi = e^{-\frac{\psi}{2}}. \qquad \square \tag{5.9}$$

Proof. We calculate

$$\begin{aligned}
\mu_t - \chi\mu_r &= \left(v + e^{-\frac{\psi}{2}} w\right)_t - e^{-\frac{\psi}{2}} \left(v + e^{-\frac{\psi}{2}} w\right)_r \\
&= v_t + e^{-\frac{\psi}{2}} w_t - \frac{1}{2} e^{-\frac{\psi}{2}} w\psi_t - e^{-\frac{\psi}{2}} \left(v_r + e^{-\frac{\psi}{2}} w_r - \frac{1}{2} e^{-\frac{\psi}{2}} w\psi_r\right) \\
&= v_t - \frac{1}{2} e^{-\frac{\psi}{2}} w\psi_t - e^{-\psi} w_r + \frac{1}{2} e^{-\psi} w\psi_r \\
&= e^{-\psi} \left(w_r + \frac{1}{r} w\right) - v^2 - \frac{1}{2} e^{-\frac{\psi}{2}} w\psi_t - e^{-\psi} w_r + \frac{1}{2} e^{-\psi} w\psi_r \\
&= -v^2 + \left(\frac{1}{r} e^{-\psi} - \frac{1}{2} v e^{-\frac{\psi}{2}} + \frac{1}{2} e^{-\psi} w\right) w.
\end{aligned} \tag{5.10}$$

In (5.10), we have made use of (5.6). Noting (5.7), we obtain from (5.10) that

$$\begin{aligned}
\mu_t - \chi\mu_r &= -\left(\frac{\mu+\nu}{2}\right)^2 + \left(\frac{1}{r} e^{-\frac{\psi}{2}} - \frac{\mu+\nu}{4} + \frac{\mu-\nu}{4}\right) \frac{\mu-\nu}{2} \\
&= -\left(\frac{\mu+\nu}{2}\right)^2 + \left(\frac{\chi}{r} - \frac{\nu}{2}\right) \frac{\mu-\nu}{2}.
\end{aligned} \tag{5.11}$$

Similarly, we can prove

$$\nu_t + \chi\nu_r = -\left(\frac{\mu+\nu}{2}\right)^2 + \left(\frac{\chi}{r} + \frac{\mu}{2}\right) \frac{\mu-\nu}{2} \tag{5.12}$$

This proves Lemma 5.1. ■

It is easy to see that the C^2 solution of (2.5) is equivalent to the corresponding C^1 solution of the following system of first order

$$\begin{cases}
\psi_t = \dfrac{\mu+\nu}{2}, \\
\mu_t - \chi\mu_r = -\left(\dfrac{\mu+\nu}{2}\right)^2 + \left(\dfrac{\chi}{r} - \dfrac{\nu}{2}\right) \dfrac{\mu-\nu}{2}, \\
\nu_t + \chi\nu_r = -\left(\dfrac{\mu+\nu}{2}\right)^2 + \left(\dfrac{\chi}{r} + \dfrac{\mu}{2}\right) \dfrac{\mu-\nu}{2},
\end{cases} \tag{5.13}$$

where χ is given by (5.9).

Similar to Lemma 2.2, we can prove

Lemma 5.2 *It holds that*

$$\begin{cases}
\mu_t - (\chi\mu)_r = -\mu\nu + \dfrac{\chi(\mu-\nu)}{2r}, \\
\nu_t + (\chi\nu)_r = -\mu\nu + \dfrac{\chi(\mu-\nu)}{2r}.
\end{cases} \qquad \square \tag{5.14}$$

Similar to (2.30), let

$$\eta = \mu_r, \quad \gamma = \nu_r. \tag{5.15}$$

By a direct calculation, we have

Lemma 5.3 *η and γ satisfy*

$$\begin{cases} \eta_t - \chi\eta_r = -\dfrac{3\mu+\nu}{4}(\eta+\gamma) - \left(\dfrac{\mu-\nu}{4r} + \dfrac{\chi}{r^2}\right)\dfrac{\mu-\nu}{2} + \left(\dfrac{\chi}{r} - \dfrac{\nu}{2}\right)\dfrac{\eta-\gamma}{2}, \\ \gamma_t + \chi\gamma_r = -\dfrac{\mu+3\nu}{4}(\eta+\gamma) - \left(\dfrac{\mu-\nu}{4r} + \dfrac{\chi}{r^2}\right)\dfrac{\mu-\nu}{2} + \left(\dfrac{\chi}{r} - \dfrac{\mu}{2}\right)\dfrac{\eta-\gamma}{2}. \end{cases} \quad \square \tag{5.16}$$

Lemmas 5.1–5.3 play an important role in the study on the global existence and blowup phenomenon of the radial solutions of the hyperbolic geometric flow. The key point is to establish the estimates on the terms with the factor $\dfrac{1}{r}$ in the equations (5.8) and (5.16). This kind of estimates is a difficult and key point in the study on the theory of hyperbolic partial differential equations, and still remains open.

6 Some geometric properties on general Riemann surfaces

In this section, we consider the geometric properties of solutions of the hyperbolic geometric flow on Riemann surface of general initial value. We first give some special explicit solutions of the reduced equation of the hyperbolic geometric flow to highlight some special forms of the solutions. Then we derive the reduced equation of the hyperbolic geometric flow in more general conformal class and give some results about certain important quantities like the total mass or volume function which illustrate some geometric obstructions to the long time existence of the reduced problems, as well as the existence of periodic solutions.

Let us consider a simple case, let $(\mathscr{M}, g)$ be a complete Riemann surface with a metric of conformal type

$$g = \rho(x,y)^2\left(dx^2 + dy^2\right).$$

Here we can assume that $\mathscr{M}$ is globally conformal to $\mathbb{R}^2$ or $\mathbb{T}^2$. If we look for the solutions of the hyperbolic geometric flow (1.1) or (1.12) in this conformal class, as in Section 1, we can reduce this system explicitly to

$$\frac{\partial^2 u}{\partial t^2} - \triangle \ln u = 0, \tag{6.1}$$

where $u = \rho^2$ and $\triangle$ is the standard Laplacian. This can be done by a simple observation that the Ricci curvature can be written in terms of the Gaussian curvature as $Ric\,(g_t) = Kg_t$, and the Gaussian curvature can be written as $K = -\frac{1}{\rho^2}\triangle \ln \rho$.

Setting $w = \ln u$, same as (2.9) we have

$$\frac{\partial^2 w}{\partial t^2} + \left|\frac{\partial w}{\partial t}\right|^2 - e^{-w}\triangle w = 0. \tag{6.2}$$

The equation (6.2) is reminiscent of the traditional wave map and semilinear hyperbolic equation in two dimension. In general, it is not easy to handle globally. We first search for some special solutions.

Example 6.1 ***(Solutions of separation variables)*** We look for solutions of (6.2) with the following form

$$w = f\,(t) + g\,(x, y)\,.$$

We have

$$\frac{d^2 f}{dt^2} + \left|\frac{df}{dt}\right|^2 - e^{-f(t)}e^{-g(x,y)} \triangle g = 0,$$

this means

$$e^{f(t)}\left(\frac{d^2 f}{dt^2} + \left|\frac{df}{dt}\right|^2\right) = e^{-g(x,y)} \triangle g = c,$$

where c is a constant independent of t, x, y. So we can solve these two equations by

$$e^{f(t)} = \frac{c}{2}t^2 + at + b,$$

and

$$\triangle g = ce^{g(x,y)}, \tag{6.3}$$

where a, b are arbitrary real numbers, and $g\,(x, y)$ is a solution of (6.3) which is the famous Liouville equation which has arisen in complex analysis and differential geometry on Riemann surfaces, in particular, in the problem of prescribing curvature. We just notice that when the Riemann surface is compact, there is no solution to (6.3) unless $c = 0$. $\square$

Example 6.2 ***(Solutions of traveling wave type)*** We look for solutions of (6.2) of the type

$$w = f\,(x - at)\,,$$

where a is a real number, then we have the following ordinary differential equation

$$e^f\left(a^2 f'' + a^2 f'^2\right) - f'' = 0,$$

which can be solved explicitly in the following implicit form

$$a^2 e^f - f = c_1 x + c_2. \qquad \square$$

Let $(\mathscr{M}, g_0)$ be a Riemann surface with metric g_0. We shall call metrics g_0 and g point-wise conformally equivalent if $g = ug_0$ for some positive function $u \in C^\infty(\mathscr{M})$, whereas we say that metrics g_0 and g are conformally equivalent if there is a diffeomorphism Φ of $\mathscr{M}$ and a positive function $u \in C^\infty(\mathscr{M})$ such that $\Phi^*(g) = ug$. Although we assume that the Riemann surface is globally conformal to $\mathbb{R}^2$ in the above simple setting, we can investigate the hyperbolic geometric flow in the same pointwise conformal class in more general setting. Let $(\mathscr{M}, g_0)$ be a Riemann surface with metric g_0, we would like to search for solutions of the hyperbolic geometric flow in the pointwise conformal class $(\mathscr{M}, u(t,\cdot)\, g_0)$, where $u(t,\cdot)$ is a function with parameter t.

To make the problem more tangible, we first prove the following lemma.

Lemma 6.1 *Let g_0 and g be point-wise conformally equivalent metrics on the Riemann surface $\mathscr{M}$ such that $g = ug_0$ for a positive smooth function u, and $k(\cdot)$ is the Gaussian curvature of $(\mathscr{M}, g_0)$, then the Gaussian curvature of $(\mathscr{M}, ug_0)$ is*

$$K(x,y) = \frac{1}{u}\left(k(x,y) - \frac{1}{2}\triangle_{g_0} \ln u\right), \tag{6.4}$$

where $(x,y) \in \mathscr{M}$ is any local parameter and $\triangle_{g_0}$ denotes the Laplacian with respect to metric g_0. $\square$

Proof. Let $\{\omega_1^0, \omega_2^0\}$ be a local oriented orthonormal coframe field on $(\mathscr{M}, g_0)$. If we set $\omega_i = \sqrt{u}\omega_i^0$, then $\{\omega_1, \omega_2\}$ is a local oriented orthonormal coframe field for g. It is worth to notice now that the Gaussian curvature k of $(\mathscr{M}, g_0)$ is determined by the following equation

$$k\omega_1^0 \wedge \omega_2^0 = d\varphi_{12}^0,$$

where φ_{12}^0 denotes the Riemannian connection form g_0.

Now we compute the connection φ_{12} form of $(\mathscr{M}, u(\cdot)\, g_0)$. Let $du = u_1\omega_1^0 + u_2\omega_2^0$. Then

$$\begin{aligned} d\omega_1 &= \left(\frac{1}{2\sqrt{u}} du \wedge \omega_1^0 - \sqrt{u}\varphi_{12}^0 \wedge \omega_2^0\right) \\ &= -\left(\frac{u_2}{2u}\omega_1^0 - \frac{u_1}{2u}\omega_2^0 + \varphi_{12}^0\right) \wedge \omega_2. \end{aligned}$$

Thus

$$\varphi_{12} = \frac{u_2}{2u}\omega_1^0 - \frac{u_1}{2u}\omega_2^0 + \varphi_{12}^0 = \varphi_{12}^0 - \frac{1}{2} * d\ln u,$$

where $*$ denotes the Hodge star operator. So we conclude

$$K\omega_1 \wedge \omega_2 = d\varphi_{12} = d\varphi_{12}^0 - \frac{1}{2}d * d\ln u = k\omega_1^0 \wedge \omega_2^0 - \frac{1}{2}\triangle(\ln u)\,\omega_1^0 \wedge \omega_2^0,$$

and the result follows immediately. ■

From (6.4) we can reduce the hyperbolic geometric flow (1.1) in a point-wise conformal class to

$$\frac{\partial^2 u}{\partial t^2} = \triangle_{g_0} \ln u - 2k(x, y), \tag{6.5}$$

and moreover we can consider the Cauchy problem for (6.5)

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = \triangle_{g_0} \ln u - 2k(x, y), \\ t = 0: \quad u = u_0(x, y), \quad u_t = u_1(x, y), \end{cases} \tag{6.6}$$

where u_0 is a positive function and u_1 is an arbitrary function on $\mathscr{M}$.

Let us now consider (6.5) or (6.6). As investigated by Kong and Liu [6], we know that the hyperbolic geometric flow on three dimension has a deep relation with the Einstein equations. It is a challenging problem to find periodic solutions and blowing up phenomena of the hyperbolic geometric flow as in the Einstein equation case. It is interesting that in two dimensional case we can prove the following theorem which gives some obstructions to the existence of smooth long time solutions and periodic solution of (6.6).

In order to state our result, we introduce

Definition 6.1 *A solution of (6.6) is regular, if it is positive and smooth.* □

Theorem 6.1 *Let $(\mathscr{M}, g_0)$ be a compact Riemannian surface with metric g_0, $\chi(\mathscr{M})$ denotes its Euler characteristic number. If $u(t, x, y)$ is a regular solution of (6.6). We have*

(a) If $\chi(\mathscr{M}) > 0$, then any solution of the reduced problem (6.6) must blow up in finite time for any initial value (u_0, u_1);

(b) If $\chi(\mathscr{M}) \neq 0$, there is no periodic regular solution of equation (6.5). Moreover in the case $\chi(\mathscr{M}) = 0$, if there is a periodic solution $u(t, x, y)$ of (6.5), then we must have $\int_{\mathscr{M}} u(t, x, y)\, dV_{g_0} = c$ for some positive constant c;

(c) If $\chi(\mathscr{M}) = 0$, and $u(t, x, y)$ is a regular solution of (6.6). Then if $\int_{\mathscr{M}} u_1(x, y)\, dV_{g_0} < 0$, $u(t, x, y)$ must blow up in finite time. □

Proof. Let $u(t,x,y)$ be a regular solution of (6.5) or (6.6). We denote the volume of $\mathscr{M}$ with respect to the metric $u(t,x,y)\, g_0$ by

$$V(t) = \int_{\mathscr{M}} u(t,x,y)\, dV_{g_0}.$$

Then taking integration on both sides of (6.5) and using Gauss-Bonnet formula, we have

$$\frac{d^2 V(t)}{dt^2} = -4\pi\chi(\mathscr{M}),$$

this means

$$V(t) = -2\pi\chi(\mathscr{M})\, t^2 + c_1 t + c_2 \tag{6.7}$$

for some constants c_1 and c_2. If we consider the Cauchy problem (6.6), we can easily get the numbers c_1 and c_2, $c_2 = \int_{\mathscr{M}} u_0(x,y)\, dV_{g_0}$ and $c_1 = \int_{\mathscr{M}} u_1(x,y)\, dV_{g_0}$. From (6.7) we conclude (a) by the assumption $\chi(\mathscr{M}) > 0$. Otherwise, for some sufficiently large number T, we have $V(T) \leq 0$, which is impossible.

To prove (b), we just notice that if $u(t,x,y)$ is a periodic solution of (6.5), then $V(t)$ must be a periodic function of parameter t, which is impossible unless $\chi(\mathscr{M}) = 0$ and $c_1 = 0$ in (6.7).

The conclusion (c) can also be concluded from the expression (6.7). In this case we have

$$V(t) = c_1 t + c_2,$$

but

$$c_1 = \int_{\mathscr{M}} u_1(x,y)\, dV_{g_0} < 0,$$

so $u(t,x,y)$ must blow up in finite time. This proves Theorem 6.1. ■

From the above theorem, we see that problem (6.6) or equation (6.5) have essential geometric obstruction to the existence of long time solution or periodic solution. However, as in the study of Ricci flow on Riemann surfaces, we consider the following normalized equation for the equation (6.5)

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = \triangle_{g_0} \ln u - 2k(x,y) + \dfrac{4\pi\chi(\mathscr{M})}{V(g_0)}, \\ t = 0: \quad u = u_0(x,y), \quad u_t = u_1(x,y), \end{cases} \tag{6.8}$$

where $V(g_0)$ denotes the volume of $\mathscr{M}$ with respect to metric g_0. We have

Theorem 6.2 *Let* $(\mathscr{M}, g_0)$ *be a compact Riemann surface with metric* g_0, $\chi(\mathscr{M})$ *denotes its Euler characteristic number. If* $u(t,x,y)$ *is a regular solution of (6.8), then we have*

(i) If $\int_{\mathscr{M}} u_1(x,y)\,dV_{g_0} < 0$, *then the volume function* $V(t)$ *is linearly contracting and* $u(t,x,y)$ *must blow up in a finite time;*

(ii) If $\int_{\mathscr{M}} u_1(x,y)\,dV_{g_0} > 0$, *then the volume function* $V(t)$ *is linearly expanding.* □

This theorem can be proved in a manner similar to the proof of (6.7), here we omit the details.

Theorem 6.2 reveals that the hyperbolic geometric flow has very different features from the traditional Ricci flow, because even for the normalized equation (6.8) we can not get the long-time existence of solutions which depends on the velocity u_1. This also gives an evidence that, by choosing the initial velocity of the hyperbolic geometric flow, we may obtain the long time existence of the corresponding solution. We believe that the above results may reveal some interesting features of the Einstein equations about the expansion of the Universe, if one could connect the foliation of solution of the Einstein equations in three dimension with the solution of (6.6) or (6.8). We will pursue this problem in a forthcoming paper.

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