Large Scale Geometry of arithmetic groups and the Integral Novikov conjecture

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The goal is to discuss the integral Novikov conjecture in K-theory and L-theory for *torsion free* groups Γ:

(1) arithmetic subgroups of linear algebraic groups $\mathbf{G}$ over $\mathbb{Q}$

(2) S-arithmetic subgroups of linear reductive or semisimple algebraic groups over number fields and function fields

(3) Finitely generated linear groups over $\mathbb{Q}$, i.e., finitely generated subgroups of $GL(n, \mathbb{Q})$.

Modified Novikov conjectures for arithmetic, finitely generated linear subgroups which may **contain torsion elements**. (The statement is different in this case.)

Plan of the talk.

1. Review of **Borel conjecture** on rigidity of aspherical manifolds, and Novikov conjectures on homotopy invariance of higher signatures and generalization.

2. Two methods to prove them

(a) via finite asymptotic dimension of Γ and finite classifying space $B\Gamma$,

(b) via finite $B\Gamma$ and compactifications of the universal cover $E\Gamma = \widetilde{B\Gamma}$.

3. Case of arithmetic subgroups by Method (a)

4. Case of S-arithmetic subgroups, and more generally of finitely generated linear groups by Methods (a) and (b).

6. **Modifications** when Γ contains torsion elements. Then there is **No** finite $B\Gamma$. Using generalizations of Methods (a) and (b).

Borel conjecture.

If two closed aspherical manifolds M and N are homotopic, then they are homeomorphic.

Asphercial: $\pi_i(M) = \{1\}$ for $i \geq 2$.

(The opposite of the sphere: $\pi_i(S^n) = \{1\}$, $i \leq n-1$, $\pi_n(S^n) = \mathbb{Z}$. For higher $i \geq n+1$, $\pi_i(S^n)$ is not so understood.)

homotopy implies **homeomorphism**. More specifically, the pi_1 determines the homeomorphism type.

Motivations.

Classical **Bieberbach** theorem: If two crystallographic groups are isomorphic as abstracts groups, they are conjugate by an affine motion.

A crystallographic group is a discrete isometry group Γ of $\mathbb{R}^n$ with compact quotient $\Gamma\backslash\mathbb{R}^n$.

Mostow's rigidity results for solvmanifolds, 1954. Two compact solvmanifolds having the same fundamental group are homeomorphic (diffeomorphic).

The universal covering space $\widetilde{M}$ is a solvable group. More general than the above abelian case.

The Borel conjecture is true for co-finite locally symmetric spaces due to the **Mostow strong rigidity**.

Mostow strong rigidity implies **isometric** with the exception of $\Gamma\backslash\mathbb{H}^2$, in which case, **homotopy** implies **diffeomorphism**

Specifically, if two irreducible locally symmetric spaces $\Gamma_1\backslash X_1$ and $\Gamma_2\backslash X_2$ have the same fundamental group $\Gamma_1 \cong \Gamma_2$, then they are isometric after suitable scaling, with the above exception.

The Borel conjecture is closely related to a homological computation of an algebraic (Hermitian) K-groups.

A lot is known about the Borel conjecture due to work of Farrell-Jones.

Recall the **Hirzebruch index theorem.**

M^{4k} a compact orientable manifold, the cup product in the middle dimension:

$$H^{2k}(M,\mathbb{Q}) \times H^{2k}(M,\mathbb{Q}) \to H^{4k}(M,\mathbb{Q}) = \mathbb{Q}$$

defines a non-degenerate quadratic form.

Poincare duality implies non-degenerate. Its signature is called the **signature** of M and denoted by $Sgn(M)$. (# of positive diagonal terms - # of negative diagonal terms). The quadratic form can be diagonalized.

Hirzebruch class: $\mathcal{L}(M)$ a power series in Pontrjagin classes with rational coefficients.

$$\mathcal{L}(M) = 1 + L_1 + L_2 + \cdots$$

$$L_1 = \frac{1}{3}P_1, \quad L_2 = \frac{1}{45}(7P_2 - P_1^2)...$$

Hirzebruch index theorem:

$Sgn(M) = \langle \mathcal{L}(M), [M] \rangle.$

The Left Side is an oriented homotopy invariant, but the Right hand side depends on the characteristic classes of the tangent bundle of M and may depend on the differential structure. (In fact, Novikov showed that the Pontrjagin classes are homeomorphism invariants).

To get more homotopy invariants, Novikov introduced higher signatures.

Definition of higher signatures

Let $\Gamma = \pi_1(M)$. Let $B\Gamma$ be a $K(\Gamma, 1)$-space.

$\pi_1(B\Gamma) = \Gamma, \quad \pi_i(B\Gamma) = \{1\}, i \geq 2$

Equivalently, $E\Gamma = \widetilde{B\Gamma}$ is contractible and Γ acts freely.

$B\Gamma = \Gamma \backslash E\Gamma$. If $\Gamma = Z$, then $B\Gamma = S^1$, $E\Gamma = \mathbb{R}$.

The universal covering $\widetilde{M} \to M$ determines a classifying map

$f : M \to B\Gamma$, unique up to homotopy.

For any $\alpha \in H^*(B\Gamma, \mathbb{Q})$, $f^*\alpha \in H^*(M, \mathbb{Q})$, and define higher signature

$$Sgn_\alpha(M) = \langle f^*\alpha \cup \mathcal{L}(M), [M] \rangle$$

When $\alpha = 1$, it gives the original signature, by Hirzebruch signature theorem.

The original **Novikov conjecture**: $Sgn_\alpha(M)$ is an oriented homotopy invariant of M.

$g : N \to M$, an orientation preserving homotopy equivalence,

$$\langle (g \circ f)^* \alpha \cup \mathcal{L}(N), [N] \rangle = \langle f^* \alpha \cup \mathcal{L}(M), [M] \rangle.$$

Wall reformulated this Novikov conjecture in terms of **rational injectivity** of the assembly map in surgery theory, i.e., the injectivity of $A \otimes \mathbb{Q}$,

Assembly map: $A : H_*(B\Gamma, \mathbb{L}(\mathbb{Z})) \to L_*(\mathbb{Z}[\Gamma])$,

it is a map from a generalized homology group to a difficult and important group.

A is *from local to global*, hence called the *assembly map*

$L_{2k}(\mathbb{Z}[\Gamma])$ is the Witt group of $(-1)^k$-Hermitian quadratic forms on finitely generated free module over the ring,

$L_{2k+1}(\mathbb{Z}[\Gamma])$ stable automorphisms of such hyperbolic forms.

The L-groups are similar to K-groups, often called Hermitian K-groups. (They are called surgery groups).

Important. The Novikov conjecture only depends on Γ, not necessarily π_1 of a compact manifold.

Ways to view the assembly map.

1. Originally defined by Quinn, *assemble* a cycle in generalized homology group into an element in a cobordism type group.

2. Use equivariant homology groups and push forward $E\Gamma \to pt$.

$H_*(B\Gamma, \mathbb{L}(\mathbb{Z})) = H_*^{\Gamma}(E\Gamma, \mathbb{L}(\mathbb{Z})) \to$

$H^{\Gamma}(pt., \mathbb{L}(\mathbb{Z})) = L_*(\mathbb{Z}[\Gamma])$

The **rational injectivity** means $A \otimes \mathbb{Q}$ is injective.

The injectivity of A is called the **integral Novikov conjecture** in surgery theory, or in L-theory.

For the integral Novikov, need to assume Γ **torsion free** in general.

If Γ contains torsion elements, need a different equivariant formulation. **(formulated later).** This modification is important, since some of the most basic groups such as $SL(n, \mathbb{Z})$ contain torsion elements.

A stronger conjecture is the map A is an isomorphism

$A : H_*(B\Gamma, \mathbb{L}(\mathbb{Z})) \cong L_*(\mathbb{Z}[\Gamma])$.

This is related to the **Borel Conjecture.**

In fact, equivalent to the cobordism Borel conjecture,

in turn equivalent to the Borel conjecture if the Whitehead group of Γ vanishes

Recall the Borel Conjecture: If two closed aspherical manifolds M, N are homotopic, they are homeomorphic.

Another reason. compute a difficult, global group by a generalized homology group.

Once formulated in terms of the assembly map, there are also other versions of the Novikov conjectures.

The **algebraic K-theory** version.

$A : H_n(B\Gamma, \mathbb{K}(\mathbb{Z})) \to K_n(\mathbb{Z}[\Gamma])$,

where $K_n(\mathbb{Z}[\Gamma])$ is algebraic K-group of group ring, and $\mathbb{K}(\mathbb{Z})$ is the algebraic K-theory spectrum of $\mathbb{Z}$,

Conjecture: When Γ is torsion free, the map A is injective (**Novikov**)

the map A is an isomorphism (called **Borel** conjecture).

Significance: Besides relation to Borel conjecture, compute a difficult, important group

$K_*(\mathbb{Z}[\Gamma])$ via a generalized homology theory, which is easier and computable.

There is also an assembly map in C^*-algebras.

$A : K_*(B\Gamma) \to K_*(C_r^*(\Gamma))$

If Γ is torsion free, A is conjectured to injective (**Novikov**)

A is an isomorphism (**Baum-Connes conjecture**)

There is also a version for algebraic K-theory, A-theory, of topological spaces X,

$H_*(X, \mathbb{A}(pt.)) \to A_*(X)$

Statements for Γ contain torsion elements formulated later.

The K-theory assembly map be defined briefly as follows.

$\Gamma \times GL_n(R) \to GL_1(R[\Gamma]) \times GL_n(R) \to GL_n(R[\Gamma])$

Apply the classifying space functor B, pass to the limit as $n \to +\infty$, and apply Quillen's +-construction

$B\Gamma_+ \wedge BGL(R)^+ \to BGL_1(R[\Gamma])_+ \wedge BGL(R)^+ \to BGL(R[\Gamma])^+$

Delooping $BGL(\cdot)^+$, obtain

$A : B\Gamma_+ \wedge \mathbb{K}(R) \to \mathbb{K}(R\Gamma)$

Taking the homotopy group π_n, obtain the above assembly map

$A : H_n(B\Gamma, \mathbb{K}(R)) \to K_n(R[\Gamma])$

Methods to prove the Novikov conjectures

Recall the Novikov conjecture only depends on the group Γ.

Method A:

Theorem If a finitely generated group Γ has finite asymptotic dimension, as-dim $\Gamma < \infty$, and has a finite CW-complex as its classifying space $B\Gamma$, then the integral Novikov conjecture in K-theory, L-theory holds for Γ, i.e., the assembly map is injective.

Proved by Carlsson-Goldfarb for K-theory, Bartels for both K-theory and L-theory, also Chang-Ferry-Yu, Dranishnikov-Ferry-Weinberger for L-theory

suggested by a result of G.Yu for the C^*-algebra Novikov conjecture.

Asymototic dimension:

Given a finite generating set S of Γ, there is a word metric d_S on Γ, a left G-invariant metric.

as-dim of Γ is defined to be the as-dim of the metric space (Γ, d_S).

(Independent of the choice of generators S)

For a noncompact metric space M, as-dim of M is the minumum n such that for any $r \gg o$, there exists a covering U_α of M by uniformly bounded sets such that any ball of radius r meets at most $n + 1$ U_i.

Large scale invariant.

Example. as-dim of $\mathbb{R}^n$ = n.

as-dim of a tree is 1 (Gromov).

Method B:

Theorem. Suppose that $B\Gamma$ is finite, and the universal cover $E\Gamma$ has a contractible, metrizable Γ-compactification $\overline{E\Gamma}$ such that the action of Γ on $\overline{E\Gamma}$ is small at infinity. Then the integral Novikov conjecture holds for Γ.

Due to Carlsson and Pederson.

Small means: If $K \subset E\Gamma$ compact subset, $g_j K$ has an accumulation near $z \in \partial\overline{E\Gamma}$, then the whole set $g_j K$ is contained in small neighborhoods of z.

Example. $\Gamma = \pi_1$ of a closed nonpositively curved manifold M. Then M is a finite $B\Gamma$, and $E\Gamma = \widetilde{M}$ is simply connected and nonpositively curved, and admits the geodesic compactification by adding the sphere at infinity $E\Gamma(\infty)$, the set of equivalence classes of geodesics.

This compactification is hmeomorphic to a closed unit ball and is small at infinity.

To give a combinatorial description of π_1 of a compact strictly negatively, Gromov introduced **hyperbolic groups**. With probability 1, a random group is hyperbolic.

Example. Γ is a finitely generated Gromov hyperbolic group. Take as $E\Gamma$ the Rips complex $P_d(\Gamma)$ for $d \gg 0$, (a k-simplex consists of $(k+1)$-tuples $(\gamma_0, \cdots, \gamma_k)$ of Γ with $d(\gamma_i, \gamma_j) \leq d$). It admits a natural compactification by its Gromov boundary, equivalence classes of sequences convergent at infinity.

According to the above method A,

Problem: Find groups with as-dim $\Gamma < +\infty$.

(1) Gromov, hyperbolic groups

(2) Dranishknikov and others, Coxeter groups, standard constructions preserve as-dim $\Gamma < +\infty$.

(3) Carlsson-Goldfarb, uniform discrete subgroups of Lie groups.

More important problem: find groups Γ with both as-dim$\Gamma < +\infty$ and finite $B\Gamma$.

Passing to any subgroup preserves as-dim$\Gamma < +\infty$. It is not so with existence of finite $B\Gamma$.

By Gromov et al, hyperbolic groups are generic ones.

Special, important ones: (rational numbers vs. generic numbers)

(1) **arithmetic subgroups**,

(2) **S-arithmetic groups**, (a nontrivial generalization); more generally, a finitely generated group of $GL(n, \mathbb{Q})$,

(3) the mapping class groups of Riemann surfaces.

An important example of arithmetic groups is $GL(n, \mathbb{Z})$ of $GL(n, \mathbb{C})$. More generally, $\mathbf{G} \subset GL(n)$ an algebraic group defined over $\mathbb{Q}$. A subgroup $\Gamma \subset \mathbf{G}(\mathbb{Q})$ is called *arithmetic* if it is commensurable with $\mathbf{G}(\mathbb{Q}) \cap GL(n, \mathbb{Z})$.

According to Margulis super-rigidity, many lattices of Lie groups are arithmetic. i.e., if G is semisimple of rank at least 2, Γ an irreducible lattice, then Γ is arithmetic.

Importance of arithmetic groups

1. Number theory, automorphic forms

2. Algebraic geometry, moduli spaces

3. topology, good cohomological finiteness properties, $\pi_0(Diff(M))$ is an arithmetic group, $\dim M \geq 6$ and $\pi_1(M) = \{e\}$.

In general, not easy to show that Γ has a finite $B\Gamma$.

In higher rank case or if the lattice is not *uniform (or cocompact)*, arithmetic groups are **not** Gromov hyperbolic. But they are certainly important groups.

The group $GL(n, \mathbb{Z})$ contains torsion elements. But it contains many torsion-free subgroups of finite index. So, there are many torsion-free arithmetic groups.

Theorem Let $\mathbf{G}$ be any linear algebraic group defined over $\mathbb{Q}$. $\Gamma \subset \mathbf{G}(\mathbf{Q})$ any torsion free arithmetic subgroup. Then as-dim $\Gamma < +\infty$ and Γ has a finite $B\Gamma$. Hence the integral Novikov conjecture in K-theory holds for Γ.

Remark. $\mathbf{G}$ is not necessarily semisimple

History of the **K-theory integral Novikov conjecture** for *discrete subgroups of Lie groups*:

(1) Γ is a cocompact discrete subgroup of a connected Lie group, Carlsson.

(2) special examples of arithmetic subgroups of **semisimple** linear algebraic groups, Goldfarb, using compactifications of $E\Gamma$ in Method (B).

Goldfarb's work was my starting point to learn this subject, to justify why studies compactifications.

The proof is based on several observations.

Proposition. If X is a proper metric space and Γ acts properly and isometrically on X. Then as-dim$\Gamma \leq$ as-dim$X$.

The point is that for any point $x_0 \in X$, the map $\Gamma \to \Gamma x_0$ is a coarse equivalence, where Γ is given a word metric.

A result of Carlsson-Goldfarb, later due to Bell-Dranashikov.

Proposition. G a connected Lie group, K a maximal compact subgroup, let $X = G/K$ be the associated homogeneous space endowed with a G-invariant Riemannian metric. Then as-dim$X = \dim X$.

Corollary. If $\Gamma \subset \mathbf{G}(\mathbb{Q})$ is an arithmetic subgroup of a linear algebraic group $\mathbb{G}$, then

as-dim$\Gamma \leq \dim X < +\infty$,

where $G = \mathbf{G}(\mathbb{R})$, $K \subset G$ a maximal compact subgroup.

Fact. as-dim$\Gamma \geq \dim X - \mathbb{Q} - rk(\mathbf{G})$. The lower bound is sharp. It is natural to conjecture that

as-dim$\Gamma = \dim X - \mathbb{Q} - rk(\mathbf{G})$

Assume Γ is torsion-free. Then Γ acts freely on X, and $\Gamma\backslash X$ is a $B\Gamma$-space.

Finite $B\Gamma$:

If $\mathbb{Q} - rk(\mathbf{G}) = 0$, then $\Gamma\backslash X$ is compact **manifold**, admitting a finite triangulation, hence a finite $B\Gamma$-space.

Otherwise, the finite $B\Gamma$ follows from the Borel-Serre compactification $\overline{\Gamma\backslash X}^{BS}$ of $\Gamma\backslash X$, a real analytic manifold with corners if Γ is torsion-free.

Since $\Gamma\backslash X$ is a $B\Gamma$-space, and the inclusion $\Gamma\backslash X \to \overline{\Gamma\backslash X}$ is a homotopy equivalence. Hence $\overline{\Gamma\backslash X}^{BS}$ is a finite $B\Gamma$-space.

Idea of construction of $\overline{\Gamma\backslash X}^{BS}$:

If $\Gamma\backslash X$ is a surface, then Borel-Serre compactification is obtained by adding a circle to each end (not just a point in complex analysis).

For every $\mathbb{Q}$-parabolic subgroup $\mathbf{P}$, attach a boundary component $e(\mathbf{P})$, a parameter space of all the geodesics in X going to infinity in the direction of $\mathbf{P}$.

Adding all these boundary components $e(\mathbf{P})$ to X, we get a partial compactification

$$\overline{X}_{\mathbb{Q}}^{BS} = X \cup \cup_{\mathbf{P}} e(\mathbf{P})$$

Γ acts properly and continuously on $\overline{X}_{\mathbb{Q}}^{BS}$ with a compact quotient $\overline{\Gamma\backslash X}^{BS}$.

In the example $G = SL(2, \mathbb{R})$, $K = SO(2)$,

$$X = \{x + iy \mid x, y \in \mathbb{R}, y > 0\}.$$

There exists a 1-1 correspondence between boundary points $\mathbb{Q} \cup i\infty \subset \mathbb{R} \cup i\infty = S^1$

and $\mathbb{Q}$-parabolic subgroups.

For each $\mathbf{P}$, the boundary component is a horocycle of the corresponding boundary point, which is equal to $\mathbf{R}$, also isomorphic to the unipotent radical of P.

$\overline{X}_{\mathbb{Q}}^{BS}$ is obtained by adding a line $\mathbb{R}$ at each rational boundary point.

The line $\mathbb{R}$ becomes a circle under dividing by $\Gamma \cap P$. So $\overline{\Gamma\backslash X}^{BS}$ is obtained from $\Gamma\backslash X$ by adding a circle to each cusp neighborhood.

S-arithmetic subgroups.

Recall that an arithmetic subgroup $\Gamma \subset \mathbf{G}(\mathbb{Q})$ is a subgroup commensurable with $\mathbf{G}(\mathbb{Z}) = \mathbf{G} \cap GL(n, \mathbb{Z})$, where $\mathbf{G} \subset GL(n, \mathbb{C})$ over $\mathbb{Q}$.

Idea. Replace $\mathbb{Z}$ by a more general ring by inverting finite many primes.

Let $S_f = \{p_1, \cdots, p_k\}$ a finite set of primes. $S = S_f \cup \{\infty\}$.

$\mathbb{Z}[\frac{1}{p_1}, \cdots, \frac{1}{p_k}]$ **ring of S-integers**, consisting of rationals $\frac{m}{n}$ where n is only divisible by the primes in S. This is a natural generalization of $\mathbb{Z}$.

Definition. A subgroup $\Gamma \subset \mathbf{G}(\mathbb{Q})$ is called S-arithmetic if it is commensurable with $\mathbf{G}(\mathbb{Z}[\frac{1}{p_1}, \cdots, \frac{1}{p_k}])$.

Example $SL(2, \mathbb{Z}[\frac{1}{p_1}, \cdots, \frac{1}{p_k}])$.

More generally, we can consider a number field F and a finite set S_f of non-zero prime ideals of the ring $\mathcal{O}_F$ of integers. Together with all the infinite (archimedean) places, it forms $S = S_f \cup \infty$.

This defines a ring $\mathcal{O}_{F,S}$ of S-integers in F, and S-arithmetic subgroups in $\mathbf{G}(F)$. (This class is strictly bigger).

The importance of S-arithmetic subgroups. A general finitely generated subgroup of $GL(n, \mathbb{Q})$ is not contained in an arithmetic subgroup, but rather in a S-arithmetic subgroup, where S_f consists of all the primes that appear in the denominators of the generators.

We can also define S-arithmetic subgroups of algebraic groups over a function field $\mathbb{F}_q(C)$, where C is a projective curve over a finite field $\mathbb{F}_q$. For example, $GL(n, \mathbb{F}_q[t])$ is an S-arithmetic subgroup of $GL(n, \mathbb{F}_q(t))$.

One important reason: over function fields, there is no arithmetic subgroup, but only S-arithmetic subgroups.

S-arithmetic groups have been studied intensively by Borel, Serre, Margulis and others. One important reason: to treat all primes (finite or not) equally at the same time.

If S consists of only infinite places, then S-arithmetic groups are arithmetic groups.

Theorem If Γ is an S-arithmetic subgroup, then as-dim $\Gamma < \infty$.

Theorem If $\mathbf{G}$ is a reductive (or semisimple) algebraic group over a number field k and Γ a torsion-free S-arithmetic subgroup of $\mathbf{G}(k)$, then it admits a finite $B\Gamma$-space, hence the integral Novokov conjectures hold for Γ.

If $\mathbf{G}$ is over a function field and of 0 rank, the same conclusion holds.

The most genereal result I could prove.

Theorem. Let k be a global field, and $\Gamma \subset GL(n, k)$ be a finitely generated subgroup. Then the integral Novikov conjecture in both K- and L-theories hold for Γ.

To apply **Method A**, we need a finite $B\Gamma$.

Number field $\mathbb{Q}$ first.

If S is nonempty, as subgroups of $\mathbf{G}(\mathbb{R})$, Γ is not a discrete subgroup. Hence Γ does not act properly on $X = G/K$.

Hence $\Gamma\backslash X$ is not Hausdorff and is not a $B\Gamma$-space for Γ.

Assume from now on that $\mathbf{G}$ is semisimple.

To get a space where Γ acts properly, we use X together with the Bruhat-Tits buildings X_p, one for each prime $p \in S_f$.

Briefly, X_p is a simplicial complex of dimension equal to the $\mathbb{Q}_p$-rk r of $\mathbf{G}$ and is union of subcomplexes $\mathbb{R}^r$. It is contractible.

When the rank $r = 1$, X_p is a tree.

X_p is the analogue of the symmetric space for p-adic group $\mathbf{G}(\mathbb{Q}_p)$

$\mathbf{G}(\mathbb{Q}_p)$ acts simplicially and properly on X_p.

Fact. If Γ is an S-arithmetic subgroup, then it is a discrete subgroup of

$G_S = \mathbf{G}(\mathbb{R}) \times \prod_{p \in S_f} \mathbf{G}(\mathbb{Q}_p)$

under diagonal embedding, hence Γ acts properly on

$X_S = X \times \prod_{p \in S_f} X_p$.

When Γ is torsion free, then $\Gamma \backslash X_S$ is a $B\Gamma$-space of Γ.

Fact. The quotient $\Gamma\backslash X \times \prod_{p\in S} X_p$ is compact if and only if the $\mathbb{Q}$-rk r of $\mathbf{G}$ is zero.

When the $\mathbb{Q}$-rank $r = 0$, $\Gamma\backslash X \times \prod_{p\in S} X_p$ is a compact space, which implies it is a finite $B\Gamma$.

Not obvious since it is not a manifold. Finite triangulization is not obvious.

If the rank $r > 0$, then we need the Borel-Serre **compactification** $\Gamma\backslash \overline{X}_{\mathbb{Q}}^{BS} \times \prod_{p\in S_f} X_p$ to get a finite $B\Gamma$. (By similar arguments)

Function field case, $k = \mathbb{F}_q(C)$

$\mathbb{F}_q$ a finite field, C a projective curve.

There is no symmetric space. We use the product $X_S = \prod_{p\in S_f} X_p$.

The quotient $\Gamma\backslash X_S$ is compact if and only if the rank of $\mathbf{G}$ is equal to 0.

If compact, has a finite CW-complex structure autimatically. The above method of using Borel-Serre compactification only works over number fields.

So we can only prove the existence of finite $B\Gamma$ when the rank of $\mathbf{G}$ is equal to 0.

If the rank is positive, there is no torsion-free S-arithmetic subgroup. Hence, no finite $B\Gamma$.

Theorem. If k is a function field and the k-rank of $\mathbf{G}$ is equal to 0, then the integral Novikov conjectures in both L-theory and K-theory hold for any torsion free S-arithmetic subgroups Γ in $\mathbf{G}(k)$.

Can also be proved by Method B.

Problem. Remove the condition, rank $r = 0$, for function fields.

Problem. If the rank $r > 0$, there is no torsion free S-arithmetic subgroups.

The integral Novikov conjecture does not hold for non-torsion free groups in general.

So for function fields, this is the best result in certain sense using the above formulation of Novikov conjecture.

Groups Γ that contain torsion groups.

In the above discussions, we assume Γ is torsion free. Otherwise, there is no finite $B\Gamma$ and the integral Novikov conjectures fail in general.

But important arithmetic groups , for example $\Gamma = SL(n, \mathbb{Z})$, and maximal Fuchsian groups contain nontrivial torsion elements. Their K-groups $K_*(\mathbb{Z}[\Gamma])$ are important.

(Passing to a torsion free subgroup Γ' of finite index loses some information).

Let $\mathcal{F}$ be the family of finite subgroups of Γ. There is a universal space $E_{\mathcal{F}}\Gamma$:

Γ acts on it with finite stabilizers (i.e. properly), and for any finite subgroup, its set of fixed points in $E_{\mathcal{F}}\Gamma$ is contractible

This is the universal space for proper actions of Γ.

If Γ is torsion free, $E_{\mathcal{F}}\Gamma$ is equal to $E\Gamma$.

If Γ is a discrete subgroup of a Lie group G, then G/K is a $E_{\mathcal{F}}\Gamma$-space.

There is an **assembly map**

$$A_{\mathcal{F}} : H_*^{\Gamma}(E_{\mathcal{F}}\Gamma, \mathbb{K}(\mathbb{Z}))) \to K_*(\mathbb{Z}[\Gamma]),$$

defined by projecting $E_{\mathcal{F}}$ to a point.

When Γ is torsion free,

$$H_*^{\Gamma}(E\Gamma, \mathbb{K}) = H_*(B\Gamma, \mathbb{K}),$$

and the $\mathcal{A}_{\mathcal{F}}$ is the usual assembly map.

Similarly, there are assembly maps for L-theory, C^*-algebras.

Generalized Integral Nivikov Conjecture. The assembly map $A_{\mathcal{F}}$ is injective.

This is an important step towards computing the right hand side $K_*(\mathbb{Z}[\Gamma])$.

In general, the map $A_{\mathcal{F}}$ is **not** *surjective.*

Already false for some virtually cyclic groups, by Farrell.

Theorem. If $\mathbf{G}$ is a reductive group over a number field or a function field k of k-rank 0, then for any S-arithmetic subgroup (may contain torsion elements), $A_{\mathcal{F}}$ is injective for K-theory, L-theory.

Theorem. If $\mathbf{G}$ is a reductive group over a number field, then for any arithmetic subgroup Γ (may contain torsion elements), the generalized integral Novikov conjecture holds for Γ.

Corollary. Under the same assumption, the assembly map in the Farrell-Jones isomorphism conjecture is rationally injective for such S-arithmetic groups.

F-J isomorphism map is the assembly map for family of all virtually cyclic subgroups of Γ.

The proof makes a result of Rosenthal

Theorem. If $E_{\mathcal{F}}\Gamma$ is compact mod Γ and admits a contractible Γ-compactification $\overline{E_{\mathcal{F}}\Gamma}$ s.t.

(1) for any finite group $F \subset \Gamma$, its fixed point set $(\overline{E_{\mathcal{F}}\Gamma})^F$ is contractible, and $(E_{\mathcal{F}}\Gamma)^F$ is dense in it.

(2) the Γ-action is small at infinity of $\overline{E_{\mathcal{F}}\Gamma}$.

Then the map $A_{\mathcal{F}}$ is injective.

This is an analogue of **Method B**.

To apply this to S-arithmetic groups, we realize that Bruhat-Tits buildings are CAT(0)-spaces.

Recall a geodesic metric space is called a CAT(0)-space if every triangle in it is thinner than the corresponding triangle in $\mathbb{R}^2$.

Nonpositively curved, simply connected Riemannian manifolds are CAT(0)-spaces. Bruhat-Tits buildings X_p are CAT(0)-spaces.

Fact. If X_1, X_2 are CAT(0)-spaces, then $X_1 \times X_2$ is also a CAT(0)-space. In particular, X_S is a CAT(0)-space.

Fact. A proper CAT(0)-space M admits a compactification $M \cup M(\infty)$ by adding the set of equivalence classes of geodesics.

If Γ acts properly and cocompactly on M, then the action of Γ on $M \cup M(\infty)$ is small at infinity.

The compactification $M \cup M(\infty)$ is also contractible (along rays from a fixed basepoint)

An analogue of **Method A**.

Theorem If as-dim$\Gamma < +\infty$ and there is a Γ-compact $E_{\mathcal{F}}\Gamma$ such that for any finite subgroup $F \subset \Gamma$, its fixed point $E_{\mathcal{F}}^{F}$ together with their finite quotients is uniformly contractible and of bounded geometry, then the map $A_{\mathcal{F}}$ is injective.

Can be used to show the generalized integral Novikov conjectures for polycyclic groups.

They provide more classes of natural groups where these assembly maps are injective.

Mapping class groups

The mapping class groups are a natural generalization of $SL(2,\mathbb{Z})$ and are closely related to arithmetic groups.

The original Novikov conjecture on the homotopy invariance of higher signatures has been proved for the mapping class groups by several people
(Peter Storm, U. Hamenstädt).

But the *integral* Novikov conjecture is not known. It is not know that if they have finite asymptotic dimension. The approach via compactification should be possible.

The mapping class groups also contain torsion elements. The generalized integral Novikov conjecture is not known either.