INTERSECTION NUMBERS ON $\overline{\mathcal{M}}_{g,n}$ AND AUTOMORPHISMS OF STABLE CURVES

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Abstract. Due to the orbifold singularities, the intersection numbers on the moduli space of curves $\overline{\mathcal{M}}_{g,n}$ are in general rational numbers rather than integers. We study the properties of the denominators of these intersection numbers and their relationship with the orders of automorphism groups of stable curves. We also present a conjecture about a multinomial type numerical property for a general class of Hodge integrals.

1. Introduction

We denote by $\overline{\mathcal{M}}_{g,n}$ the moduli space of stable n-pointed genus g complex algebraic curves. We have the forgetting the last marked point morphism

$$\pi : \overline{\mathcal{M}}_{g,n+1} \longrightarrow \overline{\mathcal{M}}_{g,n},$$

We denote by $\sigma_1, \dots, \sigma_n$ the canonical sections of π, and by $D_1, \dots, D_n$ the corresponding divisors in $\overline{\mathcal{M}}_{g,n+1}$. We let ω_π be the relative dualizing sheaf and set

$$\begin{aligned} \psi_i &= c_1(\sigma_i^*(\omega_\pi)) \\ K &= c_1\left(\omega_\pi\left(\sum D_i\right)\right) \\ \kappa_i &= \pi_*(K^{i+1}) \\ \mathbb{E} &= \pi_*(\omega_\pi) \\ \lambda_l &= c_l(\mathbb{E}), 1 \le l \le g. \end{aligned}$$

Where $\mathbb{E}$ is the *Hodge bundle*. The classes κ_i were first introduced by Mumford [15] on $\overline{\mathcal{M}}_g$, their generalization to $\overline{\mathcal{M}}_{g,n}$ here is due to Arbarello-Cornalba [1].

Hodge integrals are intersection numbers of the form

$$\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \cdots \psi_n^{d_n} \kappa_{a_1} \cdots \kappa_{a_m} \lambda_1^{k_1} \cdots \lambda_g^{k_g}$$

If $\sum_{i=1}^n d_i + \sum_{i=1}^m a_i + \sum_{i=1}^g i k_i = 3g - 3 + n$, then the above intersection number is a nonnegative rational number, otherwise it is zero.

Hodge integrals arise naturally in the localization computation of Gromov-Witten invariants. They are extensively studied by mathematicians and physicists. The Hodge integral involving only ψ classes can be computed recursively by the following famous

Witten's conjecture [18] proved by Kontsevich [13]:

$$
\begin{aligned}
\langle\tau_{k+1}\tau_{\underline{d}}\rangle &= \frac{1}{(2k+3)!!}\left[\sum_{j=1}^{n}\frac{(2k+2d_j+1)!!}{(2d_j-1)!!}\langle\tau_{d_1}\cdots\tau_{d_j+k}\cdots\tau_{d_n}\rangle\right. \\
&\quad +\frac{1}{2}\sum_{r+s=k-1}(2r+1)!!(2s+1)!!\langle\tau_r\tau_s\tau_{\underline{d}}\rangle \\
&\quad \left.+\frac{1}{2}\sum_{r+s=k-1}(2r+1)!!(2s+1)!!\sum_{I\subset\{1,\dots,n\}}\langle\tau_r\tau_{\underline{d}_I}\rangle\langle\tau_s\tau_{\underline{d}_{\mathcal{C}I}}\rangle\right]
\end{aligned}
\tag{1}
$$

for any $\underline{d}=(d_1,\dots,d_n)$, where $\langle\tau_{d_1}\cdots\tau_{d_n}\rangle_g=\int_{\overline{\mathcal{M}}_{g,n}}\psi_1^{d_1}\cdots\psi_n^{d_n}$.

Now there are several new proofs of Witten's conjecture [12, 17, 11, 14].

Let *denom*(r) denote the denominator of a rational number r in reduced form (coprime numerator and denominator, positive denominator). We define

$$D_{g,n}=lcm\left\{denom\left(\int_{\overline{\mathcal{M}}_{g,n}}\psi_1^{d_1}\cdots\psi_n^{d_n}\right)\,\Big|\,\sum_{i=1}^{n}d_i=3g-3+n\right\}$$

and for $g\geq 2$,

$$\mathcal{D}_g=lcm\left\{denom\left(\int_{\overline{\mathcal{M}}_{g}}\kappa_{a_1}\cdots\kappa_{a_m}\right)\,\Big|\,\sum_{i=1}^{m}a_m=3g-3\right\}$$

where *lcm* denotes *least common multiple.*

We know that a neighborhood of $\Sigma\in\overline{\mathcal{M}}_{g,n}$ is of the form $U/\mathrm{Aut}(\Sigma)$, where U is an open subset of $\mathbb{C}^{3g-3+n}$. This gives the orbifold structure for $\overline{\mathcal{M}}_{g,n}$. Since denominators of intersection numbers on $\overline{\mathcal{M}}_{g,n}$ all come from these orbifold quotient singularities, so the divisibility properties of $D_{g,n}$ and $\mathcal{D}_g$ should reflect overall behavior of singularities.

We are mainly interested in $\mathcal{D}_g$, which is closely related with $D_{g,n}$.

In the second section, we will study some properties of $\mathcal{D}_g$ and present an explicit multiple of $\mathcal{D}_g$. In the third section, we will discuss briefly automorphism groups of Riemann surfaces and stable curves. In the fourth section, we will study prime factors of $\mathcal{D}_g$ and its relations with automorphism groups of stable curves. In the last section, we will give a conjectural numerical property for Hodge integrals and verify it for several special cases.

2. Some properties of $\mathcal{D}_g$

If we take $k=-1$ and $k=0$ respectively in Witten-Kontsevich's formula (1), we get the string equation

$$\langle\tau_0\prod_{i=1}^{n}\tau_{k_i}\rangle_g=\sum_{j=1}^{n}\langle\tau_{k_j-1}\prod_{i\neq j}\tau_{k_i}\rangle_g$$

and the dilaton equation

$$\langle\tau_1\prod_{i=1}^{n}\tau_{k_i}\rangle_g=(2g-2+n)\langle\prod_{i=1}^{n}\tau_{k_i}\rangle_g$$

from which it's easy to prove the following

Proposition 2.1. *If $n\geq 1$, then*

a. $D_{0,n}=1$,
b. $D_{1,n}=24$,
c. $D_{g,1}=24^g\cdot g!$.

Note that $D_{0,n}=1$ is also implied by the fact that $\overline{\mathcal{M}}_{0,n}$ is a smooth manifold.

Theorem 2.2. *We have*

$$D_{g,n}\mid D_{g,n+1}.$$

Proof. Let $q^s\mid D_{g,n}$, where q is a prime number and $q^{s+1}\nmid D_{g,n}$.

We sort $\{\langle\tau_{d_1}\dots\tau_{d_n}\rangle_g\mid\sum_{i=1}^n d_i=3g-3+n, 0\leq d_1\leq\cdots\leq d_n\}$ in lexicographical order, we say $\langle\tau_{k_1}\dots\tau_{k_n}\rangle_g<\langle\tau_{m_1}\dots\tau_{m_n}\rangle_g$, if there is some i, such that $k_j=m_j, j<i$ and $k_i<m_i$.

Let $\langle\tau_{k_1}\dots\tau_{k_n}\rangle_g$ be the minimal element with respect to the lexicographical order such that its denominator is divisible by q^s.

We have

$$\begin{aligned}\langle\tau_0\tau_{k_1}\dots\tau_{k_n+1}\rangle_g &= \langle\tau_{k_1}\dots\tau_{k_n}\rangle_g+\sum_{i=1}^{n-1}\langle\tau_{k_1}\dots\tau_{k_i-1}\dots\tau_{k_{n-1}}\tau_{k_n+1}\rangle_g\\ &= \frac{c}{q^s d}+\sum_{i=1}^{n-1}\frac{b_i}{a_i}\end{aligned}$$

we require $q\nmid c, q\nmid d$ and $(a_i,b_i)=1$.

Since for $i=1,\dots,n-1$, we have $\langle\tau_{k_1}\dots\tau_{k_i-1}\dots\tau_{k_{n-1}}\tau_{k_n+1}\rangle_g<\langle\tau_{k_1}\dots\tau_{k_n}\rangle_g$, so $a_i=q^{s_i}e_i$, where $s_i<l$ and $q\nmid e_i$. We now have

$$\langle\tau_0\tau_{k_1}\dots\tau_{k_n+1}\rangle_g=\frac{c\prod_{i=1}^{n-1}e_i+qd(\sum_{j=1}^{n-1}q^{s-s_j-1}\prod_{i\neq j}e_i)}{q^s d\prod_{i=1}^{n-1}e_i}$$

we see that q can not divide the numerator, so we have proved $q^s\mid D_{g,n+1}$. Since q is arbitrary, we proved the theorem. □

Theorem 2.3. *We have $D_{g,n}\mid\mathcal{D}_g$ for all $g\geq 2, n\geq 1$. Moreover $\mathcal{D}_g=D_{g,3g-3}$.*

Proof. Let

$$\pi_n:\overline{\mathcal{M}}_{g,n}\longrightarrow\overline{\mathcal{M}}_{g,n-1},$$

be the morphism which forgets the last marked point, then we have the formulas due to Faber (see [1])

$$(\pi_1\dots\pi_n)_*(\psi_1^{a_1+1}\dots\psi_n^{a_n+1})=\sum_{\sigma\in S_n}\kappa_\sigma, \tag{2}$$

where κ_σ is defined as follows. Write the permutation σ as a product of $\nu(\sigma)$ disjoint cycles, including 1-cycles: $\sigma = \beta_1 \cdots \beta_{\nu(\sigma)}$, where we think of the symmetric group S_n as acting on the n-tuple $(a_1, \ldots, a_n)$. Denote by $|\beta|$ the sum of the elements of a cycle β. Then

$$\kappa_\sigma = \kappa_{|\beta_1|}\kappa_{|\beta_2|} \ldots \kappa_{|\beta_{\nu(\sigma)}|}.$$

From the formula (2), we get

$$\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{a_1+1} \cdots \psi_n^{a_n+1} = \sum_{\sigma \in S_n} \int_{\overline{\mathcal{M}}_g} \kappa_\sigma,$$

so we proved $D_{g,n} \mid \mathcal{D}_g$.

On the other hand, any $\int_{\overline{\mathcal{M}}_g} \kappa_{a_1} \cdots \kappa_{a_m}$ can be written as a sum of $\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \cdots \psi_n^{d_n}$'s. This can be seen by induction on the number of kappa classes, for integrals with only one kappa class, we have $\int_{\overline{\mathcal{M}}_{g,n}} \kappa_{a_1} \psi_1^{d_1} \cdots \psi_n^{d_n} = \int_{\overline{\mathcal{M}}_{g,n+1}} \psi_{n+1}^{a_1+1} \psi_1^{d_1} \cdots \psi_n^{d_n}$. We also have

$$\begin{aligned} \int_{\overline{\mathcal{M}}_{g,n}} \kappa_{a_1} \cdots \kappa_{a_m} \psi_1^{d_1} \cdots \psi_n^{d_n} \ = \ & \int_{\overline{\mathcal{M}}_{g,n+m}} \psi_{n+1}^{a_1+1} \cdots \psi_{n+m}^{a_m+1} \psi_1^{d_1} \cdots \psi_n^{d_n} \\ & - \{\text{integrals with at most } m-1 \text{ kappa classes}\}. \end{aligned}$$

thus finishing the induction argument. So we proved $\mathcal{D}_g = D_{g,3g-3}$. □

Corollary 2.4. *For $g \geq 2$,*

$$\mathcal{D}_g = lcm \left\{ denom \left(\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \cdots \psi_n^{d_n} \right) \Big| \sum_{i=1}^n d_i = 3g-3+n, d_i \geq 2, n \leq 3g-3 \right\}$$

We have used this corollary to calculate $\mathcal{D}_g$ for $g \leq 16$ in the appendix.

To proceed, we need a little knowledge of Hurwitz numbers.

Let $\pi : X_g \to \mathbb{P}^1$ be a ramified cover over the sphere $\mathbb{P}^1$ (or meromorphic function on X_g) whose only degenerate ramification point is over ∞ with ramification type μ. So we have $\deg \pi = |\mu|$. By the Riemann-Hurwitz formula, the number of simple ramification points of π is:

$$b = 2g - 2 + |\mu| + l(\mu).$$

Two covers

$$\pi : X_g \to \mathbb{P}^1, \ \pi' : X_g' \to \mathbb{P}^1$$

are isomorphic if there exits an isomorphism $\phi : X_g \to X_g'$ satisfying $\pi' \circ \phi = \pi$. Each cover π has an naturally associated automorphism group $\mathrm{Aut}(\pi)$.

Definition 2.5. The Hurwitz number $h_{g,\mu}$ is a weighted count of the distinct Hurwitz covers π of genus g with ramification μ over ∞ and simple ramification over b fixed points on $\mathbb{P}^1$. Each such cover is weighted by $1/|\mathrm{Aut}(\pi)|$.

Lemma 2.6. *Any simple Hurwitz number, if mutiplied by 2, is an integer, i.e.*

$$2h_{g,(a_1,\ldots,a_n)} \in \mathbb{Z}.$$

Proof. In the definition of simple Hurwitz numbers, each isomorphism class of cover $\pi : X_g \to P^1$ is weighted by $1/|\mathrm{Aut}(\pi)|$. So we should examine $|\mathrm{Aut}(\pi)|$.

Let $\mu = (a_1, \ldots, a_n)$ be the partition of $|\mu| = \sum_{i=1}^n a_i$, the length $l(\mu) = n$. By Riemann-Hurwitz theorem, we have $b = 2g - 2 + n + |\mu|$ simple ramification points in the target sphere.

The following table is borrowed from [17], which is not difficult to check

$h_{g,\mu}$	(1)	(2)	$(1,1)$	(3)	$(2,1)$	$(1,1,1)$
$g = 0$	1	1/2	1/2	1	4	4
$g = 1$	0	1/2	1/2	9	40	40
$g = 2$	0	1/2	1/2	81	364	364

If $g = 0$ and $b = n + |\mu| - 2 \geq 3$, then the automorphism group $\mathrm{Aut}(\pi)$ of any morphism $\pi : X_g \to \mathbb{P}^1$ with the prescribed branch type over ∞ contains only identity, since any Mobiüs transformation of the sphere fixing three points is the identity.

If $g \geq 1$, then $b = n + |\mu| + 2g - 2$, if $|\mu| = 2$, then the automorphism group $\mathrm{Aut}(\pi)$ of any morphism $\pi : X_g \to \mathbb{P}^1$ with the prescribed branch type over ∞ contains exactly two elements: the identity and the involution that transposes the sheets of the covering (fixing exactly $2g + 2$ points). If $b > 2g + 2$, i.e. $n + |\mu| > 4$, then from the following lemma 2.7, we know that the automorphism group $\mathrm{Aut}(\pi)$ of any morphism $\pi : X_g \to \mathbb{P}^1$ with the prescribed branch type over ∞ contains only identity.

Combining with the table, we proved the theorem. □

Lemma 2.7. *(See [7]) If X_g is a Riemann surface with genus $g \geq 0$ and if $1 \neq T \in Aut(X_g)$, then T has at most $2g + 2$ fixed points.*

Next we will work out an explicit multiple of $\mathcal{D}_g$.

Theorem 2.8. [11] *For any sequence of non-negative integers $d_1, \ldots, d_n$ we have*

$$\langle \tau_{d_1} \ldots \tau_{d_n} \rangle_g = \sum_{a_1=1}^{d_1+1} \cdots \sum_{a_n=1}^{d_n+1} \left(\frac{1}{(2g - 2 + n + \sum_{i=1}^n a_i)!} \prod_{i=1}^n \frac{(-1)^{d_i+1-a_i}}{(d_i + 1 - a_i)! a_i^{a_i - 1}} \right) h_{g,(a_1,\ldots,a_n)},$$

where the genus g is determined by the left-hand side, $\sum_{i=1}^n d_i = 3g - 3 + n$.

From the above formula, we get the following

Proposition 2.9. *$D_{g,n}$ divides*

$$(5g - 5 + 3n)!\ [(3g - 3 + 2n)!]^{3g-3+n} \cdot 2.$$

Proof. Note lemma 2.6 and the simple fact that $\prod_{i=1}^n k_i! \mid (\sum_{i=1}^n k_i)!$, it's easy to see that $D_{g,n}$ divides

$$\begin{aligned} & (2g - 2 + n + \sum_{i=1}^n d_i + n)! \prod_{i=1}^n [(d_i + 1)!]^{d_i} \cdot 2 \\ \mid \quad & (5g - 5 + 3n)! \left[\prod_{i=1}^n (d_i + 1)! \right]^{3g-3+n} \cdot 2 \\ \mid \quad & (5g - 5 + 3n)!\ [(3g - 3 + 2n)!]^{3g-3+n} \cdot 2 \end{aligned}$$

where "|" denotes that the latter is divisible by the former.

□

Since $\mathcal{D}_g = D_{g,3g-3}$, we have the following

Corollary 2.10. *For* $g \geq 2$, $\mathcal{D}_g$ *divides* $(14g-14)!\ [(9g-9)!]^{6g-6} \cdot 2$.

Corollary 2.11. *For* $g \geq 2$, *the denominator of intersection numbers of the form*

$$\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \cdots \psi_n^{d_n} \kappa_{a_1} \cdots \kappa_{a_m}$$

can divide $(14g-14)!\ [(9g-9)!]^{6g-6} \cdot 2$.

3. Automorphism groups of stable curves

The book [7] contains an excellent chapter on automorphisms of compact Riemann surfaces.

Let X be a compact Riemann surface of genus g and $\mathrm{Aut}(X)$ denotes the group of conformal automorphisms of X. It's a classical theorem of Hurwitz that if $g \geq 2$, then $|\mathrm{Aut}(X)| \leq 84(g-1)$.

Let $G \subset \mathrm{Aut}(X)$ be a group of automorphisms of X, consider the natural map

$$\pi : X \to X/G$$

we know that π has degree $|G|$ and X/G is a compact Riemann surface of genus g_0.

The mapping π is branched only at the fixed points of G and the branching order

$$b(P) = \mathrm{ord} G_P - 1$$

where G_P is the isotropy group at $P \in X$ which is known to be cyclic.

Let $P_1, \ldots, P_r$ be a maximal set of inequivalent fixed points of elements of $G \setminus \{1\}$. (that is, $P_i \neq h(P_j)$ for all $h \in G$ and all $j \neq k$.)

Let $n_i = \mathrm{ord} G_{P_i}$, then the total branch number of π is given by

$$B = \sum_{i=1}^{r} \frac{|G|}{n_i}(n_i - 1) = |G| \sum_{i=1}^{r} (1 - \frac{1}{n_i})$$

the Riemann-Hurwitz formula now reads

$$2g - 2 = |G| \left[2g_0 - 2 + \sum_{i=1}^{r} (1 - \frac{1}{n_i}) \right]$$

so we have

$$|G| \;\Big|\; (2g-2) \cdot lcm(n_1, \ldots, n_r), \tag{3}$$

this fact is crucial in the study of automorphism groups of compact Riemann surfaces.

We need the following theorem.

Theorem 3.1. [9] *The minimum genus* g *of compact Riemann surface which admits an automorphism of order* p^r *(*p *is prime) is given by*

$$g = \max\left\{2, \frac{p-1}{2} p^{r-1}\right\}.$$

An immediate corollary of formula (3) and theorem 3.1 is

Corollary 3.2. *Let X be a compact Riemann surface of genus $g \geq 2$, if a prime number p can divide $|Aut(X)|$, then $p \leq 2g+1$.*

We recall the definition of stable curves. A stable curve is a connected and compact nodal curve, which means that its singular points are locally analytically isomorphic to a neighborhood of the orgin of $xy = 0$ in $\mathbb{C}^2$ and satisfy the stability conditions: (i) each genus 0 component has at least three node-branches; (ii) each genus 1 component has at least one node-branch.

The stability conditions are equivalent to say that the nodal curve has finite automorphism group.

Suppose Σ is a stable curve of arithmetic genus g such that its normalization has n components $\Sigma_1, \dots, \Sigma_n$ of genus $g_1, \dots, g_n$.

Definition 3.3. We call an automorphism φ of the dual graph Γ of Σ an *admissible graph automorphism*, if there is at least one automorphism of the stable curve Σ realizing the underlying dual graph automorphism φ. All admissible automorphisms of Γ form a group, denoted by $\mathrm{Ad}(\Gamma)$, which is a subgroup of $\mathrm{Aut}(\Gamma)$.

Theorem 3.4. *Let $\widetilde{Aut}(\Sigma_i)$ be the group of automorphisms of Σ_i fixing node-branches on Σ_i. Then we have*

$$|Aut(\Sigma)| = |Ad(\Gamma)| \cdot \prod_{i=1}^{n} |\widetilde{Aut}(\Sigma_i)|$$

Proof. We note the following fact, if $f(x)$ and $g(y)$ are two holomorphic functions defined near the origin of $\mathbb{C}^1$ and satisfy $f(0) = g(0)$, then $F(x, y) = f(x) + g(y) - f(0)$ is a holomorphic function near the origin of $\mathbb{C}^2$ satisfying $F(x, 0) = f(x)$ and $F(0, y) = g(y)$. Then the theorem is obvious. □

We now generalize corollary 3.2 to stable curves.

Theorem 3.5. *Let Σ be a stable curve of arithmetic genus $g \geq 2$, if a prime number p can divide $|Aut(\Sigma)|$, then $p \leq 2g+1$.*

Proof. Let's assume that there are δ nodes on Σ and δ_i node-branches on each Σ_i. Then we have the following relations,

$$g = \sum_{i=1}^{n} (g_i - 1) + \delta + 1, \tag{4}$$

$$2g_i + \delta_i - 2 \geq 1, \tag{5}$$

$$2\delta = \sum_{i=1}^{n} \delta_i. \tag{6}$$

Sum up (5) for $i = 1$ to n and substitute (4) and (6) into (5), we get

$$n \leq 2g - 2.$$

Let e_{ij} denotes the number of edges between Σ_i and Σ_j in the dual graph of Σ, then it's obvious that $e_{ij} \leq g + 1$.

Since $|\mathrm{Aut}(\Gamma)|$ divides $n! \prod_{(i,j)} (e_{ij}!)$ which is not divisible by prime numbers greater than $2g + 1$, and $g_i \leq g$, so the theorem follows from corollary 3.2. □

4. Prime factors of $\mathcal{D}_g$

Definition 4.1. We call the following generating function

$$F(x_1,\cdots,x_n)=\sum_{g=0}^{\infty}\sum_{\sum d_i=3g-3+n}\langle\tau_{d_1}\cdots\tau_{d_n}\rangle\prod x_i^{d_i}$$

the n-point function.

In particular, 2-point function has a simple explicit form due to Dijkgraaf,

$$F(x_1,x_2)=\frac{1}{x_1+x_2}\exp\left(\frac{x_1^3}{24}+\frac{x_2^3}{24}\right)\sum_{k=0}^{\infty}\frac{k!}{(2k+1)!}\left(\frac{1}{2}x_1x_2(x_1+x_2)\right)^k.$$

Lemma 4.2. *Let p denotes a prime number and $g\geq 2$, then*

a. *If $p>2g+1$, then $p\nmid D_{g,2}$,*
b. *If $g+1\leq p\leq 2g+1$, then*

$$p\mid denom\langle\tau_{\frac{p-1}{2}}\tau_{3g-1-\frac{p-1}{2}}\rangle_g,$$

c. *If $2g+1$ is prime, then $(2g+1)\mid denom\langle\tau_d\tau_{3g-1-d}\rangle_g$ if and only if $g\leq d\leq 2g-1$.*

Proof. From the 2-point function, we get

$$\begin{aligned}\langle\tau_d\tau_{3g-1-d}\rangle &= \sum_{i=0}^{g}\sum_{k}\binom{g-k}{i}\binom{k-1}{d-3i-k}\frac{k!}{(g-k)!24^{g-k}(2k+1)!2^k}\\ &\quad+\frac{(-1)^{\mathrm{mod}(d,3)}}{g!24^g}\binom{g-1}{\lfloor\frac{d}{3}\rfloor},\end{aligned}$$

where $\mathrm{mod}(d,3)$ denotes the remainder of the division of d by 3, the summation range of k is $\max(\frac{d_1-3i+1}{2},1)\leq k\leq\min(g-i,d_1-3i)$. Then the lemma follows easily. □

Theorem 4.3. *Let p denotes a prime number, $g\geq 2$ and let $o(p,q)$ denotes the maximum integer such that $p^{o(p,q)}\mid q$, then*

a. *If $p>2g+1$, then $p\nmid\mathcal{D}_g$,*
b. *For any prime $p\leq 2g+1$, we have $p\mid\mathcal{D}_g$.*
c. *If $2g+1$ is prime, then $o(2g+1,\mathcal{D}_g)=1$,*
d. *$o(2,\mathcal{D}_g)=3g+o(2,g!)$.*

Proof. For part a., we use induction on the pair of genus and the number of marked points (g,n) to prove that denominators of all ψ class intersection numbers are not divisible by prime numbers greater than $2g+1$. We rewrite Witten-Kontsevich formula here,

$$\begin{aligned}\langle\tau_{k+1}\tau_{\underline{d}}\rangle_g &= \frac{1}{(2k+3)!!}\Bigg[\sum_{j=1}^{n}\frac{(2k+2d_j+1)!!}{(2d_j-1)!!}\langle\tau_{d_1}\ldots\tau_{d_j+k}\ldots\tau_{d_n}\rangle_g\\ &\quad+\frac{1}{2}\sum_{r+s=k-1}(2r+1)!!(2s+1)!!\langle\tau_r\tau_s\tau_{\underline{d}}\rangle_{g-1}\\ &\quad+\frac{1}{2}\sum_{r+s=k-1}(2r+1)!!(2s+1)!!\sum_{I\subset\{1,\ldots,n\}}\langle\tau_r\tau_{\underline{d}_I}\rangle_{g_1}\langle\tau_s\tau_{\underline{d}_{\mathcal{C}I}}\rangle_{g_2}\Bigg]\end{aligned}$$

where $g_1 + g_2 = g$.

For $n \geq 3$ marked points, we may take $k \leq g-1$, then by induction on (g, n) it's easy to see that the denominator of the right hand side is not divisible by prime numbers greater than $2g + 1$.

For part b., it follows from proposition 2.1 c., theorem 2.3 and lemma 4.2 b.

For part c., we again use induction on (g, n) as in the proof of part a. In view of lemma 4.2 c. and $o(2g + 1, D_{g,2}) = 1$, We need only consider the following intersection numbers,

$$\langle\tau_g\tau_g\tau_g\rangle = \frac{1}{(2g+1)!!}\left[\frac{2(4g-1)!!}{(2g-1)!!}\langle\tau_g\tau_{2g-1}\rangle + \{\text{lower genus terms}\}\right]$$

since the factor $2g+1$ in the denominator of $\langle\tau_g\tau_{2g-1}\rangle$ will be cancelled by $(4g-1)!!$. We proved part c.

For part d., since $\langle\tau_{3g-2}\rangle_g = \frac{1}{24^g g!}$, we have $o(2, \mathcal{D}_g) \geq 3g + o(2, g!)$, the reverse inequality can be seen from the Witten-Kontsevich formula by induction on (g, n) and note the following,

$$\begin{aligned} &\frac{1}{2}\sum_{r+s=k-1}(2r+1)!!(2s+1)!!\sum_{I\subset\{1,\ldots,n\}}\langle\tau_r\tau_{\underline{d}_I}\rangle_{g_1}\langle\tau_s\tau_{\underline{d}_{CI}}\rangle_{g_2} \\ =&\sum_{r+s=k-1}(2r+1)!!(2s+1)!!\sum_{I\cup J=\{1,\ldots,n\}}\langle\tau_r\tau_{\underline{d}_I}\rangle_{g_1}\langle\tau_s\tau_{\underline{d}_J}\rangle_{g_2}, \end{aligned}$$

where $\{I, J\}$ takes over unordered partitions of $\{1, \ldots, n\}$. □

Proposition 4.4. *If $p \geq 3$ is a prime number, then*

$$o(p, denom\langle\tau_{d_1}\cdots\tau_{d_n}\rangle_g) \leq o(p, \prod_{i=1}^{n}(2d_i+1)!!).$$

Proof. Following Dijkgraaf's notation [2], let

$$\langle\tilde{\tau}_{d_1}\ldots\tilde{\tau}_{d_n}\rangle_g = \left[\prod_{i=1}^{n}(2d_i+1)!!\right]\langle\tau_{d_1}\ldots\tau_{d_n}\rangle_g$$

then the Witten-Kontsevich formula can be written as,

$$\begin{aligned} \langle\tilde{\tau}_k\tilde{\tau}_{\underline{d}}\rangle_g = &\sum_{j=1}^{n}(2d_j+1)\langle\tilde{\tau}_{d_1}\ldots\tilde{\tau}_{d_j+k-1}\ldots\tilde{\tau}_{d_n}\rangle_g + \frac{1}{2}\sum_{r+s=k-2}\langle\tilde{\tau}_r\tilde{\tau}_s\tilde{\tau}_{\underline{d}}\rangle_{g-1} \\ &+\frac{1}{2}\sum_{\substack{r+s=k-2\\ I\subset\{1,\ldots,n\}}}\langle\tilde{\tau}_r\tilde{\tau}_{\underline{d}_I}\rangle_{g'}\langle\tilde{\tau}_s\tilde{\tau}_{\underline{d}_{CI}}\rangle_{g-g'} \end{aligned}$$

for any $\underline{d} = (d_1, \ldots, d_n)$.

Since $\langle\tilde{\tau}_1\rangle_1 = \frac{1}{8}$, by induction on (g, n), it's easy to prove that for any prime number $p \geq 3$,

$$p \nmid denom\langle\tilde{\tau}_{d_1}\ldots\tilde{\tau}_{d_n}\rangle_g.$$

So we proved the proposition. □

Lemma 4.5. *Let B_m denotes the Bernoulli numbers in the expansion*

$$\frac{t}{e^t-1}=\sum_{m=0}^{\infty}B_m\frac{t^m}{m!},$$

The denominator of B_{2m} is given by

$$\prod_{(p-1)|2m}p$$

where the product is taken over the primes p.

Proof. It follows easily from the Staudt's theorem (see [10]),

$$-B_{2m}\equiv\sum_{(p-1)|2m}\frac{1}{p}\pmod 1$$

where the sum is taken over the primes p. □

Theorem 4.6. *The denominator of intersection numbers of the form*

$$\int_{\overline{\mathcal{M}}_{g,n}}\psi_1^{d_1}\cdots\psi_n^{d_n}\kappa_{a_1}\cdots\kappa_{a_m}\lambda_1^{k_1}\cdots\lambda_g^{k_g}\tag{7}$$

can only contain prime factors less than or equal to $2g+1$.

Proof. Mumford [15] proved the following formula for Chern character of Hodge bundle

$$\mathrm{ch}_{2m-1}(\mathbb{E})=\frac{B_{2m}}{(2m)!}\left[\kappa_{2m-1}-\sum_{i=1}^{n}\psi_i^{2m-1}+\frac{1}{2}\sum_{\xi\in\Delta}l_{\xi*}\left(\sum_{i=0}^{2m-2}\psi_1^i(-\psi_2)^{2m-2-i}\right)\right].$$

We know that any λ_i can be expressed as a polynomial of $\mathrm{ch}_j(\mathbb{E})$'s,

$$\lambda_i=\sum_{\mu\vdash i}(-1)^{i-l(\mu)}\prod_{r\geq1}\frac{((r-1)!)^{m_r}}{m_r!}ch_\mu(\mathbb{E}),\quad i\geq1,$$

where the sum ranges over all partitions μ of i, and m_r is the number of r in μ, and $ch_\mu(\mathbb{E})=ch_{\mu_1}(\mathbb{E})\cdots ch_{\mu_l}(\mathbb{E})$.

By Faber's algorithm [3], we can reduce any Hodge integral (7) to a sum of integrals with only ψ and κ classes.

Note also that $\mathrm{ch}_0(\mathbb{E})=g$, $\mathrm{ch}_{2r}(\mathbb{E})=0$ for $r\geq1$ and $\mathrm{ch}_r(\mathbb{E})=0$ for $r\geq 2g$.

So the theorem follows from lemma 4.5 and theorem 4.3 a. □

In view of theorem 3.5, the above theorem should also follow from the definition of Chow ring of $\overline{\mathcal{M}}_{g,n}$ by Mumford [15]. We include a proof here because it's conceptually simple and direct.

Lemma 4.7. *If $g-1$ is an odd prime number, then*

$$(g-1)^2\mid denom\langle\tau_{\frac{g}{2}-1}\tau_{\frac{g}{2}-1}\tau_{2g+2}\rangle_g.$$

Proof. For $g=4$, $\langle\tau_1\tau_1\tau_{10}\rangle_4 = \frac{7}{2^{12}3^5}$. So we assume $g\geq 6$ in the following,

We need Don Zagier's marvelous 3-point function which we learned from Faber [4],

$$F(x,y,z)=e^{(x^3+y^3+z^3)/24}\sum_{r,s\geq 0}\frac{r!S_r(x,y,z)}{4^r(2r+1)!!\cdot 2}\cdot\frac{[(x+y)(y+z)(z+x)]^s}{8^s(r+s+1)!}$$

where $S_r(x,y,z)$ is the homogeneous symmetric polynomial defined by

$$S_r(x,y,z)=\frac{(xy)^r(x+y)^{r+1}+(yz)^r(y+z)^{r+1}+(zx)^r(z+x)^{r+1}}{x+y+z}\in\mathbb{Z}[x,y,z].$$

It's not difficult to prove that

$$S_r(x,y,z)=(x^r+y^r)z^{2r}+p(x,y)z^{2r-1}+\dots$$

so in the sum indexed by r,s for the coefficient of $x^{\frac{g}{2}-1}y^{\frac{g}{2}-1}z^{2g+2}$, only the denominator of terms with $r=\frac{g}{2}-1$ and $s=\frac{g}{2}-1$ can be divided by $(g-1)^2$, namely the following

$$\frac{z^6}{48}\cdot\frac{r!(x^r+y^r)z^{2r}}{4^r(g-1)!!\cdot 2}\cdot\frac{(x+y)^s z^{2s}}{8^s(g-1)!}$$

where $r=s=\frac{g}{2}-1$, then the lemma follows easily. □

Theorem 4.8. *Let X be a compact Riemann surface of genus $g\geq 2$, then $|Aut(X)|$ can divide $\mathcal{D}_g$.*

Proof. Let p denotes a prime number and $G=\mathrm{Aut}(X)$. Since

$$o(p,|G|)\leq\lfloor\log_p\frac{2pg}{p-1}\rfloor+o(p,2(g-1)),$$

it's sufficient to prove that

$$\lfloor\log_p\frac{2pg}{p-1}\rfloor+o(p,2(g-1))\leq o(p,\mathcal{D}_g)\tag{8}$$

for all prime $p\leq 2g+1$.

If $g\leq p\leq 2g+1$, then from formula (3) and theorem 3.1, we have $\lfloor\log_p\frac{2pg}{p-1}\rfloor\leq 1$, so from theorem 4.3 b., (8) holds in this case.

If $5\leq p\leq g-1$, we need to prove

$$\lfloor\log_p\frac{2pg}{p-1}\rfloor+o(p,g-1)\leq o(p,\mathcal{D}_g).$$

If $p=g-1\geq 5$ is prime, then from $g\geq 6$ and lemma 4.6, we have

$$\lfloor\log_{g-1}\frac{2g(g-1)}{g-2}\rfloor+1\leq 2\leq o(g-1,\mathcal{D}_g).$$

Otherwise if $p\nmid(g-1)$, since $g!\mid\mathcal{D}_g$, so in order to check (8), it's sufficient to prove

$$\lfloor\log_p\frac{2pg}{p-1}\rfloor\leq\lfloor\frac{g}{p}\rfloor$$

which is equivalent to prove for all $k\geq 2$,

$$p^k>\frac{2p(kp-1)}{p-1}$$

$$i.e.\qquad p^k-p^{k-1}-2kp+2>0$$

which is not difficult to check.

If $p \mid (g-1)$ and $5 \le p < g-1$, then it's sufficient to prove

$$\lfloor \log_p \frac{2pg}{p-1} \rfloor + 1 \le \lfloor \frac{g}{p} \rfloor$$

since the higher power of p divisible by $g-1$ will be subtracted by $\lfloor \frac{g}{p^2} \rfloor$, $\lfloor \frac{g}{p^3} \rfloor$, etc. in the right hand side.

Let $g = kp + 1,\ k \ge 2$, we need to prove

$$p^k > \frac{2p(kp+1)}{p-1}$$

$$i.e. \qquad p^k - p^{k-1} - 2kp - 2 > 0.$$

The above inequality holds except in the case $p = 5$ and $g = 11$ which should be treated separately. Since

$$o(5, |G|) \le \lfloor \log_5 \frac{110}{4} \rfloor + 1 = 3$$

and we know from the appendix that $o(5, \mathcal{D}_{11}) = 5$, we finished checking in this case.

Now we consider the remaining two cases, $p = 2$ and $p = 3$. Note that $24^g g! \mid \mathcal{D}_g$.

If $p = 2$, it's sufficient to prove $\log_2 4g \le 3g - 1$.

If $p = 3$, it's sufficient to prove $\log_3 3g \le g$.

Both cases are easy to check. So we conclude the proof of the theorem. □

5. A conjectural numerical property of intersection numbers

It's well-known that the intersection numbers of ψ classes on $\overline{\mathcal{M}}_{g,n}$ in genus zero is given by

$$\langle \tau_{d_1} \cdots \tau_{d_n} \rangle_0 = \binom{n-3}{d_1 \cdots d_n} = \frac{(n-3)!}{d_1! \cdots d_n!}$$

So if $d_1 < d_2$, we have

$$\langle \tau_{d_1} \tau_{d_2} \cdots \tau_{d_n} \rangle_0 \le \langle \tau_{d_1+1} \tau_{d_2-1} \cdots \tau_{d_n} \rangle_0.$$

Now we prove that the same inequality holds in genus 1.

Proposition 5.1. *For $\sum_{i=1}^n d_i = n$ and $d_1 < d_2$, we have*

$$\langle \tau_{d_1} \tau_{d_2} \cdots \tau_{d_n} \rangle_1 \le \langle \tau_{d_1+1} \tau_{d_2-1} \cdots \tau_{d_n} \rangle_1.$$

Proof. We prove the inequality by induction on n. If $n = 2$,

$$\langle \tau_0 \tau_2 \rangle_1 = \langle \tau_1 \tau_1 \rangle_1 = \frac{1}{24}$$

So we assume that the theorem for $n-1$ is proved. We may assume $d_2 - d_1 \ge 2$, otherwise it's trivial. So by the symmetry property of intersection numbers, we may assume without loss of generality that $d_n = 0$ or $d_n = 1$.

If $d_n = 1$ then by dilaton equation

$$\begin{aligned} \langle \tau_{d_1} \tau_{d_2} \cdots \tau_{d_n} \rangle_1 &= (n-1) \langle \tau_{d_1} \tau_{d_2} \cdots \tau_{d_{n-1}} \rangle_1 \\ \langle \tau_{d_1+1} \tau_{d_2-1} \cdots \tau_{d_n} \rangle_1 &= (n-1) \langle \tau_{d_1+1} \tau_{d_2-1} \cdots \tau_{d_{n-1}} \rangle_1. \end{aligned}$$

So $\langle\tau_{d_1}\tau_{d_2}\cdots\tau_{d_n}\rangle_1 \le \langle\tau_{d_1+1}\tau_{d_2-1}\cdots\tau_{d_n}\rangle_1$ holds in this case by induction.

If $d_n = 0$ then by string equation

$$\begin{aligned}\langle\tau_{d_1}\tau_{d_2}\cdots\tau_{d_n}\rangle_1 &= \langle\tau_{d_1-1}\tau_{d_2}\cdots\tau_{d_{n-1}}\rangle_1 + \langle\tau_{d_1}\tau_{d_2-1}\cdots\tau_{d_{n-1}}\rangle_1\\ &\quad+\sum_{i=3}^{n-1}\langle\tau_{d_1}\tau_{d_2}\cdots\tau_{d_i-1}\cdots\tau_{d_{n-1}}\rangle_1\\ \langle\tau_{d_1+1}\tau_{d_2-1}\cdots\tau_{d_n}\rangle_1 &= \langle\tau_{d_1}\tau_{d_2-1}\cdots\tau_{d_{n-1}}\rangle_1 + \langle\tau_{d_1+1}\tau_{d_2-2}\cdots\tau_{d_{n-1}}\rangle_1\\ &\quad+\sum_{i=3}^{n-1}\langle\tau_{d_1+1}\tau_{d_2-1}\cdots\tau_{d_i-1}\cdots\tau_{d_{n-1}}\rangle_1.\end{aligned}$$

So $\langle\tau_{d_1}\tau_{d_2}\cdots\tau_{d_n}\rangle_1 \le \langle\tau_{d_1+1}\tau_{d_2-1}\cdots\tau_{d_n}\rangle_1$ holds again by induction. □

Now we formulate the following conjecture

Conjecture 5.2. For $\sum_{i=1}^n d_i = 3g-3+n$ and $d_1 < d_2$, we have

$$\langle\tau_{d_1}\tau_{d_2}\cdots\tau_{d_n}\rangle_g \le \langle\tau_{d_1+1}\tau_{d_2-1}\cdots\tau_{d_n}\rangle_g.$$

Namely the more evenly $3g-3+n$ be distributed among indices, the larger the intersection numbers.

By the same argument of proposition 5.1, we can see that for each g, it's enough to check only those intersection numbers with $n \le 3g-1$ and $d_3 \ge 2, \ldots, d_n \ge 2$.

We have checked this conjecture to be true for $g \le 16$ with the help of Faber's Maple programm. Moreover, for $n = 2$, we have checked all $g \le 300$ (using the 2-point function); for $n = 3$, we have checked all $g \le 50$ (using Zagier's formula of the 3-point function).

Okounkov [16] has obtained general n-point functions, from which we could get some properties of intersection numbers. For example, for fixed n, the sum of all n-point intersection numbers of ψ's is finite, i.e.,

$$\sum_{g=0}^{\infty}\sum_{\sum d_i=3g-3+n}\langle\tau_{d_1}\cdots\tau_{d_n}\rangle_g < \infty.$$

However, it seems combinatorially complicated to apply the n-point functions to the above conjecture 5.2. We obtained the following partial results.

Lemma 5.3. *For $p \ge 0$ and $q \ge 0$, let $G_{p,q}(x,y)$ denotes the polynomial $(x^3+y^3)^p(x+y)^q$ and let $C(G_{p,q}, x^a y^b)$ denotes the coefficient of $x^a y^b$ in $G_{p,q}(x,y)$. Then if $b \ge a$ and $|b-a| \ge |b-a-6k|$ (i.e. $b-a \ge 3k \ge 0$), we have*

$$C(G_{p,q}, x^a y^b) \le C(G_{p,q}, x^{a+3k} y^{b-3k}) \tag{9}$$

Proof. We prove inequality (9) by induction on (p,q). We assume that (9) holds for (p,q) and prove inequality (9) for $(p+1,q)$ and $(p,q+1)$.

For the case $(p+1,q)$,

$$\begin{aligned}C(G_{p+1,q}, x^a y^b) &= C(G_{p,q}, x^{a-3} y^b) + C(G_{p,q}, x^a y^{b-3})\\ C(G_{p+1,q}, x^{a+3k} y^{b-3k}) &= C(G_{p,q}, x^{a+3k-3} y^{b-3k}) + C(G_{p,q}, x^{a+3k} y^{b-3k-3})\end{aligned}$$

We may assume that $k \ge 1$, otherwise (9) holds trivially. Note also that $b-a \ge 3k$, it's easy to prove the following,

$$|b-a+3| \ge |b-a-6k-3|, \quad i.e. \quad C(G_{p,q}, x^{a-3} y^b) \le C(G_{p,q}, x^{a+3k} y^{b-3k-3})$$

$$|b-a-3| \geq |b-a-6k+3|, \quad i.e. \quad C(G_{p,q}, x^a y^{b-3}) \leq C(G_{p,q}, x^{a+3k-3} y^{b-3k})$$

So we proved $C(G_{p+1,q}, x^a y^b) \leq C(G_{p+1,q}, x^{a+3k} y^{b-3k})$ for $b-a \geq 3k \geq 0$.

The proof for the case of $(p, q+1)$ is similar. So we proved the lemma. □

Proposition 5.4. *If $a \not\equiv 1 \pmod 3$, then for $a+b=3g-1$ and $b-a \geq 3k \geq 0$, we have*

$$\langle \tau_a \tau_b \rangle_g \leq \langle \tau_{a+3k} \tau_{b-3k} \rangle_g.$$

Proof. From the 2-point function

$$\begin{aligned} F(x,y) &= \sum_{s=1}^{g} \frac{s!}{(g-s)! 24^{g-s} (2s+1)! 2^s} (x^3+y^3)^{g-s} (x+y)^{s-1} x^s y^s \\ &\quad + \frac{1}{x+y} \exp(\frac{x^3}{24} + \frac{y^3}{24}). \end{aligned}$$

Use the notation of the above lemma, we have for $a+b=3g-1$,

$$\langle \tau_a \tau_b \rangle_g = \sum_{s=1}^{g} \frac{s!}{(g-s)! 24^{g-s} (2s+1)! 2^s} C(G_{g-s,s-1}, x^{a-s} y^{b-s}) + \frac{(-1)^{\mathrm{mod}(a,3)}}{g! 24^g} \binom{g-1}{\lfloor \frac{a}{3} \rfloor}$$

and

$$\begin{aligned} \langle \tau_{a+3k} \tau_{b-3k} \rangle_g &= \sum_{s=1}^{g} \frac{s!}{(g-s)! 24^{g-s} (2s+1)! 2^s} C(G_{g-s,s-1}, x^{a-s+3k} y^{b-s-3k}) \\ &\quad + \frac{(-1)^{\mathrm{mod}(a+3k,3)}}{g! 24^g} \binom{g-1}{\lfloor \frac{a+3k}{3} \rfloor} \end{aligned}$$

By the above lemma, we need only prove that

$$\left| \frac{g-1}{2} - \lfloor \frac{a}{3} \rfloor \right| \geq \left| \frac{g-1}{2} - \lfloor \frac{a}{3} \rfloor - k \right|$$

Since $a+b=3g-1$ and $b-a \geq 3k \geq 0$, we have $\lfloor \frac{a}{3} \rfloor \leq \lfloor \frac{3g-3k-1}{6} \rfloor \leq \frac{g-1}{2}$,

$$\left(\frac{g-1}{2} - \lfloor \frac{a}{3} \rfloor \right) - \left(\frac{g-1}{2} - \lfloor \frac{a}{3} \rfloor - k \right) = k \geq 0.$$

and

$$\left(\frac{g-1}{2} - \lfloor \frac{a}{3} \rfloor \right) - \left(-\frac{g-1}{2} + \lfloor \frac{a}{3} \rfloor + k \right) = g-1-k-2\lfloor \frac{a}{3} \rfloor \geq 0$$

since $\lfloor \frac{a}{3} \rfloor \leq \lfloor \frac{g-k}{2} - \frac{1}{6} \rfloor \leq \frac{g-k}{2} - \frac{1}{2}$. □

Corollary 5.5. *If $a \not\equiv 0 \pmod 3$, then for $a+b=3g-3$ and $b-a \geq 3k \geq 0$, we have*

$$\int_{\overline{\mathcal{M}}_g} \kappa_a \kappa_b \leq \int_{\overline{\mathcal{M}}_g} \kappa_{a+3k} \kappa_{b-3k}.$$

Proof. We know that

$$\int_{\overline{\mathcal{M}}_g} \kappa_a \kappa_b = \langle \tau_{a+1} \tau_{b+1} \rangle_g - \int_{\overline{\mathcal{M}}_g} \kappa_{3g-3}$$

So the corollary follows from the above proposition. □

Proposition 5.6. *If $a+b=3g$ and $b-a\geq 3k\geq 0$, we have*

$$\langle\tau_0\tau_a\tau_b\rangle_g \leq \langle\tau_0\tau_{a+3k}\tau_{b-3k}\rangle_g.$$

Proof. We have the following formula for the special 3-point function $F(0,x,y)$,

$$\begin{aligned} F(0,x,y) &= (x+y)F(x,y) \\ &= \sum_{s=0}^{g}\frac{s!}{(g-s)!24^{g-s}(2s+1)!2^s}(x^3+y^3)^{g-s}(x+y)^s x^s y^s. \end{aligned}$$

So the proposition follows from lemma 5.3. □

In fact, the above multinomial type numerical property is also shared by general Hodge integrals.

Conjecture 5.7. Let $f(\lambda)$ be a monomial of the form $\lambda_1^{k_1}\ldots\lambda_g^{k_g}$. Then for $\sum_{i=1}^n d_i = 3g-3+n-\sum_{i=1}^g ik_i$ and $d_1<d_2$, we have

$$\int_{\overline{\mathcal{M}}_{g,n}}\psi_1^{d_1}\psi_2^{d_2}\cdots\psi_n^{d_n}f(\lambda)\leq\int_{\overline{\mathcal{M}}_{g,n}}\psi_1^{d_1+1}\psi_2^{d_2-1}\cdots\psi_n^{d_n}f(\lambda)$$

We now give some evidence for conjecture 5.7.

For $f(\lambda)=\lambda_g$, we have the following well-known λ_g theorem proved by Faber and Pandharipande [5],

$$\int_{\overline{\mathcal{M}}_{g,n}}\psi_1^{k_1}\ldots\psi_n^{k_n}\lambda_g=\binom{2g+n-3}{k_1,\ldots,k_n}b_g.$$

So conjecture 5.7 holds in this case.

For $f(\lambda)=\lambda_{g-1}\lambda_g$, the degree 0 Virasoro conjecture for $\mathbb{P}^2$ implies that [8] if $\sum_{i=1}^n d_i = g-2+n$ and $d_i>0$, the following Faber's conjecture holds,

$$\int_{\overline{\mathcal{M}}_{g,n}}\psi_1^{d_1}\ldots\psi_n^{d_n}\lambda_g\lambda_{g-1}=\frac{(2g+n-3)!|B_{2g}|}{2^{2g-1}(2g)!\prod_{i=1}^n(2d_i-1)!!}$$

It's not difficult to prove that conjecture 5.3 holds in this case.

For any $f(\lambda)=\lambda_1^{k_1}\cdots\lambda_g^{k_g}$, where $\sum_{i=1}^g ik_i=3g-3$, we can use the same argument of proposition 5.1 to prove that conjecture 5.7 holds.

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Appendix A. Some values of $\mathcal{D}_g$

$\mathcal{D}_g$	
$g=2$	$2^7 \cdot 3^2 \cdot 5$
$g=3$	$2^{10} \cdot 3^4 \cdot 5 \cdot 7$
$g=4$	$2^{15} \cdot 3^5 \cdot 5^2 \cdot 7$
$g=5$	$2^{18} \cdot 3^6 \cdot 5^2 \cdot 7 \cdot 11$
$g=6$	$2^{22} \cdot 3^8 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13$
$g=7$	$2^{25} \cdot 3^9 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13$
$g=8$	$2^{31} \cdot 3^{10} \cdot 5^4 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17$
$g=9$	$2^{34} \cdot 3^{13} \cdot 5^4 \cdot 7^3 \cdot 11 \cdot 13 \cdot 17 \cdot 19$
$g=10$	$2^{38} \cdot 3^{14} \cdot 5^5 \cdot 7^3 \cdot 11^2 \cdot 13 \cdot 17 \cdot 19$
$g=11$	$2^{41} \cdot 3^{15} \cdot 5^5 \cdot 7^3 \cdot 11^2 \cdot 13 \cdot 17 \cdot 19 \cdot 23$
$g=12$	$2^{46} \cdot 3^{17} \cdot 5^6 \cdot 7^4 \cdot 11^2 \cdot 13^2 \cdot 17 \cdot 19 \cdot 23$
$g=13$	$2^{49} \cdot 3^{18} \cdot 5^6 \cdot 7^4 \cdot 11^2 \cdot 13^2 \cdot 17 \cdot 19 \cdot 23$
$g=14$	$2^{53} \cdot 3^{19} \cdot 5^7 \cdot 7^4 \cdot 11^2 \cdot 13^2 \cdot 17 \cdot 19 \cdot 23 \cdot 29$
$g=15$	$2^{56} \cdot 3^{21} \cdot 5^7 \cdot 7^5 \cdot 11^3 \cdot 13^2 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31$
$g=16$	$2^{63} \cdot 3^{22} \cdot 5^8 \cdot 7^5 \cdot 11^3 \cdot 13^2 \cdot 17^2 \cdot 19 \cdot 23 \cdot 29 \cdot 31$

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