Structure and Representation of Non-Balanced Quantum Doubles*

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Abstract

Given a right G-set X, S.J.Witherspoon in [11] has introduced a k-algebra $D_X(G)$ and give an equivalent description of the category $\mathbf{mod}D_X(G)$.

In this paper, the authors firstly define one special kind of quivers $Q_X(G)$ with relations ρ_X such that $D_X(G)$ is isomorphic to $k(Q_X(G), \rho_X)$, a factor of the path algebra, and hence the category $\mathbf{mod}D_X(G)$ can be described by the representation of such quivers with relations. Then, under some certain conditions when X is also a group, they show that $D_X(G)$ admits a Hopf algebra structure; isomorphically, $k(Q_X(G), \rho_X)$ admits a Hopf algebra structure although $Q_X(G)$ is not a covering quiver, which means that the quantum double $D(G)$ is isomorphic to a factor of a path algebra. They also find a sufficient and necessary condition under which $D_X(G)$ becomes a quasi-triangular Hopf algebra, generalizing the classical quantum double $D(G)$ for a finite group G. In this case, $D_X(G)$ is called a *non-balanced quantum double*. Finally, some common properties held by $D_X(G)$ and kG are considered, such as the semisimplicity, the unimodularity and the representation type, which indicated that $D_X(G)$ still keeps some nice properties which are true for kG.

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1 Introduction and Preliminaries

In this paper, k is a field, modules are always finitely generated right modules over an k-algebra and tensor products are over k unless otherwise indicated. All concepts about quantum doubles and Hopf algebras can be found in [6] and [9], while the concepts about

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path algebras and representations of quivers can be found in [1] and [2]. And, G always denotes a finite group and X a finite right G-set. For a k-algebra A, denote by **mod**A the category of finitely generated right A-modules.

Firstly, let us recall the Hopf algebra structure of the *quantum double* $D(G)$ of the group G. $D(G) = (kG^{op})^* \otimes kG$ as k-vector spaces. If denote by $\{\phi_g\}_{g\in G}$ the k-basis of $(kG^{op})^*$ dual to $\{g\}_{g\in G}$, then $D(G)$ has a k-basis consisting of all elements $\phi_g \otimes h$, which is briefly written as $\phi_g h$, for $g,\ h \in G$. On this basis, the multiplication is defined by $\phi_g h \phi_{g'} h' = \phi_g \phi_{hg'h^{-1}} hh'$, which is nonzero if and only if $g = hg'h^{-1}$. Thus the unit is $1_{D(G)} = \sum_{g\in G} \phi_g 1_G$, where 1_G is the identity of G. The comultiplication is given by

$$\Delta(\phi_g h) = \sum_{x\in G} \phi_{x^{-1}g} h \otimes \phi_x h,$$

the counit by $\epsilon(\phi_g h) = \delta_{1,g}$, and the antipode by $S(\phi_g h) = \phi_{h^{-1}g^{-1}h} h^{-1}$. Moreover, it is known that $D(G)$ is a quasi-triangular Hopf algebra with the universal R-matrix

$$R = \sum_{g\in G} g \otimes \phi_g \in D(G) \otimes D(G).$$

As well-known, the theory of quantum double and quantum group play an important role in the field of mathematical physics, especially for its quasi-triangularity which provides a way to construct solutions for the Yang-Baxter equation. In the classical quantum double $D(G) = (kG^{op})^* \otimes kG$, the two groups on both sides of $\otimes$ are coincide. In this paper, we hope to consider the possibility of the similar theory when the balance between the left and the right G's is destroyed. For this aim, we will generalize the quantum double to the so-called *non-balanced* quantum double by replacing $D(G)$ by $D_X(G)$ where X is a right G-set, and in some special cases X need to be a normal subgroup. Furthermore, as one of the motivations of this theory, we plan to consider, in our other preparing paper, how non-balanced quantum double arises in conformal field theory, specifically, in the theory of holomorphic orbifolds, as similarly in [8] for the quantum double of a finite group.

S.J.Witherspoon in [11] gave the algebraic structure on $D_X(G) = (kX)^* \otimes kG$ and found an equivalent description of **mod**$D_X(G)$ through **vect**(X, G), the category of finite dimensional G-equivalent k-vector bundles.

In Section 2, we mainly give two k-algebras which are isomorphic to $D_X(G)$, one is a factor of the path algebra $k(Q_X, \rho_X)$ and the other is the smash product algebra $(kX)^* \# kG$. Hence we get another important equivalent description of mod$D_X(G)$ by the quiver representations of (Q_X, ρ_X).

In Section 3, under some certain conditions on the right G-set X and on the G-action, we make $D_X(G)$ to become a Hopf algebra. Hence we accidently find a special class of quivers Q_X, which are not covering quivers, i.e. there is no Hopf algebra structure on the path algebra kQ_X, but there is a Hopf algebra structure on its factor $k(Q_X, \rho_X)$.

Moreover, we consider some properties of the Hopf algebras $D_X(G)$ and $k(Q_X, \rho_X)$, such as semisimplicity, unimodularity and the cohomology groups of the trivial module k. They indicate that $D_X(G)$ still keeps some nice properties which are true for kG.

In Section 4, when X is a subgroup of G, we give a sufficient and necessary condition under which it can become a quasi-triangular Hopf algebra, generalizing the classical quantum double $D(G)$ of the group G. We give such quasi-triangular Hopf algebra a new name, *non-balancd quantum double*, since X can be a proper normal subgroup of G.

In the last section, we consider the representation type of $D_X(G)$ by proving that $D_X(G)$ and kG have the same representation types when the ground field k is algebraically closed.

2 Algebraic Structure of $D_X(G)$

Throughout this section, G is a finite group and X is a finite right G-set with the action written as x^g. In [11], S.J.Witherspoon introduced the associative algebra $D_X(G)$ and found an equivalent description of $\mathbf{mod}D_X(G)$ through $\mathbf{vect}(X, G)$, the category of finite dimensional G-equivalent k-vector bundles. In this section, we firstly give two k-algebras which are both isomorphic to $D_X(G)$, one is a factor of a path algebra and the other is a smash product algebra. Hence we get another important equivalent description of the category $\mathbf{mod}D_X(G)$ via representations of quivers.

Recall that in [11], as a vector space over k, $D_X(G) = (kX)^* \otimes kG$ has a basis $\{\phi_x \otimes g\}_{x\in X, g\in G}$ where ϕ_x is the dual element of x in $(kX)^*$. Write $\phi_x \otimes g$ briefly by $\phi_x g$. Then $D_X(G)$ becomes into an associative algebra with the multiplication $\phi_x g \phi_y h = \phi_x \phi_{y^{g^{-1}}} gh$ and the unit $1_{D_X(G)} = \sum_{x\in X} \phi_x 1_G$. In case X is the G-set G under conjugation, then $D_X(G) = D(G)$, the classical quantum double of G.

A *G-equivalent k-vector bundle* U on X defined in [11] is a collection of finite dimensional vector spaces $\{U_x\}_{x\in X}$, together with a representation of G on their direct sum $\sum_{x\in X} U_x$ satisfying $U_x \cdot g = U_{x^g}$ for $x \in X, g \in G$, where U_x is called the *x-component* or *fiber* of U. If u is an element of the kG-module $\sum_{x\in X} U_x$, we write $u = \sum_{x\in X} u_x$, where $u_x \in U_x$ for each $x \in X$.

If U and V are two G-vector bundles on X, a *morphism* $f : U \to V$ is a kG-module map $f : \sum_{x\in X} U_x \to \sum_{x\in X} V_x$ which preserves fibers, that is $f(U_x) \subseteq V_x$ for all $x \in X$. An *isomorphism* is an invertible morphism.

Denote by $\mathbf{vect}(X, G)$ the category of all finite-dimensional G-equivalent k-vector bundles, which is an abelian category.

One of the main results in [11] is the following:

Theorem 2.1. ([11]) *There is an equivalence between the category* $\mathbf{mod}D_X(G)$ *and the category* $\mathbf{vect}(X, G)$*.*

Now, we give some other ways to describe the category $\mathbf{mod}D_X(G)$ by finding out two new k-algebras which are both isomorphic to $D_X(G)$ as algebras.

For a finite right G-set X, define a quiver $Q = Q_X(G)$ (write Q_X briefly) by setting the vertex set $Q_0 = X$ and the arrow set $Q_1 = \{x \longrightarrow x^g :\ x \in X,\ g \in G\}$. Here we think any two arrows $(x \longrightarrow x^g) = (y \longrightarrow y^h)$ if and only if $x = y$ and $g = h$.

Clearly, the number of arrows from x to y equals to the carditional number of the set $\{g :\ y = x^g,\ g \in G\}$. Since G is a group, it can be easily checked that:
(i) there is a loop $x \longrightarrow x^{1_G}$ for every vertex $x \in Q_0$;
(ii) there is an arrow $x \longrightarrow y$ if and only if there is an arrow $y \longrightarrow x$;
(iii) if there are two arrows $x \longrightarrow y$ and $y \longrightarrow z$, then there is an arrow $x \longrightarrow z$,
(iv) the number of arrows starting from (or ending at) any vertex is equal to $|G|$.

By (i), (ii) and (iii), the arrow $\longrightarrow$ in fact defines an equivalent relation on X. Each equivalent class is corresponding to one G-orbit, and is corresponding to a connected component of the quiver Q. Hence Q is connected if and only if there is a unique G-orbit.

Moreover, given a G-set X and a G'-set X', denote their corresponding quivers by $Q = Q_X(G)$ and $Q' = Q_{X'}(G')$ respectively, then we have:

Proposition 2.2. *$Q \cong Q'$ as quivers if and only if there is a bijection $\varphi : G \to G'$ and a bijection $\psi : X \to X'$ satisfying $\psi(x^g) = \psi(x)^{\varphi(g)}$.*

Proof. ($\Leftarrow$): ψ is considered as a bijective map between the vertex sets Q_0 and Q'_0, then we can extend ψ onto the whole quiver Q by defining

$$\psi(x \longrightarrow x^g) = \psi(x) \longrightarrow \psi(x^g) = \psi(x) \longrightarrow \psi(x)^{\varphi(g)} \in Q'_1,$$

for any arrow $x \longrightarrow x^g$ in Q_1. Then, it is easy to check that ψ is also bijective on Q_1, then it is an isomorphism from Q to Q' as quivers.

($\Rightarrow$): Let π be a bijection from Q to Q', then $\psi = \pi\mid_{Q_0}$ is a bijection from X to X'. Since $\pi(x \longrightarrow x^g) = \pi(x) \longrightarrow \pi(x^g) \in Q'_1$ for any arrow $x \longrightarrow x^g$ in Q_1, there must exist a unique $g' \in G'$ such that $\pi(x^g) = \pi(x)^{g'}$. Define a map $\varphi : G \to G'$ by setting $\varphi(g) = g'$, then $\psi(x^g) = \psi(x)^{\varphi(g)}$. Obviously, φ is a bijection from G to G' since π is an isomorphism between Q and Q'.

The End.

Define a set of relations as

$$\rho_X = \{(x \longrightarrow x^g \longrightarrow x^{gh}) - (x \longrightarrow x^{gh}),\ (x \longrightarrow x^{1_G}) - e_x :\ x \in X,\ g, h \in G\},$$

where e_x is the trivial path of length 0 corresponding to the vertex $x \in Q_0$. In the following, we call this kind quivers with such relations (Q_X, ρ_X) *the associated quiver to the G-set X*. Consider the k-algebra $k(Q_X, \rho_X) = kQ_X / < \rho_X >$, where $< \rho_X >$ denotes the ideal of the path algebra kQ_X generated by the set of relations ρ_X. For simplicity, for a path p in kQ_X, we still write p to stand for its image $\bar{p}$ in $k(Q_X, \rho_X)$. Thus, we find the following fact:

Theorem 2.3. *For a finite group G and its finite right G-set X, $D_X(G) \cong k(Q_X, \rho_X)$ as algebras, hence the category* $\mathbf{mod}D_X(G)$ *is isomorphic to the category* $\mathbf{mod}k(Q_X, \rho_X)$.

Proof. Define a k-map $F : D_X(G) \to k(Q_X, \rho_X)$ by sending $\phi_x g$ to the arrow $x \longrightarrow x^g$, then $F(\phi_x g \phi_y h) = F(\phi_x g)F(\phi_y h)$, for any $x, y \in X$ and $g, h \in G$. Indeed,

$$\begin{aligned} F(\phi_x g \phi_y h) &= F(\phi_x \phi_{y^{g^{-1}}} gh) \\ &= \begin{cases} F(\phi_x gh), & x = y^{g^{-1}} \\ 0, & x \neq y^{g^{-1}} \end{cases} \\ &= \begin{cases} x \longrightarrow x^{gh}, & x^g = y \\ 0, & x^g \neq y \end{cases} \end{aligned}$$

$$\begin{aligned} F(\phi_x g)F(\phi_y h) &= (x \longrightarrow x^g)(y \longrightarrow y^h) \\ &= \begin{cases} x \longrightarrow x^g \longrightarrow x^{gh}, & x^g = y \\ 0, & x^g \neq y \end{cases} \\ &= \begin{cases} x \longrightarrow x^{gh}, & x^g = y \\ 0, & x^g \neq y \end{cases} \end{aligned}$$

And $F(1_{D_X(G)}) = F(\sum_{x\in X} \phi_x 1_G) = \sum_{x\in X}(x \longrightarrow x^{1_G}) = \sum_{x \in X} e_x = 1_{k(Q_X,\, \rho_X)}$.
So F is an algebra homomorphism.

Clearly, F is an isomorphism with the inverse homomorphism

$$F^{-1}(x \longrightarrow x^g) = \phi_x g.$$

The End.

Corollary 2.4. *If there is a group isomorphism $\varphi : G \to G'$ and a bijection $\psi : X \to X'$ such that $\psi(x^g) = \psi(x)^{\varphi(g)}$, then $k(Q_X(G), \rho_X) \cong k(Q_{X'}(G'), \rho_{X'})$ as algebras, equivalently $D_X(G) \cong D_{X'}(G')$ as algebras.*

Proof. We only prove the result $k(Q_X(G), \rho_X) \cong k(Q_{X'}(G'), \rho_{X'})$ as algebras. By Proposition 2.2, $Q_X(G) \cong Q_{X'}(G')$ as quivers under the extended map of ψ. It can also be extended to an isomorphism of algebras from $kQ_X(G)$ to $kQ_{X'}(G')$, which we also denote by ψ. The left thing we need to do is proving the isomorphism preserves relations. Indeed,

$$\psi(x \longrightarrow x^{1_G}) = \psi(x) \longrightarrow \psi(x)^{\varphi(1_G)} = \psi(x) \longrightarrow \psi(x)^{1_{G'}} = e_{\psi(x)} = \psi(e_x),$$

and

$$\begin{aligned}
\psi(x \longrightarrow x^g \longrightarrow x^{gh}) &= \psi(x \longrightarrow x^g)\psi(x^g \longrightarrow x^{gh}) \\
&= (\psi(x) \longrightarrow \psi(x)^{\varphi(g)})(\psi(x)^{\varphi(g)} \longrightarrow \psi(x)^{\varphi(g)\varphi(h)}) \\
&= \psi(x) \longrightarrow \psi(x)^{\varphi(g)} \longrightarrow \psi(x)^{\varphi(g)\varphi(h)} \\
&= \psi(x) \longrightarrow \psi(x)^{\varphi(g)\varphi(h)} \\
&= \psi(x) \longrightarrow \psi(x)^{\varphi(gh)} \\
&= \psi(x \longrightarrow x^{gh})
\end{aligned}$$

The End.

Denote by $\mathbf{rep}(Q_X, \rho_X)$ the category of finite dimensional representations of (Q_X, ρ_X). It is well-known [1] and [2] that $\mathbf{rep}(Q_X, \rho_X)$ is equivalent to the category $\mathbf{mod}k(Q_X, \rho_X)$. Therefore we have the second corollary of Theorem 2.3, which gives an interesting way to describe $D_X(G)$-modules:

Corollary 2.5. *The category* $\mathbf{mod}D_X(G)$ *is equivalent to the category* $\mathbf{rep}(Q_X, \rho_X)$*, where* (Q_X, ρ_X) *is the quiver with relations associated to the* G*-set* X*.*

To see the correspondence clearly, we give the equivalent functor on objects. For any $D_X(G)$-module U, define a k-space $U_x = U \cdot \phi_x$ for each vertex $x \in Q_0$, and a k-map $U_{x \longrightarrow x^g} : U_x \longrightarrow U_{x^g}$ by $U_{x\longrightarrow x^g}(u_x) = u_x \cdot g$ for each arrow $x \longrightarrow x^g$. Then one can check that $U_{x\longrightarrow x^{1_G}} = id_{U_x}$ and $U_{x^g \longrightarrow x^{gh}} U_{x\longrightarrow x^g} = U_{x\longrightarrow x^{gh}}$. Hence $U_{rep} = \{(U_x),\ (U_{x\longrightarrow x^g})\}$ is a representation of (Q_X, ρ_X). Conversely, for any representation $U = \{(U_x),\ (U_{x\longrightarrow x^g})\}$ of (Q_X, ρ_X), let $U_{D_X(G)} = \sum_{x\in X} U_x$ with the action $(\sum_{x\in X} u_x) \cdot \phi_y h = U_{y\longrightarrow y^h}(u_y)$. Then $U_{D_X(G)}$ is the corresponding module in $\mathbf{mod}D_X(G)$.

In the final part of this section, we give another equivalent description of the algebra $D_X(G)$ by smash product algebra.

Theorem 2.6. *There is a left* kG*-module algebra structure on* $(kX)^*$ *such that* $D_X(G) \cong (kX)^* \# kG$ *as algebras. Hence, the category* $\mathbf{mod}D_X(G)$ *is isomorphic to the category* $\mathbf{mod}(kX)^* \# kG$*.*

Proof. Define a left kG-action on $(kX)^*$ by $g \cdot \phi_x = \phi_{x^{g^{-1}}}$. Then it is easy to check that:
(i) $1_G \cdot \phi_x = \phi_x$,
(ii) $(gh) \cdot \phi_x = g \cdot (h \cdot \phi_x)$,
(iii) $g \cdot 1_{(kX)^*} = \epsilon(g) 1_{(kX)^*}$,
(iv) $g \cdot (\phi_x \phi_y) = (g \cdot \phi_x)(g \cdot \phi_y)$.
Indeed, (i) and (ii) are trivial; (iii) and (iv) are from the following equalities:

$$g \cdot 1_{(kX)^*} = g \cdot (\sum_{x\in X} \phi_x) = \sum_{x\in X} \phi_{x^{g^{-1}}} = \sum_{x\in X} \phi_x = 1_{(kX)^*} = \epsilon(g) 1_{(kX)^*}$$

and

$$(g\cdot\phi_x)(g\cdot\phi_y)=\phi_{x^{g^{-1}}}\phi_{y^{g^{-1}}}=\delta_{x^{g^{-1}},\,y^{g^{-1}}}\phi_{x^{g^{-1}}}=\delta_{x,\,y}(g\cdot\phi_x)=g\cdot(\phi_x\phi_y),$$

then $(kX)^*$ becomes a left kG-module algebra. The equality

$$\phi_x g\phi_y h=\phi_x\phi_{y^{g^{-1}}}gh=\phi_x(g\cdot\phi_y)gh$$

implies that the multiplication in $D_X(G)$ coincides with that in $(kX)^*\#kG$, which means that $D_X(G)\cong(kX)^*\#kG$ as algebras.

The End.

3 Hopf Algebra Structure of $D_X(G)$

In this section, for a finite group G and its right G-set X, we will find a Hopf algebra structure on $D_X(G)$ and then on $k(Q_X,\rho_X)$. For this aim, we always suppose in this section, the right G-set X is also a finite group, and any g-action is a group automorphism of X. That is, $(xy)^g=x^gy^g$, $1_X^g=1_X$ for any $x,y\in X$, $g\in G$ due to the invertibility of the group action.

A classical example will be given in Section 4, when X is a normal subgroup of G and the action of G on X is the conjugation.

Under the assumption as above, we have:

Theorem 3.1. *$D_X(G)$ has a Hopf algebra structure with*

(a) the multiplication: $\phi_xg\phi_yh=\phi_x\phi_{y^{g^{-1}}}gh$,

(b) the unit: $1_{D_X(G)}=\sum_{x\in X}\phi_x1_G$,

(c) the comultiplication: $\Delta(\phi_xg)=\sum_{a\in X}\phi_{a^{-1}x}g\otimes\phi_ag$,

(d) the counit: $\epsilon(\phi_xg)=\delta_{1_X,x}$,

(e) the antipode: $S(\phi_xg)=\phi_{(x^{-1})^g}g^{-1}=\phi_{(x^g)^{-1}}g^{-1}$.

Proof. It is easy to check that $D_X(G)=(kX^{op})^*\otimes kG$ becomes into a coalgebra with the comultiplication and the counit given in (c) and (d). Here, we need only to prove the following:

(i) Δ is an algebra map.

Indeed, for any $x,y\in X$ and $g,h\in G$,

$$
\begin{aligned}
\Delta(\phi_x g)\Delta(\phi_y h) &= (\sum_{a\in X}\phi_{a^{-1}x}g\otimes\phi_a g)(\sum_{b\in X}\phi_{b^{-1}y}h\otimes\phi_b h)\\
&= \sum_{a\in X}\sum_{b\in X}\phi_{a^{-1}x}g\phi_{b^{-1}y}h\otimes\phi_a g\phi_b h\\
&= \sum_{a\in X}\sum_{b\in X}\phi_{a^{-1}x}\phi_{(b^{-1}y)^{g^{-1}}}gh\otimes\phi_a\phi_{b^{g^{-1}}}gh\\
&= \sum_{a\in X}\sum_{b\in X}\phi_{a^{-1}x}\phi_{(b^{-1})^{g^{-1}}y^{g^{-1}}}gh\otimes\phi_a\phi_{b^{g^{-1}}}gh\\
&= \sum_{a\in X}\phi_{a^{-1}x}\phi_{a^{-1}y^{g^{-1}}}gh\otimes\phi_a gh\\
&= \begin{cases}\sum_{a\in X}\phi_{a^{-1}x}gh\otimes\phi_a gh, & x=y^{g^{-1}}\\ 0, & x\neq y^{g^{-1}}\end{cases}\\
&= \begin{cases}\Delta(\phi_x gh), & x=y^{g^{-1}}\\ 0, & x\neq y^{g^{-1}}\end{cases}\\
&= \Delta(\phi_x\phi_{y^{g^{-1}}}gh)\\
&= \Delta(\phi_x g\phi_y h)
\end{aligned}
$$

where the forth and fifth equalities hold due to the assumption that any g-action is a group automorphism of X.

$$
\begin{aligned}
\Delta(1_{D_X(G)}) &= \Delta(\sum_{x\in X}\phi_x 1_G)\\
&= \sum_{x\in X}\sum_{a\in X}\phi_{a^{-1}x}1_G\otimes\phi_a 1_G\\
&= \sum_{a\in X}(\sum_{x\in X}\phi_{a^{-1}x}1_G)\otimes\phi_a 1_G\\
&= \sum_{a\in X}(\sum_{b\in X}\phi_b 1_G)\otimes\phi_a 1_G\\
&= (\sum_{b\in X}\phi_b 1_G)\otimes(\sum_{a\in X}\phi_a 1_G)\\
&= 1_{D_X(G)}\otimes 1_{D_X(G)}.
\end{aligned}
$$

(ii) ϵ is an algebra map.

Indeed, for any $x,y\in X$ and $g,h\in G$,

$$\epsilon(1_{D_X(G)})=\epsilon(\sum_{x\in X}\phi_x 1_G)=\sum_{x\in X}\delta_{1_X,x}=1,$$

and

$$\begin{aligned}
\epsilon(\phi_x g\phi_y h) &= \begin{cases} \epsilon(\phi_x gh), & x = y^{g^{-1}} \\ 0, & x \neq y^{g^{-1}} \end{cases} \\
&= \begin{cases} \delta_{1_X,x}, & x = y^{g^{-1}} \\ 0, & x \neq y^{g^{-1}} \end{cases} \\
&= \begin{cases} 1, & x = y^{g^{-1}} = 1_X \\ 0, & \text{otherwise} \end{cases} \\
&= \begin{cases} 1, & x = y = 1_X \\ 0, & \text{otherwise} \end{cases} \\
&= \delta_{1_X,x}\delta_{1_X,y} \\
&= \epsilon(\phi_x g)\epsilon(\phi_y h).
\end{aligned}$$

(iii) $S * id = id * S = \eta\epsilon$.

Indeed, For any $x \in X$ and $g \in G$,

$$\begin{aligned}
(S * id)(\phi_x g) &= \sum_{a\in X} S(\phi_{a^{-1}x} g)\phi_a g \\
&= \sum_{a\in X} \phi_{(x^{-1}a)^g} g^{-1}\phi_a g \\
&= \sum_{a\in X} \phi_{(x^{-1}a)^g}\phi_{a^g} 1_G \\
&= \sum_{a\in X} \phi_{(x^{-1})^g a^g}\phi_{a^g} 1_G \\
&= \delta_{1_X,(x^{-1})^g} \sum_{a\in X} \phi_{a^g} 1_G \\
&= \delta_{1_X,x} \sum_{a\in X} \phi_a 1_G \\
&= \epsilon(\phi_x g) 1_{D_X(G)}.
\end{aligned}$$

Similarly, one can prove that $(id * S)(\phi_x g) = \epsilon(\phi_x g)1_{D_X(G)}$.

The End.

Under the algebra isomorphism F defined in Theorem 2.3, we can define a Hopf algebra structure on the factor $k(Q_X, \rho_X)$ of the path algebra kQ_X, and show that it is isomorphic to $D_X(G)$ as Hopf algebras.

Theorem 3.2. *Isomorphic to $D_X(G)$ as Hopf algebras, there is a Hopf algebra structure on $k(Q_X, \rho_X)$ as follows:*

$$\Delta(x \longrightarrow x^g) = \sum_{a\in X} (a^{-1}x \longrightarrow (a^{-1}x)^g) \otimes (a \longrightarrow a^g)$$

$$\Delta(e_x) = \sum_{a\in X} e_{a^{-1}x} \otimes e_a$$

$$\epsilon(x \longrightarrow x^g) = \delta_{1_X, x}$$

$$\epsilon(e_x) = \delta_{1_X, x}$$

$$S(x \longrightarrow x^g) = (x^{-1})^g \longrightarrow x^{-1} = (x^g)^{-1} \longrightarrow x^{-1}$$

$$S(e_x) = e_{x^{-1}}$$

where (Q_X, ρ_X) is the quiver with relations associated to the G-set X.

Proof. Using the algebra isomorphism F defined in Theorem 2.3, we need only to prove the following facts:

(i) F is a coalgebra map.

Indeed, For any $x \in X$ and $g \in G$,

$$\begin{aligned}
\Delta F(\phi_x g) &= \Delta(x \longrightarrow x^g) \\
&= \sum_{a \in X} (a^{-1}x \longrightarrow (a^{-1}x)^g) \otimes (a \longrightarrow a^g) \\
&= (F \otimes F)(\sum_{a \in X} \phi_{a^{-1}x} g \otimes \phi_a g) \\
&= (F \otimes F)\Delta(\phi_x g),
\end{aligned}$$

$$\epsilon F(\phi_x g) = \epsilon(x \longrightarrow x^g) = \delta_{1_X, x} = \epsilon(\phi_x g)$$

(ii) $FS = SF$.

Indeed, For any $x \in X$ and $g \in G$,

$$FS(\phi_x g) = F(\phi_{(x^{-1})^g} g^{-1}) = (x^{-1})^g \longrightarrow x^{-1} = S(x \longrightarrow x^g) = SF(\phi_x g).$$

The End.

Remark 3.3. *Similarly, there is an isomorphic Hopf algebra structure on smash product algebra $(kX^{op})^* \# kG$.*

Next, we consider some common properties held by $D_X(G)$, $k(Q_X(G), \rho_X)$ and kG, such as semisimplicity, unimodularity and the cohomology groups of the trivial module.

It is well known that kG is a semisimple Hopf algebra if and only if the characteristic of k does not divide the order $\mid G \mid$ of G. Similarly, we have the same result for $D_X(G)$.

Proposition 3.4. *(a) The Hopf algebra $D_X(G)$ defined in Theorem 3.1 is semisimple if and only if the characteristic of k does not divide the order $\mid G \mid$ of G.*

(b) The Hopf algebra $k(Q_X, \rho_X)$ defined in Theorem 3.2 is semisimple if and only if the characteristic of k does not divide the order $\mid G \mid$ of G.

Proof. We need only to prove the case for $D_X(G)$.

Take $t = \sum_{g \in G} \phi_1 g$, where 1 is the identity 1_X of the group X. Then for any $\phi_x h \in D_X(G)$, we have $t(\phi_x h) = \sum_{g \in G} \phi_1 g \phi_x h = \sum_{g \in G} \phi_1 \phi_{x^{g^{-1}}} gh = \delta_{1,x} \sum_{g \in G} \phi_1 gh =$

$\delta_{1,x}\sum_{g\in G}\phi_1 g = \epsilon(\phi_x h)t$, so t is a right integral in $D_X(G)$. Furthermore, $\epsilon(t)=\epsilon(\sum_{g\in G}\phi_1 g)$ $=\sum_{g\in G}\epsilon(\phi_1 g) =\mid G\mid$. By Maschke's theorem for finite-dimensional Hopf algebra, $D_X(G)$ is semisimple if and only if $\epsilon(t)\neq 0$ for some right integral t in $D_X(G)$, equivalently, if and only if the characteristic of k does not divide the order $\mid G\mid$ of G, since the space of right integals is of dimension one.

The End.

By [9], a finite dimensional semisimple Hopf algebra must be unimodular. But here, without any restriction, kG is always unimodular with $t=\sum_{g\in G} g$ generating both the space of left integrals and the space of right integals. Similarly we have the same result for $D_X(G)$.

Proposition 3.5. *The Hopf algebras $D_X(G)$ and $k(Q_X,\rho_X)$ are unimodular.*

Proof. We need only to prove the case for $D_X(G)$. Take $t=\sum_{g\in G}\phi_1 g$, where 1 is the identity 1_X of the group X. Then for any $\phi_x h\in D_X(G)$, we have $(\phi_x h)t = \sum_{g\in G}\phi_x h\phi_1 g = \sum_{g\in G}\phi_x\phi_1 hg = \delta_{1,x}\sum_{g\in G}\phi_1 hg = \delta_{1,x}\sum_{g\in G}\phi_1 g = \epsilon(\phi_x h)t$, so t is a left integral in $D_X(G)$. From the proof of Proposition 3.4, t is also a right integral in $D_X(G)$. Then the space of left integrals coincides with the space of right integrals since they are both of one dimension. Therefore the Hopf algebra $D_X(G)$ is unimodular.

The End.

Finally, we discuss the cohomology groups $H^n(D_X(G),\, k)$ and $H^n(k(Q_X,\rho_X),\ k)$ of the trivial modules k.

Proposition 3.6. *For each natural number n,*

(a) the cohomology groups $H^n(D_X(G),\, k)$ and $H^n(kG,\, k)$ are isomorphic, where the former k is the trivial $D_X(G)$-module and the latter is the trivial kG-module;

(b) the cohomology groups $H^n(k(Q_X,\rho_X),\ k)$ and $H^n(kG,\, k)$ are isomorphic, where the former k is the trivial $k(Q_X,\rho_X)$-module and the latter is the trivial kG-module.

Proof. We need only to prove the case for $D_X(G)$.

Let I_1 be the subspace $\sum_{g\in G} k\phi_{1_X} g$ and I_2 the subspace $\sum_{1_X\neq x\in X,\, h\in G} k\phi_x h$. Then I_1 and I_2 are two ideals of $D_X(G)$ such that $D_X(G)=I_1\oplus I_2$. Obviously, I_1 is isomorphic to kG as algebras. In this way, any kG-module M has a $D_X(G)$-module structure as

$$m\cdot\phi_x g = \begin{cases} m\cdot g, & x=1_X\\ 0, & \text{otherwise}\end{cases}$$

Indeed, it can be checked that

$$m\cdot 1_{D_X(G)} = m\cdot\sum_{x\in X}\phi_x 1_G = m\cdot 1_G = m$$

and

$$\begin{aligned}
m\cdot(\phi_x g\phi_y h) &= m\cdot\phi_x\phi_{y^{g^{-1}}}gh \\
&= \begin{cases} m\cdot\phi_x gh, & x=y^{g^{-1}} \\ 0, & \text{otherwise} \end{cases} \\
&= \begin{cases} m\cdot gh, & x=y^{g^{-1}}=1_X \\ 0, & \text{otherwise} \end{cases} \\
&= \begin{cases} (m\cdot g)\cdot h, & x=y=1_X \\ 0, & \text{otherwise} \end{cases} \\
&= (m\cdot\phi_x g)\cdot\phi_y h
\end{aligned}$$

So **mod**kG can be embedded into **mod**$D_X(G)$ as a full subcategory and this embedding preserves projectivity. Indeed, for P a projective kG-module, there exist a kG-module Q and a positive integer n such that $P\oplus Q\cong kG^{(\oplus n)}$ as kG-module, and hence as the induced $D_X(G)$-module. Then we have

$$P\oplus Q\oplus I_2^{(\oplus n)}\cong kG^{(\oplus n)}\oplus I_2^{(\oplus n)}\cong I_1^{(\oplus n)}\oplus I_2^{(\oplus n)}\cong D_X(G)^{(\oplus n)}$$

as the induced $D_X(G)$-module. So the induced $D_X(G)$-module P is also projective.

Under this embedding the trivial $D_X(G)$-module k is the image of the trivial kG-module k. Indeed for any $\lambda\in k$ and $\phi_x g\in D_X(G)$,

$$\begin{aligned}
\lambda\cdot\phi_x g &= \epsilon(\phi_x g)\lambda \\
&= \begin{cases} \lambda, & x=1_X \\ 0, & \text{otherwise} \end{cases} \\
&= \begin{cases} \epsilon(g)\lambda, & x=1_X \\ 0, & \text{otherwise} \end{cases} \\
&= \begin{cases} \lambda\cdot g, & x=1_X \\ 0, & \text{otherwise} \end{cases}.
\end{aligned}$$

Thus a projective resolution of the trivial kG-module may be considered to be a projective resolution of the trivial $D_X(G)$-module, hence the cohomology groups $H^n(D_X(G),\, k)$ and $H^n(kG,\, k)$ are isomorphic for each natural number n.

The End.

4 Non-Balanced Quantum Double $D_X(G)$

As said in the introduction, we hope to consider on $D(G)=(kG^{op})^*\otimes kG$ what would happen when the balance between the left and right G's is destroyed. In this section, we shall find out some suitable conditions under which the Hopf algebra structures of $D_X(G)$ and $k(Q_X, \rho_X)$ are quasi-triangular. We call such quasi-triangular Hopf algebras *Non-balanced Quantum Double* when X is not isomorphic to G.

Hence, throughout this section we suppose that the right G-set X is also a group and any g-action is a group automorphism of X such that $D_X(G)$ and $k(Q_X, \rho_X)$ respectively has a Hopf algebra structure as in Theorem 3.1 and Theorem 3.2.

Theorem 4.1. *When X is a subgroup of G, then*

(a) the Hopf algebra structure on $D_X(G)$ given in Theorem 3.1 is quasi-triangular with the universal $R-$matrix $R_1 = \sum_{x\in X} x \otimes \phi_x$ if and only if X is a normal subgroup of G and the G-action is given by conjugation;

(b) the Hopf algebra structure on $k(Q_X, \rho_X)$ given in Theorem 3.2 is quasi-triangular with the universal $R-$matrix $R_2 = \sum_{x,y\in X}(y \longrightarrow y^x) \otimes e_x$ if and only if X is a normal subgroup of G and the G-action is given by conjugation.

Moreover, they are isomorphic as quasi-triangular Hopf algebras since $R_2 = (F \otimes F)R_1$.

Proof. We only need to show the case for $D_X(G)$.

Firstly $R = R_1 \in D_X(G) \otimes D_X(G)$ since X is a subgroup of G. Moreover,

(i) R is invertible with $R^{-1} = \sum_{x\in X} x^{-1} \otimes \phi_x$.

Indeed,

$$\begin{aligned} RR^{-1} &= (\sum_{x\in X} x \otimes \phi_x)(\sum_{x\in X} x^{-1} \otimes \phi_x) \\ &= \sum_{x\in X}\sum_{y\in X} xy^{-1} \otimes \phi_x\phi_y \\ &= \sum_{x\in X} 1_X \otimes \phi_x \\ &= 1_{D_X(G)} \otimes 1_{D_X(G)}, \end{aligned}$$

since $1_X = 1_G$. Similarly, $R^{-1}R = 1_{D_X(G)} \otimes 1_{D_X(G)}$.

(ii) $(\Delta \otimes id)R = R^{13}R^{23}$, $(id \otimes \Delta)R = R^{13}R^{12}$, where $R^{12} = \sum_{x\in X} x \otimes \phi_x \otimes 1_{D_X(G)}$, $R^{13} = \sum_{x\in X} x \otimes 1_{D_X(G)} \otimes \phi_x$, and $R^{23} = \sum_{x\in X} 1_{D_X(G)} \otimes x \otimes \phi_x$.

Indeed,

$$\begin{aligned} (id \otimes \Delta)R &= \sum_{a\in X}\sum_{x\in X} x \otimes \phi_{a^{-1}x} \otimes \phi_a \\ &= \sum_{a\in X}\sum_{x\in X} aa^{-1}x \otimes \phi_{a^{-1}x} \otimes \phi_a \\ &= \sum_{a\in X}\sum_{y\in X} ay \otimes \phi_y \otimes \phi_a \\ &= (\sum_{a\in X} a \otimes 1_{D_X(G)} \otimes \phi_a)(\sum_{y\in X} y \otimes \phi_y \otimes 1_{D_X(G)}) \\ &= R^{13}R^{12} \end{aligned}$$

$$\begin{aligned}(\Delta\otimes id)R &= \sum_{x\in X} x\otimes x\otimes\phi_x\\ &= \sum_{x\in X}\sum_{y\in X} x\otimes y\otimes\phi_x\phi_y\\ &= (\sum_{x\in X} x\otimes 1_{D_X(G)}\otimes\phi_x)(\sum_{y\in X} 1_{D_X(G)}\otimes y\otimes\phi_y)\\ &= R^{13}R^{23}.\end{aligned}$$

(iii) The left thing we need to do is to find out the condition under which $R\Delta(\phi_y h) = \Delta^{op}(\phi_y h)R$ for any $\phi_y h\in D_X(G)$.

Indeed,

$$\begin{aligned}\Delta^{op}(\phi_y h)R &= (\sum_{a\in X}\phi_a h\otimes\phi_{a^{-1}y}h)(\sum_{x\in X} x\otimes\phi_x)\\ &= \sum_{a\in X}\sum_{x\in X}\phi_a hx\otimes\phi_{a^{-1}y}\phi_{x^{h^{-1}}}h\\ &= \sum_{a\in X}\phi_a h(a^{-1}y)^h\otimes\phi_{a^{-1}y}h\\ &= \sum_{x\in X}\phi_{yx^{-1}}hx^h\otimes\phi_x h.\end{aligned}$$

$$\begin{aligned}R\Delta(\phi_y h) &= (\sum_{x\in X} x\otimes\phi_x)(\sum_{a\in X}\phi_{a^{-1}y}h\otimes\phi_a h)\\ &= \sum_{x\in X}\sum_{a\in X}\phi_{(a^{-1}y)^{x^{-1}}}xh\otimes\phi_x\phi_a h\\ &= \sum_{x\in X}\phi_{(x^{-1}y)^{x^{-1}}}xh\otimes\phi_x h\end{aligned}$$

Hence, since $\{\phi_x g : x\in X, g\in G\}$ is a k-basis of $D_X(G)$, $R\Delta(\phi_y h) = \Delta^{op}(\phi_y h)R$ for any $\phi_y h\in D_X(G)$ if and only if $\phi_{(x^{-1}y)^{x^{-1}}}xh = \phi_{yx^{-1}}hx^h$ for any $\phi_y h\in D_X(G)$ and $x\in X$, if and only if $x^h = h^{-1}xh$ for any $x\in X$ and $h\in G$.

The End.

Since $\{1_G\}$ and G is two special G-sets under conjugation, note that:

(i) When $X=\{1_G\}$, the non-balanced quantum double is just the group algebra kG, which is a trivial quasi-triangular Hopf algebra with the universal R-matrix $R_1 = 1_G\otimes 1_G$. Correspondingly, $k(Q_{\{1_G\}},\rho_{\{1_G\}})$ is also a trivial quasi-triangular Hopf algebra with the universal R-matrix $R_2 = (1_G\to 1_G)\otimes(1_G\to 1_G)$, where the quiver $(Q_{\{1_G\}},\rho_{\{1_G\}})$ is just a unique vertex with $|G|$ loops.

(ii) When $X=G$, $D_X(G)$ is just the classical quantum double $D(G)$ with the universal R-matrix $R_1=\sum_{g\in G} g\otimes\phi_g$. Correspondingly, $k(Q_G,\rho_G)$ is also a quasi-triangular Hopf algebra with the universal R-matrix $R_2=\sum_{g\in Q_0}\sum_{h\in Q_0}(h\longrightarrow g^{-1}hg)\otimes e_g$.

We know [10] that a quasi-triangular Hopf algebra is called *minimal* if it has no proper quasi-triangular Hopf subalgebra. It is obvious by [10] that the unique minimal quasi-triangular Hopf subalgebra contained in $D_X(G)$ is the quantum double $D(X)$. Thus, we have:

Corollary 4.2. *The quasi-triangular Hopf algebra $D_X(G)$ given in Theorem 4.1 is minimal if and only if $X = G$.*

As a part of the theory of representations of non-balanced quantum double, we have the following property about its representation ring:

Corollary 4.3. *When X is a normal subgroup of G and the the action is given by $x^g = g^{-1}xg$, the representation rings $R(D_X(G))$ and $R(k(Q_X, \rho_X))$ are both commutative.*

Proof. We only need to show the case for $D_X(G)$.

For any two right $D_X(G)$-modules U and V, define a k-map $\Phi : U \otimes V \to V \otimes U$ by sending $u \otimes v$ to $\sum_{x\in X} v \cdot x \otimes u \cdot \phi_x$. It is bijective since $R = \sum_{x\in X} x \otimes \phi_x$ is invertible. Furthermore, Φ is a $D_X(G)$-module homomorphism. In fact, for any $\phi_y h \in D_X(G)$, we have:

$$\begin{aligned}
\Phi((u \otimes v) \cdot \phi_y h) &= \Phi(\sum_{a\in X} u \cdot \phi_{a^{-1}y} h \otimes v \cdot \phi_a h) \\
&= \sum_{a\in X}\sum_{x\in X} (v \cdot \phi_a h) \cdot x \otimes (u \cdot \phi_{a^{-1}y} h) \cdot \phi_x \\
&= \sum_{x\in X}\sum_{a\in X} (v \cdot x) \cdot \phi_{a^{-1}y} h \otimes (u \cdot \phi_x) \cdot \phi_a h \\
&= (\sum_{x\in X} v \cdot x \otimes u \cdot \phi_x) \cdot \phi_y h \\
&= \Phi(u \otimes v) \cdot \phi_y h
\end{aligned}$$

where the third equality is from the fact that $R\Delta(\phi_y h) = \Delta^{op}(\phi_y h)R$ by Theorem 4.1. The End.

About the non-balanced quantum double $D_X(G)$ in Theorem 4.1, we only consider the case that X is a subgroup of G. It is natural for us to discuss the opposite side, i.e. when G is a subgroup of X. Unfortunately, we can not find a suitable universial R-matrix. Actually, we have the following statements as remarks:

Remark 4.4. *When G is a subgroup of X, set $R = \sum_{g\in G} g \otimes \phi_g \in D_X(G) \otimes D_X(G)$.*

(a) Although R is non-invertible, there exists an element $\bar{R} = \sum_{g\in G} g^{-1} \otimes \phi_g \in D_X(G) \otimes D_X(G)$ satisfying $R\bar{R}R = R$ and $\bar{R}R\bar{R} = \bar{R}$, that is, R is regular under the meaning of von Neumann's.

(b) Similarly as the proof of Theorem 4.1, it can be checked that $D_X(G)$ is almost cocommutative with the regular element $R \in D_X(G) \otimes D_X(G)$, or say $R\Delta(\phi_y h) = \Delta^{op}(\phi_y h)R$ for any $\phi_y h \in D_X(G)$, if and only if the G-action is given by conjugation when restricted onto G.

(c) In general, $(\Delta \otimes id)R = R^{13}R^{23}$ and $(id \otimes \Delta)R = R^{13}R^{12}$ do not hold. However we can show that $R^{12}R^{13}R^{23} = R^{23}R^{13}R^{12}$ if and only if G is an abelian group and the restriction of the G-action on G is just the identity map on G, which means that under this special case R is a solution of the quantum Yang-Baxter equation.

Of course, the similar conclusions on $k(Q_X(G), \rho_X)$ also hold under isomorphism.

5 Representation Type of $D_X(G)$

In this section we shall consider the relation between the representation types of $D_X(G)$ and of kG, under the assumption that the right G-set X is also a group and any g-action is a group automorphism of X, i.e. such that $D_X(G)$ and $k(Q_X, \rho_X)$ respectively has a Hopf algebra structure as in Theorem 3.1 and Theorem 3.2.

Firstly, let us recall some concepts from [4] and [5]:

(a) A k-algebra A is said to be of *finite representation type* if there are only finitely number of non-isomorphic finitely generated indecomposable A-modules;

(b) A k-algebra A is said to be of *tame type* if it is not of finite representation type, and for any positive integer d, there are a finite number of $k[T]$-A-bimodules M_i which are free as left $k[T]$-modules such that all but a finite number of indecomposable A-modules of dimension d are isomorphic to some $k[T]/(T-\lambda) \otimes_{k[T]} M_i$ as A-modules for $\lambda \in k$;

(c) A k-algebra A is said to be of *wild type* if there is a finitely generated $k < X, Y >$-A-bimodule M which is free as a left $k < X, Y >$-module such that the functor $- \otimes_{k<X,Y>} M$ from the category **mod**$k < X, Y >$ to the category **mod**A preserves indecomposability and reflects isomorphisms.

The famous tame-and-wild theorem of Drozd's in [4] and [5] states that a finite dimensional algebra over an algebraically closed field k, which is not of finite representation type, is either of tame type or of wild type, but not of the both types. Therefore it gives the classification of finite dimensional algebras over an algebraically closed field k due to the representation type.

In order to discuss when $D_X(G)$ is of finite representation type, we cite the following three results from [2]. About the definitions of *hereditary*, *preinjective* and *preprojective*, one can also find in [2].

Lemma 5.1. ([2]) *A finite dimension hereditary k-algebra A is of finite representation type if and only if all A-modules are both preinjective and preprojective.*

Lemma 5.2. ([2]) *Let A be a k-subalgebra of a finite dimensional k-algebra B.*
(a) Suppose A is a two-sided summand of B, i.e. $B = A \oplus C$ as two-sided A-modules for some C. Then A is of finite representation type if B is of finite representation type.
(b) Suppose M is a B-summand of $M \otimes_A B$ for any M in **mod**B, *i.e.* $M \otimes_A B =$

$M \oplus N$ as B-module for some N. Then B is of finite representation type if A is of finite representation type.

Lemma 5.3. ([2]) *Suppose G is a finite group whose order is dividing by* $\mathbf{char}k = p$. *Then kG is of finite representation type if and only if every Sylow p-subgroup of G is a cyclic group.*

Firstly, we give the following result about finite representation type:

Theorem 5.4. *$D_X(G)$ is of finite representation type if and only if kG is of finite representation type.*

Proof. From [9], we know that every finite-dimensional Hopf algebra is free over any of its Hopf subalgebra. Consider $kG \cong \epsilon kG$ as a Hopf subalgebra of $D_X(G)$, then $D_X(G) \cong kG^{(\otimes n)}$ as kG-bimodules for some positive integer n. By Lemma 5.2 (a), kG is of finite representation type if $D_X(G)$ is of finite representation type.

Conversely for any M in $\mathbf{mod}D_X(G)$, it has a induced kG-module structure by restriction. We claim that M is a $D_X(G)$-direct summand of $M \otimes_{kG} D_X(G)$, where the $D_X(G)$-action on $M \otimes_{kG} D_X(G)$ is given by right multiplication.

In fact, define a k-map $\varphi : M \otimes_{kG} D_X(G) \to M$ by sending $m \otimes \phi_x g$ to $m \cdot \phi_x g$. Clearly φ is $D_X(G)$-epimorphism. Define a k-map $\psi : M \to M \otimes_{kG} D_X(G)$ by sending m to $\sum_{x\in X} m \cdot \phi_x \otimes \phi_x$. Then for any $\phi_y h \in D_X(G)$,

$$\begin{aligned}
\psi(m \cdot \phi_y h) &= \sum_{x\in X} (m \cdot \phi_y h) \cdot \phi_x \otimes \phi_x \\
&= \sum_{x\in X} m \cdot (\phi_y h \phi_x) \otimes \phi_x \\
&= \sum_{x\in X} m \cdot (\phi_y \phi_{x^{h^{-1}}} h) \otimes \phi_x \\
&= m \cdot \phi_y h \otimes \phi_{y^h} \\
&= m \cdot \phi_y \otimes h \phi_{y^h} \\
&= m \cdot \phi_y \otimes \phi_y h,
\end{aligned}$$

$$\begin{aligned}
\psi(m) \cdot \phi_y h &= (\sum_{x\in X} m \cdot \phi_x \otimes \phi_x) \cdot \phi_y h \\
&= \sum_{x\in X} m \cdot \phi_x \otimes \phi_x \phi_y h \\
&= m \cdot \phi_y \otimes \phi_y h.
\end{aligned}$$

Hence ψ is also a $D_X(G)$-homomorphism. Furthermore,

$$\varphi\psi(m) = \varphi(\sum_{x\in X} m \cdot \phi_x \otimes \phi_x) = \sum_{x\in X} (m \cdot \phi_x) \cdot \phi_x = \sum_{x\in X} m \cdot (\phi_x \phi_x) = \sum_{x\in X} m \cdot \phi_x = m,$$

so ψ is a split monomorphism and M is a $D_X(G)$-direct summand of $M \otimes_{kG} D_X(G)$.

By Lemma 5.2 (b), $D_X(G)$ is of finite representation type if kG is of finite representation type.

The End.

Corollary 5.5. *(a) If the characteristic of k does not divide $\mid G \mid$, then $D_X(G)$ is of finite representation type,*

(b) If the characteristic p of k divides $\mid G \mid$, then $D_X(G)$ is of finite representation type if and only if every Sylow p-subgroup of G is a cyclic group.

Proof. (a) By Proposition 3.4, $D_X(G)$ is semisimple. So all $D_X(G)$-modules are both projective and injective, hence also both preprojective and preinjective. In particular its radical $\mathbf{rad}D_X(G) = 0$ is projective. Therefore $D_X(G)$ is hereditary algebra. By Lemma 5.1, $D_X(G)$ is of finite representation type.

(b) It is directly from Lemma 5.3 and Theorem 5.4.

The end.

About the tame type, we need the following result in [7]:

Lemma 5.6. ([7]) *For a finite dimensional k-algebra A, let M be $k[T]$-A-bimodule which is free over $k[T]$. Then M is indecomposable as $k[T]$-A-bimodule if and only if $k[T]/(T-\lambda) \otimes_{k[T]} M$ is indecomposable as $k[T]/(T-\lambda)$-A-bimodule for $\lambda \in k$.*

Theorem 5.7. *$D_X(G)$ is of tame type if and only if kG is of tame type.*

Proof. (i) kG is of tame type if $D_X(G)$ is of tame type.

For any indecomposable right kG-module M in $\mathbf{mod}kG$, it has a right $D_X(G)$-module structure, denoted by $M_{D_X(G)}$, given by $m \cdot \phi_x g = \delta_{1,x} m \cdot g$. If $M_{D_X(G)} = M' \oplus M''$ is a nontrivial decomposition in $\mathbf{mod}D_X(G)$, restricting onto kG, we have

$$M = (M_{D_X(G)}) \downarrow_{kG} = M' \downarrow_{kG} \oplus M'' \downarrow_{kG}$$

a nontrivial decomposition of M in $\mathbf{mod}kG$ since kG, as a Hopf subalgebra, has the same unit to $D_X(G)$. This contradicts to that M is indecomposable as kG-module. Thus $M_{D_X(G)}$ is indecomposable in $\mathbf{mod}D_X(G)$.

For any positive integer d, since $D_X(G)$ is of tame type, there are a finite number of $k[T]$-$D_X(G)$-bimodules $M_1, M_2, \cdots, M_n$ which are free as left $k[T]$-module such that all, but a finite number, of indecomposable $D_X(G)$-modules of dimension d are isomorphic to $k[T]/(T-\lambda) \otimes_{k[T]} M_j$ as $D_X(G)$-modules for some $1 \leq j \leq n$ and some $\lambda \in k$. Hence all, but a finite number, of indecomposable kG-modules M of dimension d satisfying $M_{D_X(G)} \cong k[T]/(T-\lambda) \otimes_{k[T]} M_j$ as $D_X(G)$-modules for some $1 \leq j \leq n$ and some $\lambda \in k$. Restricting onto kG, we have $M = (M_{D_X(G)}) \downarrow_{kG} \cong k[T]/(T-\lambda) \otimes_{k[T]} (M_j \downarrow_{kG})$ as kG-modules. Each $M_j \downarrow_{kG}$ must be free over $k[T]$ since the left module structure has not been changed. Then $\{M_j \downarrow_{kG}: j = 1, 2, \cdots, n\}$ is a finite set of $k[T]$-kG-bimodules which are free as left $k[T]$-modules such that all, but a finite number, of indecomposable kG-modules of dimension d are isomorphic to $k[T]/(T-\lambda)\otimes_{k[T]}(M_j\downarrow_{kG})$ as kG-modules for some $1\le j\le n$ and some $\lambda\in k$. Hence kG is of tame type.

(ii) $D_X(G)$ is of tame type if kG is of tame type.

For any positive integer d', since kG is of tame type, there are a finite set of $k[T]$-kG-bimodules $\{M_j^{d'}: j\in J_{d'}\}$ indexed by a finite set $J_{d'}$, all of which are free as left $k[T]$-module such that all but a finite number of indecomposable kG-modules of dimension d' are isomorphic to $k[T]/(T-\lambda)\otimes_{k[T]}M_j^{d'}$ as kG-modules for some $j\in J_{d'}$ and some $\lambda\in k$.

Since any $M_j^{d'}$ is finitely generated over kG, $M_j^{d'}\otimes_{kG}D_X(G)$ is finitely generated over $D_X(G)$. From the freeness of $M_j^{d'}$ over $k[T]$, one can deduce the freeness of $M_j^{d'}\otimes_{kG}D_X(G)$ over $k[T]$ since $D_X(G)$ is free over kG. Hence we get a finite direct sum decomposition $M_j^{d'}\otimes_{kG}D_X(G)=\bigoplus_{l\in L_j}M_{j,l}^{d'}$ with $M_{j,l}^{d'}$ indecomposable $k[T]$-$D_X(G)$-bimodules and free over $k[T]$. Then $\{M_{j,l}^{d'}: j\in J_{d'},\ l\in L_j\}$ is a finite set of indecomposable $k[T]$-$D_X(G)$-bimodules which are free as left $k[T]$-modules. By Lemma 5.6, each $k[T]/(T-\lambda)\otimes_{k[T]}M_{j,l}^{d'}$ is indecomposable as $k[T]/(T-\lambda)$-$D_X(G)$-bimodules, hence is indecomposable as $D_X(G)$-module since $k[T]/(T-\lambda)\cong k$ as algebras.

For any indecomposable N in $\mathbf{mod}D_X(G)$ of dimension d, let $N\downarrow_{kG}=N_1\oplus\cdots\oplus N_m$ be a direct sum of indecomposable kG-modules. By the claim in the proof of Theorem 5.4, N is a $D_X(G)$-direct summand of $N\otimes_{kG}D_X(G)$. But $N\otimes_{kG}D_X(G)=(N_1\otimes_{kG}D_X(G))\oplus\cdots\oplus(N_m\otimes_{kG}D_X(G))$, by Krull-Schmidt theorem, there exists a t $(1\le t\le m)$ such that N is a $D_X(G)$-direct summand of $N_t\otimes_{kG}D_X(G)$. Let $\mathbf{dim}N_t=\mu(t)$, then $\mu(t)\le d$. Since kG is tame, for all but a finite number of such N and N_t, it satisfies that $N_t\cong k[T]/(T-\lambda)\otimes_{k[T]}M_{j_t}^{\mu(t)}$ for some $j_t\in J_{\mu(t)}$ and $\lambda\in k$. Moreover,

$$\begin{aligned} N_t\otimes_{kG}D_X(G) &\cong k[T]/(T-\lambda)\otimes_{k[T]}M_{j_t}^{\mu(t)}\otimes_{kG}D_X(G)\\ &= k[T]/(T-\lambda)\otimes_{k[T]}(\bigoplus_{l\in L_{j_t}}M_{j_t,l}^{\mu(t)})\\ &= \bigoplus_{l\in L_{j_t}}(k[T]/(T-\lambda)\otimes_{k[T]}M_{j_t,l}^{\mu(t)}). \end{aligned}$$

We have known above that each $k[T]/(T-\lambda)\otimes_{k[T]}M_{j_t,l}^{\mu(t)}$ is indecomposable as $D_X(G)$-module, hence there must exist some $s\in L_{j_t}$ such that $N\cong k[T]/(T-\lambda)\otimes_{k[T]}M_{j_t,s}^{\mu(t)}$ as $D_X(G)$-modules.

Then for any positive integer d, $\{M_{j,l}^i \mid 1\le i\le d,\ j\in J_i,\ l\in L_j\}$ is a finite set of $k[T]$-$D_X(G)$-bimodules whose members are all free as left $k[T]$-modules and such that all but a finite number of indecomposable $D_X(G)$-modules of dimension d are isomorphic to some $k[T]/(T-\lambda)\otimes_{k[T]}M_{j,l}^i$ for $\lambda\in k$. Hence $D_X(G)$ is of tame type.

The End.

By Theorem 5.4 and 5.7, due to the Drozd's tame-and-wild theorem, we obtain the following:

Corollary 5.8. *When k is a algebraically closed field, $D_X(G)$ and kG has the same representation type. In particular, the classical quantum double $D(G)$ and the non-balanced quantum doubles $D_X(G)$ have the same representation type as the group algebra kG.*

An interesting fact given from the above discussion is that the representation type of the Hopf algebra $D_X(G)$ is independent of the property of the finite group X. The same result also hold for $k(Q_X(G), \rho_X)$ under isomorphism.

References

[1] I.Assem, D.Simson and A.Skowronski, Elements of the Representation Theory of Associative Algebras, Volume I Techniques of Representation Theory, *London Mathematical Society Student Texts 65*, *Cambridge University Press*, Cambridge, 2005

[2] M.Auslander, I.Reiten and S.O.Smalø, Representation Theory of Artin Algebra, *Cambridge University Press*, Cambridge, 1995

[3] M.Cohen and D.Fishman, Hopf Algebra Actions, *Journal of Algebra* 100, 363-379, 1986

[4] W.W.Crawley-Boevey, on Tame Algebras and Bocses, *Proc. London Math. Soc.* 5, 451-483, 1988

[5] Yu.A.Drozd, Tame and Wild Matrix Problems, Representations and Quadratic Forms, *Inst. Math., Acad. Sciences Ukrainian SSR,* Kiev 1979, 39-74, *Amer. Math. Soc. Transl.* 128, 31-55, 1986

[6] C.Kassel, Quantum Groups, Springer-Verlag New York, Inc. 1995

[7] X.G.Liu, Classification of finite dimensional basic Hopf algebras and related topics, Ph.D.Disseration, Zhejiang University, China, 2005

[8] G.Mason, The quantum double of a finite group and its role in conformal field theory, Groups'93 Galway/St.Andrews, Vol.2, Cambridge Univ. Press, Cambridge, 405-417, 1995

[9] S.Montgomery, Hopf Algebras and Their Actions on Rings, CBMS, *Lecture in Math, Providence,* RI, Vol.82, 1993

[10] D.E.Radford, Minimal Quasitriangular Hopf Algenras, *Journal of Algebra* 157, 285-315, 1993

[11] S.J.Witherspoon, The Representation Ring of the Quantum Double of a Finite Group, *Journal of Algebra*, 179, 305-329, 1996