# NEW RESULTS OF INTERSECTION NUMBERS ON MODULI SPACES OF CURVES

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Abstract. We present a series of new results we obtained recently about the intersection numbers of tautological classes on moduli spaces of curves, including a simple formula of the n-point functions for Witten's $\tau$ classes, an effective recursion formula to compute higher Weil-Petersson volumes, several new recursion formulae of intersection numbers and our proof of a conjecture of Itzykson and Zuber concerning denominators of intersection numbers. We also present Virasoro and KdV properties of generating functions of general mixed $\kappa$ and $\psi$ intersections.

## Introduction

Let $\overline{\mathcal{M}}_{g,n}$ denote the Deligne-Mumford moduli stack of stable curves of genus $g$ with $n$ marked points. Let $\psi_i$ be the first Chern class of the line bundle whose fiber over each pointed stable curve is the cotangent line at the $i$-th marked point. Let $\lambda_i$ be the $i$-th Chern class of the Hodge bundle $\mathbb{E}$, whose fiber over each pointed stable curves is $H^0(C, \omega_C)$.

We also have the $\kappa$ classes originally defined by Mumford [1], Morita [2] and Miller [3]. A more natural variation was later given by Arbarello-Cornalba [4]. It is known that the $\kappa$ and $\psi$ classes generate the tautological cohomology ring of the moduli spaces, and most of the known cohomology classes are tautological.

The following intersection numbers

$$\langle \tau_{d_1} \cdots \tau_{d_n} \prod_{j\geq 1} \kappa_j^{b_j} \rangle_g := \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \cdots \psi_n^{d_n} \prod_{j\geq 1} \kappa_j^{b_j}.$$

are called the higher Weil-Petersson volumes [5]. These are important invariants of moduli spaces of curves.

In 1990, Witten [6] made the remarkable conjecture that the generating function of intersection numbers of $\psi$ classes on moduli spaces are governed by KdV hierarchy. Witten's conjecture (first proved by Kontsevich [7]) is among the deepest known properties of moduli spaces of curves and motivated a surge of subsequent developments.

The intersection theory of tautological classes on the moduli space of curves is a very important subject and has close connections to string theory, quantum gravity and many branches of mathematics.

## The n-point functions for intersection numbers

**Definition 1.** We call the following generating function

$$F(x_1,\dots,x_n)=\sum_{g=0}^{\infty}\sum_{\sum d_j=3g-3+n}\langle\tau_{d_1}\cdots\tau_{d_n}\rangle_g\prod_{j=1}^{n}x_j^{d_j}$$

the $n$-point function.

Consider the following "normalized" $n$-point function

$$G(x_1,\dots,x_n)=\exp\left(\frac{-\sum_{j=1}^n x_j^3}{24}\right)\cdot F(x_1,\dots,x_n).$$

Starting from 1-point function $G(x)=\frac{1}{x^2}$, we can obtain any $n$-point function recursively by the following theorem.

**Theorem 2.** [8] *For $n\geq 2$,*

$$G(x_1,\dots,x_n)=\sum_{r,s\geq 0}\frac{(2r+n-3)!!}{4^s(2r+2s+n-1)!!}P_r(x_1,\dots,x_n)\Delta(x_1,\dots,x_n)^s,$$

*where $P_r$ and $\Delta$ are homogeneous symmetric polynomials defined by*

$$\Delta(x_1,\dots,x_n)=\frac{(\sum_{j=1}^n x_j)^3-\sum_{j=1}^n x_j^3}{3},$$

$$P_r(x_1,\dots,x_n)=\left(\frac{1}{2\sum_{j=1}^n x_j}\sum_{\underline{n}=I\coprod J}(\sum_{i\in I}x_i)^2(\sum_{i\in J}x_i)^2G(x_I)G(x_J)\right)_{3r+n-3}$$

$$=\frac{1}{2\sum_{j=1}^n x_j}\sum_{\underline{n}=I\coprod J}(\sum_{i\in I}x_i)^2(\sum_{i\in J}x_i)^2\sum_{r'=0}^{r}G_{r'}(x_I)G_{r-r'}(x_J),$$

*where $I,J\neq\emptyset$, $\underline{n}=\{1,2,\dots,n\}$ and $G_g(x_I)$ denotes the degree $3g+|I|-3$ homogeneous component of the normalized $|I|$-point function $G(x_{k_1},\dots,x_{k_{|I|}})$, where $k_j\in I$.*

Thus we have an elementary and more efficient algorithm to calculate all intersection numbers of $\psi$ classes other than the celebrated Witten-Kontsevich theorem.

Since $P_0(x,y)=\frac{1}{x+y}$, $P_r(x,y)=0$ for $r>0$ and

$$P_r(x,y,z)=\frac{r!}{2^r(2r+1)!}\frac{(xy)^r(x+y)^{r+1}+(yz)^r(y+z)^{r+1}+(zx)^r(z+x)^{r+1}}{x+y+z},$$

we recover Dijkgraaf's 2-point function and Zagier's 3-point function obtained more than ten years ago.

There is another slightly different formula of the $n$-point functions. When $n=3$, this has also been obtained by Zagier.

**Theorem 3.** [8] *For $n\geq 2$,*

$$F(x_1,\dots,x_n)=\exp\frac{(\sum_{j=1}^n x_j)^3}{24}\sum_{r,s\geq 0}\frac{(-1)^sP_r(x_1,\dots,x_n)\Delta(x_1,\dots,x_n)^s}{8^s(2r+2s+n-1)s!}$$

*where $P_r$ and $\Delta$ are the same polynomials as defined in Theorem 2.*

Okounkov [9] obtained an analytic expression of the $n$-point functions using $n$-dimensional error-function-type integrals. Brézin and Hikami [10] use correlation functions of GUE ensemble to find explicit formulae of $n$-point functions.

## Recursion formulae of higher Weil-Petersson volumes

We have discovered a general recursion formula of higher Weil-Petersson volumes [12], which is a vast generalization of the Mirzakhani recursion formula [11].

First we fix notations as in [5].

Consider the semigroup $N^\infty$ of sequences $\mathbf{m} = (m_1, m_2, \dots)$ where $m_i$ are non-negative integers and $m_i = 0$ for sufficiently large $i$.

Let $\mathbf{m}, \mathbf{t}, \mathbf{a_1}, \dots, \mathbf{a_n} \in N^\infty$, $\mathbf{m} = \sum_{i=1}^n \mathbf{a_i}$, and $\mathbf{s} := (s_1, s_2, \dots)$ be a family of independent formal variables.

$$|\mathbf{m}| := \sum_{i\geq 1} i m_i, \quad ||\mathbf{m}|| := \sum_{i\geq 1} m_i, \quad \mathbf{s}^{\mathbf{m}} := \prod_{i\geq 1} s_i^{m_i}, \quad \mathbf{m}! := \prod_{i\geq 1} m_i!,$$

$$\binom{\mathbf{m}}{\mathbf{t}} := \prod_{i\geq 1} \binom{m_i}{t_i}, \quad \binom{\mathbf{m}}{\mathbf{a_1}, \dots, \mathbf{a_n}} := \prod_{i\geq 1} \binom{m_i}{a_1(i), \dots, a_n(i)}.$$

Let $\mathbf{b} \in N^\infty$, we denote a formal monomial of $\kappa$ classes by

$$\kappa(\mathbf{b}) := \prod_{i\geq 1} \kappa_i^{b_i}.$$

**Theorem 4.** [12] *Let $\mathbf{b} \in N^\infty$ and $d_j \geq 0$.*

$$\begin{aligned}
&(2d_1+1)!!\langle \kappa(\mathbf{b}) \prod_{j=1}^n \tau_{d_j} \rangle_g \\
&= \sum_{j=2}^n \sum_{\mathbf{L}+\mathbf{L}'=\mathbf{b}} \alpha_{\mathbf{L}} \binom{\mathbf{b}}{\mathbf{L}} \frac{(2(|\mathbf{L}|+d_1+d_j)-1)!!}{(2d_j-1)!!} \langle \kappa(\mathbf{L}')\tau_{|\mathbf{L}|+d_1+d_j-1} \prod_{i\neq 1,j} \tau_{d_i} \rangle_g \\
&+ \frac{1}{2} \sum_{\mathbf{L}+\mathbf{L}'=\mathbf{b}} \sum_{r+s=|\mathbf{L}|+d_1-2} \alpha_{\mathbf{L}} \binom{\mathbf{b}}{\mathbf{L}} (2r+1)!!(2s+1)!! \langle \kappa(\mathbf{L}')\tau_r\tau_s \prod_{i\neq 1} \tau_{d_i} \rangle_{g-1} \\
&\quad + \frac{1}{2} \sum_{\substack{\mathbf{L}+\mathbf{e}+\mathbf{f}=\mathbf{b} \\ I \coprod J = \{2,\dots,n\}}} \sum_{r+s=|\mathbf{L}|+d_1-2} \alpha_{\mathbf{L}} \binom{\mathbf{b}}{\mathbf{L}, \mathbf{e}, \mathbf{f}} (2r+1)!!(2s+1)!! \\
&\qquad\qquad \times \langle \kappa(\mathbf{e})\tau_r \prod_{i\in I} \tau_{d_i} \rangle_{g'} \langle \kappa(\mathbf{f})\tau_s \prod_{i\in J} \tau_{d_i} \rangle_{g-g'}.
\end{aligned}$$

*These tautological constants $\alpha_{\mathbf{L}}$ can be determined recursively from the following formula*

$$\sum_{\mathbf{L}+\mathbf{L}'=\mathbf{b}} \frac{(-1)^{||\mathbf{L}||}\alpha_{\mathbf{L}}}{\mathbf{L}!\mathbf{L}'!(2|\mathbf{L}'|+1)!!} = 0, \qquad \mathbf{b} \neq 0,$$

*namely*

$$\alpha_{\mathbf{b}} = \mathbf{b}! \sum_{\substack{\mathbf{L}+\mathbf{L}'=\mathbf{b} \\ \mathbf{L}'\neq \mathbf{0}}} \frac{(-1)^{||\mathbf{L}'||-1}\alpha_{\mathbf{L}}}{\mathbf{L}!\mathbf{L}'!(2|\mathbf{L}'|+1)!!}, \qquad \mathbf{b} \neq 0,$$

*with the initial value $\alpha_{\mathbf{0}} = 1$.*

The proof of the above theorem is to use Witten-Kontsevich theorem, a combinatorial formula in [5] expressing $\kappa$ classes by $\psi$ classes and the following elementary but crucial lemma [12].

**Lemma 5.** *Let $F(\mathbf{L},n)$ and $G(\mathbf{L},n)$ be two functions defined on $N^\infty\times\mathbb{N}$, where $\mathbb{N}=\{0,1,2,\dots\}$ is the set of nonnegative integers. Let $\alpha_{\mathbf{L}}$ and $\beta_{\mathbf{L}}$ be real numbers depending only on $\mathbf{L}\in N^\infty$ that satisfy $\alpha_{\mathbf{0}}\beta_{\mathbf{0}}=1$ and*

$$\sum_{\mathbf{L}+\mathbf{L}'=\mathbf{b}}\alpha_{\mathbf{L}}\beta_{\mathbf{L}'}=0,\qquad \mathbf{b}\neq 0.$$

*Then the following two identities are equivalent.*

$$G(\mathbf{b},n)=\sum_{\mathbf{L}+\mathbf{L}'=\mathbf{b}}\alpha_{\mathbf{L}}F(\mathbf{L}',n+|\mathbf{L}|),\quad \forall\ (\mathbf{b},n)\in N^\infty\times\mathbb{N}$$
$$F(\mathbf{b},n)=\sum_{\mathbf{L}+\mathbf{L}'=\mathbf{b}}\beta_{\mathbf{L}}G(\mathbf{L}',n+|\mathbf{L}|),\quad \forall\ (\mathbf{b},n)\in N^\infty\times\mathbb{N}$$

When $\mathbf{b}=(l,0,0,\dots)$, Theorem 4 recovers Mirzakhani's recursion formula of Weil-Petersson volumes for moduli spaces of bordered Riemann surfaces [13, 14, 15, 16].

Theorem 4 also provides an effective algorithm to compute higher Weil-Petersson volumes recursively.

In fact we can use the main formula in [5] to generalize almost all pure $\psi$ intersections to identities of higher Weil-Petersson volumes which share similar structures as Theorem 4. For example, the identities in the following theorem are generalizations of the string and dilaton equations.

**Theorem 6.** [12] *For $\mathbf{b}\in N^\infty$ and $d_j\ge 0$,*

$$\sum_{\mathbf{L}+\mathbf{L}'=\mathbf{b}}(-1)^{||\mathbf{L}||}\binom{\mathbf{b}}{\mathbf{L}}\langle\tau_{|\mathbf{L}|}\prod_{j=1}^n\tau_{d_j}\kappa(\mathbf{L}')\rangle_g=\sum_{j=1}^n\langle\tau_{d_j-1}\prod_{i\neq j}\tau_{d_i}\kappa(\mathbf{b})\rangle_g,$$

*and*

$$\sum_{\mathbf{L}+\mathbf{L}'=\mathbf{b}}(-1)^{||\mathbf{L}||}\binom{\mathbf{b}}{\mathbf{L}}\langle\tau_{|\mathbf{L}|+1}\prod_{j=1}^n\tau_{d_j}\kappa(\mathbf{L}')\rangle_g=(2g-2+n)\langle\prod_{j=1}^n\tau_{d_j}\kappa(\mathbf{b})\rangle_g.$$

Note that Theorem 6 generalizes the results in [17].

## New identities of intersection numbers

The next two theorems follow from a detailed study of coefficients of the $n$-point functions in Theorem 2.

**Theorem 7.** [8] *We have*

(1) *Let $k>2g$, $d_j\ge 0$ and $\sum_{j=1}^n d_j=3g+n-k$.*

$$\sum_{\underline{n}=I\coprod J}\sum_{j=0}^k(-1)^j\langle\tau_j\tau_0^2\prod_{i\in I}\tau_{d_i}\rangle_{g'}\langle\tau_{k-j}\tau_0^2\prod_{i\in J}\tau_{d_i}\rangle_{g-g'}=0.$$

(2) *Let* $d_j \geq 1$ *and* $\sum_{j=1}^n d_j = g+n$.

$$\sum_{\underline{n}=I\coprod J}\sum_{j=0}^{2g}(-1)^j\langle\tau_j\tau_0^2\prod_{i\in I}\tau_{d_i}\rangle_{g'}\langle\tau_{2g-j}\tau_0^2\prod_{i\in J}\tau_{d_i}\rangle_{g-g'}=\frac{(2g+n+1)!}{4^g(2g+1)!\prod_{j=1}^n(2d_j-1)!!}.$$

**Theorem 8.** [18, 8] *We have*

(1) *Let* $k > 2g$, $d_j \geq 0$ *and* $\sum_{j=1}^n d_j = 3g+n-k-1$.

$$\sum_{j=0}^{k}(-1)^j\langle\tau_{k-j}\tau_j\prod_{i=1}^n\tau_{d_i}\rangle_g=0.$$

(2) *Let* $d_j \geq 1$ *and* $\sum_{j=1}^n (d_j-1) = g-1$.

$$\sum_{j=0}^{2g}(-1)^j\langle\tau_{2g-j}\tau_j\prod_{i=1}^n\tau_{d_i}\rangle_g=\frac{(2g+n-1)!}{4^g(2g+1)!\prod_{j=1}^n(2d_j-1)!!}.$$

In fact, it's easy to see that Theorems 7 and 8 imply each other through the following proposition.

**Proposition 9.** [8] *Let* $d_j \geq 0$ *and* $\sum_{j=1}^n d_j = g+n$.

$$\sum_{\underline{n}=I\coprod J}\sum_{j=0}^{2g}(-1)^j\left(\langle\tau_j\tau_0^2\prod_{i\in I}\tau_{d_i}\rangle\langle\tau_{2g-j}\tau_0^2\prod_{i\in J}\tau_{d_i}\rangle+\langle\tau_j\tau_{2g-j}\tau_0^2\prod_{i\in I}\tau_{d_i}\rangle\langle\tau_0^2\prod_{i\in J}\tau_{d_i}\rangle\right)$$
$$=(2g+n+1)\sum_{j=0}^{2g}(-1)^j\langle\tau_0\tau_j\tau_{2g-j}\prod_{i=1}^n\tau_{d_i}\rangle_g$$

Since $\mathrm{ch}_k(\mathbb{E})=0$ for $k>2g$, $\lambda_g\lambda_{g-1}=(-1)^{g-1}(2g-1)!\mathrm{ch}_{2g-1}(\mathbb{E})$, by Mumford's formula [1] of the Chern character of Hodge bundles, it's not difficult to see that Theorem 8 implies the following theorem.

**Theorem 10.** [18, 8] *Let* $k$ *be an even number and* $k \geq 2g$, $d_j \geq 0$, $\sum_{j=1}^n d_j = 3g+n-k-2$.

$$(1)\quad\langle\prod_{j=1}^n\tau_{d_j}\tau_k\rangle_g=\sum_{j=1}^n\langle\tau_{d_j+k-1}\prod_{i\neq j}\tau_{d_i}\rangle_g$$
$$-\frac{1}{2}\sum_{\underline{n}=I\coprod J}\sum_{j=0}^{k-2}(-1)^j\langle\tau_j\prod_{i\in I}\tau_{d_i}\rangle_{g'}\langle\tau_{k-2-j}\prod_{i\in J}\tau_{d_i}\rangle_{g-g'}.$$

Note that when $k=2g$, the above theorem is equivalent to the following Hodge integral identity [19] (also known as Faber's intersection number conjecture [20])

$$\int_{\overline{\mathcal{M}}_{g,n}}\psi_1^{d_1}\dots\psi_n^{d_n}\lambda_g\lambda_{g-1}=\frac{(2g-3+n)!|B_{2g}|}{2^{2g-1}(2g)!\prod_{j=1}^n(2d_j-1)!!}.$$

where $\sum_{j=1}^n(d_j-1)=g-2$ and $d_j \geq 1$.

The above $\lambda_g\lambda_{g-1}$ integral follows from degree 0 Virasoro constraints for $\mathbb{P}^2$ announced by Givental [21]. However it is very desirable to have a direct proof of identity (1) when $k=2g$, possibly using our explicit formulae of the $n$-point functions (see also [22]).

As pointed out in the last section, we can generalize all of the above new recursion formulae of $\psi$ classes to identities of higher Weil-Petersson volumes. For example, we may generalize Proposition 9 and Theorem 10 to the following

**Proposition 11.** [12] *Let* $\mathbf{b}\in N^\infty$, $d_j\geq 0$.

$$\begin{aligned}
&\sum_{j=0}^{2g}(-1)^j\langle\tau_0\tau_1\tau_j\tau_{2g-j}\prod_{j=1}^n\tau_{d_j}\kappa(\mathbf{b})\rangle_g\\
&\quad=\sum_{\substack{\mathbf{L}+\mathbf{L}'=\mathbf{b}\\ \underline{n}=I\coprod J}}\sum_{j=0}^{2g}(-1)^j\binom{\mathbf{b}}{\mathbf{L}}\Bigg(\langle\tau_j\tau_0^2\prod_{i\in I}\tau_{d_i}\kappa(\mathbf{L})\rangle\langle\tau_{2g-j}\tau_0^2\prod_{i\in J}\tau_{d_i}\kappa(\mathbf{L}')\rangle\\
&\qquad\qquad+\langle\tau_j\tau_{2g-j}\tau_0^2\prod_{i\in I}\tau_{d_i}\kappa(\mathbf{L})\rangle\langle\tau_0^2\prod_{i\in J}\tau_{d_i}\kappa(\mathbf{L}')\rangle\Bigg)
\end{aligned}$$

**Theorem 12.** [12] *Let* $\mathbf{b}\in N^\infty$, $M\geq 2g$ *be an even number and* $d_j\geq 0$.

$$\begin{aligned}
&\sum_{\mathbf{L}+\mathbf{L}'=\mathbf{b}}(-1)^{||\mathbf{L}||}\binom{\mathbf{b}}{\mathbf{L}}\langle\tau_{|\mathbf{L}|+M}\prod_{j=1}^n\tau_{d_j}\kappa(\mathbf{L}')\rangle_g=\sum_{j=1}^n\langle\tau_{d_j+M-1}\prod_{i\neq j}\tau_{d_i}\kappa(\mathbf{b})\rangle_g\\
&\quad-\frac{1}{2}\sum_{\substack{\mathbf{L}+\mathbf{L}'=\mathbf{b}\\ \underline{n}=I\coprod J}}\sum_{j=0}^{M-2}(-1)^j\binom{\mathbf{b}}{\mathbf{L}}\langle\tau_j\prod_{i\in I}\tau_{d_i}\kappa(\mathbf{L})\rangle_{g'}\langle\tau_{M-2-j}\prod_{i\in J}\tau_{d_i}\kappa(\mathbf{L}')\rangle_{g-g'}.
\end{aligned}$$

We also found the following conjectural identity experimentally, which is amazing if compared with Theorems 8 and 10.

**Conjecture 13.** [18] *Let* $g\geq 2$, $d_j\geq 1$, $\sum_{j=1}^n(d_j-1)=g$.

$$\begin{aligned}
\frac{(2g-3+n)!}{2^{2g+1}(2g-3)!\prod_{j=1}^n(2d_j-1)!!}&=\langle\prod_{j=1}^n\tau_{d_j}\tau_{2g-2}\rangle_g-\sum_{j=1}^n\langle\tau_{d_j+2g-3}\prod_{i\neq j}\tau_{d_i}\rangle_g\\
&\quad+\frac{1}{2}\sum_{\underline{n}=I\coprod J}\sum_{j=0}^{2g-4}(-1)^j\langle\tau_j\prod_{i\in I}\tau_{d_i}\rangle_{g'}\langle\tau_{2g-4-j}\prod_{i\in J}\tau_{d_i}\rangle_{g-g'}.
\end{aligned}$$

Since $(2g-3)!\mathrm{ch}_{2g-3}(\mathbb{E})=(-1)^{g-1}(3\lambda_{g-3}\lambda_g-\lambda_{g-1}\lambda_{g-2})$, it's easy to see that the above identity is equivalent to the following identity of Hodge integrals.

**Conjecture 14.** *Let* $g\geq 2$, $d_j\geq 1$, $\sum_{j=1}^n(d_j-1)=g$.

$$\begin{aligned}
&\frac{2g-2}{|B_{2g-2}|}\left(\langle\prod_{j=1}^n\tau_{d_j}\mid\lambda_{g-1}\lambda_{g-2}\rangle_g-3\langle\prod_{j=1}^n\tau_{d_j}\mid\lambda_{g-3}\lambda_g\rangle_g\right)\\
&\quad=\frac{1}{2}\sum_{j=0}^{2g-4}(-1)^j\langle\tau_{2g-4-j}\tau_j\prod_{i=1}^n\tau_{d_i}\rangle_{g-1}+\frac{(2g-3+n)!}{2^{2g+1}(2g-3)!\prod_{j=1}^n(2d_j-1)!!}.
\end{aligned}$$

**Virasoro constraints and KdV hierarchy**

From Theorem 4, we found new Virasoro constraints and KdV hierarchy for generating functions of higher Weil-Petersson volumes which vastly generalize the Witten conjecture and the results of Mulase and Safnuk [15].

Let $\mathbf{s} := (s_1, s_2, \dots)$ and $\mathbf{t} := (t_0, t_1, t_2, \dots)$, we introduce the following generating function

$$G(\mathbf{s}, \mathbf{t}) := \sum_g \sum_{\mathbf{m},\mathbf{n}} \langle \kappa_1^{m_1} \kappa_2^{m_2} \cdots \tau_0^{n_0} \tau_1^{n_1} \cdots \rangle_g \frac{\mathbf{s}^{\mathbf{m}}}{\mathbf{m}!} \prod_{i=0}^{\infty} \frac{t_i^{n_i}}{n_i!},$$

where $\mathbf{s}^{\mathbf{m}} = \prod_{i \geq 1} s_i^{m_i}$.

We introduce the following family of differential operators for $k \geq -1$,

$$V_k = -\frac{1}{2} \sum_{\mathbf{L}} (2(|\mathbf{L}| + k) + 3)!! \gamma_{\mathbf{L}} \mathbf{s}^{\mathbf{L}} \frac{\partial}{\partial t_{|\mathbf{L}|+k+1}} + \frac{1}{2} \sum_{j=0}^{\infty} \frac{(2(j+k)+1)!!}{(2j-1)!!} t_j \frac{\partial}{\partial t_{j+k}}$$
$$+ \frac{1}{4} \sum_{d_1+d_2=k-1} (2d_1+1)!!(2d_2+1)!! \frac{\partial^2}{\partial t_{d_1} \partial t_{d_2}} + \frac{\delta_{k,-1} t_0^2}{4} + \frac{\delta_{k,0}}{48},$$

where $\gamma_{\mathbf{L}}$ are defined by

$$\gamma_{\mathbf{L}} = \frac{(-1)^{||\mathbf{L}||}}{\mathbf{L}!(2|\mathbf{L}|+1)!!}.$$

**Theorem 15.** [12, 15] *We have $V_k \exp(G) = 0$ for $k \geq -1$ and the operators $V_k$ satisfy the Virasoro relations*

$$[V_n, V_m] = (n-m) V_{n+m}.$$

The Witten-Kontsevich theorem states that the generating function for $\psi$ class intersections

$$F(t_0, t_1, \dots) = \sum_g \sum_{\mathbf{n}} \langle \prod_{i=0}^{\infty} \tau_i^{n_i} \rangle_g \prod_{i=0}^{\infty} \frac{t_i^{n_i}}{n_i!}$$

is a $\tau$-function for the KdV hierarchy.

Since Virasoro constraints uniquely determine the generating functions $G(\mathbf{s}, t_0, t_1, \dots)$ and $F(t_0, t_1, \dots)$, we have the following theorem.

**Theorem 16.** [12, 15]

$$G(\mathbf{s}, t_0, t_1, \dots) = F(t_0, t_1, t_2 + p_2, t_3 + p_3, \dots),$$

*where $p_k$ are polynomials in $\mathbf{s}$ given by*

$$p_k = \sum_{|\mathbf{L}|=k-1} \frac{(-1)^{||\mathbf{L}||-1}}{\mathbf{L}!} \mathbf{s}^{\mathbf{L}}.$$

*In particular, for any fixed values of $\mathbf{s}$, $G(\mathbf{s}, \mathbf{t})$ is a $\tau$-function for the KdV hierarchy.*

Theorem 16 also generalized results in [23].

## Denominators of intersection numbers

Let *denom*$(r)$ denotes the denominator of a rational number $r$ in reduced form (coprime numerator and denominator, positive denominator). We define

$$D_{g,n} = lcm \left\{ denom \left( \langle \prod_{j=1}^{n} \tau_{d_j} \rangle_g \right) \Bigg| \sum_{j=1}^{n} d_j = 3g - 3 + n \right\}$$

and for $g \geq 2$,

$$\mathcal{D}_g = lcm\left\{denom\left(\int_{\overline{\mathcal{M}}_g} \kappa(\mathbf{b})\right) \Big| \; |\mathbf{b}| = 3g-3\right\}$$

where *lcm* denotes *least common multiple*.

Since denominators of intersection numbers on $\overline{\mathcal{M}}_{g,n}$ all come from orbifold quotient singularities, the divisibility properties of $D_{g,n}$ and $\mathcal{D}_g$ should reflect overall behavior of singularities.

We have the following properties of $D_{g,n}$ and $\mathcal{D}_g$.

**Proposition 17.** [24] *We have* $D_{g,n} \mid D_{g,n+1}$, $D_{g,n} \mid \mathcal{D}_g$ *and* $\mathcal{D}_g = D_{g,3g-3}$.

**Theorem 18.** [24] *For* $1 < g' \leq g$*, the order of any automorphism group of a Riemann surface of genus* $g'$ *divides* $D_{g,3}$.

The following corollary of Theorem 18 is a conjecture raised by Itzykson and Zuber [25] in 1992.

**Corollary 19.** *For* $1 < g' \leq g$*, the order of any automorphism group of an algebraic curve of genus* $g'$ *divides* $\mathcal{D}_g$.

The proof of Theorem 18 needs the following two lemmas (see [24]).

**Lemma 20.** *If* $p \leq g+1$ *is a prime number, then* $\mathrm{ord}(p, D_{g,3}) \geq 2$.

**Lemma 21.** [26] *Let* $X$ *be a Riemann Surface of genus* $g \geq 2$*, then for any prime number* $p$,

$$\mathrm{ord}(p, |Aut(X)|) \leq \lfloor \log_p \frac{2pg}{p-1} \rfloor + \mathrm{ord}(p, 2(g-1)).$$

We have also obtained conjectural exact values of $\mathcal{D}_g$ for all $g$ in [18].

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