Large scale geometry, compactifications and the integral Novikov conjectures for arithmetic groups

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Abstract

The original Novikov conjecture concerns the (oriented) homotopy invariance of higher signatures of manifolds and is equivalent to the rational injectivity of the assembly map in surgery theory. The integral injectivity of the assembly map is important for other purposes and is called the integral Novikov conjecture. There are also assembly maps in other theories and hence related Novikov and integral Novikov conjectures. In this paper, we discuss several results on the integral Novikov conjectures for all torsion free arithmetic subgroups of linear algebraic groups and all S-arithmetic subgroups of reductive linear algebraic groups over number fields. For reductive linear algebraic groups over function fields of rank 0, the integral Novikov conjecture also holds for all torsion-free S-arithmetic subgroups. Since groups containing torsion elements occur naturally and frequently, we also discuss a generalized integral Novikov conjecture for groups containing torsion elements, and prove it for all arithmetic subgroups of reductive linear algebraic groups over number fields and S-arithmetic subgroups of reductive algebraic groups of rank 0 over function fields.

1 Introduction

In the study of topology of manifolds, an important conjecture is the Novikov conjecture on oriented homotopy invariance of higher signatures. This conjecture is equivalent to the *rational injectivity* of the *assembly map* in surgery theory (or L-theory). Assembly maps occur naturally in other theories, and the *rational* injectivity of the assembly map is called the *Novikov conjecture* in that theory, and the *(integral) injectivity* of the assembly map is called the *integral Novikov conjecture*. Each of them is important for different purposes, but the assembly map is the common thread which unites all these conjectures.

In general, the Novikov conjectures are formulated in terms of discrete groups, which can be taken to be the fundamental group of manifolds. For each abstract group Γ (usually assumed to be finitely generated), there are several versions of Novikov conjecture corresponding to different theories.

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An important class of groups consists of arithmetic groups of linear algebraic groups defined over $\mathbb{Q}$. They are finitely presented and also enjoy many other finiteness properties (see [Bo1] [Bo5]). Their cohomology groups and other properties have been studied by many people (see [Bo3], [Sch] and [LS]).

The Novikov conjectures have been studied by different methods. One approach consists of large scale geometry (or geometry at infinity) of Γ. One important large scale invariant of an infinite group Γ is its asymptotic dimension, introduced by Gromov in [Gr]. An important result in [Yu] [Ba] [CG] [CFY] [DFW] (see Proposition 3.1 below) says that if Γ has finite asymptotic dimension and a finite $B\Gamma$-space, then the integral Novikov conjecture holds for Γ.

In [Ji1], we showed the following result.

Theorem 1.1 *Let $\mathbf{G}$ be any linear algebraic group defined over $\mathbb{Q}$, and $\Gamma \subset \mathbf{G}(\mathbb{Q})$ any arithmetic subgroup. Then the asymptotic dimension* asdim Γ *of Γ is finite.*

Combined with the Borel-Serre compactification in [BS1], we obtained the following result as a corollary.

Corollary 1.2 *Assume that Γ is a torsion free arithmetic subgroup of a linear algebraic group defined over $\mathbb{Q}$. Then the integral Novikov conjectures hold for Γ.*

In the above corollary, the assumption that Γ is torsion free is important. Otherwise, the integral injectivity of the assembly map will fail in general (see Remark 2.7). It might be helpful to note that the existence of a finite $B\Gamma$ implies that Γ is torsion-free [Br2].

For groups that contain nontrivial torsion elements, there is a generalized integral Novikov conjecture for the surgery theory, K-theory etc (see Equation 2.9 near the end of §2), which is closely related to the Farrell-Jones isomorphism conjecture (see [FJ1] [LR] and the discussions after Equation 2.9).

By combining Corollary 1.2 with [LR, Proposition 66] (or Proposition 2.10 in its preprint form), we obtain the following result.

Corollary 1.3 *Assume that Γ is a torsion free arithmetic subgroup of a linear algebraic group defined over $\mathbb{Q}$. Then the map in the Farrell-Jones isomorphism conjecture in both K-theory and L-theory is injective.*

An important generalization of arithmetic subgroups is the class of S-arithmetic subgroups (see [Bo2] [Bo4] [BS2] [PR]). For example, $SL(n,\mathbb{Z})$ and its congruence subgroups are typical examples of arithmetic subgroups. For a finite set of primes $p_1, \cdots, p_m$, the subgroup $SL(n, \mathbb{Z}[\frac{1}{p_1}, \cdots, \frac{1}{p_m}])$ and its subgroups of finite index are S-arithmetic subgroups. One obvious difference between them is that $SL(n,\mathbb{Z})$ is a discrete subgroup of the real Lie group $SL(n,\mathbb{R})$, but $SL(n, \mathbb{Z}[\frac{1}{p_1}, \cdots, \frac{1}{p_m}])$ is not. Instead, $SL(n, \mathbb{Z}[\frac{1}{p_1}, \cdots, \frac{1}{p_m}])$ is a discrete subgroup of $SL(n,\mathbb{R}) \times SL(n,\mathbb{Q}_{p_1}) \times \cdots \times SL(n,\mathbb{Q}_{p_m})$ under the diagonal embedding.

For many problems such as those concerning rigidity questions in [Mar] and other applications in number theory [PR], it is important and fruitful to treat S-arithmetic subgroups on the same footing as arithmetic subgroups.

Arithmetic subgroups can not be defined for algebraic groups over the function field of a projective curve over a finite field. On the other hand, S-arithmetic subgroups can also be defined for such algebraic groups.

Unlike the case of arithmetic subgroups, if the group $\mathbf{G}$ over a number field is not reductive, its S-arithmetic subgroups may not have some finiteness properties (see [Ab2] [Br1, Chap. VII]). If the group **G** is over a function field, its S-arithmetic subgroups have even fewer finiteness properties (see [Br1, Chap. VII], [Be1] [Be2] [Ab1] [Ab2]).

In the above discussion of arithmetic groups, we have assumed that the field of definition is equal to $\mathbb{Q}$. The reason is that consideration of algebraic groups defined over general number fields, i.e., finite extensions of $\mathbb{Q}$, does not give more examples of arithmetic groups. On the other hand, for S-arithmetic subgroups, the consideration of algebraic groups over number fields leads to a larger class of S-arithmetic subgroups.

Another approach to Novikov conjecture uses suitable compactifications of $E\Gamma$, where $E\Gamma$ is the universal covering space of the classifying space $B\Gamma$, i.e., a contractible space with a free Γ-action. By using this approach, we proved in [Ji3] the following result.

Theorem 1.4 *Let* $\mathbf{G}$ *be a linear reductive algebraic group over a global field* k*, i.e., either a number field or the function field of a projective curve over a finite field. If the* k*-rank of* $\mathbf{G}$ *is equal to 0, then the integral Novikov conjectures hold for every torsion free S-arithmetic subgroup of* $\mathbf{G}(k)$*, and the generalized integral Novikov conjectures hold for S-arithmetic subgroups of* $\mathbf{G}(k)$ *that contain torsion elements.*

In the function field case, the zero rank assumption in Theorem 1.4 is necessary in order to get torsion free S-arithmetic subgroups. On the other hand, it is natural to expect that the rank 0 condition in the case of number field can be removed. In fact, in [Ji4], we proved the following result.

Theorem 1.5 *Let* $\mathbf{G}$ *be a linear reductive algebraic group over a number field* k*. Then the integral Novikov conjectures hold for every torsion free S-arithmetic subgroup of* $\mathbf{G}(k)$*.*

For the generalized integral Novikov conjectures, we proved the following result in [Ji6].

Theorem 1.6 *If* $\mathbf{G}$ *is a reductive algebraic group defined over a number field* k *and* $\Gamma \subset \mathbf{G}(k)$ *is any arithmetic subgroup that may contain torsion elements, then the generalized integral Novikov conjectures hold for* Γ*.*

We also observed [Ji6] that the main theorem in [BHM] implies the following result.

Proposition 1.7 *Let* $\mathbf{G}$ *be a linear reductive group as in the above theorem, and* $\Gamma \subset \mathbf{G}(k)$ *an S-arithmetic subgroup that contains torsion elements. Then the (rational) Novikov conjecture in* K*-theory holds for* Γ*.*

In fact, Γ contains a torsion-free subgroup Γ' of finite index. Using the Borel-Serre compactification, it can be shown that Γ' admits a finite $B\Gamma'$ and hence is of type FL, in particular of type FP_∞. By [Br2, Proposition VIII.5.1], Γ is also of type FP_∞, which implies that for every i, $H_i(\Gamma, \mathbb{Z})$ is finitely generated.

It is natural to conjecture that *the generalized integral Novikov conjectures also hold for every S-arithmetic subgroup of* $\mathbf{G}(k)$ *when* $\mathbf{G}$ *is reductive and* k *is a number field.* In fact, when the symmetric space associated with $\mathbf{G}$ is the real hyperbolic space, the generalized integral conjectures are true and proved in [Ji4, Theorem 3.2].

Remark 1.8 As in Corollary 1.3, the map in the Farrell-Jones conjecture is also injective for torsion-free Γ satisfying the conditions in Theorems 1.4 and 1.5.

Remark 1.9 After this paper was submitted and revised, the author proved in [Ji7] that the generalized integral Novikov conjecture holds for every finitely generated subgroup of $GL(n, \overline{k})$, where k is a global field and $\overline{k}$ its algebraic closure, and also for every S-arithmetic subgroup of a reductive algebraic group over a global field. In particular, the conjecture above Remark 1.8 is true.

The Novikov conjectures have been proved for various classes of groups. The Novikov conjecture in the surgery theory is often called the Novikov conjecture in L-theory. For surveys of the status of the Novikov conjectures and the Farrell-Jones isomorphism conjecture, see [FRR], [LR] and [We]. For the class of discrete subgroups of Lie groups, the earlier results on the integral Novikov conjectures are listed as follows:

1. The integral analytic (or C^*-algebras) Novikov conjecture for all discrete subgroups of connected Lie groups was proved by Kasparov (see [FRR]), finitely generated subgroups of $GL(n, \overline{\mathbb{Q}})$ by Kasparov and Skandalis [KS], and all finitely generated subgroups of $GL(n, K)$, where K is any field, by Guentner, Higson and Weinberger [GHW]. Hence, the original Novikov conjecture holds for all these groups.

2. The integral Novikov conjecture in L-theory for all torsion-free discrete subgroups of $GL(n, \mathbb{C})$ and the fundamental group of nonpositively curved manifolds was proved by Farrell-Hsiang, Ferry-Weinberger, and Farrell-Jones (see [FJ2, p. 217, p. 220]).

3. The integral K-theoretic Novikov conjecture was known only for the following classes of discrete subgroups of Lie groups:

   (a) co-compact lattices in connected Lie groups by Carlsson [Ca] (see also [FJ1]),

   (b) arithmetic subgroups of semisimple linear algebraic groups $\mathbf{G}$ defined over $\mathbb{Q}$ of $\mathbb{R}$-rank equal to 1 by Goldfarb [Go2],

   (c) arithmetic subgroups of $\mathbf{G} = SL(3)$, and more general arithmetic subgroups of $\mathbf{G}$ when $\mathbf{G}$ is semisimple and the $\mathbb{Q}$-rank of $\mathbf{G}$ is equal to the $\mathbb{R}$-rank of $\mathbf{G}$ by Goldfarb [Go3] [Go1]. (See Remark 6.9 below for comments on the proofs in these papers.)

4. The (rational) Novikov conjecture in K-theory holds for every group Γ such that in every degree $i \geq 0$, $H_i(\Gamma, \mathbb{Z})$ is finitely generated, by the celebrated result in [BHM]. An important class of groups that satisfy this condition consists of arithmetic subgroups of any linear algebraic groups defined over number fields and S-arithmetic subgroups of *reductive* linear algebraic groups defined over number fields. Hence, the Novikov conjecture in K-theory holds for these groups.

The rest of this paper is organized as follows. In §2, we recall the original Novikov conjecture and the reformulation in terms of the assembly map in L-theory. Then we state the other Novikov conjectures in K-theory, A-theory and the theory of C^*-algebras; we also mention the Farrell-Jones isomorphism conjecture. In §3, we discuss two approaches to prove the integral Novikov conjectures for torsion-free groups and the generalized integral Novikov conjecture for groups containing torsion elements. In §4, we define and discuss basic properties of arithmetic and S-arithmetic subgroups. In §5, we outline proofs of Theorem 1.1 and Corollary 1.2, and a generalization to geometrically finite Kleinian groups. In §6, we outline proofs of Theorem 1.4, Theorem 1.5 and Theorem 1.6.

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the results [Br2, Proposition VIII. 5.1] and [Br3, Proposition 1.1]. I would also like to thank the anonymous referees for their careful reading of a preliminary version of this paper and several constructive suggestions.

2 Statements of Novikov conjectures

In this section, we recall the original Novikov conjecture and state other versions of Novikov conjecture. In particular, we explain the important notions of assembly maps. After studying the case for torsion-free groups, we state the generalized integral Novikov conjectures for groups containing torsion elements.

To motivate the Novikov conjectures, we first recall the Hirzebruch index theorem. Let M^{4k} be a compact oriented manifold (without boundary) of dimension $4k$. The cup product defines a non-degenerate quadratic form on the middle dimension cohomology group:

$$Q : H^{2k}(M, \mathbb{Q}) \times H^{2k}(M, \mathbb{Q}) \to H^{4k}(M, \mathbb{Q}) = \mathbb{Q}. \tag{2.1}$$

This quadratic form can be diagonalized over $\mathbb{R}$ to the form $\mathrm{Diag}(1, \cdots, 1; -1, \cdots, -1)$, and the number of $+1$ minus the number of -1 is called the signature of M and denoted by $Sgn(M)$. Since the identification $H^{4k}(M, \mathbb{Q}) = \mathbb{Q}$ depends on the orientation of M, the signature $Sgn(M)$ depends on the orientation and is an oriented homotopy invariant of M.

The Hirzebruch class $\mathcal{L}(M)$ is a power series in Pontrjagin classes $P_1, P_2, \cdots$, with rational coefficients,

$$\mathcal{L}(M) = 1 + L_1 + L_2 + \cdots,$$

where $L_1 = \frac{1}{3}P_1, \quad L_2 = \frac{1}{45}(7P_2 - P_1^2), \cdots$.

Then the *Hirzebruch index theorem* is the following equality:

$$Sgn(M) = \langle \mathcal{L}(M), [M] \rangle, \tag{2.2}$$

where the right hand side is the evaluation of $\mathcal{L}(M)$ on the fundamental class $[M]$.

The Hirzebruch class $\mathcal{L}(M)$ depends on the characteristic classes of the tangent bundle of M and a priori depends on the differential structure of M. (In fact, these rational classes in $H^*(M, \mathbb{Q})$ are homeomorphism invariants of M). As pointed out earlier, the left side is an oriented homotopy invariant, and hence the above equality shows that $\langle \mathcal{L}(M), [M] \rangle$ only depends on the oriented homotopy type of M.

To get more homotopy invariants, Novikov introduced the higher signatures. Let $\Gamma = \pi_1(M)$. Let $B\Gamma$ be a classifying space of the discrete group Γ, i.e., a $K(\Gamma, 1)$-space,

$$\pi_1(B\Gamma) = \Gamma, \quad \pi_i(B\Gamma) = \{1\}, \quad i \geq 2.$$

The universal covering space $E\Gamma$ of $B\Gamma$ is contractible and admits a free Γ-action. Equivalently, we can reverse this process and define first $E\Gamma$ as a contractible space with a free Γ-action, and then define $B\Gamma$ as the quotient $\Gamma \backslash E\Gamma$. For example, when $\Gamma = \mathbb{Z}$, it acts freely by translation on $\mathbb{R}$ and hence $E\Gamma = \mathbb{R}$ and $B\Gamma = \mathbb{R}/\mathbb{Z} = S^1$.

For each group Γ, the spaces $E\Gamma$ and $B\Gamma$ are unique up to homotopy. The universal covering map $\tilde{M} \to M$ determines a classifying map $f : M \to B\Gamma$, which is unique up to homotopy.

For any $\alpha \in H^*(B\Gamma, \mathbb{Q})$, $f^*\alpha \in H^*(M, \mathbb{Q})$, and define a *higher signature*

$$Sgn_\alpha(M) = \langle f^*\alpha \cup \mathcal{L}(M), [M] \rangle. \tag{2.3}$$

The original Novikov conjecture is stated as follows:

Conjecture 2.1 (Novikov conjecture) *For any $\alpha \in H^*(B\Gamma, \mathbb{Q})$, the higher signature $Sgn_\alpha(M)$ is an oriented homotopy invariant of M, i.e., if N is another oriented manifold and $g : N \to M$ is an orientation preserving homotopy equivalence, then*

$$\langle (g \circ f)^* \alpha \cup \mathcal{L}(N), [N] \rangle = \langle f^* \alpha \cup \mathcal{L}(M), [M] \rangle.$$

Wall [Wa, §17 H] [Ra3, §24] [Ra2, p.274] reformulated this Novikov conjecture in terms of rational injectivity of the assembly map in surgery theory.

The surgery obstruction groups $L_*(\mathbb{Z}[\Gamma])$, or L-groups of $\mathbb{Z}[\Gamma]$, were first introduced in [Wa] and we will use the free quadratic L-groups as in [Ra2]. Briefly, for $m = 2k$, $L_m(\mathbb{Z}[\Gamma])$ is the Witt group of stable isomorphism classes $(-1)^k$-quadratic forms on finitely generated free modules over the group ring $\mathbb{Z}[\Gamma]$, and $L_{2k+1}(\mathbb{Z}[\Gamma])$ a stable automorphism group of hyperbolic $(-1)^k$-quadratic forms on finitely generated free modules over $\mathbb{Z}[\Gamma]$. Since $(-1)^k$ is 4-periodic in m, the groups $L_m(\mathbb{Z}[\Gamma])$ are 4-periodic in m.

Remarks 2.2 (1) In fact, the group $L_{2k+1}(\mathbb{Z}[\Gamma])$ is more conveniently defined in terms of $(-1)^k$-quadratic formations on finitely generated free modules over $\mathbb{Z}[\Gamma]$. The relation to the description in terms of automorphisms of hyperbolic $(-1)^k$-quadratic forms is that each such automorphism gives a $(-1)^k$-quadratic formation. See [Lu, §4.5] and [Ra1, §12.3] for details.

(2) A uniform definition of the groups $L_m(\mathbb{Z}[\Gamma])$ in terms of m-dimensional quadratic Poincaré chain complexes in the category of finitely generated free $\mathbb{Z}[\Gamma]$-modules is given in [Ra3, p.32].

Remark 2.3 To understand these groups, it might be helpful to recall that the algebraic K-group $K_0(\mathbb{Z}[\Gamma])$ is the group obtained from the monoid of stable equivalence classes of finitely projective (or free) modules over $\mathbb{Z}[\Gamma]$, and $K_1(\mathbb{Z}[\Gamma])$ is defined in terms of automorphisms of finitely generated free modules over $\mathbb{Z}[\Gamma]$, i.e., $K_1(\mathbb{Z}[\Gamma]) = GL(\mathbb{Z}[\Gamma])/[GL(\mathbb{Z}[\Gamma]), GL(\mathbb{Z}[\Gamma])]$. Due to this similarity, the L-groups are often called Hermitian K-groups. On the other hand, the definition of higher algebraic K-groups $K_m(\mathbb{Z}[\Gamma])$ is more complicated, and $K_m(\mathbb{Z}[\Gamma])$ are not periodic in m.

Let $\mathbb{L}(\mathbb{Z})$ be the surgery spectrum:

$$\pi_m(\mathbb{L}(\mathbb{Z})) = L_m(\mathbb{Z}), \quad m \in \mathbb{Z}.$$

The spectrum $\mathbb{L}(\mathbb{Z})$ defines a general homology theory with coefficient in $\mathbb{L}(\mathbb{Z})$. For any topological space X, there are general homology groups $H_*(X; \mathbb{L}(\mathbb{Z})) = \pi_*(X_+ \wedge \mathbb{L}(\mathbb{Z}))$, where X_+ is the disjoint union of X and a point.

There is an important notion of assembly map:

$$A : H_*(X; \mathbb{L}(\mathbb{Z})) \to L_*(\mathbb{Z}[\pi_1(X)]). \tag{2.4}$$

Proposition 2.4 *The Novikov conjecture, i.e., the oriented homotopy invariance of the higher signatures in Conjecture 2.1, is equivalent to the rational injectivity of the assembly map in Equation (2.4), i.e., the following rational assembly map is injective:*

$$A \otimes \mathbb{Q} : H_*(B\Gamma; \mathbb{L}(\mathbb{Z})) \otimes \mathbb{Q} \to L_*(\mathbb{Z}[\Gamma]) \otimes \mathbb{Q}.$$

For a proof of this statement, see [Wa, §17 H] [Ra3, §24] [Ra2, Proposition 6.6]. The surgery exact sequence is used in the proof. We explain briefly why the rational injectivity of the assembly map implies the Novikov conjecture on the homotopy invariance of higher signatures. The manifold

M has a canonical $\mathbb{L}(\mathbb{Z})$-homology fundamental class $[M]_{\mathbb{L}(\mathbb{Z})} \in H_*(M; \mathbb{L}(\mathbb{Z}))$. Under the classifying map $f : M \to B\Gamma$ and the identification

$$H_*(B\Gamma; \mathbb{L}(\mathbb{Z})) \otimes \mathbb{Q} = \oplus_{k \in \mathbb{Z}} H_{*-4k}(B\Gamma; \mathbb{Q}),$$

the class $[M]_{\mathbb{L}(\mathbb{Z})}$ is mapped to $f_*([M] \cap \mathcal{L}(M))$. Clearly, the homotopy invariance of all higher signatures is equivalent to homotopy invariance of the class $f_*([M] \cap \mathcal{L}(M))$. So the class $[M]_{\mathbb{L}(\mathbb{Z})}$ is an integral version of the $\mathcal{L}$-class of M, or rather $[M] \cap \mathcal{L}(M)$. An important fact about the assembly map is that the image $A([M]_{\mathbb{L}(\mathbb{Z})})$ in $L_*(\mathbb{Z}[\pi_1(X)])$ is an oriented homotopy invariant of M. Hence the injectivity of the rational assembly map $A \otimes \mathbb{Q}$ implies that $f_*([M] \cap \mathcal{L}(M))$ and hence all the higher signatures are oriented homotopy invariants of M.

Remarks 2.5 (1) The notion of assembly map was first introduced by Quinn in his thesis [Qu]. Elements in $H_*(X; \mathbb{L}(\mathbb{Z}))$ are represented by certain cycles, and the image under A is obtained (or assembled) from these cycles by gluing along faces. So this map is an assemblage from local parts to global invariants, and is hence called an assembly map.

(2) There are several other ways to view or define the assembly map. In [Ra2] [Ra3], it is defined as follows. Let $p : \tilde{X} \to X$ be the projection map from the universal cover. An element in $H_*(X; \mathbb{L}(\mathbb{Z}))$ is represented by a cycle c of quadratic Poincaré chain complex over X. Under the map p, it is pulled back to $p^! c$, a cycle over $\tilde{X}$. Let $pt.$ be the trivial Γ-space consisting of one point. The pushforward $q_! p^! c$ under the map $q : \tilde{X} \to pt.$ gives a quadratic Poincaré chain complex over $\mathbb{Z}[\Gamma]$, which is mapped to an element in $L_*(\mathbb{Z}[\pi_1(X)])$.

(3) Another point of view is to use equivariant homology groups (see [LR] for details and references). Briefly, the homology group $H_*(B\Gamma; \mathbb{L}(\mathbb{Z}))$ can be identified with the equivariant homology group $H_*^\Gamma(E\Gamma; \mathbb{L}(\mathbb{Z}))$. Note that $H_*^\Gamma(pt.; \mathbb{L}(\mathbb{Z})) = L_*(\mathbb{Z}[\Gamma])$. The projection $E\Gamma \to pt.$ defines a map $H_*^\Gamma(E\Gamma; \mathbb{L}(\mathbb{Z})) \to H_*^\Gamma(pt.; \mathbb{L}(\mathbb{Z}))$, and the composition

$$H_*(B\Gamma; \mathbb{L}(\mathbb{Z})) \cong H_*^\Gamma(E\Gamma; \mathbb{L}(\mathbb{Z})) \to H_*^\Gamma(pt.; \mathbb{L}(\mathbb{Z})) = L_*(\mathbb{Z}[\Gamma]) \tag{2.5}$$

is the assembly map.

(4) It will be seen later that the assembly maps in other theories can be defined more directly or interpreted concretely.

Conjecture 2.6 (Integral Novikov conjecture) *If Γ is torsion free, then the assembly map $A : H_*(B\Gamma; \mathbb{L}(\mathbb{Z})) \to L_*(\mathbb{Z}[\Gamma])$ is injective.*

This conjecture is also called the L-theory (or surgery theory) integral Novikov conjecture.

Remark 2.7 In the integral Novikov conjecture, the torsion free assumption on Γ is important. In fact, it is known that the conjecture is often false for finite groups. One reason is that if G is a finite group of odd order, $H_*(B\Gamma; \mathbb{L}(\mathbb{Z}))$ often contains torsion elements of odd order, but $L_*(\mathbb{Z}[\Gamma])$ only contains 2-torsion elements. Hence A can not be injective. See [We, p. 286] [Ra1, Remark 7.4. (i)].

Remark 2.8 Conjecture 2.6 also implies the oriented homotopy invariance of the image in $K_*(B\Gamma)$ under the classifying map $f : M \to B\Gamma$ of the element in $K_*(M)$ determined by the signature operator of M, and of the higher Morgan-Sullivan L-classes in the 2-local homology group $H_*(B\Gamma, \mathbb{Z}_{(2)})$. On the other hand, for linear lens spaces, this homotopy invariance fails. This is another indication that Conjecture 2.6 often fails for groups containing torsion elements.

Clearly, the integral Novikov conjecture implies the rational Novikov conjecture and gives an integral version of homotopy invariance of higher signature. There are also several other reasons to consider the integral, rather than the original (rational) Novikov conjecture:

1. Relation to the rigidity of manifolds, in particular, the Borel conjecture for rigidity of aspherical manifolds.

2. Computation of the L-groups $L_*(\mathbb{Z}[\Gamma])$ in terms of a generalized homology theory.

The *Borel conjecture* states that if M, N are two closed aspherical manifolds, i.e., $\pi_i(M) = \pi_i(N) = \{1\}$, for $i \geq 2$, then any homotopy equivalence between M and N is homotopic to a homeomorphism. The h-cobordism version of the Borel conjecture [FRR, p. 17] says that any homotopy equivalence between two closed aspherical manifolds M and N is h-corbordant to a homeomorphism. The Borel conjecture was related to and in fact motivated by the Mostow strong rigidity of locally symmetric spaces (or rather his earlier work on the rigidity of solvable manifolds).

The following equivalence is known [FRR, p.28]:

Proposition 2.9 *The h-corbordism Borel conjecture is equivalent to the conjecture that the integral assembly map* $A : H_*(B\Gamma; \mathbb{L}(\mathbb{Z})) \to L_*(\mathbb{Z}[\Gamma])$ *is an isomorphism for* $\Gamma = \pi_1(M)$.

Basically this follows from the surgery exact sequence. It is also known that the injectivity of A implies that any manifold homotopy equivalent to M with $\pi_1(M) = \Gamma$ is normally cobordant to it [We, p. 286].

The left-hand side $H_*(B\Gamma; \mathbb{L}(\mathbb{Z}))$ is a generalized homology theory and hence can be computed relatively easily. On the other hand, the groups $L_*(\mathbb{Z}[\Gamma])$ are important but difficult to compute. If the assembly map is an isomorphism, this allows one to compute $L_*(\mathbb{Z}[\Gamma])$ using $H_*(B\Gamma; \mathbb{L}(\mathbb{Z}))$. Clearly the injectivity of the assembly map is an important step in this direction. In fact, it is worthwhile to point out that in many cases where the injectivity of the assembly map A is proved to true, the proof, for example using Theorems 3.1, 3.2 below (and also Theorems 3.5, 3.6 for the generalized integral Novikov conjectures), yields the stronger conclusion that A is split injective, and hence $H_*(B\Gamma; \mathbf{L}(\mathbb{Z}))$ is a *direct summand* of $L_*(\mathbb{Z}[\Gamma])$.

Assembly map in algebraic K-theory.

Once formulated in terms of the assembly map, there are also other versions of Novikov conjecture. For any associative ring with unit R, there is a family of algebraic K-groups $K_i(R)$, $i \in \mathbb{Z}$. For example, as mentioned above in Remark 2.3, $K_0(R)$ is defined by stable equivalence classes of finitely generated projective modules, and $K_1(R) = GL(R)/[GL(R), GL(R)]$. The higher K-groups $K_i(R)$, $i \geq 2$, are defined to be the homotopy groups of the space $BGL(R)^+$, where $BGL(R)$ is the classifying space of $GL(R)$ considered as a discrete group, and $BGL(R)^+$ is the space by applying the Quillen +-construction to the perfect subgroup $E(R) = [GL(R), GL(R)]$, in particular, the homology groups of $BGL(R)^+$ and $BGL(R)$ are equal to each other under the inclusion (see [Ro2, §5.2] [Lo, §11.2]). The K-theory spectrum $\mathbb{K}(R)$ with $\pi_i(\mathbb{K}(R)) = K_i(R)$, $i \in \mathbb{Z}$, is given by the delooping of the infinite loop space $BGL(R)^+ \times K_0(R)$. (See [Ro2, p. 269] [Lo, p. 355, p.396] [Lu] and the references there.)

Let Γ be a group as above, and $H_*(B\Gamma; \mathbb{K}(\mathbb{Z}))$ the generalized homology of $B\Gamma$ with coefficient in $\mathbb{K}(R)$. There is also an assembly map

$$A : H_*(B\Gamma; \mathbb{K}(\mathbb{Z})) \to K_*(\mathbb{Z}[\Gamma]). \qquad (2.6)$$

Besides the approach via equivariant homology groups in [LR] as mentioned in Remarks 2.5,

$$H_*(B\Gamma; \mathbb{K}(\mathbb{Z})) = H_*^\Gamma(E\Gamma; \mathbb{K}(\mathbb{Z})) \to H_*^\Gamma(pt.; \mathbb{K}(\mathbb{Z})) = K_*(\mathbb{Z}[\Gamma]),$$

this assembly map can be described explicitly as follows [Lo, p. 396] [Ca, p. 6]. Multiplication by elements of Γ defines an inclusion $\Gamma \to GL_1(\mathbb{Z}[\Gamma])$, which in turn defines a map

$$j : \Gamma \times GL_n(\mathbb{Z}) \to GL_1(\mathbb{Z}[\Gamma]) \times GL_n(\mathbb{Z}) \to GL_n(\mathbb{Z}[\Gamma]).$$

Passing to the limit as $n \to +\infty$, we obtain a map

$$(j, id.) : \Gamma \times GL(\mathbb{Z}) \to GL(\mathbb{Z}[\Gamma]) \times GL(\mathbb{Z}).$$

Applying the classifying space functor B and Quillen's +-construction gives

$$B\Gamma_+ \wedge BGL(\mathbb{Z})^+ \to BGL(\mathbb{Z}[\Gamma])^+ \wedge BGL(\mathbb{Z})^+ \to BGL(\mathbb{Z}[\Gamma])^+,$$

where the second map follows from the multiplicative structure of algebraic K-theory [Lo, p. 396, p. 357]. Delooping $BGL(\cdot)^+$, we obtain the assembly map on the spectrum level,

$$A : B\Gamma_+ \wedge \mathbb{K}(\mathbb{Z}) \to \mathbb{K}(\mathbb{Z}\Gamma).$$

Taking the homotopy groups gives the assembly map

$$A : H_*(B\Gamma; \mathbb{K}(\mathbb{Z})) \to K_*(\mathbb{Z}[\Gamma]).$$

It is clear from this description that both parts $B\Gamma$ and $\mathbb{K}(\mathbb{Z})$ are used together to produce the assembly map.

Conjecture 2.10 (Integral Novikov conjecture in algebraic K-theory) *Assume that Γ is torsion free. Then the assembly map*

$$A : H_*(B\Gamma; \mathbb{K}(\mathbb{Z})) \to K_*(\mathbb{Z}[\Gamma])$$

is injective.

As pointed out in Remark 2.7, the torsion free assumption is important. On the other hand, for any group, it is conjectured that the rational assembly map

$$A : H_*(B\Gamma; \mathbb{K}(\mathbb{Z})) \otimes \mathbb{Q} \to K_*(\mathbb{Z}[\Gamma]) \otimes \mathbb{Q}$$

is injective. It is called the *Novikov conjecture in algebraic K-theory.*

If Γ is torsion free, it is believed that the assembly map A in the above conjecture is an isomorphism,

$$H_*(B\Gamma; \mathbb{K}(\mathbb{Z})) \cong K_*(\mathbb{Z}[\Gamma]).$$

This is also called the *Borel conjecture* in algebraic K-theory in view of Proposition 2.9.

If the map A is an isomorphism, then the group $K_*(\mathbb{Z}[\Gamma])$ can be described in terms of two parts: the space $B\Gamma_+$ and the spectrum $\mathbb{K}(\mathbb{Z})$. As mentioned earlier, generalized homology theories such as $H_*(B\Gamma; \mathbb{K}(\mathbb{Z}))$ are relatively easier to compute. On the other hand, if Γ is non-abelian, $\mathbb{Z}[\Gamma]$ is a non-abelian ring, and there are very few general techniques to compute its algebraic K-groups, unlike the case of commutative rings (see [Ca, p. 5]).

Assembly map in A-theory of topological spaces

Closely related to the algebraic K-theory of rings is the A-theory, i.e., the algebraic K-theory of topological spaces. In fact, when the topological space is $E\Gamma$, the A-groups $A_*(B\Gamma)$ are rationally isomorphic to the algebraic K-groups of the ring $\mathbb{Z}[\Gamma]$:

$$A_*(B\Gamma) \otimes \mathbb{Q} = K_*(\mathbb{Z}[\Gamma]) \otimes \mathbb{Q}.$$

See [Ro3] for a summary of the A-theory and its applications.

Besides the above relation, there are several reasons why this is called the algebraic K-theory of spaces. First, for each topological space, there is a ring up to homotopy; and the A-groups are basically the algebraic K-groups of this ring up to homotopy. Second, the A-groups can be constructed from a category with cofibration and weak equivalence which is built up from the topological space. This construction is closely related to the Q-construction of Quillen for algebraic K-groups of rings.

From this description, the algebraic K-groups of a topological space are clearly different from the topological K-groups of the space, which are constructed from the monoid of equivalence classes of vector bundles on the topological space and its suspensions. On the other hand, these A-groups (or algebraic K-groups) of manifolds can be used to describe the space of pseudo-isotopies of the manifolds, an important notion in geometric topology.

Let W be a topological space and W_+ the union of W with a disjoint point. Then there is an assembly map:

$$A : H_*(W_+; \mathbb{A}(pt.)) \to A_*(W).$$

This map is shown to be split injective when W is a complete Riemannian manifold of nonpositive sectional curvature in [FW2].

The A-groups are easier to compute due to the cyclotomic trace, which maps the A-groups to the topological cyclic homology groups. In fact, this technique is used in [BHM] to prove that for any discrete group Γ, if in every degree i, $H_i(B\Gamma, \mathbb{Z})$ is finitely generated, then the (rational) Novikov conjecture in algebraic K-theory holds for Γ.

Assembly map in C^*-algebras

There is also an assembly map in the theory of C^*-algebras. The group algebra $\mathbb{C}[\Gamma]$ acts on $\ell^2(\Gamma)$ by the left translation. The completion of $\mathbb{C}[\Gamma]$ in the Hilbert space of bounded operators on $\ell^2(\Gamma)$ with respect to the norm topology is the reduced C^*-algebra $C_r^*(\Gamma)$ associated with Γ.

Let $K_*(C_r^*(\Gamma))$ be the topological K-groups of $C_r^*(\Gamma)$. There is an assembly map

$$A : K_*(B\Gamma) \to K_*(C_r^*(\Gamma)). \tag{2.7}$$

Besides a description in terms of equivariant homology groups in [LR] (see Remarks 2.5),

$$K_*(B\Gamma) = H_*(B\Gamma; \mathbb{K}^{top}) = H_*^\Gamma(E\Gamma; \mathbb{K}^{top}) \to H_*^\Gamma(pt.; \mathbb{K}^{top}) = K_*(C_r^*(\Gamma)),$$

this assembly map is essentially given by taking the index of elliptic pseudo-differential operators. Briefly, the K-homology groups $K_*(B\Gamma)$ are the duals of the more common K-cohomology groups $K^*(B\Gamma)$, and cycles in these homology groups can be described in terms of elliptic pseudo-differential operators on $B\Gamma$. The tensor product of these operators with $C_r^*(\Gamma)$ gives operators on $C_r^*(\Gamma)$-bundles, and their indexes give elements in $K_*(C_r^*(\Gamma))$ (see [LR] and the references there for details).

Conjecture 2.11 (Integral analytic Novikov conjecture) *If Γ is torsion free, then the assembly map*

$$A : K_*(B\Gamma) \to K_*(C_r^*(\Gamma))$$

is injective.

Similarly, there is also a rational version about the injectivity of $A \otimes \mathbb{Q}$. This analytic Novikov conjecture implies the original Novikov conjecture on homotopy invariance of higher signatures. In fact, the rational injectivity of this assembly map is equivalent to the rational injectivity of the assembly map in surgery theory. Because of this, the analytic Novikov conjecture is often called *strong Novikov conjecture*. On the other hand, the integral analytic Novikov conjecture does not imply the integral Novikov conjecture in surgery (i.e., L-) theory, and there are no other direct implications between these many versions of Novikov conjectures.

A stronger conjecture is the following one.

Conjecture 2.12 (Baum-Connes conjecture) *If Γ is torsion free, then the assembly map*

$$A : K_*(B\Gamma) \to K_*(C_r^*(\Gamma))$$

is an isomorphism.

This is an analogue of the Borel conjecture (see Proposition 2.9), and also implies some rigidity results. For many applications of the analytic Novikov conjecture and the Baum-Connes conjecture, see [LR] and the references cited there.

Generalized Integral Novikov Conjecture for groups with torsion.

Many natural groups such as (maximal) arithmetic groups $SL(n,\mathbb{Z})$, S-arithmetic subgroups $SL(n,\mathbb{Z}[\frac{1}{p_1},\cdots,\frac{1}{p_l}])$, and the mapping class groups etc all contain torsion elements. In general, suppose a group Γ contains a torsion-free subgroup Γ' of finite index. It is not easy to compute the global groups such as $L_*(\mathbb{Z}[\Gamma])$ and $K_*(\mathbb{Z}[\Gamma])$ from $L_*(\mathbb{Z}[\Gamma'])$ and $K_*(\mathbb{Z}[\Gamma'])$. Since one motivation of the integral Novikov conjecture is to compute such global groups, a natural question is to formulate a generalized integral Novikov conjecture in each theory for groups containing torsion elements.

Let $E\Gamma$ be the universal covering space of $B\Gamma$. Then it is the universal space for proper and fixed-point free actions of Γ. It is known that

$$H_*(B\Gamma) = H_*^{\Gamma}(E\Gamma).$$

Assume that Γ contains torsion elements. Let $\mathcal{F}$ be the family of all finite subgroups of Γ. Then there is a universal (or classifying) space $E_{\mathcal{F}}\Gamma$, unique up to homotopy, which is a Γ-space satisfying the following properties:

1. For any element $H \in \mathcal{F}$, its set of fixed points $(E_{\mathcal{F}}\Gamma)^H$ is nonempty and contractible. In particular, $E_{\mathcal{F}}\Gamma$ is contractible.

2. For any point $x \in E_{\mathcal{F}}\Gamma$, its stabilizer in Γ belongs to $\mathcal{F}$.

Note that $E_{\mathcal{F}}\Gamma$ is the universal space for proper actions of Γ.

Then the *generalized integral Novikov conjecture* in L-theory says that the following assembly map is injective:

$$A : H_*^{\Gamma}(E_{\mathcal{F}}; \mathbb{L}(\mathbb{Z})) \to H_*^{\Gamma}(pt.; \mathbb{L}(\mathbb{Z})) = L_*(\mathbb{Z}[\Gamma]), \tag{2.8}$$

which is defined by the equivariant projection from $E_{\mathcal{F}}\Gamma$ to the trivial one point Γ-space pt. It should be pointed out that the rational injectivity of this assembly map is the same as the rational injectivity map of the earlier usual assembly map $H_*(B\Gamma; \mathbb{L}(\mathbb{Z})) \to L_*(\mathbb{Z}[\Gamma])$.

For any family $\mathcal{C}$ of subgroups of Γ which are closed under taking subgroups and conjugates, there is an associated classifying space $E_{\mathcal{C}}\Gamma$ characterized by the same properties as above when $\mathcal{F}$ is replaced by $\mathcal{C}$. Similarly, there is an assembly map

$$A : H_*^{\Gamma}(E_{\mathcal{C}}\Gamma; \mathbb{L}(\mathbb{Z})) \to L_*(\mathbb{Z}[\Gamma]).$$

Another important family of subgroups is the the family $\mathcal{VCY}$ of virtually cyclic sbgroups of Γ. The conjecture that this assembly map for the family $\mathcal{VCY}$ is an isomorphism when the L-theory is given the $-\infty$ decoration is called the *Farrell-Jones isomorphism conjecture* [FJ1] [LR, Conjecture 2.2].

The generalized integral Novikov conjectures in algebraic K-theory and the theory of C^*-algebras can be similarly formulated in terms of $E_{\mathcal{F}}\Gamma$, for example, in K-theory,

$$A : H_*^\Gamma(E_{\mathcal{F}}\Gamma; \mathbb{K}(\mathbb{Z})) \to K_*(\mathbb{Z}[\Gamma]). \tag{2.9}$$

There are also analogues of the Farrell-Jones isomorphism conjecture.

In both L-theory and K-theory, the assembly maps in Equations (2.8) and (2.9) are in general not expected to be surjective. In fact, they already fail for some virtually cyclic groups [Fa]. This is the reason to use the larger family $\mathcal{VCY}$ instead of $\mathcal{F}$ in the Farrell-Jones isomorphism conjecture. (Note that if $\mathcal{C}$ is taking to be the family of all subgroups of Γ, then $E_{\mathcal{C}}\Gamma = pt.$, and the associated assembly map is the identity map, which is not helpful to understand or compute $L_*(\mathbb{Z}[\Gamma])$. Hence, it is better in some sense to choose a smaller family $\mathcal{C}$.)

On the other hand, in the theory of C^*-algebras, the assembly map

$$A : K_*^\Gamma(E_{\mathcal{F}}\Gamma) \to K_*(C_r^*(\Gamma))$$

is conjectured to be both injective and surjective. This is the *Baum-Connes conjecture* for groups containing torsion elements.

3 Approaches to proving Novikov conjectures

In this section, we first recall two methods (or criterions) to prove the integral Novikov conjectures. They are related to global (or large scale) geometry of the group Γ and its related spaces $B\Gamma$ and $E\Gamma$. Then we recall similar criterions for the generalized integral Novikov conjectures.

Motivated by a result of Yu in [Yu] for the analytic Novikov conjecture, Carlsson and Goldfarb [CG1, main Theorem], Bartels [Ba, Theorems 1.1 and 7.2], Dranishnikov, Ferry and Weinberger [DFW], and Chang, Ferry and Yu [CFY] proved the following result.

Theorem 3.1 *If a finitely generated group* Γ *has finite asymptotic dimension,* asdim $\Gamma < \infty$*, and has a finite* $B\Gamma$*, i.e., its classifying space* $B\Gamma$ *can be realized as a finite CW-complex, then the integral Novikov conjectures in K-theory and L-theory hold for* Γ*.*

We recall that the definition of as-dim Γ from [Gr]. First, given a non-compact metric space (M, d), its asymptotic dimension, asdim (M, d), is defined as the smallest integer n, which could be ∞, such that for every $r > 0$, there exists a cover $\mathcal{C} = \{U_i\}_{i \in I}$ of M by uniformly bounded sets U_i with the r-multiplicity less than or equal to $n+1$, i.e., every ball in M of radius r intersects at most $n+1$ sets in $\mathcal{C}$. (Note that this is similar to the definition of dimension of a topological space via coverings of smaller and smaller sets).

For any finitely generated group Γ, a choice of a symmetric generating set S, i.e., $S^{-1} = S$, defines a word metric d_S on Γ. Define asdim Γ = asdim (Γ, d_S), which is known to be independent of the choice of the generating set S.

Briefly, in the proof of Theorem 3.1, the assumption that asdim $\Gamma < +\infty$ is used to construct a suitable family of coverings of $E\Gamma$. The assumption that $B\Gamma$ can be taken to be a finite CW complex is also very important in the proof.

Another method by Carlsson-Pedersen [CP3] is to use a suitable compactification of $E\Gamma$.

Theorem 3.2 *Suppose that $B\Gamma$ is finite CW-complex, and the universal cover $E\Gamma$ has a contractible, metrizable Γ-compactification $\overline{E\Gamma}$ such that the action of Γ on $\overline{E\Gamma}$ is small at infinity. Then the integral Novikov conjecture in K-theory and L-theory holds for Γ.*

Recall that by a small action of Γ at the infinity of a compactification $\overline{E\Gamma}$ of $E\Gamma$, it means that for every boundary point $y \in \overline{E\Gamma} - E\Gamma$ and every compact subset $K \subset E\Gamma$, and every small neighborhood U of y in $\overline{E\Gamma}$, there exists another small neighborhood V of y such that if $g \in \Gamma$ and $gK \cap V \neq \emptyset$, then $gK \subset U$.

Basically it says that for a sequence $g_j \in \Gamma$ going to infinity, if a point of $g_j K$ is contained in a boundary neighborhood, then the whole set $g_j K$ is contained a neighborhood of the same boundary point. Hence, $g_j K$ are shrunk to points near the boundary.

Suppose that $E\Gamma$ is given a Γ-invariant metric d, and the compactification $\overline{E\Gamma}$ has the property that sequences within bounded distance converges to the same limit, i.e., for any two unbounded sequences x_j, x_j' in $E\Gamma$ with $\lim_{j\to+\infty} x_j = y$ and $\lim_{j\to+\infty} x_j' = y$, if $\limsup_{j\to+\infty} d(x_j, x_j') < +\infty$, then $y = y'$. Then the action of Γ on $\overline{E\Gamma}$ is small at infinity. Such a compactification exists when $B\Gamma$ is given by a compact manifold M with nonpositive curvature. In this case, $E\Gamma = \tilde{M}$ is simply connected and nonpositively curved and admits the known geodesic compactification $\tilde{M} \cup \tilde{M}(\infty)$, where $\tilde{M}(\infty)$ is the set of equivalence classes of geodesics. The above condition that sequences within bounded distance converges to the same limit point is easily seen to be satisfied. It is clear from the definition of the topology (or convergence of unbounded sequences in $\tilde{M}$) that the Γ-action on $\tilde{M}$ extends to $\tilde{M} \cup \tilde{M}(\infty)$. Hence it is a compactification with small Γ-action at infinity. It is also clear that $\tilde{M} \cup \tilde{M}(\infty)$ is homeomorphic to a closed unit ball and hence is contractible. (We note that the one point compactification $E\Gamma \cup \{\infty\}$ is always a Γ-compactification with small Γ-action at infinity, but it is not contractible in general.)

The basic idea of the proof of Theorem 3.2 is that the assembly map can be interpreted as a "forget some control" functor from a continuously controlled category to a boundedly controlled category [Pe, Proposition 10, Theorem 15]. The assumption that the action of Γ at the compactification $\overline{E\Gamma}$ is small allows us to regain control, i.e., to get a continuous control from a bounded control. See [Pe] and [Ro1] for details of such explanations of the proof in [CP3].

Remark 3.3 A related result is given by Ferry-Weinberger [FW1], where the boundary $\partial\overline{E\Gamma}$ of a compactification $\overline{E\Gamma}$ is required to be a Z-set, i.e., there exists a homotopy $h_t : \partial\overline{E\Gamma} \to \overline{E\Gamma}$, $t \in [0, 1]$, such that h_0 is the identity map (or the inclusion), and $h_t(\partial\overline{E\Gamma}) \subset E\Gamma$ for $t > 0$; and $\overline{E\Gamma}$ is small in the sense that every continuous bounded map $f : E\Gamma \to E\Gamma$ extends by the identity map to a continuous map $f : \overline{E\Gamma} \to \overline{E\Gamma}$.

Remark 3.4 Carlsson-Pedersen [CP1] [Go2] has a generalization of Theorem 3.2, relaxing the smallness of the Γ-action at infinity, which requires a Γ-equivariant compactification $\overline{E\Gamma}$ which is Cech-acyclic whose boundary is covered by a Γ-invariant collection of boundedly saturated open sets satisfying a weak homotopy equivalence between the inverse limit of the nerve and the boundary.

Generalized Integral Novikov conjectures.

To study the generalized integral Novikov conjectures for groups containing torsion elements, the above criterions in Theorems 3.1 and 3.2 can not be used. In fact, the existence of a finite CW-complex $B\Gamma$ implies that Γ is torsion-free. We present generalizations of these criterions in Theorems 3.1 and 3.2.

Recall that a Γ-CW-complex E is called a Γ-*cofinite CW-complex* (or Γ-*cofinite*) if the quotient $\Gamma\backslash E$ is a finite CW-complex. Rosenthal [Ros] generalized the method of [CP3] and proved the following generalization of Theorem 3.2.

Theorem 3.5 *Assume that $E_{\mathcal{F}}\Gamma$ is a cofinite Γ-CW-complex and admits a compactification $\overline{E_{\mathcal{F}}\Gamma}$ such that*

1. *The Γ-action on $E_{\mathcal{F}}\Gamma$ extends to a continuous action on the compactification $\overline{E_{\mathcal{F}}\Gamma}$.*
2. *$\overline{E_{\mathcal{F}}\Gamma}$ is metrizable.*
3. *For any finite subgroup H of Γ, the fixed point set $\overline{E_{\mathcal{F}}\Gamma}^H$ is nonempty and contractible and contains the fixed point set $E_{\mathcal{F}}\Gamma^H$ in $E_{\mathcal{F}}\Gamma$ as a dense subset. In particular, the compactification $\overline{E_{\mathcal{F}}\Gamma}$ is contractible.*
4. *The Γ-action of Γ on $\overline{E_{\mathcal{F}}\Gamma}$ is small at infinity.*

Then the generalized integral Novikov conjecture in K- and L-theories hold for Γ, i.e., the assembly maps in Equations (2.8) and (2.9) are injective.

A generalization of Theorem 3.1 is given in [Ji6]

Theorem 3.6 *Assume that Γ has finite asymptotic dimension and admits a Γ-co-finite $E_{\mathcal{F}}\Gamma$. For any pair of finite subgroups H, I of Γ, $I \subset H$, let $N_H(I)$ be the normalizer of I in H. Assume that for any such a pair H, I, the set of fixed points $(E_{\mathcal{F}}\Gamma)^I$ and the quotient $N_H(I)\backslash(E_{\mathcal{F}}\Gamma)^I$ are uniformly contractible and of bounded geometry. Then the generalized integral Novikov conjecture in both K- and L-theories holds for Γ.*

Comments.

In order to apply these methods to study the integral Novikov conjectures, it is important to find groups satisfying the assumptions. In fact, the known classes are quite limited. For example, groups Γ with asdim $\Gamma < +\infty$ have been intensively studied. The following is a list of such groups:

1. hyperbolic groups [Gr],
2. Coxeter groups, standard constructions from groups with finite asymptotic dimension (see [BD] and [Dr]),
3. uniform discrete subgroups of Lie groups [CG2],
4. arithmetic groups [Ji1].
5. S-arithmetic subgroups of reductive algebraic groups over global fields [Mat] [Ji4].

It is a nontrivial problem to decide when a group Γ has a finite CW-complex as $B\Gamma$, i.e., Γ is of type F. If Γ is the fundamental group of an aspherical closed manifold, it is of type F. There are also necessary and sufficient cohomological conditions. See [Br2] and [Da] for details. The further condition that the universal covering space $E\Gamma$ of a finite $B\Gamma$ admits a contractible Γ-compactification with small Γ-action at infinity is not easy to satisfy. See [Go2] for some discussions about this issue.

A basic point of this paper is that the class of arithmetic subgroups of linear algebraic groups and the class of S-arithmetic groups of *reductive* algebraic groups are natural and important classes which satisfy the conditions in Theorems 3.1, 3.2, 3.5 and 3.6.

4 Definition of arithmetic and S-arithmetic subgroups

In this section, we recall very briefly definitions and basic properties of arithmetic groups and S-arithmetic subgroups.

Let $\mathbf{G} \subset GL(n)$ be a linear algebraic group defined over a number field k. Let $\mathcal{O}_k$ be the ring of integers of k. Any subgroup $\Gamma \subset \mathbf{G}(k)$ commensurable with $\mathbf{G}(\mathcal{O}_k) = \mathbf{G}(k) \cap GL(n, \mathcal{O}_k)$ is called an *arithmetic subgroup* of $\mathbf{G}$.

Remark 4.1 For arithmetic subgroups, there is no loss of generality in considering only the case $k = \mathbb{Q}$. In fact, the functor of restriction of scalars reduces the general case of number fields to the special case of $\mathbb{Q}$. On the other hand, for S-arithmetic subgroups, the general number fields give a larger class of S-arithmetic groups [Se1]. That's why we start with general number fields.

An important example of arithmetic subgroups is given by $\Gamma = SL(n, \mathbb{Z})$ and its subgroups of finite index.

Some of the basic properties of arithmetic groups are listed in the following (see [Bo1] [Bo2] [Bo5]).

Proposition 4.2 *Let Γ be an arithmetic subgroup of a linear algebraic group $\mathbf{G}$ defined over k.*

1. *Γ is finitely presented.*
2. *Γ admits a torsion free subgroup of finite index.*

Now we recall the definition of S-arithmetic groups. Let k be a global field, i.e., either a number field (a finite extension of $\mathbb{Q}$), or the function field of a smooth projective curve over a finite field $\mathbb{F}_q$ (a finite separable extension of $\mathbb{F}_q(t)$).

For each place $\mathfrak{p}$, let $\nu_{\mathfrak{p}}$ be the associated valuation of k. Let S be a finite set of places of k including all archimedean places, which exist if and only if k is a number field. Define the ring $\mathcal{O}_{k,S}$ of S-integers by

$$\mathcal{O}_{k,S} = \{x \in k \mid \nu_{\mathfrak{p}}(x) \geq 0, \mathfrak{p} \notin S\}.$$

If k is a number field and S consists of precisely all the archimedean places, then $\mathcal{O}_{k,S} = \mathcal{O}_k$, the usual ring of integers in k. If $k = \mathbb{Q}$ and $S = \{p_1, \cdots, p_m\}$, then

$$\mathcal{O}_{\mathbb{Q},S} = \mathbb{Z}[\frac{1}{p_1}, \cdots, \frac{1}{p_m}].$$

If k is a function field over a finite field $\mathbb{F}_q$ and $S = \emptyset$, then $\mathcal{O}_{k,S} = \mathbb{F}_q$ and hence is finite. (Recall that any regular function on a smooth projective curve is constant.) To get nontrivial rings of S-integers, we need to assume that S is nonempty, or equivalently contains non-archimedean places. Then $\mathcal{O}_{k,S}$ is the ring of functions on the curve regular outside the set of points corresponding to S.

Let $\mathbf{G} \subset GL(n)$ be a linear algebraic group defined over a global field k. A subgroup Γ of $\mathbf{G}(k)$ is called an *S-arithmetic subgroup* if it is commensurable with $\mathbf{G}(k) \cap GL(n, \mathcal{O}_{k,S})$.

Important examples are given by $SL(n, \mathbb{Z}[\frac{1}{p_1}, \cdots, \frac{1}{p_m}])$ and its subgroups of finite index. When k is a number field and S consists of precisely all the archimedean places, an S-arithmetic subgroup is an arithmetic subgroup as defined earlier. From this point of view, it is natural to consider S-arithmetic subgroups together with the rather special arithmetic subgroups.

In the following, by an S-arithmetic subgroup, we assume that S contains some non-archimedean places unless indicated otherwise.

Unlike arithmetic subgroups, S-arithmetic subgroups do not have the usual cohomological finiteness unless $\mathbf{G}$ is reductive. Over function fields, the cohomological finiteness results are even more restricted (see [Br1, Chap. VII], [Be1] [Be2] [Ab1] [Ab2]).

Proposition 4.3 *(1) Assume that $\mathbf{G}$ is a reductive algebraic group defined over a number field. Then any S-arithmetic subgroup Γ is finitely presented and admits torsion free subgroups of finite index. (2) Assume that $\mathbf{G}$ is a reductive algebraic group defined over a function field k of k-rank 0. Then any S-arithmetic subgroup Γ is finitely presented and admits torsion free subgroups of finite index.*

In the above proposition, the rank 0 assumption in the case of function fields is important. See [Se2] [Bo2] [Br1, Chap. VII] for more discussions about S-arithmetic subgroups.

5 Novikov conjectures for arithmetic subgroups

In this section, we first outline a proof of Theorem 1.1 and Corollary 1.2 by applying Theorem 3.1. Then we discuss a generalization to the class of geometrically finite Kleinian groups.

To prove Corollary 1.2, we need to check the following two conditions:

1. There exists a finite $B\Gamma$, i.e., given by a finite CW-complex.

2. asdim $\Gamma < +\infty$, i.e., Theorem 1.1 holds.

We first discuss Condition (1). Let $G = \mathbf{G}(\mathbb{R})$ be the real locus, and $K \subset G$ a maximal compact subgroup. Then the homogeneous space $X = G/K$ is diffeomorphic to a euclidean space and hence is contractible. Give X a G-invariant metric. If $\mathbf{G}$ is reductive, then X is a symmetric space not containing any compact factor. If $\mathbf{G}$ is semisimple, then X is a symmetric space of noncompact type.

The arithmetic group Γ is a discrete subgroup of G and hence acts properly on X. If Γ is torsion free, it acts freely on X, and the quotient $\Gamma\backslash X$ is a manifold. Since X is contractible, $\Gamma\backslash X$ is a $K(\Gamma, 1)$-space, or a $B\Gamma$-space. In the following, we assume that Γ is torsion free.

If the $\mathbb{Q}$-rank of $\mathbf{G}$ is equal to 0, then $\Gamma\backslash X$ is compact. Then $\Gamma\backslash X$ is a compact manifold (without boundary) and hence admits a finite triangulation, which implies that $\Gamma\backslash X$ is a finite $B\Gamma$.

On the other hand, if the $\mathbb{Q}$-rank of $\mathbf{G}$ is positive, $\Gamma\backslash X$ is non-compact. Without loss of generality, we can assume that $\Gamma\backslash X$ has finite volume with respect to the invariant metric, i.e., Γ is a lattice subgroup of G. (This can be achieved by dividing out the $\mathbb{Q}$-isotropic part of the center of $\mathbf{G}$.) Then there exists a compactification $\overline{\Gamma\backslash X}^{BS}$, called the Borel-Serre compactification, which is a manifold with corners. The interior of $\overline{\Gamma\backslash X}^{BS}$ is equal to $\Gamma\backslash X$ and hence the inclusion $\Gamma\backslash X \to \overline{\Gamma\backslash X}^{BS}$ is a homotopy equivalence. Since $\overline{\Gamma\backslash X}^{BS}$ admits a finite triangulation, this implies that $\overline{\Gamma\backslash X}^{BS}$ is a finite $K(\Gamma, 1)$-space.

We briefly outline the construction of $\overline{\Gamma\backslash X}^{BS}$. For simplicity, we assume that $\mathbf{G}$ is semisimple and hence X is a symmetric space. It is constructed in the following steps:

1. For every $\mathbb{Q}$-parabolic subgroup $\mathbf{P}$, attach a boundary component $e(\mathbf{P})$, which is roughly a parameter space of all the geodesics in X going to infinity in the direction of $\mathbf{P}$.

2. Add all these boundary components $e(\mathbf{P})$ to X to obtain a partial compactification $\overline{X}^{BS}_{\mathbb{Q}} = X \cup \coprod_{\mathbf{P}} e(\mathbf{P})$.

3. Show that Γ acts properly and continuously on $\overline{X}^{BS}_{\mathbb{Q}}$ with a compact quotient, which gives $\overline{\Gamma\backslash X}^{BS}$.

By assumption, Γ is torsion-free, and hence Γ acts freely on $\overline{X}^{BS}_{\mathbb{Q}}$. Since $\overline{X}^{BS}_{\mathbb{Q}}$ is a manifold with corners, the quotient $\overline{\Gamma\backslash X}^{BS}$ is also a manifold with corners. Consider the example $\mathbf{G} = SL(2)$. Then $G = SL(2, \mathbb{R})$ and $K = SO(2)$, and the symmetric space $X = SL(2, \mathbb{R})/SO(2)$ can be identified with the upper half plane $\mathbf{H} = \{x + iy \mid x, y \in \mathbb{R}, y > 0\}$.

The inclusion of $\mathbf{H}$ in $\mathbb{C}P^1$ naturally gives a boundary $\mathbb{R} \cup \{\infty\}$. The set $\mathbb{Q} \cup \{\infty\}$ of rational boundary points corresponds bijectively to the set of $\mathbb{Q}$-parabolic subgroups of $SL(2)$.

For each $\mathbf{P}$, the boundary component $e(\mathbf{P})$ is a horocycle of the corresponding boundary point in $\mathbb{Q} \cup \{\infty\}$ minus the boundary point, which can be identified with $\mathbf{R}$, also isomorphic to the unipotent radical N_P of P. Hence $\overline{X}^{BS}_{\mathbb{Q}}$ is obtained by adding a line $\mathbb{R}$ at each rational boundary point.

The quotient of $e(\mathbf{P}) = \mathbb{R}$ by $\Gamma \cap P$ becomes a circle. So $\overline{\Gamma\backslash \mathbf{H}}^{BS}$ is obtained from $\Gamma\backslash\mathbf{H}$ by adding a circle to each cusp neighborhood. (Note that there is an 1-1 correspondence between the ends of $\Gamma\backslash\mathbf{H}$ and the Γ-equivalence classes of $\mathbb{Q}$-parabolic subgroups of $SL(2)$.)

This completes the discussion about Condition (1) on the existence of finite $B\Gamma$. For Condition (2), we use the following results in [Ji1].

Proposition 5.1 *Let X be a proper metric space. If a finitely generated group Γ acts properly and isometrically on X, then* $\operatorname{asdim} \Gamma \leq \operatorname{asdim} X$.

The point is that for any point $x_0 \in X$, the map $\Gamma \to \Gamma x_0$ is a coarse equivalence, where Γ is given a word metric.

The coarse equivalence is defined as follows. Let (M_1, d_1) and (M_2, d_2) be two metric spaces. Two maps $\varphi_1, \varphi_2 : M_1 \to M_2$ are called coarsely equivalent if there exists a constant C such that $d_2(\varphi_1(x), \varphi_2(x)) \leq C$ for all $x \in M_1$. Two metric spaces (M_1, d_1) and (M_2, d_2) are called *coarsely equivalent* if there exist coarsely uniform, (metric) proper maps $\varphi : M_1 \to M_2$ and $\psi : M_2 \to M_1$ such that $\varphi\psi$ and $\psi\varphi$ are coarsely equivalent to the identity maps on M_2 and M_1 respectively, where a map φ is coarsely uniform if there is a function $f(r)$ with $\lim_{r\to+\infty} f(r) = +\infty$ such that $d_2(\varphi(x), \varphi(y)) \leq f(d_1(x, y))$ for all $x, y \in M_1$. A basic observation is that two coarsely equivalent metric spaces have the same asymptotic dimension. See [Roe] for details.

To prove the above proposition, both the properness and the isometry of the Γ action are used to show that the map $\Gamma \to \Gamma x_0$ is coarsely uniform. If the stabilizer of x_0 in Γ is trivial, certainly there is a canonical inverse map from Γx_0 to Γ. Otherwise, the properness of the action shows the stabilizers of points in Γx_0 are finite. Since these points belong to one Γ-orbit, these stabilizers are conjugate and hence are uniformly bounded. This allows one to get a coarse inverse map from $\Gamma x_0 \to \Gamma$ which is also coarsely uniform. This proves the coarse equivalence between Γ and the orbit Γx_0.

In our case, $X = G/K$ is a homogeneous space with a left invariant metric, which is clearly a proper metric space. The finiteness of $\operatorname{asdim} X$ is proved in [CG2].

Proposition 5.2 *Let G a real Lie group with finitely many components, and K a maximal compact subgroup. Let $X = G/K$ be the associated homogeneous space endowed with a G-invariant Riemannian metric. Then* $\operatorname{asdim} X = \dim X < +\infty$.

The combination of these two results gives the following result, and hence the second condition in Theorem 3.1 is satisfied.

Corollary 5.3 *If* $\Gamma \subset \mathbf{G}(\mathbb{Q})$ *is an arithmetic subgroup of a linear algebraic group* $\mathbf{G}$*, then* $\operatorname{asdim} \Gamma \leq \dim G/K < +\infty$.

Geometrically finite Kleinian groups.

We briefly discuss another natural class of discrete subgroups of Lie groups, the class of geometrically finite Kleinian groups, which satisfy both conditions in Theorem 3.1 and hence the integral Novikov conjectures hold for them. For more details and references for some of the results discussed below, see [Ji2].

When G is semisimple, arithmetic subgroups Γ are lattices or lattice subgroups, i.e., $vol(\Gamma\backslash X) < +\infty$. But there are other important discrete subgroups which are not necessarily lattices, for example, geometrically finite groups acting on noncompact symmetric spaces of rank 1, in particular, the real hyperbolic spaces. Discrete subgroups acting on symmetric spaces of rank 1 are often called Kleinian groups. They have been studied extensively in complex analysis, topology, harmonic analysis. The class of geometrically finite Kleinian groups strictly contains the class of arithmetic subgroups of rank 1 semisimple Lie groups.

We recall the definition. Assume for the rest of this section that G is a connected semisimple Lie group of rank 1. Then $X = G/K$ is a symmetric space of noncompact type of rank 1, i.e., a symmetric space of strictly negative curvature. Let $X(\infty)$ be the set of equivalence classes of geodesics in X, which can be identified with the unit sphere in the tangent space $T_{x_0}X$ for any basepoint $x_0 \in X$, and hence called the sphere at infinity. Let $X \cup X(\infty)$ be the geodesic compactification. It is known that $X \cup X(\infty)$ is a real analytic manifold with boundary.

Let $\Gamma \subset G$ be a discrete subgroup. Then Γ acts isometrically on X. The action extends to $X \cup X(\infty)$ by real analytic diffeomorphisms.

Let $\Lambda(\Gamma) \subset X(\infty)$ be the set of limit points of Γ, i.e., the set of accumulation points of any orbit $\Gamma \cdot x$ in $X \cup X(\infty)$, for any $x \in X$. The complement $\Omega(\Gamma) = X(\infty) - \Lambda(\Gamma)$ is called the domain of discontinuity, and Γ acts properly on $\Omega(\Gamma)$.

Assume that Γ is torsion free. Then Γ acts properly and freely on $X \cup \Omega(\Gamma)$, and the quotient $\Gamma\backslash X \cup \Omega(\Gamma)$ is a real analytic manifold with boundary. If this quotient is compact, Γ is called convex cocompact. Otherwise, Γ contains parabolic elements.

Definition 5.4 A discrete subgroup Γ of G is called *geometrically finite* if $\Gamma\backslash X \cup \Omega(\Gamma)$ has finitely many ends, and each end is of certain standard form.

There are several equivalent definitions of geometrically finite groups in terms of the nature of limit points (conical limit points), convex core and thick-thin decomposition. When X is the real hyperbolic space $\mathbf{H}^n$, $n = \dim X$, there is also a definition in terms of shapes of fundamental domains.

As mentioned earlier, arithmetic subgroups (or more general lattice subgroups) are geometrically finite. In fact, if Γ is an arithmetic subgroup, then $\Lambda(\Gamma) = X(\infty)$. The reduction theory of lattice subgroups shows that $\Gamma\backslash X$ has finitely many ends and each of which is a topological cylinder.

But there are many other geometrically finite Kleinian groups which are not lattice subgroups. In fact, there are several general constructions of geometrically finite groups from reflections associated with polyhedrons and combinations of simpler geometrically finite groups. For example, Fuchsian groups in $SL(2, \mathbb{R})$ considered as Kleinian groups acting on $\mathbf{H}^3$ are geometrically finite.

Proposition 5.5 *If* Γ *is geometrically finite and torsion free, then* $\Gamma\backslash X \cup \Omega(\Gamma)$ *admits a compactification as a real analytic compactification with corners, which gives a finite* $B\Gamma$*-space.*

This result is essentially due to Apanasov and Xie [AX]. The compactification in the above proposition gives a compactification of $\Gamma\backslash X$, which is an exact analogue of the Borel-Serre compactification for locally symmetric spaces of finite volume.

Corollary 5.6 *The integral Novikov conjectures in K-theory and L-theory hold for torsion free geometrically finite Kleinian groups* Γ*.*

To prove this, we note that asdim $\Gamma \leq \dim X$ as above for arithmetic groups and apply Theorem 3.1. See [Ji2] for details and references.

6 Novikov conjectures for S-arithmetic subgroups

In this section, we outline the proof of Theorems 1.4, 1.5 and 1.6. We assume $\mathbf{G}$ is a linear reductive algebraic group defined over a global field k, and Γ an S-arithmetic subgroup, where S contains non-archimedean places. We also assume that the center of $\mathbf{G}$ has rank 0 over k, which is automatically satisfied if $\mathbf{G}$ is semisimple. The basic reason for this assumption is that Γ will be a lattice in the group $\prod_{\mathfrak{p}\in S}\mathbf{G}(k_\mathfrak{p})$ defined below. We apply Theorem 3.2 to prove Theorem 1.4, Theorem 3.1 to prove Theorem 1.5, and the generalization of Theorem 3.1 given in Theorem 3.6 to prove Theorem 1.6.

To prove Theorem 1.4, we need to check the following two conditions:

1. Γ admits a finite $B\Gamma$.

2. $E\Gamma$ admits a Γ-equivariant compactification with small Γ-action at infinity.

For each place $\mathfrak{p}$ of k, let $k_\mathfrak{p}$ be the completion of k with respect to the norm associated with $\mathfrak{p}$. When $\mathfrak{p}$ is archimedean, $k_\mathfrak{p}$ is isomorphic to either $\mathbb{R}$ or $\mathbb{C}$. When $\mathfrak{p}$ is non-archimedean, $k_\mathfrak{p}$ is a local field with a finite residue field.

For each archimedean place $\mathfrak{p}$, let $K_\mathfrak{p}$ be a maximal compact subgroup of $\mathbf{G}(k_\mathfrak{p})$ and

$$X_\mathfrak{p} = \mathbf{G}(k_\mathfrak{p})/K_\mathfrak{p}$$

the associated symmetric space. For each non-archimedean place $\mathfrak{p}$, let $X_\mathfrak{p}$ be the Bruhat-Tits building associated to the reductive group $\mathbf{G}(k_\mathfrak{p})$ over a local field. It is known that $X_\mathfrak{p}$ has a Tits metric whose restriction to each apartment is isometric to $\mathbb{R}^r$, where r is the $k_\mathfrak{p}$-rank of $\mathbf{G}$, and $\mathbf{G}(k_\mathfrak{p})$ acts isometrically and properly on $X_\mathfrak{p}$ (see [BS2] [Se1] [Se2] [Ji5] for definitions of the Bruhat-Tits buildings, the Tits metric, and other properties). In the fololwing, $X_\mathfrak{p}$ is considered as a metric space with respect to the Tits metric.

Define

$$X_\infty = \prod_{\text{archimedean } \mathfrak{p}\in S} X_\mathfrak{p}, \qquad X_{S,f} = \prod_{\text{non-archimedean } \mathfrak{p}\in S} X_\mathfrak{p}.$$

Then X_∞ is a symmetric space of nonpositive curvature, and $X_{S,f}$ is an euclidean building. Define

$$X_S = \prod_{\mathfrak{p}\in S} X_\mathfrak{p}.$$

When k is a number field,

$$X_S = X_\infty \times X_{S,f};$$

when k is a function field,

$$X_S = X_{S,f}.$$

Since the Bruhat-Tits buildings and symmetric spaces $X_{\mathfrak{p}}$ are all contractible, all three spaces X_S, $X_{S,f}$ and X_∞ are contractible.

It is known that Γ can be embedded diagonally into $\prod_{\mathfrak{p}\in S}\mathbf{G}(k_{\mathfrak{p}})$ as a discrete subgroup, for example, when $\mathbf{G}=G_m=GL(1)$, $\left(\mathbb{Z}\left[\frac{1}{p_1},\cdots,\frac{1}{p_m}\right]\right)^\times$ can be embedded diagonally into $\mathbb{R}^\times\times\mathbb{Q}_{p_1}^\times\times\cdots\times\mathbb{Q}_{p_m}^\times$ as a discrete subgroup. Arithmetic subgroups such as $SL(n,\mathbb{Z})$ are discrete subgroups of the Lie groups such as $SL(n,\mathbb{R})$ because $\mathbb{Z}$ is a discrete subgroup of $\mathbb{R}$. Similarly, the basic reason for the above discrete embedding of S-arithmetic subgroups over number fields is that $\mathbb{Z}[\frac{1}{p_1},\cdots,\frac{1}{p_m}]$ is a discrete subgroup of $\mathbb{R}\times\mathbb{Q}_{p_1}\times\cdots\times\mathbb{Q}_{p_m}$.

Since each $\mathbf{G}(k_{\mathfrak{p}})$ acts properly on $X_{\mathfrak{p}}$, Γ acts properly on X_S. Therefore, if Γ is torsion free, Γ acts freely on X_S, and hence $\Gamma\backslash X_S$ is a $K(\Gamma,1)$-space.

Under the conditions in Proposition 4.3, Γ admits torsion free subgroups of finite index. We can assume that Γ is torsion free if necessary.

It is known that $\Gamma\backslash\prod_{\mathfrak{p}\in S}\mathbf{G}(k_{\mathfrak{p}})$ is compact if and only if the k-rank of $\mathbf{G}$ is equal to zero, which is also equivalent to that the quotient $\Gamma\backslash X_S$ is compact. (Note that though $\mathbf{G}(k_{\mathfrak{p}})$ does not act transitively on $X_{\mathfrak{p}}$ when $k_{\mathfrak{p}}$ is a local field, the quotient of $X_{\mathfrak{p}}$ by $\mathbf{G}(k_{\mathfrak{p}})$ is compact.)

Proposition 6.1 *When the k-rank of* $\mathbf{G}$ *is equal to 0, the quotient* $\Gamma\backslash X_S$ *has the structure of a finite CW-complex.*

1. *If* Γ *is torsion free, it admits a finite* $B\Gamma$ *given by* $\Gamma\backslash X_S$.
2. *If* Γ *contains torsion elements, then* X_S *is a cofinite* Γ*-CW-complex* $E_{\mathcal{F}}\Gamma$.

Since S contains non-archimedean places, X_S and hence $\Gamma\backslash X_S$ are not manifolds. Therefore it is not obvious or automatically true that it is a finite CW-complex. The basic idea of the proof is that $X_{S,f}$ has a simplicial structure induced from the Bruhat-Tits buildings. For each simplex σ in $X_{S,f}$, its stabilizer Γ_σ in Γ is an arithmetic subgroup acting on X_∞. Using triangulations of X_∞ equivariant with respect to these arithmetic subgroups and suitable refinements, we can get an Γ-equivariant CW-complex structure on X_S which gives the structure of a finite CW-complex $\Gamma\backslash X_S$. The second statement is more complicated. We need to use the fact that X_S is a CAT(0)-space discussed below in Proposition 6.3 in order to study the set of fixed points of finite groups in X_S and hence show that X_S is a $E_{\mathcal{F}}\Gamma$-space. We also need to use triangulation of orbifolds to show that it is a cofinite Γ-CW-complex $E_{\mathcal{F}}\Gamma$. See [Ji3] for details.

We point out that when k is a function field, its characteristic is positive. If the k-rank of $\mathbf{G}$ is positive, then any S-arithmetic group Γ contains nontrivial unipotent elements, which are torsion. Hence, the rank 0 assumption for function fields is necessary for the existence of torsion free S-arithmetic subgroups and also for the existence of finite $B\Gamma$ in the above proposition. (As mentioned in Remark 2.7, the integral Novikov conjectures are expected to fail in general for groups containing nontrivial torsion elements.)

On the other hand, the rank 0 assumption can be removed for number fields.

Proposition 6.2 *Assume that* $\mathbf{G}$ *is a linear reductive group defined over a number field. If* Γ *a torsion free S-arithmetic subgroup, then* Γ *admits a finite* $B\Gamma$.

As in the case of arithmetic subgroups, we enlarge X_∞ to the Borel-Serre partial compactification $\overline{X_\infty}_{\mathbb{Q}}^{BS}$. For simplicity of notation, we denote X_∞ by X. Then it can be shown that $\Gamma\backslash\overline{X}_{\mathbb{Q}}^{BS}\times X_{S,f}$ is compact (see [Se2], [BS2]). Similar arguments as in the above proposition shows that $\Gamma\backslash\overline{X}_{\mathbb{Q}}^{BS}\times X_{S,f}$ has the structure of a finite CW-complex.

We need to check the second condition on the existence of a compactification of $E\Gamma$ with small Γ-action at infinity.

By assumption, the k-rank of $\mathbf{G}$ is equal to 0. Then $\Gamma\backslash X_S$ is a finite $B\Gamma$-space, and the corresponding $E\Gamma$-space is given by X_S. To describe the desired compactification, we recall that a *geodesic metric space is called a CAT(0)-space* if every triangle in it is thinner than the corresponding triangle of same side lengths in $\mathbb{R}^2$.

CAT(0)-spaces occur naturally. For example, nonpositively curved, simply connected Riemannian manifolds are CAT(0)-spaces. In fact, CAT(0)-spaces were motivated by the study of such nonpositively curved manifolds. It is also known that Euclidean buildings and hence Bruhat-Tits buildings are CAT(0)-spaces (see [Br1]).

A simple but crucial observation is the following:

Proposition 6.3 *If X_1, X_2 are CAT(0)-spaces, then $X_1 \times X_2$ is also a CAT(0)-space. In particular, $X_S = \prod_{\mathfrak{p}\in S} X_{\mathfrak{p}}$ is a CAT(0)-space.*

It is known that a CAT(0)-space is contractible. Another important fact is the existence of the geodesic compactification.

Proposition 6.4 *Let X be a CAT(0)-space and $X(\infty)$ the set of equivalence classes of geodesics (or rays) in X. If X is proper, then it can be compactified by adding $X(\infty)$. The compactification $X \cup X(\infty)$ satisfies the following property:*

1. *$X \cup X(\infty)$ is contractible.*
2. *Any isometric action of a group Γ on X extends to an action on $X \cup X(\infty)$.*
3. *If Γ acts properly on X with a compact quotient, then the action of Γ on $X \cup X(\infty)$ is small at infinity.*

When X is a simply connected nonpositively curved manifold, these conclusions are well-known. In fact, parts (2) and (3) follow from the definition of the topology in terms of convergences of geodesics; and part (1) follows from the fact that $X \cup X(\infty)$ is homeomorphic to the closed unit ball in the tangent space $T_{x_0}X$ for any basepoint $x_0 \in X$. For a general CAT(0)-space X, parts (2) and (3) can be proved in the same way, and part (1) can be proved by contracting along rays from a fixed basepoint.

Corollary 6.5 *Assume that $\mathbf{G}$ is a linear reductive algebraic group defined over a global field k with k-rank equal to 0. If Γ a torsion free S-arithmetic subgroup of $\mathbf{G}(k)$, then its $E\Gamma = X_S$ admits a Γ-equivariant, metrizable compactification $X_S \cup X_S(\infty)$ such that the Γ-action on the compactification is small at infinity. Similarly, if Γ contains torsion elements, then $E_{\mathcal{F}}\Gamma$ admits a Γ-equivariant small compactification.*

Together with Proposition 6.1, this shows that the conditions in Theorem 3.2 and Theorem 3.5 are satisfied, and hence Theorem 1.4 is proved.

In Theorem 1.5, $\mathbf{G}$ is a reductive algebraic group defined over a number field k. To apply Theorem 3.1, we need to show that the asymptotic dimension of Γ, denoted by asdim Γ, is finite and Γ admits a finite $B\Gamma$.

By Proposition 6.2, the condition of $B\Gamma$ being realized by a finite CW-complex is satisfied. In fact, we can take $\Gamma\backslash\overline{X}_{\mathbb{Q}}^{BS} \times X_{S,f}$ as a finite $B\Gamma$-space. Then $E\Gamma = \overline{X}_{\mathbb{Q}}^{BS} \times X_{S,f}$.

To show asdim $\Gamma < +\infty$, we note that Γ acts properly and isometric on X_S. Hence by the subadditivity of the asymptotic dimension, it suffices to prove that for each $\mathfrak{p} \in S_f$, asdim $X_{\mathfrak{p}} < +\infty$. Since $X_{\mathfrak{p}}$ can be isometrically embedded into the Bruhat-Tits building of the group $SL(n, k_{\mathfrak{p}})$, it suffices to prove that the asymptotic dimension of the latter is finite, which is proved in [Mat, Theorem 3.21].

To prove Theorem 1.6 using the criterion in Theorem 3.6, the condition asdim $\Gamma < +\infty$ follows from the above discussions. The difficulty is to get an explicit model of $E_{\mathcal{F}}\Gamma$ and to understand the geometry of the fixed point sets in $E_{\mathcal{F}}\Gamma$ by finite subgroups of Γ and finite quotients of such fixed point sets.

The following result on $E_{\mathcal{F}}\Gamma$ is known and stated in literature without proof. We gave a proof in [Ji6]. See [Ji6] for details and other results as well.

Proposition 6.6 *Let* $\mathbf{G}$ *be a reductive algebraic group defined over a number field* k*, and* Γ *an arithmetic subgroup of* $\mathbf{G}(k)$ *as above. Then the Borel-Serre partial compactification* $\overline{X_\infty}^{BS}$ *is a* Γ*-cofinite* $E_{\mathcal{F}}\Gamma$*-space.*

Remark 6.7 As pointed out earlier, if $\mathbf{G}$ is a reductive linear algebraic group of positive rank over a function field, then every S-arithmetic subgroup Γ contains nontrivial torsion elements and hence does not admit a finite $B\Gamma$-space. In fact, by [Be1] [Be2], such S-arithmetic subgroups are not of type FP_∞. On the other hand, the existence of a cofinite Γ-CW-complex $E_{\mathcal{F}}\Gamma$ implies that Γ is of type FP_∞ [Br3, Proposition 1.1]. Therefore, Γ does not likely admit a Γ-cofinite CW-complex $E_{\mathcal{F}}\Gamma$.

Remark 6.8 When a preliminary version of this paper was written, the paper [Mat] was not available. In that version, only a special case of Theorem 1.5 was proved and the proof was modeled after [Go2] by using a criterion more relaxed than Theorem 3.2. Since this original approach is probably applicable to prove the integral Novikov conjecture in both K- and L-theories for the mapping class groups (or rather its torsion-free subgroups), we keep an outline of this approach in the current version.

As pointed out earlier, the existence of a finite $B\Gamma$ follows from Proposition 6.2. To apply Theorem 3.2, the problem is to find a good compactification of $E\Gamma = \overline{X_\infty}^{BS} \times X_{S,f}$. Note that $\overline{X_\infty}$ is a partial compactification of X_∞ determined by the rational structure of $\mathbf{G}$, such a compactification of $\overline{X_\infty}^{BS}$ and hence of X_∞ should make use of the real structure of $\mathbf{G}$, and the construction will depend on the interplay between the rational and real structures and is hence complicated.

For simplicity, the symmetric space at infinity X_∞ is denoted by X. When S does not contain any non-archimedean place, Γ is an arithmetic subgroup and $X_S = X$. If the rank of X is equal to 1, this problem was solved in [Go2]. It turns out that in general it is difficult to construct a compactification of $\overline{X}_{\mathbb{Q}}^{BS}$ with a small Γ-action at infinity. Instead, a Γ-equivariant compactification $\overline{X}^*$ of $\overline{X}_{\mathbb{Q}}^{BS}$ was constructed in [Go2] and a generalization of Theorem 3.2 in [CP1] as mentioned in Remark 3.4 can be applied to prove the integral Novikov conjectures.

We briefly describe the compactification $\overline{X}^*$ when X is the Poincaré disc, or the upper half plane. In this case, we add a line to every point in $\mathbb{R} \cup i\infty$, and obtain $\overline{X}_{\mathbb{R}}^{BS}$.

The resulting space $\overline{X}_{\mathbb{R}}^{BS}$ is still noncompact. In fact, it is naturally mapped onto the geodesic compactification $X \cup X(\infty)$, and the fibers on the boundary are $\mathbb{R}$ and hence noncompact. We can compactify $\mathbb{R} \cong (-1, 1)$ to $[-1, 1]$. The resulting space is $\overline{X}^*$. Note that the boundary of this compactification has dimension 2, which is also the dimension of X.

When S contains non-archimedean places, we combine the compactification $\overline{X}^*$ with the Borel-Serre compactification of Bruhat-Tits buildings $X_{\mathfrak{p}}$ by spherical Tits buildings in [BS2], or equivalently, the compactification of the Bruhat-Tits buildings as CAT(0)-spaces, to obtain a compactification of $E\Gamma = \overline{X}_{\mathbb{Q}}^{BS} \times X_{S,f}$. Then apply the methods in [Go2] using the result in [CP1] to prove the integral Novikov conjecture for torsion-free S-arithmetic subgroups when the rank of $X = X_\infty$ is equal to 1.

Remark 6.9 When the symmetric space at infinity $X = X_\infty$ has rank strictly greater than 1, the construction is even more complicated.

From the brief description in the previous remark, there are two steps in the construction of $\overline{X}^*$:

1. Fill in all the missing irrational directions to enlarge the Borel-Serre partial compactification $\overline{X}_{\mathbb{Q}}^{BS}$ by adding boundary components of $\mathbb{R}$-parabolic subgroups.

2. Partially compactify the boundary components, both rational and irrational ones.

The first step can be carried out using the real Borel-Serre partial compactification $\overline{X}_{\mathbb{R}}^{BS}$, which has a boundary component for every $\mathbb{R}$-parabolic subgroup [BS1]. But there is a problem if the the $\mathbb{R}$-rank of $\mathbf{G}$ is not equal to $\mathbb{Q}$-rank of $\mathbf{G}$. In fact, in $\overline{X}_{\mathbb{R}}^{BS}$, the real Langlands decomposition of parabolic subgroups is used for $\mathbb{Q}$-parabolic subgroups; while in $\overline{X}_{\mathbb{Q}}^{BS}$, the rational Langlands decomposition of parabolic subgroups is used for $\mathbb{Q}$-parabolic subgroups. These two decompositions are not the same, and there is no inclusion $\overline{X}_{\mathbb{Q}}^{BS} \hookrightarrow \overline{X}_{\mathbb{R}}^{BS}$ if the $\mathbb{R}$-rank is strictly greater than the $\mathbb{Q}$-rank of $\mathbf{G}$. This is the reason that it is assumed in [Go1] that these ranks are equal. This problem with non-equal ranks can be solved by blowing up the boundary components of real parabolic subgroups in $\overline{X}_{\mathbb{R}}^{BS}$. Briefly, let $\mathbf{P}$ be a $\mathbb{Q}$-parabolic subgroup of $\mathbf{G}$, and $A_{\mathbf{P}}$ the $\mathbb{Q}$-split component of $P = \mathbf{P}(\mathbb{R})$ and $A_{\mathbb{R}}$ the $\mathbb{R}$-split component of P. Then $A_{\mathbb{Q}} \subseteq A_{\mathbb{R}}$. Assume that $A_{\mathbb{Q}} \neq A_{\mathbb{R}}$. Then we can use an orthogonal complement of $\mathfrak{a}_{\mathbb{Q}}$ in $\mathfrak{a}_{\mathbb{R}}$ (or rather a compactification of this complement) to enlarge (or blow up) the boundary of P in $\overline{X}_{\mathbb{R}}^{BS}$ so that the resulting space contains $\overline{X}_{\mathbb{Q}}^{BS}$.

The next problem is to partially compactify the boundary components. In the example of the Poincaré disc, the boundary component $e(P)$ is equal to the unipotent radical N_P of P. For general G and X, the boundary component is the product of the unipotent radical N_P and a lower dimensional symmetric space X_P associated with P (or a suitable blow-up as explained above). It turns out that it suffices to compactify the factor N_P. Since the compactification of $E\Gamma = \overline{X}_{\mathbb{Q}}^{BS}$ needs to be Γ-equivariant, the compactification of N_P needs to be $\Gamma \cap P$-equivariant rather than *only* $\Gamma \cap N_P$-equivariant as required in [Go1]. Since the Γ-equivariance of the compactification is crucial for the application to the Novikov conjecture but not clear from the construction, it seems that there might be some problems with the compactification $\overline{X}^*$ in [Go1]. (When the rank of X is equal to 1 and Γ is torsion-free, by passing to a subgroup Γ' of Γ of finite index if necessary, it should be true that $\Gamma' \cap P = \Gamma' \cap N_P$, and the above problem can be avoided in this special case in [Go2].)

Specifically, the idea in [Go1] and [Go2, §4] is to choose a Malcev set of generators of $\Gamma \cap N_P$ and hence a Malcev set of generators of the Lie algebra $\mathfrak{n}_P$, which induces a filtration of $\mathfrak{n}_P$ (or N_P). This filtration is used to obtain a compactification $\overline{N_P}$. By the choice of the Maclev generators, this filtration is invariant under the action of N_P and hence $\Gamma \cap N_P$, which implies that the compactification $\overline{N_P}$ is a $\Gamma \cap N_P$-equivariant compactification. But the problem is that there is no canonical choice of such Maclev generators of $\mathfrak{n}_P$ which are invariant under the automorphism group of N_P, for example, under the conjugation of M_P and $\Gamma \cap P$. A possible way to solve this problem is to use the *canonical* filtration of N_P induced from the decreasing commuting series, which is invariant under the automorphism group of N_P. Now each successive quotient is an abelian group, diffeomorphic to an Euclidean space, and can be compactified by the sphere at infinity. These compactifications can then be put together as in [Go2, §4] to obtain a compactification of N_P which is equivariant with respect to the automorphism group of N_P.

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