

# The Rational Part of QCD Amplitude III: the Six-Gluon

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July 3, 2006

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## Abstract

The rational part of the 6-gluon one-loop amplitude with scalar circulating in the loop is computed by using the newly developed method for computing the rational part directly from Feynman integrals. We present the analytic results for the two MHV helicity configurations:  $(1^-2^+3^+4^-5^+6^+)$  and  $(1^-2^+3^-4^+5^+6^+)$ , and the two NMHV helicity configurations:  $(1^-2^-3^+4^-5^+6^+)$  and  $(1^-2^+3^-4^+5^-6^+)$ . Combined with the previously computed results for the cut-constructible part, our results are the last missing pieces for the complete partial helicity amplitudes of the 6-gluon one-loop QCD amplitude.

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# 1 Introduction

In two previous papers [1, 2], we developed a method of computing the rational part of the one-loop amplitude directly from the Feynman integral and applied it to compute the rational part of the 5-gluon amplitude. The result agrees with the well-known result of Bern, Dixon and Kosower [3] obtained firstly by using string-inspired method. In this paper we use our method to compute explicitly the rational part of the 6-gluon amplitude in QCD. For all the 8 independent helicity configurations, we computed explicitly all cases except the all plus helicity case, as this is a well-known finite amplitude [4]. For the three helicity configurations which have explicit results published [5, 6, 7, 8], we found complete analytic agreement. For the remaining two MHV cases we compared with result of Berger, Bern, Dixon, Forde and Kosower [9] obtained by using the bootstrap recursive relations. We find complete agreement. The results obtained by Berger, Bern, Dixon, Forde and Kosower in [9] are more general all-multiplicity one-loop MHV amplitudes. Our method shows its power by firstly obtaining also the analytic results for the rational part of the rest 2 non-MHV helicity configurations of the one-loop 6-gluon amplitude. Our method and the bootstrap recursive approach complement each other quite nicely.

The constant effort to calculate higher-point one-loop amplitudes in general and 6-point amplitude in particular lies in the application to the forthcoming experimental program at CERN's Large Hadron Collider (LHC), as there are lots of processes with many particles as final states [10]. We refer the reader to [1] for a discussion and extensive references for the recent efforts in computing the multi-particle one-loop amplitudes and the recent developments inspired by twistor string theory [11, 12]. Here we will concentrate on the 6-gluon one-loop amplitude in QCD.

In principle any one-loop amplitude in QCD can be computed by using Feynman diagrams and Feynman rules. The 4-parton amplitude was computed in this way by computing the many Feynman diagrams [13]. However using this brute-force method to compute even the 5-gluon one-loop amplitude was quite a challenge. Starting from 1987, Bern and Kosower developed the method to compute one-loop amplitude by string theory [14]. The 5-gluon one-loop amplitude was computed firstly in this way by Bern, Dixon and Kosower [3]. It turns out that the lessons learnt from string theory are quite useful. In fact one can forget about string theory and only keeps the

tricks learnt in these lessons [16, 17, 18]. Subsequently other 5-parton amplitudes were later computed by using either standard Feynman diagrammatic technique [19] or supersymmetric decomposition and perturbative unitarity [20].

However the string theory (inspired) method is still not powerful enough and the analytic computation of the 6-gluon one-loop amplitude is quite a challenge. Only for special helicity and special models, some analytic results are known [4, 5, 21].

To make progress, it is a good strategy to decompose the QCD amplitude into simpler ones by using supersymmetric decomposition:

$$A^{QCD} = A^{N=4} - 4A^{N=1 \text{ chiral}} + A^{N=0 \text{ or scalar}}, \quad (1)$$

where  $A^{QCD}$  denotes an amplitude with only a gluon circulating in the loop,  $A^{N=4,1}$  have the full  $N = 4, 1$  multiplets circulating in the loop, and  $A^{N=0}$  has only a complex scalar in the loop.

By using the general properties of the one-loop amplitude, Bern, Dunbar, Dixon and Kosower proved that the supersymmetric amplitudes  $A^{N=4,1}$  are completely determined by 4-dimensional unitarity [15], i.e. the amplitude is completely cut-constructible and the rational part is vanishing (see [1] for more detail explanation). For MHV helicity configurations, explicit results were obtained for  $A^{N=4}$  in [15]. The recent development of using MHV vertices to compute one-loop amplitudes leads to many new results for the cut-constructible part [22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32]. In particular, Bedford, Brandhuber, Spence and Travaglini [22, 26] applied the the MHV vertices to one-loop calculations. Britto, Buchbinder, Cachazo, Feng and Mastrolia [31, 32, 33] developed efficient technique for evaluating the rational coefficients in an expansion of the one-loop amplitude in terms of scalar box, triangle and bubble integrals (the cut-constructible part, see [1] for details). By using their technique, it is much easier to calculate the coefficient of box integrals without doing any integration. Recently, Britto, Feng and Mastrolia completed the computation of the cut-constructible terms for all the 6-gluon helicity amplitudes [33].

In order to complete the QCD calculation for the 6-gluon helicity amplitudes, the remaining challenge is to compute the rational part of the helicity amplitudes with scalar circulating in the loop.

As we reviewed in [1], there are various approaches [34, 35, 36, 37, 6, 7] to compute the rational part. In particular, Bern, Dixon and Kosower [6, 7]

developed the bootstrap recursive approach which has leads to quite general results [38, 8, 9]. In this paper we will use the approach as developed in [1] and apply it to compute the rational part of the 6-gluon one-loop amplitude in QCD (see also [9]) which are the last missing pieces for the complete partial helicity amplitudes of the 6-gluon one-loop QCD amplitude. The usefulness of this approach was checked in [2] by reproducing (the rational part of) the well-known 5-gluon one-loop amplitude of Bern, Dixon and Kosower [3]. See [16] for a general review.

For the helicity configurations  $(1^+2^+3^+4^+5^+6^+)$  and  $(1^-2^+3^+4^+5^+6^+)$ , the cut-constructible parts are zero and the rational parts were known already [4, 5]. These rational parts have also been derived by using the bootstrap recursive method [8]. We also computed the rational part of  $(1^-2^+3^+4^+5^+6^+)$  by using our new method and found the same analytic result.

For the so call “split-helicity” configurations  $(1^-2^-3^+4^+5^+6^+)$  and  $(1^-2^-3^-4^+5^+6^+)$ , the computation of the rational part is a recent achievement [7, 8]. By using our method, we also computed the rational part for these two helicity configurations. The results obtained agree with their results after taking into account the (rational) terms absorbed in  $L_n$ . The proof is done by using Mathematica by expressing all spinor products in terms of 12 independent spinor products after explicitly solving the equations from momenta conservation. We will not give any details in this paper.

A general strategy for computing the remaining 4 helicity configuration was outlined in [6]. Explicit results for the all-multiplicity MHV one-loop amplitudes, including the remaining two MHV helicity configurations of the 6 gluons  $(1^-2^+3^-4^+5^+6^+)$  and  $(1^-2^+3^+4^-5^+6^+)$ , were given in [9]. Independently we also obtained the explicit results of the rational parts for these two MHV helicity configurations of the 6 gluons, see Sects. 4 and 5.

Due to the appearance of the 3 mass triangle integrals in the remaining 2 non-MHV helicity configurations  $(1^-2^-3^+4^-5^+6^+)$  and  $(1^-2^+3^-4^+5^-6^+)$ , the analytic results are much more complicated. The results can be actually presented in a compact form by exploiting the symmetry of the amplitude. As we showed in [1], the 2-mass hard box rational part can be expressed in terms of 2-mass and 3-mass triangle integrals. If we write the final results in terms of the 2-mass and 3-mass triangle integrals, the analytic result is actually not too complicated and would be well suited for inputs into any program for computing physically interesting quantities. One of main goal of this paper is to obtain explicit analytic results for the rational parts of

these two non-MHV helicity configurations. They are the last missing pieces for the complete partial helicity amplitudes of the 6-gluon one-loop QCD amplitude. Although our computation was done by hand, we have also input our analytic results into Mathematica codes. These codes would be useful for others to check our results and as inputs to compute physically interesting quantities.

This paper is organized as follows: in the next section we recall briefly the Feynman diagrams and the Feynman rules, tailored for the purpose of computing the rational part of the 6-gluon helicity amplitude. A brief review of tensor reduction was given in Sect. 3. The rest 4 sections of the paper present results with some intermediate steps. In Sect. 8 we briefly discuss the factorization properties of the one-loop amplitudes and the checks done for the NMHV results. We collect some explicit results for the rational part of Feynman integrals in the Appendix.

## 2 Notation, the Feynman diagrams and the Feynman rules

A word about notation: we use the same notation as given in [1]. We use  $\epsilon_{i(i+1)\dots(i+n)}$  to denote the composite polarization vectors for sewing trees to the loop.

For the purpose of this paper we consider only the Feynman diagrams and Feynman rules for the one-loop gluon amplitude with scalar circulating in the loop. We don't follow the usual convention of differentiating different particles by different kinds of lines because there are only two kinds of particles: gluons and scalars, and scalars only appear in the loop.

For explicit calculation of the one-loop amplitude by the usual Feynman diagram technique, we can first collect all terms with the same loop structure into one entity. Generally a few cyclicly consecutive external lines are joined in tree diagrams and connected to the same point on the loop (sewing trees to loop). We denote the sum of all these contributions by  $P_{i(i+1)\dots(i+m-1)}$  for  $m$  such external lines. For  $m = 1, 2, 3$ , the relevant Feynman diagrams were shown in Fig. 1. Explicitly we have:

$$P_i(p) = (\epsilon_i, p) = (\epsilon_i, p - k_i), \quad (2)$$

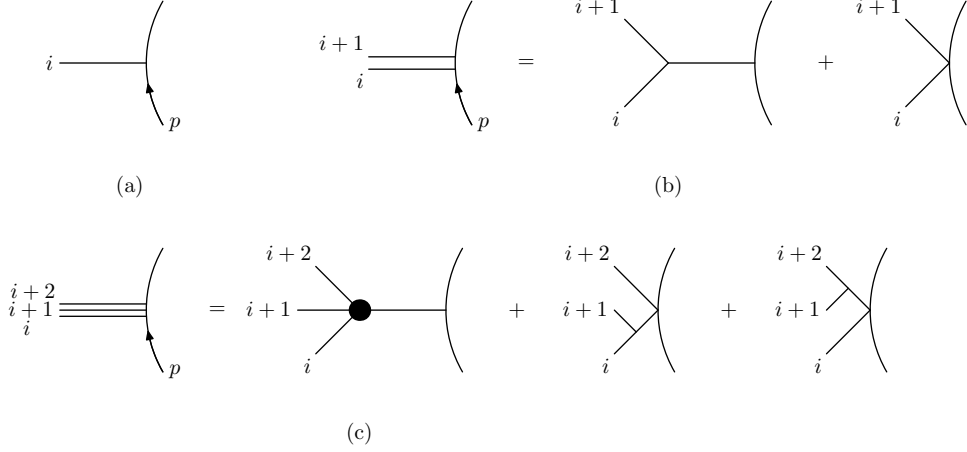


Figure 1: The Feynman rules for sewing trees to loop. The blob denotes an expansion of tree amplitude.

$$P_{i(i+1)}(p) = (\epsilon_{i(i+1)}, p) - \frac{1}{2}(\epsilon_i, \epsilon_{i+1}), \quad (3)$$

$$P_{i(i+1)(i+2)}(p) = (\epsilon_{i(i+1)(i+2)}, p) - \frac{1}{2}((\epsilon_{i(i+1)}, \epsilon_{i+2}) + (\epsilon_i, \epsilon_{(i+1)(i+2)})), \quad (4)$$

where  $\epsilon_{\dots}$ 's are composite polarization vectors introduced in [1]. The computation of these composite polarization vectors is a simplified version of the general recursive calculation of the tree-level  $n$ -gluon amplitudes [39].

Considering all these different tree diagrams just as the same diagram and denoting them by multiple parallel lines attached to the loop, we have only 51 different Feynman diagrams for the 6-gluon one-loop amplitude (with scalars circulating the loop). Some representative diagrams are given in Fig. 2. The counting goes as follows:

- 1 hexagon diagram, the diagram (a);
- 6 pentagon diagrams because there are 6 different ways of combining two consecutive external lines, diagram (b) with  $i = 1, \dots, 6$ ;
- 15 box diagrams which are further divided into 6 two mass hard box diagrams (diagram (c)), 3 two mass easy diagrams (diagram (d)) and 6 one mass diagrams (diagram (e));

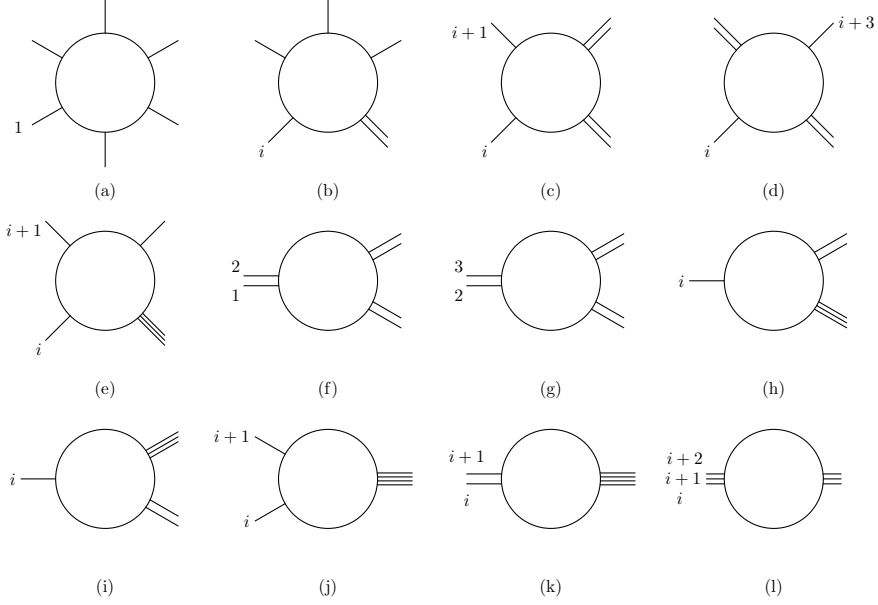


Figure 2: All the possible one-loop Feynman diagrams for six gluons. The index  $i$  can run from 1 to 3 in (d) and (l), and 1 to 6 in the rests if there is an index  $i$ .

- 20 triangle diagrams: 2 three mass triangle diagrams (diagrams (f) and (g)), 6 + 6 2 mass triangle diagrams (diagrams (h) and (i)) and 6 one mass triangle diagrams (diagram (j));
- 9 bubble diagrams (diagrams (k) and (l)).

We use the notation  $I_4^{2mh(i)}$ ,  $I_3^{2m(i)}$  and  $\tilde{I}_3^{2m(i)}$  to denote the rational part of the Feynman integrals from the diagrams (c), (h) and (i). The rational parts from the diagrams (f) and (g) are denoted by  $\tilde{I}_3^{3m}$  and  $I_3^{3m}$  respectively. Explicit results from [1] are collected in the Appendix for quick reference.

It is straightforward to compute the rational part from each diagram. A better way to organize the computation is to use the simple tensor reduction formula derive in [1]. To simplify things further we also organize the result into a summation over the symmetry group. This is possible if the choice of the reference momenta also preserves the symmetry of the (rational part of the) amplitude. The results for different Feynman diagrams connected by symmetry operation can be obtained easily by using symmetry operation

on the rational part directly. In this way we need only to compute the different Feynman diagrams not connected by symmetry operation. This roughly reduced the number of different diagrams (needed to be computed explicitly) by a factor of the rank of the (finite) symmetry group. Even for Feynman diagrams which are invariant under the symmetry operation, it's also possible to split the results into several pieces which are related by symmetry operations.

After we briefly review the tensor reduction formulas used in this paper in the next section, we will then present some intermediate steps and the results for the rational parts of the various (sets of) Feynman diagrams for the 4 different helicity configurations in 4 sections. At the beginning of each section we spell out the symmetry group of the helicity configurations. The final result for the rational part is written mostly as a summation over the symmetry group. We gave only the explicit result for the identity group element. Others are obtained by symmetry action. We will indicate which diagram(s) gave the relevant contributions. We also gave the explicit choice of the polarization vectors for each helicity configuration. Our choice of the polarization vectors preserves the symmetry of the helicity configurations.

### 3 Review of Tensor reduction of the one-loop amplitude

There is a vast literature on this subject [40]. The tensor reduction relations we will use for our calculation of the 5- and 6-gluon amplitudes are quite simple. It was based on the BDK trick [34] of multiplying and dividing by spinor square roots. We purposely made the specific choice of the reference momenta in this paper to make all tensor reduction simple enough to obtain relatively compact analytic results for (the rational part of) the 6-gluon amplitude.

For tensor reduction with only 2 neighboring same helicity external gluons, it is possible to choose the reference momenta to be each other's momenta, i.e.  $\epsilon_1 = \lambda_1 \tilde{\lambda}_2$ ,  $\epsilon_2 = \lambda_2 \tilde{\lambda}_1$ . The tensor reduction is done by considering the contributions from 2 diagrams together and the result formula is shown



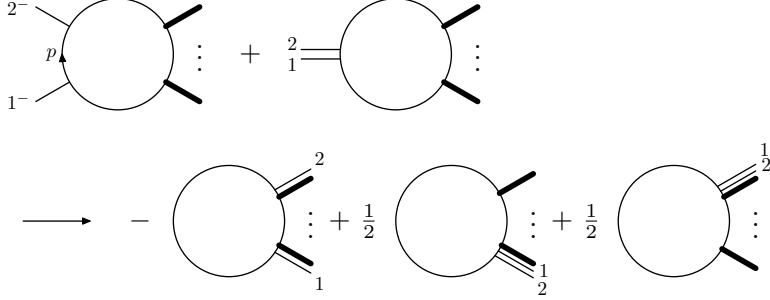


Figure 3: For two adjacent same helicity, the tensor reduction for the combination of two diagrams is even simpler.

pictorially in Fig. 3. The exact algebraic formula is:

$$\begin{aligned} & \frac{(\epsilon_1, p + k_1)(\epsilon_2, p)}{(p + k_1)^2 p^2 (p - k_2)^2} + \frac{(\epsilon_{12}, p + k_1) - (\epsilon_1, \epsilon_2)/2}{(p + k_1)^2 (p - k_2)^2} \\ &= -\frac{1}{p^2} + \frac{1/2}{(p + k_1)^2} + \frac{1/2}{(p - k_2)^2}. \end{aligned} \quad (5)$$

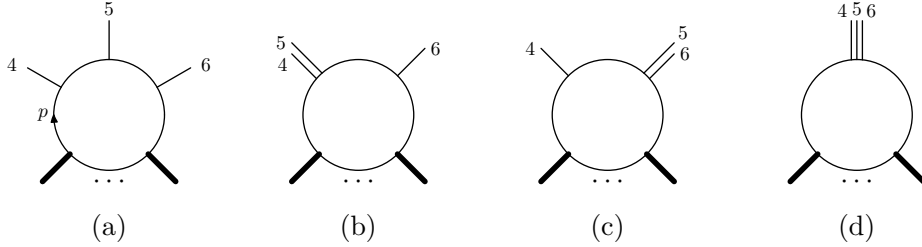


Figure 4: For three adjacent same helicity, the tensor reduction for the combination of these four diagrams is also quite simple.

For three adjacent particles with the same helicity, we choose the following polarization vectors (omitting an overall factor for each polarization vector):

$$\epsilon_4 = \lambda_5 \tilde{\lambda}_4, \quad \epsilon_5 = \eta \tilde{\lambda}_5, \quad \epsilon_6 = \lambda_5 \tilde{\lambda}_6. \quad (6)$$

Then the tensor reduction of the 4 diagrams shown in Fig. 4 is given as follows:

$$A_a + A_b + A_c + A_d = \frac{\langle \eta 5 \rangle}{2 \langle 6 5 \rangle} \frac{P_4(p)}{p^2 (p - k_4)^2}$$

$$+ \frac{\langle \eta 5 \rangle}{2 \langle 4 5 \rangle} \frac{P_6(p - k_{45})}{(p - k_{45})^2 (p - k_{456})^2}. \quad (7)$$

which has nice symmetric property under the flipping operation  $4 \leftrightarrow 6$ .

Sometimes we also need the reduction formulas for  $A_a + A_b$  or  $A_a + A_c$ :

$$\begin{aligned} A_a + A_b &= \frac{P_4(p) P_5(p - k_4)}{p^2 (p - k_4)^2 (p - k_{45})^2} + \frac{P_{45}(p)}{p^2 (p - k_{45})^2} \\ &= -\frac{(\eta \tilde{\lambda}_4, p)}{p^2 (p - k_4)^2} + \frac{\langle \eta 5 \rangle}{2 \langle 4 5 \rangle} \left[ \frac{1}{(p - k_{45})^2} - \frac{1}{p^2} \right], \end{aligned} \quad (8)$$

$$\begin{aligned} A_a + A_c &= \frac{P_5(\tilde{p}) P_6(\tilde{p} - k_5)}{\tilde{p}^2 (\tilde{p} - k_5)^2 (\tilde{p} - k_{56})^2} + \frac{P_{56}(\tilde{p})}{\tilde{p}^2 (\tilde{p} - k_{56})^2} \\ &= \frac{(\eta \tilde{\lambda}_6, \tilde{p} - k_5)}{(\tilde{p} - k_5)^2 (\tilde{p} - k_{56})^2} + \frac{\langle \eta 5 \rangle}{2 \langle 6 5 \rangle} \left[ \frac{1}{\tilde{p}^2} - \frac{1}{(\tilde{p} - k_{56})^2} \right], \end{aligned} \quad (9)$$

where  $\tilde{p} = p - k_4$ .

## 4 MHV: $A_6(1^- 2^+ 3^+ 4^- 5^+ 6^+)$

For this helicity configuration, the symmetry group is

$$G = \{1, \sigma, \tau, \sigma\tau\},$$

where  $\sigma = (2 \leftrightarrow 6, 3 \leftrightarrow 5)$  and  $\tau = (i \rightarrow i + 3)$  are the two generators of the symmetry group. We have  $\sigma\tau = (1 \leftrightarrow 4, 2 \leftrightarrow 3, 5 \leftrightarrow 6)$ . Under the symmetry action  $\sigma$  (or  $\sigma\tau$ ) the ordering of the external lines is reversed. The reversed ordering can be easily transformed to the standard ordering (clockwise in our convention) by using the symmetric or anti-symmetric property of the composite polarization vectors. From this change of the ordering, there is an overall sign of  $(-)^n$  for  $n$ -point amplitude which matched the same factor from the reversing of the ordering of the color factor.

The polarization vectors for this helicity configuration can be chosen as follows:

$$\epsilon_2 = \frac{\lambda_3 \tilde{\lambda}_2}{\langle 3 2 \rangle}, \quad \epsilon_3 = \frac{\lambda_2 \tilde{\lambda}_3}{\langle 2 3 \rangle}, \quad (10)$$

$$\epsilon_5 = \frac{\lambda_6 \tilde{\lambda}_5}{\langle 6 5 \rangle}, \quad \epsilon_6 = \frac{\lambda_5 \tilde{\lambda}_6}{\langle 5 6 \rangle}. \quad (11)$$

We can leave the reference momenta of  $\epsilon_{1,4}$  arbitrary. To be specific, an explicit symmetric choice is  $\epsilon_1 = \frac{\lambda_1 \tilde{\lambda}_4}{[14]}$  and  $\epsilon_4 = \frac{\lambda_4 \tilde{\lambda}_1}{[41]}$ . To ease the writing, we will omit all the denominators of these polarization vectors and simply use  $\epsilon_2 = \lambda_3 \tilde{\lambda}_2$ , etc. The overall factor will be reinstated in the final result.

By using the above symmetry group  $G$ , the 51 Feynman diagrams are classified into 10 sets. Representatives (or all Feynman diagrams) from each set are shown in Figs. 5 to 14. The diagrams are ordered from higher point diagrams to lower point diagrams because the tensor reduction of higher point diagram often gives pieces which are cancelled by lower point diagrams. Our strategy is to make this (local) cancellation manifest by grouping these diagrams together. We now compute the rational part of each set of diagrams in turn.

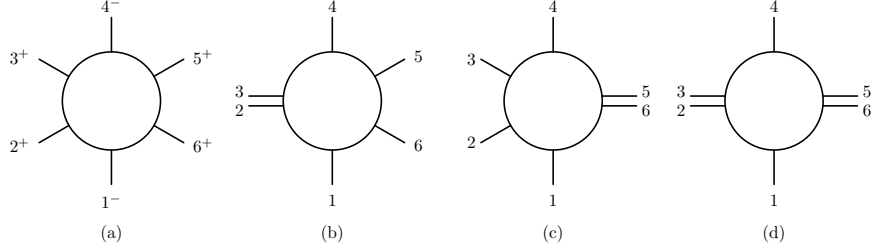


Figure 5: These 4 diagrams are tensor reduced simply with the two pairs of external lines  $k_{2,3}$  and  $k_{5,6}$  with the same helicities.

The 1st set consists of 4 Feynman diagrams as shown in Fig. 5. The rational part can be computed quite easily by using tensor reduction with the two pairs of external lines  $k_{2,3}$  and  $k_{5,6}$  with the same helicities. The Feynman diagrams reduced to only bubble diagrams. The rational part is:

$$\begin{aligned}
R_1 &= -\frac{1}{72} \left( 4(\epsilon_1, k_{612}) (\epsilon_4, k_{345}) + 2 \sum_G (\epsilon_1, k_{12}) (\epsilon_4, k_{12}) \right. \\
&\quad \left. + (\epsilon_1, k_{123}) (\epsilon_4, k_{456}) + (\epsilon_1, k_{561}) (\epsilon_4, k_{234}) \right) \\
&\quad + \frac{1}{36} (2(s_{12} + s_{34} + s_{45} + s_{61}) - 4s_{345} - s_{123} - s_{234}) (\epsilon_1, \epsilon_4). \quad (12)
\end{aligned}$$

The correction terms are cancelled between the 3 diagrams obtained from the 1st step of tensor reduction from hexagon (+ pentagon) to box. By analyzing

the symmetry property of the various terms, we can write this result as a summation over the symmetry group  $G$ . Explicitly we have

$$R_1(1) = -\frac{1}{36} \left( (\epsilon_1, k_2) (\epsilon_4, k_{12}) + \frac{1}{2} (\epsilon_1, k_{612}) (\epsilon_4, k_{345}) \right. \\ \left. + \frac{1}{4} (\epsilon_1, k_{123}) (\epsilon_4, k_{456}) \right) + \frac{1}{72} (4s_{12} - 2s_{345} - s_{123}) (\epsilon_1, \epsilon_4), \quad (13)$$

$$R_1(g) = g(R_1(1)), \quad (14)$$

$$R_1 = \sum_{g \in G} R_1(g). \quad (15)$$

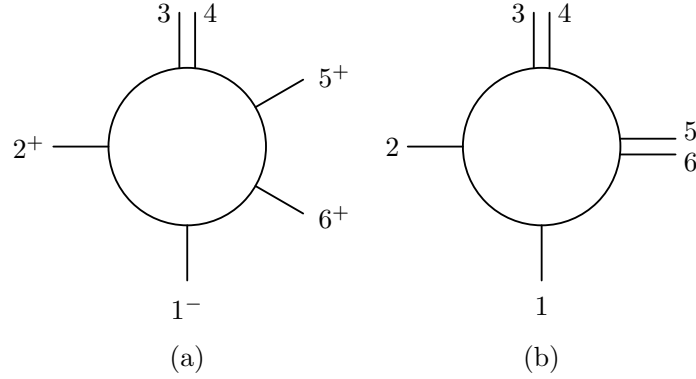


Figure 6: This set has 8 diagrams. The other 6 diagrams are obtained by symmetry operations. The above 2 diagrams can be reduced simply with  $k_{5,6}$ .

The 2nd set consists of 8 diagrams and two representatives are shown in Fig. 6. The rational part from these two diagrams gives  $R_2(1)$  and we have:

$$R_2(1) = \frac{(\epsilon_3, \epsilon_4)(\epsilon_1, k_2)}{8} \left( \frac{2(\epsilon_2, k_{61})}{s_{61} - s_{345}} - \frac{(\epsilon_2, k_{34})}{s_{34} - s_{234}} + \frac{(\epsilon_2, k_1)}{s_{12}} \right) \\ - \frac{1}{18} ((\epsilon_2, \epsilon_{34})\epsilon_1 + (\epsilon_2, \epsilon_1)\epsilon_{34} + (\epsilon_1, \epsilon_{34})\epsilon_2, k_5 - k_6) \\ - \frac{1}{24} ((\epsilon_2, \epsilon_{34})(\epsilon_1, k_2) + (\epsilon_2, \epsilon_1)(\epsilon_{34}, k_2)) \\ + \frac{1}{12} (\epsilon_2, \epsilon_1) (\epsilon_{34}, k_1 + 2k_2)$$

$$\begin{aligned}
& - \frac{s_{61} + s_{345}}{6(s_{61} - s_{345})^2} (\epsilon_2, k_{61})(\epsilon_{34}, k_2)(\epsilon_1, k_2) \\
& - \frac{s_{34} + s_{234}}{12(s_{34} - s_{234})^2} (\epsilon_2, k_{34})(\epsilon_{34}, k_2)(\epsilon_1, k_2) \\
& + \frac{(\epsilon_2, k_{61})}{6(s_{61} - s_{345})} ((\epsilon_{34}, k_2)(\epsilon_1, k_6) - (\epsilon_{34}, k_5)(\epsilon_1, k_2)) \\
& - \frac{(\epsilon_2, k_{34})}{12(s_{34} - s_{234})} (\epsilon_{34}, k_2)(\epsilon_1, k_{56}) \\
& - \frac{s_{61} + s_{345}}{12(s_{61} - s_{345})} ((\epsilon_2, \epsilon_{34})(\epsilon_1, k_2) + (\epsilon_2, \epsilon_1)(\epsilon_{34}, k_2)) \\
& - \frac{s_{34} + s_{234}}{24(s_{34} - s_{234})} ((\epsilon_2, \epsilon_{34})(\epsilon_1, k_2) + (\epsilon_2, \epsilon_1)(\epsilon_{34}, k_2)). \quad (16)
\end{aligned}$$

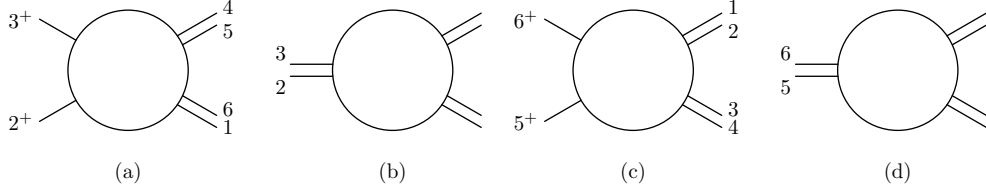


Figure 7: This set has 4 diagrams. The first 2 diagrams and the last 2 are related by the symmetry operation  $\tau$ .

The computation of the rational part from the 4 diagrams shown in Fig. 7 is also easy by using tensor reduction with  $k_{2,3}$  or  $k_{5,6}$ . Here we need a correction term to the naive tensor reduction in  $D = 4$  (see [1] for explicit computation of the correction term). The result is:

$$\begin{aligned}
R_3(1) &= R_3(\sigma\tau) = \frac{1}{36} (\epsilon_{45}, k_3) (\epsilon_{61}, k_2) \\
&+ \frac{1}{36} (2s_{345} - s_{45} - s_{61} - 3s_{23}) (\epsilon_{45}, \epsilon_{61}). \quad (17)
\end{aligned}$$

The tensor reduction for the two diagrams shown in Fig. 8 is also easy with the 2 external momenta  $k_{2,3}$  because of the same adjacent helicities. They are reduced to bubble diagrams. As in the last case with the reduction of the 2 mass hard diagram, we also need a correction term to the naive

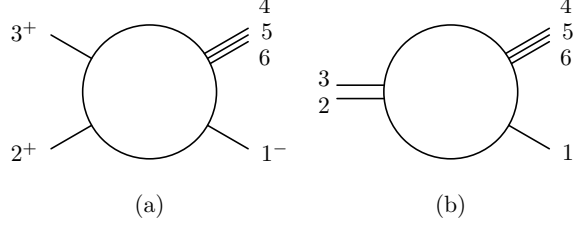


Figure 8: This set has 8 diagrams. The other 6 diagrams are obtained by symmetry operations.

tensor reduction in  $D = 4$ . By including the correction term we have

$$R_4(1) = \frac{1}{18} (\epsilon_1, k_2) (\epsilon_{456}, k_3) + \frac{1}{18} (2s_{12} - s_{123} - 3s_{23}) (\epsilon_1, \epsilon_{456}). \quad (18)$$

In the above we have used the physical condition of the (composite) polarization vectors. The term with the factor  $s_{23}$  comes from the correction term in the tensor reduction from the (one mass) box diagram. The other 6 diagrams (in 3 pairs) give the contributions  $R_4(\sigma)$ ,  $R_4(\tau)$  and  $R_4(\sigma\tau)$ .

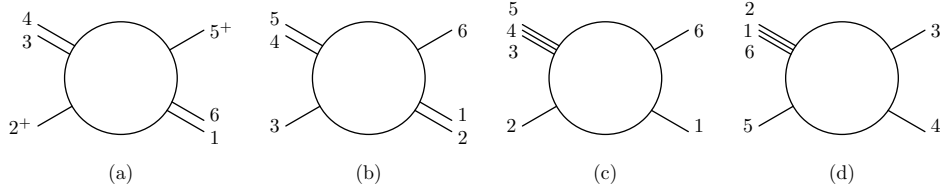


Figure 9: This set has 4 diagrams. (a) and (b) are related by symmetry operation  $\sigma$ , whereas (c) and (d) are related by symmetry operation  $\tau$ .

The 4 diagrams shown in Fig. 9 gave the most complicated algebraic expressions for this helicity configuration. We gave the results in 2 parts:

$$\begin{aligned} R_{5,1}(1) &= \frac{\langle 26 \rangle}{4\langle 25 \rangle} \left[ \frac{(\epsilon_2, k_{34})}{s_{34} - s_{234}} - \frac{(\epsilon_2, k_{61})}{s_{61} - s_{345}} \right] (\epsilon_3, \epsilon_4)(\epsilon_{61}, k_2) \\ &+ \frac{\langle 26 \rangle}{4\langle 25 \rangle} \left[ \frac{(\epsilon_2, k_{34})}{s_{34} - s_{234}} - \frac{(\epsilon_2, k_{61})}{s_{61} - s_{345}} \right] (\epsilon_1, \epsilon_6)(\epsilon_{34}, k_2) \\ &+ \frac{\langle 25 \rangle}{4\langle 26 \rangle} \left[ \frac{(\epsilon_2, k_{61})}{s_{61} - s_{345}} + \frac{(\epsilon_2, k_1)}{s_{12}} \right] (\epsilon_1, k_2)((\epsilon_3, \epsilon_{45}) + (\epsilon_{34}, \epsilon_5)) \end{aligned}$$

$$\begin{aligned}
& + \frac{\langle 26 \rangle}{9\langle 25 \rangle} ((\epsilon_2, \epsilon_{34})\epsilon_{61} + (\epsilon_2, \epsilon_{61})\epsilon_{34} + (\epsilon_{34}, \epsilon_{61})\epsilon_2, k_5) \\
& + \frac{\langle 25 \rangle}{\langle 26 \rangle} \left[ \frac{1}{9} ((\epsilon_2, \epsilon_{345})\epsilon_1 + (\epsilon_2, \epsilon_1)\epsilon_{345} + (\epsilon_{345}, \epsilon_1)\epsilon_2, k_6) \right. \\
& - \left. \frac{1}{12} ((\epsilon_2, \epsilon_{345})\epsilon_1 + (\epsilon_2, \epsilon_1)\epsilon_{345}, k_2) + \frac{1}{6} (\epsilon_1, \epsilon_2) (\epsilon_{345}, k_2 - k_6) \right] \\
& + \frac{\langle 26 \rangle}{\langle 25 \rangle} \left[ \frac{1}{12} \left[ \frac{s_{34} + s_{234}}{s_{34} - s_{234}} + \frac{s_{61} + s_{345}}{s_{61} - s_{345}} \right] ((\epsilon_2, \epsilon_{34})\epsilon_{61} + (\epsilon_2, \epsilon_{61})\epsilon_{34}, k_2) \right. \\
& + (\epsilon_{34}, k_2)(\epsilon_{61}, k_2) \left[ \frac{(s_{34} + s_{234})(\epsilon_2, k_{34})}{6(s_{34} - s_{234})^2} + \frac{(s_{61} + s_{345})(\epsilon_2, k_{61})}{6(s_{61} - s_{345})^2} \right] \\
& + \left. \left[ \frac{(\epsilon_2, k_{34})(\epsilon_{34}, k_2)(\epsilon_{61}, k_5)}{6(s_{34} - s_{234})} + \frac{(\epsilon_2, k_{61})(\epsilon_{34}, k_5)(\epsilon_{61}, k_2)}{6(s_{61} - s_{345})} \right] \right] \\
& - \frac{\langle 25 \rangle}{\langle 26 \rangle} \left[ \frac{s_{61} + s_{345}}{12(s_{61} - s_{345})} ((\epsilon_2, \epsilon_{345})\epsilon_1 + (\epsilon_2, \epsilon_1)\epsilon_{345}, k_2) \right. \\
& + (\epsilon_2, k_{61})(\epsilon_{345}, k_2) \left. \left[ \frac{(s_{61} + s_{345})(\epsilon_1, k_2)}{6(s_{61} - s_{345})^2} - \frac{(\epsilon_1, k_6)}{6(s_{61} - s_{345})} \right] \right], \quad (19)
\end{aligned}$$

and

$$\begin{aligned}
R_{5,2}(1) & = (k_2, k_5) (\epsilon_{34}, \epsilon_{61}) \left[ \frac{5}{18} \frac{\langle 23 \rangle \langle 56 \rangle}{\langle 25 \rangle^2} - \frac{1}{18} \frac{\langle 35 \rangle \langle 26 \rangle}{\langle 25 \rangle^2} \right] \\
& + ((\epsilon_{34}, k_2)(\epsilon_{61}, k_5) + (\epsilon_{34}, k_5)(\epsilon_{61}, k_2)) \\
& \quad \times \left[ -\frac{2}{9} \frac{\langle 23 \rangle \langle 56 \rangle}{\langle 25 \rangle^2} + \frac{1}{36} \frac{\langle 35 \rangle \langle 26 \rangle}{\langle 25 \rangle^2} \right] \\
& + (k_2, k_6)(\epsilon_1, \epsilon_{345}) \left[ -\frac{5}{18} \frac{\langle 23 \rangle \langle 56 \rangle}{\langle 26 \rangle^2} - \frac{1}{18} \frac{\langle 36 \rangle \langle 25 \rangle}{\langle 26 \rangle^2} \right] \\
& + (\epsilon_1, k_2)(\epsilon_{345}, k_6) \left[ \frac{4}{9} \frac{\langle 23 \rangle \langle 56 \rangle}{\langle 26 \rangle^2} + \frac{1}{18} \frac{\langle 36 \rangle \langle 25 \rangle}{\langle 26 \rangle^2} \right] \\
& + (\epsilon_1, k_2)(\epsilon_{345}, k_2) \frac{\langle 23 \rangle \langle 56 \rangle}{4 \langle 26 \rangle^2} + \frac{s_{61} + s_{345}}{s_{61} - s_{345}} \\
& \quad \times \left[ \frac{\langle 23 \rangle \langle 56 \rangle}{4 \langle 25 \rangle^2} (\epsilon_{34}, k_2)(\epsilon_{61}, k_2) + \frac{\langle 23 \rangle \langle 56 \rangle}{4 \langle 26 \rangle^2} (\epsilon_{345}, k_2)(\epsilon_1, k_2) \right] \\
& + \frac{s_{34} + s_{234}}{s_{34} - s_{234}} \frac{\langle 23 \rangle \langle 56 \rangle}{4 \langle 25 \rangle^2} (\epsilon_{34}, k_2)(\epsilon_{61}, k_2). \quad (20)
\end{aligned}$$

The complete rational part is  $R_5(1) = R_{5,1}(1) + R_{5,2}(1)$ .

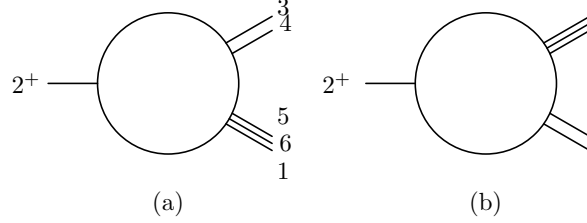


Figure 10: This set has 8 diagrams. Others are obtained from the above 2 diagrams by symmetry operations.

The rational part from the 2 mass triangle diagrams shown in Fig. 10 can be written directly by using eqs. (106) and (107). We have

$$\begin{aligned}
R_{6,1}(1) &= \frac{1}{36} (7(\epsilon_2, \epsilon_{34})(\epsilon_{561}, k_2) - 7(\epsilon_2, \epsilon_{561})(\epsilon_{34}, k_2) + 4(\epsilon_{34}, \epsilon_{561})(\epsilon_2, k_{34})) \\
&- \frac{s_{34} + s_{234}}{12(s_{34} - s_{234})} ((\epsilon_2, \epsilon_{34})(\epsilon_{561}, k_2) + (\epsilon_2, \epsilon_{561})(\epsilon_{34}, k_2)) \\
&- \frac{s_{34} + s_{234}}{6(s_{34} - s_{234})^2} (\epsilon_2, k_{34})(\epsilon_{34}, k_2)(\epsilon_{561}, k_2) \\
&- \frac{1}{4}(\epsilon_3, \epsilon_4) \left[ (\epsilon_2, \epsilon_{561}) + \frac{(\epsilon_2, k_{34})(\epsilon_{561}, k_2)}{s_{34} - s_{234}} \right] \\
&- \frac{1}{4}((\epsilon_5, \epsilon_{61}) + (\epsilon_{56}, \epsilon_1)) \left[ (\epsilon_2, \epsilon_{34}) + \frac{(\epsilon_2, k_{34})(\epsilon_{34}, k_2)}{s_{34} - s_{234}} \right], \quad (21)
\end{aligned}$$

$$\begin{aligned}
R_{6,2}(1) &= \frac{1}{36} (7(\epsilon_2, \epsilon_{345})(\epsilon_{61}, k_2) - 7(\epsilon_2, \epsilon_{61})(\epsilon_{345}, k_2) - 4(\epsilon_{345}, \epsilon_{61})(\epsilon_2, k_{61})) \\
&+ \frac{s_{61} + s_{345}}{12(s_{61} - s_{345})} ((\epsilon_2, \epsilon_{345})(\epsilon_{61}, k_2) + (\epsilon_2, \epsilon_{61})(\epsilon_{345}, k_2)) \\
&+ \frac{s_{61} + s_{345}}{6(s_{61} - s_{345})^2} (\epsilon_2, k_{61})(\epsilon_{345}, k_2)(\epsilon_{61}, k_2) \\
&- \frac{1}{4}(\epsilon_6, \epsilon_1) \left[ (\epsilon_2, \epsilon_{345}) + \frac{(\epsilon_2, k_{61})(\epsilon_{345}, k_2)}{s_{61} - s_{345}} \right] \\
&- \frac{1}{4}((\epsilon_3, \epsilon_{45}) + (\epsilon_{34}, \epsilon_5)) \left[ (\epsilon_2, \epsilon_{61}) + \frac{(\epsilon_2, k_{61})(\epsilon_{61}, k_2)}{s_{61} - s_{345}} \right], \quad (22)
\end{aligned}$$

$$R_6(1) = R_{6,1}(1) + R_{6,2}(1). \quad (23)$$



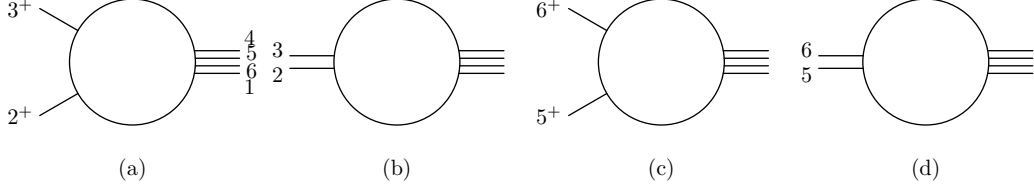


Figure 11: This set also has 4 diagrams. The last 2 are obtained from the first two diagrams by the  $\tau$  symmetry operation.

The diagrams in Fig. 11 can be easily computed either directly or by using the tensor reduction formula with correction terms. We have:

$$\begin{aligned}
 R_7(1) = R_7(\sigma\tau) &= \frac{1}{8} s_{23} ((\epsilon_{45}, \epsilon_{61}) + (\epsilon_4, \epsilon_{561}) + (\epsilon_{456}, \epsilon_1)) \\
 &\quad - \frac{1}{12} s_{23}(\epsilon_{4561}, k_2).
 \end{aligned} \tag{24}$$

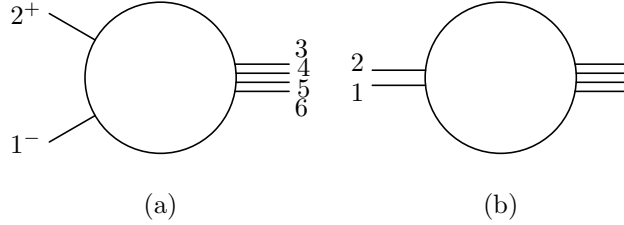


Figure 12: This set has 8 diagrams. The other 6 diagrams are obtained from these two diagrams by symmetry operations. The rational part from diagram (a) cancels the rational part from diagram (b).

The rational parts from the two diagrams in Fig. 12 have very nice property. By using the relation  $(\epsilon_1, k_2)(\epsilon_2, k_1) = s_{12}(\epsilon_1, \epsilon_2)$  for a pair of neighboring lines with opposite helicities, one shows that they cancelled each other and the sum of the rational parts is identically equal to zero, i.e.,

$$R_8(1) = 0. \tag{25}$$

This result is also true for the other helicity configurations as long as the 2 massless external lines in the 1 mass triangle diagram have opposite helicities.

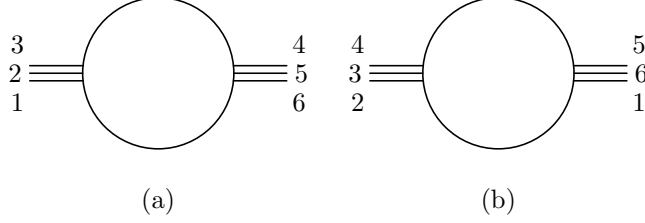


Figure 13: This set has 2 diagrams. The 2 diagrams are related by the  $\sigma$  symmetry operation.

For the diagrams in Fig. 13, we have

$$R_9(1) = R_9(\tau) = -\frac{1}{18} s_{123}(\epsilon_{123}, \epsilon_{456}), \quad (26)$$

$$R_9(\sigma) = R_9(\sigma\tau) = R_9(1)|_{2\leftrightarrow 6, 3\leftrightarrow 5} = -\frac{1}{18} s_{234}(\epsilon_{234}, \epsilon_{561}). \quad (27)$$

In the above, the indicated permutations are applied to the polarization vectors, momenta and kinematic variables.

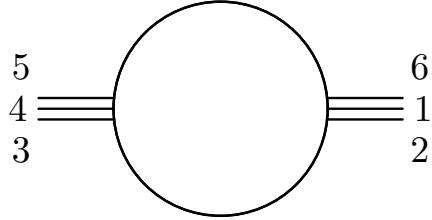


Figure 14: This diagram is the most symmetric one. It leads to a rational part which is invariant under all symmetric operations.

The diagram shown in Fig. 14 is invariant under all symmetry operations. The rational part is:

$$R_{10} = -\frac{1}{9} s_{345}(\epsilon_{345}, \epsilon_{612}). \quad (28)$$

Dividing by the rank of the (finite) symmetry group gives  $R_{10}(g)$ :

$$R_{10}(g) = -\frac{1}{36} s_{345}(\epsilon_{345}, \epsilon_{612}), \quad g = 1, \sigma, \tau, \text{ or } \sigma\tau. \quad (29)$$

Combining all the above results together, the complete rational part is given as follows:

$$R = F \sum_{g \in G} R(g) = F \sum_{g \in G} \sum_{i=1}^{10} R_i(g), \quad (30)$$

where

$$F = -\frac{1}{(\langle 23 \rangle \langle 56 \rangle [14])^2}, \quad (31)$$

is an overall factor from the omitted factors of the polarization vectors. The explicit form of  $R$  is not quite illuminating and we will not write down it here. To be sure that the above result is correct we have chosen another completely different set of reference momenta for the polarization vectors and obtained the same analytic result (in a quite different form of course). To do some consistency checks, let us consider the double pole terms from  $R(1)$ :

$$\begin{aligned} A_2(1) &= \frac{s_{61} + s_{345}}{6(s_{61} - s_{345})^2} \left[ (\epsilon_2, k_{61})(\epsilon_{345}, k_2)(\epsilon_{61}, k_2) - (\epsilon_2, k_{61})(\epsilon_{34}, k_2)(\epsilon_1, k_2) \right. \\ &\quad \left. - \frac{\langle 25 \rangle}{\langle 26 \rangle} (\epsilon_2, k_{61})(\epsilon_{345}, k_2)(\epsilon_1, k_2) + \frac{\langle 26 \rangle}{\langle 25 \rangle} (\epsilon_{34}, k_2)(\epsilon_{61}, k_2)(\epsilon_2, k_{61}) \right] \\ &\quad + \frac{(s_{34} + s_{234})}{6(s_{34} - s_{234})^2} \left[ \frac{\langle 26 \rangle}{\langle 25 \rangle} (\epsilon_{34}, k_2)(\epsilon_{61}, k_2)(\epsilon_2, k_{34}) \right. \\ &\quad \left. - \frac{1}{2} (\epsilon_2, k_{34})(\epsilon_{34}, k_2)(\epsilon_1, k_2) - (\epsilon_2, k_{34})(\epsilon_{34}, k_2)(\epsilon_{561}, k_2) \right]. \quad (32) \end{aligned}$$

We have checked that these terms agree with the result of cut-constructible part [26] ( $T_3$  terms) as required by the absence of spurious poles in the complete amplitude (cut-constructible part + rational part), up to simple pole terms. These residual simple pole terms adding together with the simple pole terms from our explicit computations also agree with the cut-constructible part of [26] ( $T_2$  terms).

In comparing with the notation of Bern, Dixon and Kosower [6],  $R$  is one half of what they called  $R(z)$ . The exact relation is:

$$R = \frac{1}{2} \left( R(z)|_{z=0} + \frac{2}{9} A^{\text{tree}} \right), \quad (33)$$

by stripping away a factor of  $i c_\Gamma$ . By using the Mathematica code provided by them for their result, we have done a numerical check by comparing our

result with the result of Berger, Bern, Dixon, Forde and Kosower [9]. The check is done by randomly assigning a set of complex rational numbers to the 12 independent  $\langle ij \rangle$  and  $[ij]$ . We found exact match without suffering the usual real number approximation because all computations are done with rational numbers.

We note that this numerical check actually gives a proof of the equivalence of our result and the result of [9]. This is because a rational function with a limited number of different denominators and a fixed degree can only depend on a finite number of free parameters. Although we have no exact estimation of the number of these free parameters, the number (over a few thousands) of checks we have done should be quite enough.

## 5 MHV: $A_6(1^-2^+3^-4^+5^+6^+)$

For this helicity configuration, the symmetry group is

$$G = \{1, \sigma\},$$

where  $\sigma = (1 \leftrightarrow 3, 4 \leftrightarrow 6)$ . The polarization vectors are:

$$\epsilon_1 = \frac{\lambda_1 \tilde{\lambda}_2}{[12]}, \quad \epsilon_3 = \frac{\lambda_3 \tilde{\lambda}_2}{[32]}, \quad (34)$$

$$\epsilon_2 = \frac{\lambda_5 \tilde{\lambda}_2}{\langle 52 \rangle}, \quad \epsilon_5 = \frac{\lambda_2 \tilde{\lambda}_5}{\langle 25 \rangle}, \quad (35)$$

$$\epsilon_4 = \frac{\lambda_5 \tilde{\lambda}_4}{\langle 54 \rangle}, \quad \epsilon_6 = \frac{\lambda_5 \tilde{\lambda}_6}{\langle 56 \rangle}. \quad (36)$$

We note that the above choice of reference momenta preserves the symmetry of the amplitude manifestly.

For this helicity configuration, there are some diagrams which are  $G$ -invariant. Even for these diagrams it still simplifies to split the result into 2 parts which are related by the symmetry operation  $\sigma$ . In the following we will present the results of the rational part from the various diagrams in a uniform way such that the complete result is obtained by summing over various individual terms and over the symmetry group, although the symmetry group  $G$  only has rank 2.

The 51 Feynman diagrams are split into 12 sets. We now give the results for the rational part from each set in turn, with some explanations of the tricks we used.

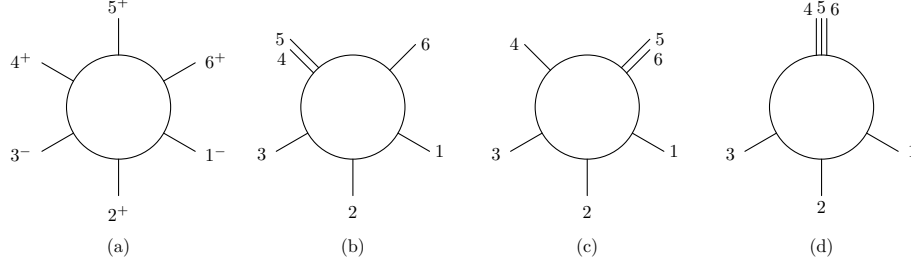


Figure 15: This set has 4 Feynman diagrams. See text for the logic of this combination.

The first set shown in Fig. 15 has 4 Feynman diagrams. This includes the sole 6-point diagram. The logic for considering these four diagrams together is that the tensor reduction gives only two one mass easy box diagrams which can be easily computed following the method elucidated in [2]. The rational part is:

$$\begin{aligned}
R_1(1) &= \frac{\langle 25 \rangle^2 \langle 56 \rangle}{8 \langle 26 \rangle^2 \langle 45 \rangle} \left[ 1 + \frac{s_{12} + s_{345}}{s_{12} - s_{345}} \right] (\epsilon_1, k_6)(\epsilon_3, k_6) \\
&+ \frac{\langle 25 \rangle \langle 56 \rangle}{6 \langle 26 \rangle \langle 45 \rangle} \left( \frac{(\epsilon_1, k_6)}{s_{12} - s_{345}} (\epsilon_3, k_{45})(\epsilon_6, k_{12}) \right. \\
&\quad - \frac{1}{2} \frac{s_{12} + s_{345}}{(s_{12} - s_{345})^2} (\epsilon_3, k_6)(\epsilon_6, k_{12})(\epsilon_1, k_6) \\
&\quad - \frac{1}{4} \frac{s_{12} + s_{345}}{s_{12} - s_{345}} ((\epsilon_3, k_6)(\epsilon_6, \epsilon_1) + (\epsilon_1, k_6)(\epsilon_6, \epsilon_3)) \\
&\quad - \frac{1}{2} (\epsilon_1, \epsilon_6)(\epsilon_3, 2k_1 + k_6) \\
&\quad \left. - \frac{1}{4} ((\epsilon_1, \epsilon_6)(\epsilon_3, k_6) + (\epsilon_3, \epsilon_6)(\epsilon_1, k_6)) \right) \quad (37)
\end{aligned}$$

The 4 diagrams shown in Fig. 16 can be reduced similarly as in the above and the rational part is:

$$R_2(1) = \frac{\langle 25 \rangle}{6 \langle 45 \rangle} \left( \frac{(\epsilon_3, k_{45})}{s_{12} - s_{345}} (\epsilon_6, k_{12})(\epsilon_{12}, k_6) \right)$$

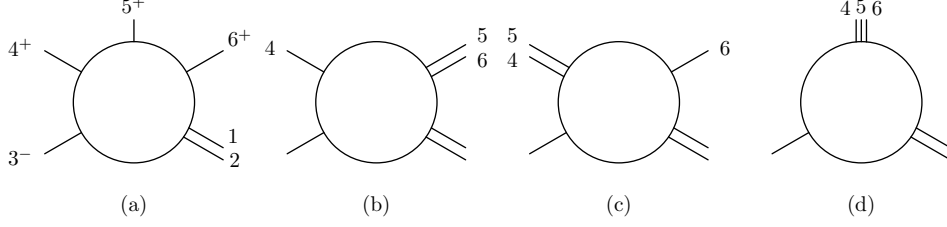


Figure 16: This set has 8 diagrams. The other 4 are obtained by the symmetry operation  $\sigma$ .

$$\begin{aligned}
& -\frac{1}{2} \frac{s_{12} + s_{345}}{(s_{12} - s_{345})^2} (\epsilon_3, k_6)(\epsilon_6, k_{12})(\epsilon_{12}, k_6) \\
& -\frac{1}{4} \frac{s_{12} + s_{345}}{s_{12} - s_{345}} ((\epsilon_6, \epsilon_{12})(\epsilon_3, k_6) + (\epsilon_6, \epsilon_3)(\epsilon_{12}, k_6)) \\
& -\frac{7}{12} ((\epsilon_6, \epsilon_{12})(\epsilon_3, 2k_1 + k_6) + (\epsilon_6, \epsilon_3)(\epsilon_{12}, k_6)). \quad (38)
\end{aligned}$$

by omitting a one mass triangle diagram which will be cancelled by a bubble diagram from the tensor reduction of Fig. 19.

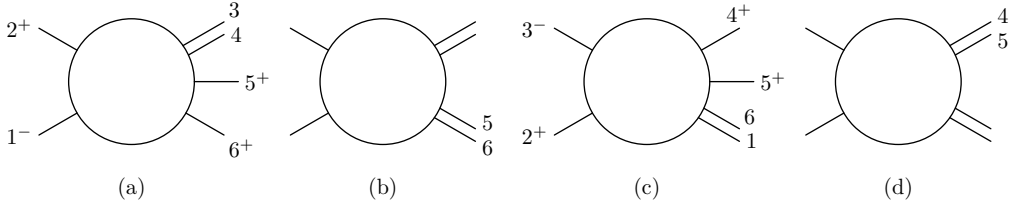


Figure 17: This set has 4 diagrams. The first 2 and the last 2 are related by symmetry operation  $\sigma$ .

The rational part from the diagrams in Fig. 17 is:

$$\begin{aligned}
R_3(1) &= -\frac{\langle 25 \rangle}{\langle 26 \rangle} \left[ -\frac{1}{4} \left( 1 + \frac{s_{12} + s_{345}}{s_{12} - s_{345}} \right) (\epsilon_1, k_6)(\epsilon_{34}, k_6) \right. \\
&\quad \left. + \frac{1}{9} (5(k_2, k_6)(\epsilon_{34}, \epsilon_1) - 4(\epsilon_1, k_6)(\epsilon_{34}, k_2)) \right] \\
&+ \frac{\langle 56 \rangle}{3\langle 26 \rangle} \left[ \frac{(\epsilon_{34}, k_5)}{s_{12} - s_{345}} (\epsilon_1, k_6)(\tilde{\epsilon}_6, k_1) \right]
\end{aligned}$$

$$\begin{aligned}
& -\frac{1}{2} \frac{s_{12} + s_{345}}{(s_{12} - s_{345})^2} (\epsilon_1, k_6) (\epsilon_{34}, k_6) (\tilde{\epsilon}_6, k_1) \\
& -\frac{1}{4} \left( \frac{s_{12} + s_{345}}{s_{12} - s_{345}} + 1 \right) \left( (\epsilon_{34}, k_6) (\tilde{\epsilon}_6, \epsilon_1) + (\epsilon_1, k_6) (\tilde{\epsilon}_6, \epsilon_{34}) \right) \\
& -\frac{1}{2} (\tilde{\epsilon}_6, \epsilon_1) (\epsilon_{34}, k_1 - k_5) - \frac{2}{3} (\tilde{\epsilon}_6, \epsilon_1) (\epsilon_{34}, k_2) \Big] \\
& - \frac{\langle 56 \rangle}{4 \langle 26 \rangle} (\epsilon_3, \epsilon_4) (\tilde{\epsilon}_6, \epsilon_1) \left( 1 + \frac{s_{61}}{s_{12} - s_{345}} \right) \\
& - \frac{\langle 25 \rangle}{12 \langle 45 \rangle} \left[ \frac{2}{3} \left( (\epsilon_2, \epsilon_{61}) (\epsilon_3, k_{61}) + (\epsilon_2, k_{61}) (\epsilon_3, \epsilon_{61}) \right) \right. \\
& \left. + \frac{(\epsilon_3, k_{61})}{s_{61} - s_{345}} (\epsilon_2, k_{61}) (\epsilon_{61}, k_2) \right], \tag{39}
\end{aligned}$$

by also omitting a one mass triangle diagram which will be cancelled by a bubble diagram from the tensor reduction of Fig. 19. Here  $\tilde{\epsilon}_6 = \lambda_2 \tilde{\lambda}_6$ .

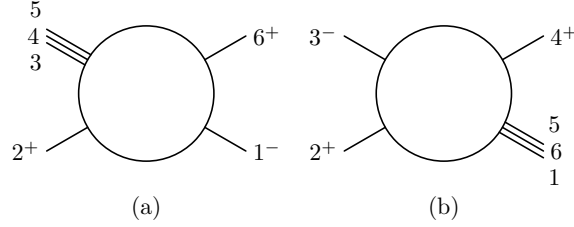


Figure 18: This set has 2 diagrams which are related by the symmetry operation  $\sigma$ .

The 4th set shown in Fig. 18 has 2 diagrams which are related by the symmetry operation  $\sigma$ . The rational part from the first diagram is:

$$\begin{aligned}
R_4(1) &= -\frac{\langle 56 \rangle}{4 \langle 26 \rangle} \left[ \frac{(\epsilon_1, k_6) (\epsilon_6, k_{12})}{s_{12} - s_{345}} + (\epsilon_1, \epsilon_6) \right] \left( (\epsilon_3, \epsilon_{45}) + (\epsilon_{34}, \epsilon_5) \right) \\
&- \frac{\langle 25 \rangle \langle 56 \rangle}{4 \langle 26 \rangle^2} \left[ -\frac{s_{12} + s_{345}}{s_{12} - s_{345}} (\epsilon_1, k_6) (\epsilon_{345}, k_6) \right. \\
&\quad \left. + \frac{8}{3} (k_2, k_6) (\epsilon_1, \epsilon_{345}) - (\epsilon_1, k_6) (\epsilon_{345}, 2k_2 + k_6) \right] \\
&+ \frac{\langle 56 \rangle}{3 \langle 26 \rangle} \left[ -\frac{1}{2} \frac{s_{12} + s_{345}}{(s_{12} - s_{345})^2} (\epsilon_1, k_6) (\epsilon_6, k_{12}) (\epsilon_{345}, k_6) \right]
\end{aligned}$$

$$\begin{aligned}
& -\frac{1}{4} \left( \frac{s_{12} + s_{345}}{s_{12} - s_{345}} + 1 \right) ((\epsilon_6, \epsilon_{345})(\epsilon_1, k_6) + (\epsilon_6, \epsilon_1)(\epsilon_{345}, k_6)) \\
& - \frac{1}{2} (\epsilon_6, \epsilon_1)(\epsilon_{345}, k_1 + 2k_2) + \frac{1}{3} ((\epsilon_1, \epsilon_{345})(\epsilon_6, k_2) + (\epsilon_6, \epsilon_1)(\epsilon_{345}, k_2)) \Big] \\
& + \frac{\langle 25 \rangle}{3 \langle 26 \rangle} \left[ \frac{1}{3} ((\epsilon_1, \epsilon_{345})(\epsilon_2, k_6) + (\epsilon_2, \epsilon_{345})(\epsilon_1, k_6)) \right. \\
& \left. + \frac{1}{2} \frac{(\epsilon_{345}, k_2)}{(s_{61} - s_{345})} (\epsilon_1, k_6)(\epsilon_2, k_{61}) \right]. \tag{40}
\end{aligned}$$

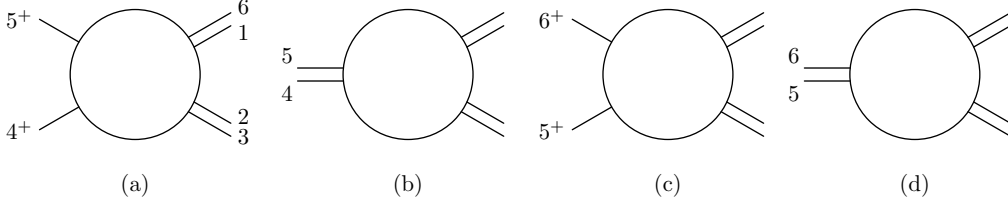


Figure 19: This set has 4 diagrams. The first 2 and the last 2 are related by symmetry operation  $\sigma$ .

The rational part from the diagrams in Fig. 19 is:

$$\begin{aligned}
R_5(1) &= -\frac{1}{6} \langle 25 \rangle [56] (\epsilon_{12}, \epsilon_{34}) - \frac{(\tilde{\epsilon}_6, k_1)(\epsilon_{12}, k_6)}{12(s_{12} - s_{345})} (3(\epsilon_3, \epsilon_4) - 4(\epsilon_{34}, k_5)) \\
& - \frac{1}{6} \frac{s_{12} + s_{345}}{(s_{12} - s_{345})^2} (\tilde{\epsilon}_6, k_1)(\epsilon_{12}, k_6)(\epsilon_{34}, k_6) \\
& - \frac{1}{12} \frac{s_{12} + s_{345}}{s_{12} - s_{345}} ((\tilde{\epsilon}_6, \epsilon_{34})(\epsilon_{12}, k_6) + (\tilde{\epsilon}_6, \epsilon_{12})(\epsilon_{34}, k_6)) \\
& + \frac{1}{36} (-7(\tilde{\epsilon}_6, \epsilon_{34})(\epsilon_{12}, k_6) + 7(\tilde{\epsilon}_6, \epsilon_{12})(\epsilon_{34}, 2k_5 + k_6) \\
& + 4(\tilde{\epsilon}_6, k_1)(\epsilon_{12}, \epsilon_{34}) - 9(\tilde{\epsilon}_6, \epsilon_{12})(\epsilon_3, \epsilon_4)), \tag{41}
\end{aligned}$$

where  $\tilde{\epsilon}_6 = \lambda_2 \tilde{\lambda}_6$  as defined before. As we said before, the bubble diagrams from tensor reduction cancel the rational part from the 1 mass triangle diagrams from tensor reduction in Figs. 16 and 17.

The 4 diagrams shown in Fig. 20 can be reduced by using the two adjacent helicity legs  $k_{4,5}$  or  $k_{5,6}$  and the rational part from the first two diagrams is:

$$R_6(1) = \frac{1}{9} ((\epsilon_1, \epsilon_{234})(\tilde{\epsilon}_6, k_1) - (\tilde{\epsilon}_6, \epsilon_{234})(\epsilon_1, k_6) + \frac{1}{2} (\epsilon_{234}, k_6 - k_1)(\tilde{\epsilon}_6, \epsilon_1))$$



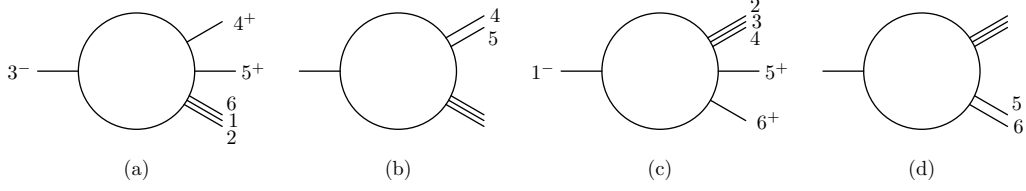


Figure 20: This set has 4 diagrams. The tensor reduction was by using the two adjacent same helicity legs  $k_{4,5}$  for (a) + (b), and  $k_{5,6}$  for (c) + (d).

$$-\frac{\langle 25 \rangle}{18 \langle 45 \rangle} s_{345} (\epsilon_3, \epsilon_{612}) - \frac{1}{6} \langle 25 \rangle [56] (\epsilon_1, \epsilon_{234}). \quad (42)$$

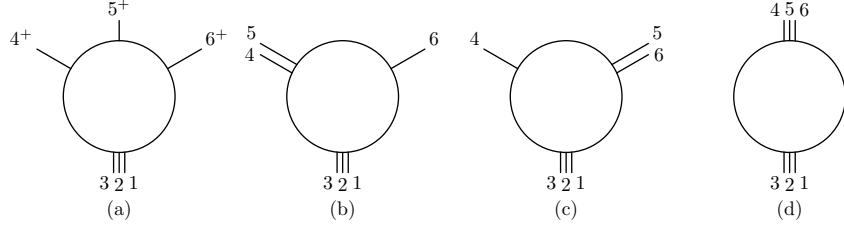


Figure 21: These diagrams have the same upper parts as the diagrams shown in Fig. 15. Here the bottom part has  $k_{1,2,3}$  pinched together.

The 4 diagrams shown in Fig. 21 have the same upper parts as the 4 diagrams shown in Fig. 15 after from the bottom part which have  $k_{1,2,3}$  pinched together. The rational part can be easily computed and we have:

$$R_7(1) = -\frac{1}{6} \langle 25 \rangle [56] (\epsilon_4, \epsilon_{123}) + \frac{1}{12} \langle 25 \rangle [6 | k_{123} \epsilon_{123} | 4]. \quad (43)$$

The diagram shown in Fig. 22 is a two mass easy box diagram which is invariant under the symmetry operation  $\sigma$ . The rational part is:

$$\begin{aligned} R_8(1) &= -\frac{5}{18} (k_2, k_5) (\epsilon_{34}, \epsilon_{61}) - \frac{1}{4} (\epsilon_{34}, k_5) (\epsilon_{61}, k_5) \frac{s_{34} + s_{345}}{s_{34} - s_{345}} \\ &+ \frac{4}{9} (\epsilon_{34}, k_5) (\epsilon_{61}, k_2) - \frac{1}{4} (\epsilon_{34}, k_2) (\epsilon_{61}, k_2) \frac{s_{61} + s_{345}}{s_{61} - s_{345}}. \end{aligned} \quad (44)$$

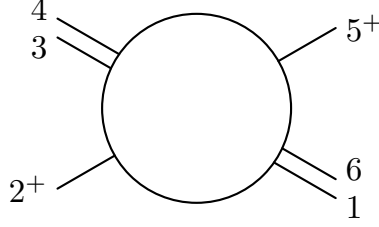


Figure 22: A two mass easy box diagram which is invariant under the symmetry operation  $\sigma$ .

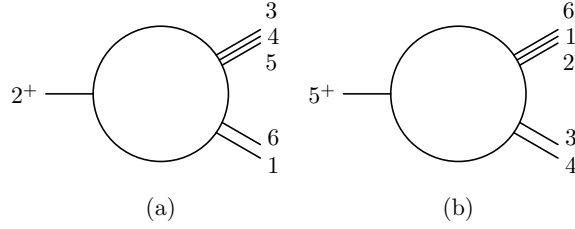


Figure 23: This set has 4 Feynman diagrams. The other 2 diagrams are obtained by symmetry operation  $\sigma$  from the above 2 diagrams.

The rational part from the first diagram in Fig. 23 is:

$$\begin{aligned}
R_{9,1}(1) &= \frac{1}{6} \frac{s_{61} + s_{345}}{(s_{61} - s_{345})^2} (\epsilon_2, k_{61})(\epsilon_{61}, k_2)(\epsilon_{345}, k_2) \\
&+ \frac{1}{12} \frac{s_{61} + s_{345}}{s_{61} - s_{345}} ((\epsilon_2, \epsilon_{345})(\epsilon_{61}, k_2) + (\epsilon_2, \epsilon_{61})(\epsilon_{345}, k_2)) \\
&- \frac{1}{4} \frac{(\epsilon_2, k_{61})}{s_{61} - s_{345}} ((\epsilon_{61}, k_2)((\epsilon_3, \epsilon_{45}) + (\epsilon_{34}, \epsilon_5)) + (\epsilon_6, \epsilon_1)(\epsilon_{345}, k_2)) \\
&- \frac{1}{36} (9((\epsilon_2, \epsilon_{345})(\epsilon_6, \epsilon_1) + (\epsilon_2, \epsilon_{61})(\epsilon_3, \epsilon_{45}) + (\epsilon_2, \epsilon_{61})(\epsilon_{34}, \epsilon_5)) \\
&+ 7(-(\epsilon_2, \epsilon_{345})(\epsilon_{61}, k_2) + (\epsilon_2, \epsilon_{61})(\epsilon_{345}, k_2)) \\
&+ 4(\epsilon_{61}, \epsilon_{345})(\epsilon_2, k_{61})), \tag{45}
\end{aligned}$$

and the rational part from the 2nd diagram in Fig. 23 is:

$$R_{9,2}(1) = \frac{1}{6} \frac{s_{34} + s_{345}}{(s_{34} - s_{345})^2} (\epsilon_5, k_{34})(\epsilon_{34}, k_5)(\epsilon_{612}, k_5)$$

$$\begin{aligned}
& + \frac{1}{12} \frac{s_{34} + s_{345}}{s_{34} - s_{345}} ((\epsilon_5, \epsilon_{612})(\epsilon_{34}, k_5) + (\epsilon_5, \epsilon_{34})(\epsilon_{612}, k_5)) \\
& - \frac{1}{4} \frac{(\epsilon_5, k_{34})}{s_{34} - s_{345}} ((\epsilon_{34}, k_5)((\epsilon_6, \epsilon_{12}) + (\epsilon_{61}, \epsilon_2)) + (\epsilon_3, \epsilon_4)(\epsilon_{612}, k_5)) \\
& - \frac{1}{36} (9((\epsilon_5, \epsilon_{612})(\epsilon_3, \epsilon_4) + (\epsilon_5, \epsilon_{34})((\epsilon_6, \epsilon_{12}) + (\epsilon_{61}, \epsilon_2))) \\
& - 7((\epsilon_5, \epsilon_{612})(\epsilon_{34}, k_5) - (\epsilon_5, \epsilon_{34})(\epsilon_{612}, k_5)) \\
& + 4(\epsilon_{34}, \epsilon_{612})(\epsilon_5, k_{34})). \tag{46}
\end{aligned}$$

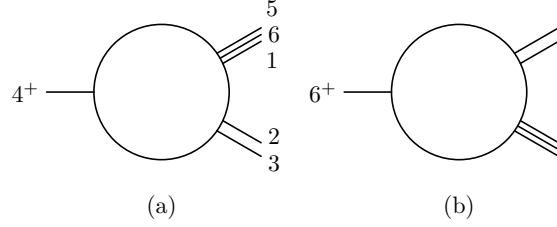


Figure 24: These 2 diagrams are related by the symmetry operation  $\sigma$ .

The rational part from the second diagram in Fig. 24 is:

$$\begin{aligned}
R_{9,3}(1) & = -\frac{1}{6} \frac{s_{12} + s_{345}}{(s_{12} - s_{345})^2} (\epsilon_6, k_{12})(\epsilon_{12}, k_6)(\epsilon_{345}, k_6) \\
& - \frac{1}{12} \frac{s_{12} + s_{345}}{s_{12} - s_{345}} ((\epsilon_6, \epsilon_{345})(\epsilon_{12}, k_6) + (\epsilon_6, \epsilon_{12})(\epsilon_{345}, k_6)) \\
& - \frac{1}{4} \frac{(\epsilon_6, k_{12})(\epsilon_{12}, k_6)}{s_{12} - s_{345}} ((\epsilon_3, \epsilon_{45}) + (\epsilon_{34}, \epsilon_5)) \\
& - \frac{1}{36} (9(\epsilon_6, \epsilon_{12})((\epsilon_3, \epsilon_{45}) + (\epsilon_5, \epsilon_{34})) \\
& + 7((\epsilon_6, \epsilon_{345})(\epsilon_{12}, k_6) - (\epsilon_6, \epsilon_{12})(\epsilon_{345}, k_6)) \\
& - 4(\epsilon_{12}, \epsilon_{345})(\epsilon_6, k_{12})). \tag{47}
\end{aligned}$$

Adding these three results together, we get:

$$R_9(1) = R_{9,1}(1) + R_{9,2}(1) + R_{9,3}(1). \tag{48}$$

The rational part from the first 2 Feynman diagrams is:

$$R_{10}(1) = \frac{1}{6} (\epsilon_5, \epsilon_6)(\epsilon_{1234}, k_5)$$

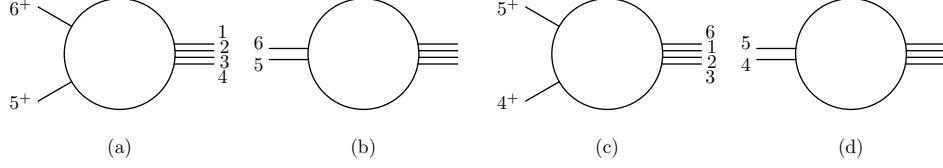


Figure 25: This set has 4 diagrams. Tensor reduction is easy with  $k_{5,6}$  or  $k_{4,5}$ .

$$- \frac{1}{4}(\epsilon_5, \epsilon_6)((\epsilon_1, \epsilon_{234}) + (\epsilon_{12}, \epsilon_{34}) + (\epsilon_{123}, \epsilon_4)). \quad (49)$$

There are other 8 Feynman diagrams which has the same form as given in Fig. 25, but with the 2 massless legs of the 1 mass triangle having opposite helicities. These gave 0 by using the previous result, eq. (25). We will not show these diagrams here.

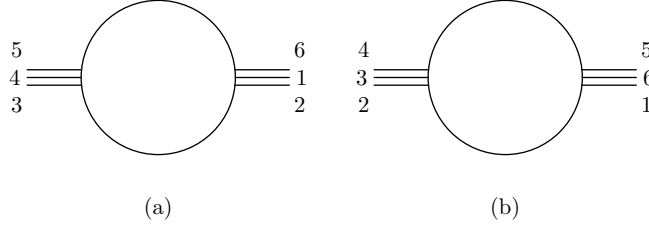


Figure 26: 2 bubble diagrams.

The last 2 diagrams were shown in Fig. 26 and the rational part is:

$$R_{11}(1) = -\frac{1}{9}s_{345}(\epsilon_{345}, \epsilon_{612}). \quad (50)$$

Combining all the above results together, the complete rational part is given as follows:

$$R = F \sum_{g \in G} R(g) = F \sum_{g \in G} \sum_{i=1}^{11} R_i(g), \quad (51)$$

where

$$F = -\frac{1}{[12][23]\langle 25 \rangle^2 \langle 45 \rangle \langle 56 \rangle}, \quad (52)$$

is an overall factor from the omitted factors of the polarization vectors.

The above result was also cross-checked with the explicit results of Berger, Bern, Dixon, Forder and Kosower [9] and we found exact (rational number) agreement.

## 6 NMHV: $A_6(1^-2^-3^+4^-5^+6^+)$

Starting from this section we will compute the rational part for the two non-MHV helicity configurations. These will give much more complicated algebraic expressions if one wrote down the complete expressions. The strategy we used is to use another level of definition and express the result either explicitly or implicitly through the (rational part of the) 2 mass and 3 mass triangle integrals, and the 2 mass hard box integrals. The explicit results in terms of the more elementary quantities, i.e., spinor products and kinematic variables, are collected in an appendix for quick reference.

For the NMHV( $1^-2^-3^+4^-5^+6^+$ ), the symmetry group is  $G = \{1, \sigma\}$  where  $\sigma = \{i \leftrightarrow 7 - i\}$  followed by conjugation. We can choose the following polarization vectors:

$$\epsilon_1 = \frac{\lambda_1 \tilde{\lambda}_2}{[12]}, \quad \epsilon_6 = \frac{\lambda_5 \tilde{\lambda}_6}{\langle 56 \rangle}, \quad (53)$$

$$\epsilon_2 = \frac{\lambda_2 \tilde{\lambda}_1}{[21]}, \quad \epsilon_5 = \frac{\lambda_6 \tilde{\lambda}_5}{\langle 65 \rangle}, \quad (54)$$

$$\epsilon_4 = \frac{\lambda_4 \tilde{\lambda}_2}{[42]}, \quad \epsilon_3 = \frac{\lambda_5 \tilde{\lambda}_3}{\langle 53 \rangle}. \quad (55)$$

We note again that we omitted the overall factors for all polarization vectors.

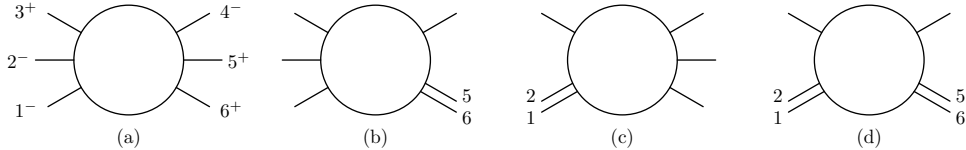


Figure 27: Diagrams (a) and (d) are invariant under the symmetry operation  $\sigma$ . Diagram (b) changed to diagram (c) under the symmetry operation  $\sigma$ .

The first set of diagrams are given in Fig. 27 which includes the only hexagon Feynman diagram. The tensor reduction can be done either with

$k_{5,6}$  or  $k_{1,2}$ , but we can not do tensor reduction with both for all terms. By doing tensor reduction firstly with  $k_{5,6}$ , we have:

$$R_1 = \left( -I^{(1)} I^{(5)} + \frac{1}{2} I^{(5)} I^{(6)} + \frac{1}{2} I^{(1)} I^{(6)} \right) (\epsilon_3, p - k_{12})(\epsilon_4, p - k_{123}) \\ \times \left( (\epsilon_1, p)(\epsilon_2, p - k_1) + I^{(2)} \left( (\epsilon_{12}, p) + \frac{s_{12}}{2} \right) \right), \quad (56)$$

where  $I^{(i)}$  denotes the inverse propagator between  $k_{i-1}$  and  $k_i$ . The last factor can be reduced further with the external momentum  $k_2$  by noting the following relation:

$$(\epsilon_1, p)(\epsilon_2, p - k_1) + I^{(2)} \left( (\epsilon_{12}, p) + \frac{s_{12}}{2} \right) = \frac{I^{(2)}}{2} (k_{12}, p - \frac{k_{12}}{2}) - I^{(3)}(k_1, p). \quad (57)$$

By using these results the rational part can be computed easily. We have:

$$R_1 = I_3^{3m}(k_1, \epsilon_3, \epsilon_4) + (\epsilon_4, k_5) I_3^{3m}(k_1, \epsilon_3) \\ - \frac{1}{2} \left[ \tilde{I}_3^{2m(4)}(\epsilon_4, k_1, \epsilon_3) + (s_{56} - s_{234}) \tilde{I}_3^{2m(4)}(\epsilon_4, \epsilon_3) \right] \\ - \frac{1}{2} \left[ I_3^{2m(3)}(\epsilon_3, \epsilon_4, k_{12}) + s_{12} I_3^{2m(3)}(\epsilon_3, \epsilon_4) \right] \\ + \frac{1}{36} s_{34} (\epsilon_{34}, k_{12}) - \frac{1}{36} ((\epsilon_3, k_2)(\epsilon_4, k_3) - 2s_{23}(\epsilon_3, \epsilon_4)) \\ - \frac{1}{72} ((\epsilon_3, k_{12})(\epsilon_4, k_{56}) + 2s_{123}(\epsilon_3, \epsilon_4)). \quad (58)$$

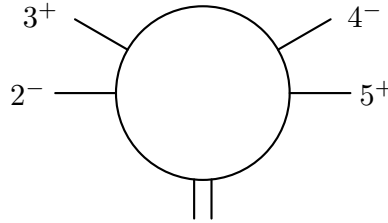


Figure 28: This diagram is invariant under the symmetry operation  $\sigma: 1 \leftrightarrow 6, 2 \leftrightarrow 5, 3 \leftrightarrow 4$  (with conjugation).

The diagram shown in Fig. 28 is invariant under the symmetry operation  $\sigma$ . Its tensor reduction can be done for  $k_2$  and  $k_5$  because of our special

choice of the reference momenta for  $\epsilon_{3,4}$ . We have:

$$\begin{aligned}
R_2 &= -\langle 4\ 3 \rangle [3\ 2] (I^{(5)} - I^{(6)}) I^{(1)}(\lambda_2 \tilde{\lambda}_1, p) (\lambda_6 \tilde{\lambda}_3, p) (\epsilon_{61}, p) \\
&+ \langle 5\ 4 \rangle [4\ 3] (I^{(2)} - I^{(3)}) I^{(1)}(\lambda_4 \tilde{\lambda}_1, p) (\lambda_6 \tilde{\lambda}_5, p) (\epsilon_{61}, p) \\
&+ (I^{(2)}(\lambda_4 \tilde{\lambda}_1, p - k_{12}) - I^{(3)}(\lambda_4 \tilde{\lambda}_1, p - k_1)) \\
&\quad \times (I^{(5)}(\lambda_6 \tilde{\lambda}_3, p + k_6) - I^{(6)}(\lambda_6 \tilde{\lambda}_3, p + k_{56})) I^{(1)} P_{61}(p), \quad (59)
\end{aligned}$$

where

$$P_{61}(p) = (\epsilon_{61}, p + k_6) - \frac{1}{2}(\epsilon_6, \epsilon_1). \quad (60)$$

The rational part is computed by using the formulas for box and triangle integrals and we have:

$$\begin{aligned}
R_2 &= -\langle 4\ 3 \rangle [3\ 2] (I_4^{2mh(2)}(\lambda_2 \tilde{\lambda}_1, \lambda_6 \tilde{\lambda}_3, \epsilon_{61}) - I_4^{1m(2)}(\lambda_2 \tilde{\lambda}_1, \lambda_6 \tilde{\lambda}_3, \epsilon_{61})) \\
&+ \langle 5\ 4 \rangle [4\ 3] (I_4^{1m(3)}(\lambda_4 \tilde{\lambda}_1, \lambda_6 \tilde{\lambda}_5, \epsilon_{61}) - I_4^{2mh(4)}(\lambda_4 \tilde{\lambda}_1, \lambda_6 \tilde{\lambda}_5, \epsilon_{61})) \\
&+ I_3^{2m(3)}(\lambda_6 \tilde{\lambda}_3, \epsilon_{61}, \lambda_4 \tilde{\lambda}_1) - \langle 6 | k_{45} | 3 \rangle I_3^{2m(3)}(\lambda_4 \tilde{\lambda}_1, \epsilon_{61}) \\
&- ((\epsilon_{61}, k_{45}) + \frac{1}{2}(\epsilon_6, \epsilon_1)) I_3^{2m(3)}(\lambda_6 \tilde{\lambda}_3, \lambda_4 \tilde{\lambda}_1) \\
&+ \tilde{I}_3^{2m(4)}(\lambda_4 \tilde{\lambda}_1, \lambda_6 \tilde{\lambda}_3, \epsilon_{61}) + \langle 4 | k_{23} | 1 \rangle \tilde{I}_3^{2m(4)}(\lambda_6 \tilde{\lambda}_3, \epsilon_{61}) \\
&+ ((\epsilon_{61}, k_{23}) - \frac{1}{2}(\epsilon_6, \epsilon_1)) \tilde{I}_3^{2m(4)}(\lambda_6 \tilde{\lambda}_3, \lambda_4 \tilde{\lambda}_1) \\
&- I_3^{3m}(\lambda_4 \tilde{\lambda}_1, \epsilon_{61}, \lambda_6 \tilde{\lambda}_3) - ((\epsilon_{61}, k_{23}) - \frac{1}{2}(\epsilon_6, \epsilon_1)) I_3^{3m}(\lambda_4 \tilde{\lambda}_1, \lambda_6 \tilde{\lambda}_3) \\
&+ \frac{7}{18} s_{34}(\tilde{\epsilon}_{34}, \epsilon_{61}) + \frac{1}{4} \langle 6\ 4 \rangle [1\ 3] (\epsilon_{61}, k_3 - k_4), \quad (61)
\end{aligned}$$

where

$$\begin{aligned}
\tilde{\epsilon}_{34} &= \epsilon_{34} |_{\epsilon_3 \rightarrow \lambda_6 \tilde{\lambda}_3, \epsilon_4 \rightarrow \lambda_4 \tilde{\lambda}_1} \\
&= \frac{\langle 6\ 4 \rangle}{\langle 3\ 4 \rangle} \lambda_4 \tilde{\lambda}_1 - \frac{[1\ 3]}{[4\ 3]} \lambda_6 \tilde{\lambda}_3 + \frac{\langle 6\ 4 \rangle [1\ 3]}{2 \langle 3\ 4 \rangle [4\ 3]} (k_3 - k_4). \quad (62)
\end{aligned}$$

The four Feynman diagrams in Fig. 29 are related by symmetry operation  $\sigma$ . The rational part from diagrams (a) and (b) is:

$$R_3(1) = \frac{1}{2} I_3^{2m(3)}(\epsilon_3, \epsilon_{45}, \epsilon_6) + \frac{1}{4} I_3^{2m(3)}(\epsilon_3, 2(\epsilon_6, k_{12})\epsilon_{45} - (\epsilon_4, \epsilon_5)\epsilon_6)$$

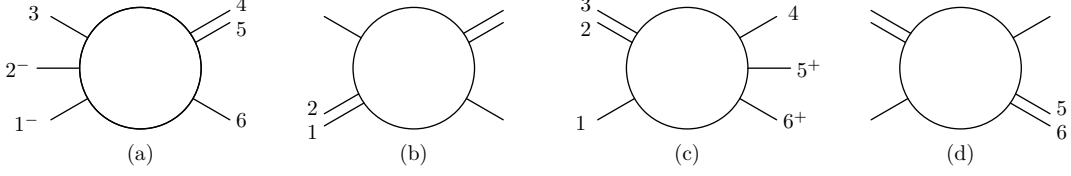


Figure 29: 4 Feynman diagrams which can be reduced easily with  $k_{1,2}$  ((a) and (b)) or  $k_{5,6}$  ((c) and (d)). (c) and (d) are obtained from (a) and (b) by symmetry operation  $\sigma$ .

$$\begin{aligned}
& + \frac{1}{2} \tilde{I}_3^{2m(6)}(\epsilon_6, \epsilon_3, \epsilon_{45}) - \frac{1}{4} \tilde{I}_3^{2m(6)}(\epsilon_6, 2(\epsilon_3, k_{12})\epsilon_{45} + (\epsilon_4, \epsilon_5)\epsilon_3) \\
& - I_3^{3m}(\epsilon_{45}, \epsilon_3, \epsilon_6) + \left( (\epsilon_{45}, k_{23}) + \frac{1}{2}(\epsilon_4, \epsilon_5) \right) I_3^{3m}(\epsilon_3, \epsilon_6). \quad (63)
\end{aligned}$$

The rational part from diagrams (c) and (d) is denoted as  $R_3(\sigma)$ . It is obtained from  $R_3(1)$  by the symmetry operation  $\sigma$ .

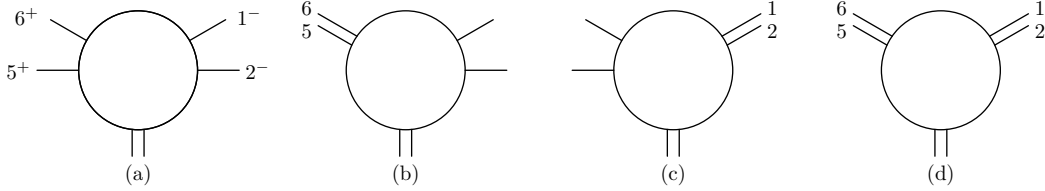


Figure 30: These 4 diagrams have the same structure as those in Fig. 27 with  $k_{3,4}$  replaced by  $k_{34}$ .

For the 4 diagrams shown in Fig. 30, the rational part can be obtained by using the same tensor reduction techniques as for Fig. 27 and the rational part is:

$$\begin{aligned}
R_4 & = \frac{1}{36}((\epsilon_{34}, k_2)(k_1, k_{56}) - (\epsilon_{34}, k_5)(k_6, k_{12})) \\
& + \frac{1}{18}((\epsilon_{34}, k_1)(k_5, k_{61}) - (\epsilon_{34}, k_6)(k_2, k_{61})) \\
& - \frac{1}{72} s_{34}(\epsilon_{34}, k_{12} - k_{56}) - \frac{1}{6}(s_{12}(\epsilon_{34}, \epsilon_{56}) + s_{56}(\epsilon_{34}, \epsilon_{12})). \quad (64)
\end{aligned}$$

The rational part for the 3 two mass hard box diagrams and the one three



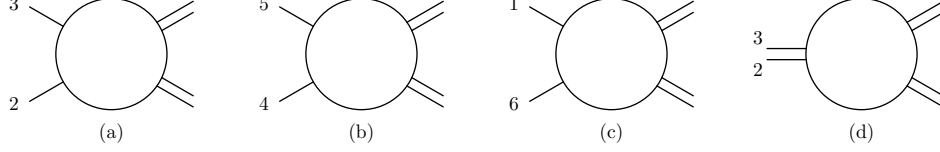


Figure 31: Four diagrams which should be computed individually.

mass triangle diagram is:

$$\begin{aligned}
R_5 &= I_4^{2mh(2)}(\lambda_2 \tilde{\lambda}_1, \lambda_5 \tilde{\lambda}_3, \epsilon_{45}, \epsilon_{61}; -(\epsilon_4, \epsilon_5)/2, -(\epsilon_6, \epsilon_1)/2) \\
&+ I_4^{2mh(4)}(\lambda_4 \tilde{\lambda}_2, \lambda_6 \tilde{\lambda}_5, \epsilon_{61}, \epsilon_{23}; -(\epsilon_6, \epsilon_1)/2, -(\epsilon_2, \epsilon_3)/2) \\
&+ I_4^{2mh(6)}(\lambda_5 \tilde{\lambda}_6, \lambda_1 \tilde{\lambda}_2, \epsilon_{23}, \epsilon_{45}; -(\epsilon_2, \epsilon_3)/2, -(\epsilon_4, \epsilon_5)/2) \\
&+ I_3^{3m}(\epsilon_{23}, \epsilon_{45}, \epsilon_{61}) - \frac{1}{2}(\epsilon_2, \epsilon_3) I_3^{3m}(\epsilon_{45}, \epsilon_{61}) \\
&- \frac{1}{2}(\epsilon_4, \epsilon_5) I_3^{3m}(\epsilon_{61}, \epsilon_{23}) - \frac{1}{2}(\epsilon_6, \epsilon_1) I_3^{3m}(\epsilon_{23}, \epsilon_{45}), \tag{65}
\end{aligned}$$

just by using the definition of  $I_4^{2mh(i)}$  and  $I_3^{3m}$ . See the appendix for explicit formulas.

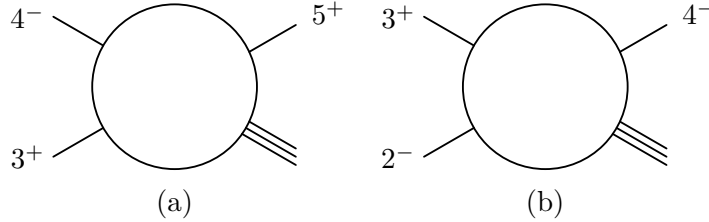


Figure 32: 2 Feynman diagrams. The 2nd can be obtained from the first one by the symmetry operation  $\sigma$ .

The first diagram in Fig. 32 gives the following rational part:

$$\begin{aligned}
R_6(1) &= \frac{\langle 65 \rangle}{\langle 35 \rangle} \left[ \frac{4}{9} ((\epsilon_4, k_3)(\epsilon_{612}, k_5) + (\epsilon_4, k_5)(\epsilon_{612}, k_3)) \right. \\
&- \frac{5}{9} (k_3, k_5)(\epsilon_4, \epsilon_{612}) + \frac{1}{4} ((\epsilon_4, k_3)(\epsilon_{612}, k_3) + (\epsilon_4, k_5)(\epsilon_{612}, k_5)) \\
&\left. + \frac{1}{4} \left[ \frac{s_{45} + s_{345}}{s_{45} - s_{345}} (\epsilon_4, k_3)(\epsilon_{612}, k_3) + \frac{s_{34} + s_{345}}{s_{34} - s_{345}} (\epsilon_4, k_5)(\epsilon_{612}, k_5) \right] \right]
\end{aligned}$$

$$\begin{aligned}
& + \frac{\langle 63 \rangle}{\langle 53 \rangle} \left[ -\frac{1}{9} s_{34}(\epsilon_{34}, \epsilon_{612}) + I_3^{2m(3)}(\epsilon_3, \epsilon_4, \epsilon_{612}) \right. \\
& \quad \left. - \frac{1}{2} ((\epsilon_6, \epsilon_{12}) + (\epsilon_{61}, \epsilon_2)) I_3^{2m(3)}(\epsilon_3, \epsilon_4) \right]. \tag{66}
\end{aligned}$$

The rational part of the other diagram Fig. 32 is  $R_6(\sigma)$  and it is obtained from  $R_6(1)$  by symmetry operation  $\sigma$ .

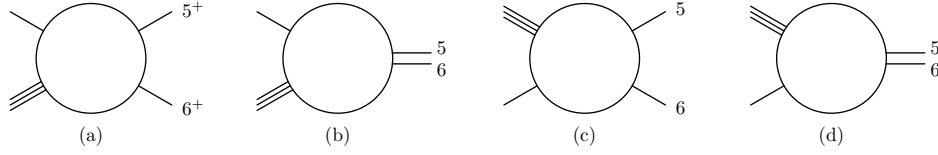


Figure 33: This set has 8 Feynman diagrams. Four more diagrams are obtained from these four diagrams by the symmetry operation  $\sigma$ . The above four diagrams can be reduced easily with  $k_{5,6}$ .

The four diagrams shown in Fig. 33 gave the following rational part:

$$\begin{aligned}
R_7(1) & = \frac{1}{18} (\epsilon_4, k_5) (\epsilon_{123}, k_6) + \frac{1}{18} (2s_{45} - s_{123} - 3s_{56}) (\epsilon_{123}, \epsilon_4) \\
& + \frac{1}{18} (\epsilon_1, k_6) (\epsilon_{234}, k_5) + \frac{1}{18} (2s_{61} - s_{234} - 3s_{56}) (\epsilon_1, \epsilon_{234}). \tag{67}
\end{aligned}$$

$R_7(\sigma)$  is obtained from  $R_7(1)$  by symmetry operation  $\sigma$  (with conjugation).

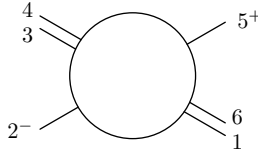


Figure 34: A 2 mass easy box diagram.

The rational part from the diagram shown in Fig. 34 decomposes as  $R_8(1) + R_8(\sigma)$  and we have:

$$R_8(1) = \frac{[12]\langle 65 \rangle}{[52]\langle 25 \rangle} \left[ \frac{5}{18} \langle 2|\epsilon_{34}|5 \rangle \langle 2|\epsilon_{61}|5 \rangle + \frac{\langle 2|k_{34}|5 \rangle^2 (\epsilon_{34}, k_2) (\epsilon_{61}, k_2)}{3 (s_{34} - s_{234}) (s_{61} - s_{345})} \right]$$

$$\begin{aligned}
& + \frac{\langle 6\ 2 \rangle}{\langle 5\ 2 \rangle} \left[ \tilde{I}_3^{2m(2)}(\lambda_2 \tilde{\lambda}_1, \epsilon_{34}, \epsilon_{61}) - \frac{1}{2} \tilde{I}_3^{2m(2)}(\lambda_2 \tilde{\lambda}_1, (\epsilon_6, \epsilon_1)\epsilon_{34} + (\epsilon_3, \epsilon_4)\epsilon_{61}) \right. \\
& \quad \left. - I_3^{2m(2)}(\lambda_2 \tilde{\lambda}_1, \epsilon_{34}, \epsilon_{61}) + \frac{1}{2} I_3^{2m(2)}(\lambda_2 \tilde{\lambda}_1, (\epsilon_6, \epsilon_1)\epsilon_{34} + (\epsilon_3, \epsilon_4)\epsilon_{61}) \right] \\
& - \frac{[1\ 5]\langle 6\ 2 \rangle}{18[2\ 5]\langle 5\ 2 \rangle} ((k_2, k_5)(\epsilon_{34}, \epsilon_{61}) - (\epsilon_{34}, k_2)(\epsilon_{61}, k_5)), \tag{68}
\end{aligned}$$

whereas  $R_8(\sigma)$  is obtained from  $R_8(1)$  by symmetry operation  $\sigma$  (with conjugation).

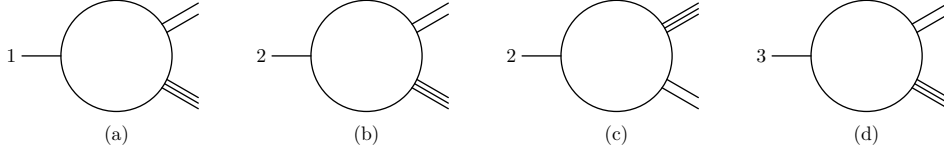


Figure 35: Four 2 mass triangle diagrams which should be computed individually.

The diagrams in Fig. 35 gives the following rational part:

$$\begin{aligned}
R_9(1) & = I_3^{2m(1)}(\epsilon_1, \epsilon_{23}, \epsilon_{456}) + I_3^{2m(3)}(\epsilon_3, \epsilon_{45}, \epsilon_{612}) \\
& + I_3^{2m(2)}(\epsilon_2, \epsilon_{34}, \epsilon_{561}) + \tilde{I}_3^{2m(2)}(\epsilon_2, \epsilon_{345}, \epsilon_{61}) \\
& - \frac{1}{2} I_3^{2m(1)}(\epsilon_1, (\epsilon_2, \epsilon_3)\epsilon_{456} + ((\epsilon_4, \epsilon_{56}) + (\epsilon_{45}, \epsilon_6))\epsilon_{23}) \\
& - \frac{1}{2} I_3^{2m(3)}(\epsilon_3, (\epsilon_4, \epsilon_5)\epsilon_{612} + ((\epsilon_6, \epsilon_{12}) + (\epsilon_{61}, \epsilon_2))\epsilon_{45}) \\
& - \frac{1}{2} I_3^{2m(2)}(\epsilon_2, (\epsilon_3, \epsilon_4)\epsilon_{561} + ((\epsilon_5, \epsilon_{61}) + (\epsilon_{56}, \epsilon_1))\epsilon_{34}) \\
& - \frac{1}{2} \tilde{I}_3^{2m(2)}(\epsilon_2, ((\epsilon_3, \epsilon_{45}) + (\epsilon_{34}, \epsilon_5))\epsilon_{61} + (\epsilon_6, \epsilon_1)\epsilon_{345}). \tag{69}
\end{aligned}$$

The rational part from the first two diagrams in Fig. 36 is:

$$R_{10}(1) = \frac{1}{4} s_{12} ((\epsilon_3, \epsilon_{456}) + (\epsilon_{34}, \epsilon_{56}) + (\epsilon_3, \epsilon_{456})) - \frac{1}{6} s_{12}(\epsilon_{3456}, k_1). \tag{70}$$

The last contribution to the rational part is from the remaining three bubble diagrams shown in Fig. 38 and it is given by:

$$R_{11} = -\frac{1}{9}(s_{123}(\epsilon_{123}, \epsilon_{456}) + s_{234}(\epsilon_{234}, \epsilon_{561}) + s_{345}(\epsilon_{345}, \epsilon_{612})). \tag{71}$$

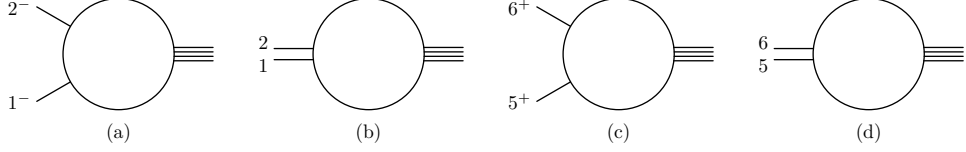


Figure 36: Four 1 mass triangle diagrams which can be computed either directly or by tensor reduction with  $k_{1,2}$  or  $k_{5,6}$ .

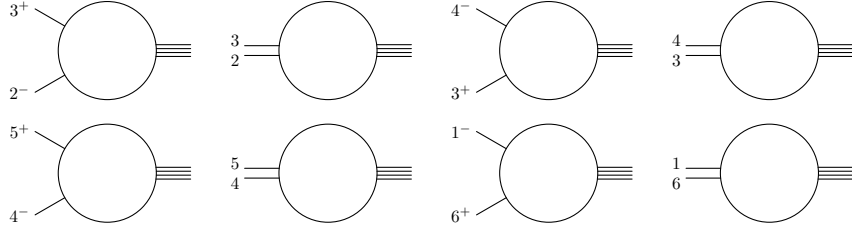


Figure 37: These 8 diagrams gave vanishing contribution to the rational part.

The complete result for the rational part is:

$$R = F \left[ \sum_{i=1,2,4,5,11} R_i + \sum_{i=3,6,7,8,9,10} \sum_g R_i(g) \right], \quad (72)$$

where

$$F = \frac{1}{\langle 35 \rangle [24] ([12] \langle 56 \rangle)^2}, \quad (73)$$

is an overall factor.

## 7 NMHV: $A_6(1^-2^+3^-4^+5^-6^+)$

In this case we chose the following polarization vectors:

$$\epsilon_1 = \frac{\lambda_1 \tilde{\lambda}_5}{[15]}, \quad \epsilon_2 = \frac{\lambda_6 \tilde{\lambda}_2}{\langle 62 \rangle}, \quad (74)$$

$$\epsilon_3 = \frac{\lambda_3 \tilde{\lambda}_1}{[31]}, \quad \epsilon_4 = \frac{\lambda_2 \tilde{\lambda}_4}{\langle 24 \rangle}, \quad (75)$$

$$\epsilon_5 = \frac{\lambda_5 \tilde{\lambda}_3}{[53]}, \quad \epsilon_6 = \frac{\lambda_4 \tilde{\lambda}_6}{\langle 46 \rangle}. \quad (76)$$

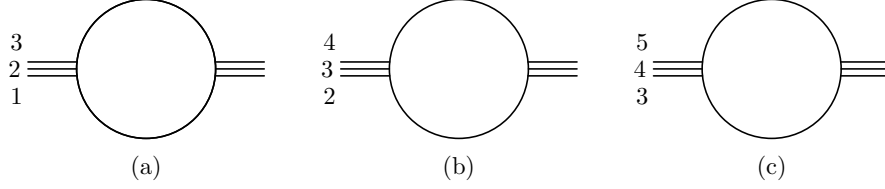


Figure 38: The three remaining bubble diagrams.

This helicity configuration has the biggest symmetry group which is  $Z_6$  generated by  $\tau : i \rightarrow i + 1$  accompanied by conjugation. Because of this symmetry group, we need only to compute a few diagrams from each set and the complete result will be obtained by a summation over the symmetry group. For an invariant diagram, tensor reduction often (partially) destroyed the symmetry and the symmetry is not manifest in the resulting expression of the rational part. In this case it is often simpler to give the complete result for this invariant diagram as a whole or as a summation over only a subgroup of the symmetry group  $Z_6$ .

Let's first start with the diagram (a) in Fig. 2. We can do the tensor reduction with  $k_{1,4}$  and we have

$$\begin{aligned}
R_1 &= -\langle 45 \rangle [56] (I^{(1)} - I^{(2)}) (I^{(3)} - I^{(4)}) (I^{(5)} - I^{(6)}) (\lambda_2 \tilde{\lambda}_4, p) (\lambda_6 \tilde{\lambda}_2, p) \\
&\quad - \langle 32 \rangle [21] (I^{(2)} - I^{(3)}) (I^{(4)} - I^{(5)}) (I^{(6)} - I^{(1)}) (\lambda_5 \tilde{\lambda}_3, p) (\lambda_1 \tilde{\lambda}_5, p) \\
&\quad - I^{(1)} I^{(5)} (\lambda_3 \tilde{\lambda}_5, p - k_1) (\lambda_6 \tilde{\lambda}_2, p - k_{12}) (\lambda_2 \tilde{\lambda}_6, p - k_{123}) (\lambda_5 \tilde{\lambda}_3, p + k_6) \\
&\quad - I^{(2)} I^{(4)} (\lambda_2 \tilde{\lambda}_6, p + k_{56}) (\lambda_5 \tilde{\lambda}_3, p + k_6) (\lambda_3 \tilde{\lambda}_5, p) (\lambda_6 \tilde{\lambda}_2, p - k_{12}) \\
&\quad + I^{(1)} I^{(4)} (\lambda_3 \tilde{\lambda}_5, p - k_1) (\lambda_6 \tilde{\lambda}_2, p - k_{12}) (\lambda_2 \tilde{\lambda}_6, p + k_{56}) (\lambda_5 \tilde{\lambda}_3, p + k_6) \\
&\quad + I^{(2)} I^{(5)} (\lambda_6 \tilde{\lambda}_2, p - k_{12}) (\lambda_2 \tilde{\lambda}_6, p - k_{123}) (\lambda_5 \tilde{\lambda}_3, p + k_6) (\lambda_3 \tilde{\lambda}_5, p). \quad (77)
\end{aligned}$$

We denote the contribution (to the rational part) of each term by  $R_1^{(i)}$ ,  $i = 1, \dots, 6$ . Let's compute the various terms explicitly. We have

$$\begin{aligned}
R_1^{(1)} &= -\langle 45 \rangle [56] \left[ \frac{1}{2} \left[ \frac{\langle 2|k_{56}|4\rangle}{s_{56} - s_{456}} - \frac{\langle 2|k_{23}|4\rangle}{s_{23} - s_{234}} \right] (\lambda_6 \tilde{\lambda}_2, k_4) \right. \\
&\quad + I_3^{3m}(\lambda_2 \tilde{\lambda}_4, \lambda_6 \tilde{\lambda}_2) - \tilde{I}_3^{3m}(\lambda_2 \tilde{\lambda}_4, \lambda_6 \tilde{\lambda}_2) \\
&\quad \left. + \frac{1}{2} \left[ \frac{\langle 6|k_{12}|2\rangle}{s_{12} - s_{345}} - \frac{\langle 6|k_{45}|2\rangle}{s_{45} - s_{123}} \right] (\lambda_2 \tilde{\lambda}_4, k_6) \right]. \quad (78)
\end{aligned}$$

$R_1^{(2)}$  can be obtained from  $R_1^{(1)}$  by the symmetry operation  $\tau^3$ , i.e.:

$$R_1^{(2)} = (R_1^{(1)}|_{i \rightarrow i+3})^*, \quad (79)$$

where  $*$  denotes conjugation:  $\langle \rangle \leftrightarrow [ ]$ .

The contribution from the 3rd term in eq. (77) is:

$$\begin{aligned} R_1^{(3)} &= -\frac{1}{6} (k_2, k_6)(k_3, k_5) - \langle 3|k_2|5 \rangle I_3^{2m(3)}(\lambda_5 \tilde{\lambda}_3, k_6) \\ &+ \langle 5|k_4|3 \rangle I_3^{2m(3)}(\lambda_3 \tilde{\lambda}_5, k_6) - I_3^{2m(3)}(\lambda_3 \tilde{\lambda}_5, \lambda_5 \tilde{\lambda}_3, k_6) \\ &+ (k_6, k_{23}) I_3^{3m}(\lambda_3 \tilde{\lambda}_5, \lambda_5 \tilde{\lambda}_3) + I_3^{3m}(\lambda_3 \tilde{\lambda}_5, k_6, \lambda_5 \tilde{\lambda}_3) \\ &+ \langle 2|k_3|6 \rangle \left[ I_3^{3m}(\lambda_6 \tilde{\lambda}_2, k_5) - \frac{1}{2} \left[ \langle 6|k_5|2 \rangle + \frac{\langle 6|k_1|2 \rangle (k_2, k_5)}{s_{61} - s_{345}} \right] \right]. \end{aligned} \quad (80)$$

$R_1^{(4)}$  can be obtained from  $R_1^{(3)}$  by the symmetry operation  $\tau^3$  just as for the term  $R_1^{(2)}$  from  $R_1^{(1)}$ .

The rational parts from the last two terms in eq. (77) are

$$\begin{aligned} R_1^{(5)} &= \frac{1}{2} (k_3, k_5) (k_2, k_6) + \frac{1}{18} (2(k_3, k_6)(k_2, k_5) \\ &\quad - (k_5, k_3)(k_2, k_6) - (k_5, k_6)(k_2, k_3)) \\ &- \frac{1}{2} \left[ \frac{\langle 2|k_{34}k_6k_{61}|2 \rangle}{s_{61} - s_{345}} - \frac{\langle 2|k_{34}k_6k_{34}|2 \rangle}{s_{34} - s_{234}} \right] (k_3, k_2) \\ &- \frac{1}{2} \left[ \frac{\langle 3|k_{61}k_5k_{34}|3 \rangle}{s_{34} - s_{345}} - \frac{\langle 3|k_{61}k_5k_{61}|3 \rangle}{s_{61} - s_{234}} \right] (k_6, k_5), \end{aligned} \quad (81)$$

$$\begin{aligned} R_1^{(6)} &= \frac{1}{2} (k_3, k_5) (k_2, k_6) + \frac{1}{18} (2(k_3, k_6)(k_2, k_5) \\ &\quad - (k_5, k_3)(k_2, k_6) - (k_5, k_6)(k_2, k_3)) \\ &+ \frac{1}{2} \left[ \frac{\langle 2|k_{45}k_6(k_{45} - k_{12})|2 \rangle}{s_{45} - s_{123}} - \frac{\langle 2|(k_{45} - k_{12})k_6k_{12}|2 \rangle}{s_{12} - s_{345}} \right] (k_5, k_6) \\ &+ \frac{1}{2} \left[ \frac{\langle 5|k_{45}k_3(k_{45} - k_{12})|5 \rangle}{s_{45} - s_{345}} - \frac{\langle 5|(k_{45} - k_{12})k_3k_{12}|5 \rangle}{s_{12} - s_{123}} \right] (k_2, k_3). \end{aligned} \quad (82)$$

In summary the rational part from the six-point diagram (a) in Fig. 2 is:

$$R_1 = \sum_{i=1}^6 R_1^{(i)}. \quad (83)$$

It is manifestly invariant only under a subgroup of the symmetry group, because the reduction procedure breaks the symmetry group and keeps only a subgroup manifest. Of course,  $R_1$  must have a hidden symmetry as the final result doesn't depend on how one proceeds with the tensor reduction. We have checked that  $R_1$  do have the full  $Z_6$  symmetry.

For the diagram (b) with  $i = 2$  in Fig. 2, the computation proceeded much as the computation of the 6-point diagram as given in the above. The result is:

$$\begin{aligned}
R_2(1) &= -I_4^{2mh(4)}(\lambda_6 \tilde{\lambda}_4, \lambda_5 \tilde{\lambda}_3, \epsilon_{61}, \lambda_3 \tilde{\lambda}_1; -(\epsilon_6, \epsilon_1)/2, -\langle 3|k_2|1 \rangle) \\
&- \langle 6|k_{23}|4 \rangle I_4^{2mh(4)}(\lambda_5 \tilde{\lambda}_3, \epsilon_{61}, \lambda_3 \tilde{\lambda}_1) \\
&+ \frac{[1\ 3]}{[5\ 3]} \left[ \frac{4}{9} (\langle 6|k_3|4 \rangle (\epsilon_{61}, k_5) + \langle 6|k_5|4 \rangle (\epsilon_{61}, k_3)) \right. \\
&\quad - \frac{5}{9} (k_3, k_5) \langle 6|\epsilon_{61}|4 \rangle + \frac{1}{4} (\langle 6|k_3|4 \rangle (\epsilon_{61}, k_3) + \langle 6|k_5|4 \rangle (\epsilon_{61}, k_5)) \\
&\quad \left. + \frac{1}{4} \left[ \frac{s_{45} + s_{345}}{s_{45} - s_{345}} \langle 6|k_3|4 \rangle (\epsilon_{61}, k_3) + \frac{s_{34} + s_{345}}{s_{34} - s_{345}} \langle 6|k_5|4 \rangle (\epsilon_{61}, k_5) \right] \right] \\
&+ \frac{[1\ 5]}{[3\ 5]} \left[ \frac{[6\ 3][3\ 4]}{2} \left( (\lambda_5 \tilde{\lambda}_3, \epsilon_{61}) - \frac{(\lambda_5 \tilde{\lambda}_3, k_4) (\epsilon_{61}, k_5)}{s_{45}} \right) \right. \\
&+ \left. \frac{1}{9} s_{45} (\tilde{\epsilon}_{45}, \epsilon_{61}) - \tilde{I}_3^{2m(5)}(\lambda_5 \tilde{\lambda}_3, v_4) - \tilde{I}_3^{2m(5)}(\lambda_5 \tilde{\lambda}_3, \epsilon_{61}, \lambda_6 \tilde{\lambda}_4) \right] \\
&- \langle 2|k_3|4 \rangle \left[ \frac{[1\ 4]}{[5\ 4]} (I_3^{3m}(\lambda_6 \tilde{\lambda}_2, \epsilon_{61}) - \tilde{I}_3^{2m(4)}(\lambda_6 \tilde{\lambda}_2, \epsilon_{61})) \right. \\
&+ \left. \frac{[1\ 5]}{[4\ 5]} I_4^{2mh(4)}(\lambda_5 \tilde{\lambda}_4, \lambda_6 \tilde{\lambda}_2, \epsilon_{61}) - I_4^{2me(2)}(\lambda_6 \tilde{\lambda}_2, \lambda_5 \tilde{\lambda}_1, \epsilon_{61}) \right], \quad (84)
\end{aligned}$$

where

$$v_4 = (\lambda_6 \tilde{\lambda}_4, k_3) \epsilon_{61} - \frac{1}{2} (\epsilon_6, \epsilon_1) \lambda_6 \tilde{\lambda}_4, \quad (85)$$

$$\begin{aligned}
\tilde{\epsilon}_{45} &= \epsilon_{45}|_{\epsilon_4 \rightarrow \lambda_6 \tilde{\lambda}_4} = \\
&= \frac{\langle 6\ 5 \rangle}{\langle 4\ 5 \rangle} \lambda_5 \tilde{\lambda}_3 - \frac{[3\ 4]}{[5\ 4]} \lambda_6 \tilde{\lambda}_4 + \frac{\langle 6\ 5 \rangle [3\ 4]}{2 \langle 4\ 5 \rangle [5\ 4]} (k_4 - k_5). \quad (86)
\end{aligned}$$

There are three different kinds of box diagrams shown in Fig. 2. First, the rational part of the 2 mass hard box with two massless external lines  $k_{2,3}$

is:

$$R_3(1) = I_4^{2mh(2)}(\epsilon_2, \epsilon_3, \epsilon_{45}, \epsilon_{61}; -(\epsilon_4, \epsilon_5)/2, -(\epsilon_6, \epsilon_1)/2). \quad (87)$$

The explicit expression in terms of the rational part of triangle integrals and other more primitive quantities is given in the appendix.

The rational part of the two-mass easy box with massless external lines  $k_{2,5}$  is decomposed into two parts  $R_4(1) + R_4(4)$  and we have:

$$\begin{aligned} R_4(1) &= \frac{\langle 6\ 2 \rangle [3\ 5]}{\langle 5\ 2 \rangle [2\ 5]} \left[ \frac{\langle 5|k_{34}|2\rangle^2(\epsilon_{34}, k_2)(\epsilon_{61}, k_2)}{3(s_{34} - s_{234})(s_{61} - s_{345})} + \frac{5}{18} \langle 5|\epsilon_{34}|2\rangle \langle 5|\epsilon_{61}|2\rangle \right] \\ &+ \frac{[3\ 2]}{[5\ 2]} \left[ I_4^{2me(2)}(\epsilon_2, \epsilon_{34}, k_5, \epsilon_{61}) \right. \\ &+ \left. \frac{1}{4} \left[ \frac{(\epsilon_2, k_{34})}{s_{34} - s_{234}} - \frac{(\epsilon_2, k_{61})}{s_{61} - s_{345}} \right] ((\epsilon_3, \epsilon_4)\epsilon_{61} + (\epsilon_6, \epsilon_1)\epsilon_{34}, k_2) \right] \\ &- \frac{\langle 6\ 5 \rangle [3\ 2]}{36 \langle 2\ 5 \rangle [5\ 2]} (2(k_2, k_5)(\epsilon_{34}, \epsilon_{61}) \\ &- ((\epsilon_{34}, k_2)(\epsilon_{61}, k_5) + (\epsilon_{34}, k_5)(\epsilon_{61}, k_2))). \end{aligned} \quad (88)$$

$R_4(4)$  is obtained from  $R_4(1)$  by the symmetry operation  $\tau^3$ . The rational part from one of the one mass box diagram, i.e. diagram (e) with  $i = 2$  in Fig. 2, is:

$$\begin{aligned} R_5(1) &= \frac{\langle 6\ 2 \rangle}{\langle 4\ 2 \rangle} \left[ \frac{4}{9} ((\epsilon_3, k_2)(\epsilon_{561}, k_4) + (\epsilon_3, k_4)(\epsilon_{561}, k_2)) \right. \\ &- \frac{5}{9} (k_2, k_4)(\epsilon_3, \epsilon_{561}) + \frac{1}{4} ((\epsilon_3, k_2)(\epsilon_{561}, k_2) + (\epsilon_3, k_4)(\epsilon_{561}, k_4)) \\ &+ \left. \frac{1}{4} \left[ \frac{s_{34} + s_{234}}{s_{34} - s_{234}} (\epsilon_3, k_2)(\epsilon_{561}, k_2) + \frac{s_{23} + s_{234}}{s_{23} - s_{234}} (\epsilon_3, k_4)(\epsilon_{561}, k_4) \right] \right] \\ &+ \frac{\langle 6\ 4 \rangle}{\langle 2\ 4 \rangle} \left[ \frac{1}{2} ((\epsilon_5, \epsilon_{61}) + (\epsilon_{56}, \epsilon_1)) \tilde{I}_3^{2m(4)}(\epsilon_4, \epsilon_3) \right. \\ &+ \left. \frac{1}{9} s_{34}(\epsilon_{34}, \epsilon_{561}) - \tilde{I}_3^{2m(4)}(\epsilon_4, \epsilon_{561}, \epsilon_3) \right]. \end{aligned} \quad (89)$$

The rational part for the three mass triangle with external momenta  $(k_{23}, k_{45}, k_{61})$ , i.e. diagram (g) in Fig. 2, is:

$$R_0 = I_3^{3m}(\epsilon_{23}, \epsilon_{45}, \epsilon_{61}) - \frac{1}{2}(\epsilon_2, \epsilon_3) I_3^{3m}(\epsilon_{45}, \epsilon_{61})$$



$$- \frac{1}{2} (\epsilon_4, \epsilon_5) I_3^{3m}(\epsilon_{61}, \epsilon_{23}) - \frac{1}{2} (\epsilon_6, \epsilon_1) I_3^{3m}(\epsilon_{23}, \epsilon_{45}). \quad (90)$$

The rational part from the 2 two-mass triangle diagrams, i.e. diagrams (h) and (i) with  $i = 2$  in Fig. 2, is:

$$\begin{aligned} R_6(1) &= I_3^{2m(2)}(\epsilon_2, \epsilon_{34}, \epsilon_{561}) - \frac{1}{2} I_3^{2m(2)}(\epsilon_2, v_6) \\ &+ \tilde{I}_3^{2m(2)}(\epsilon_2, \epsilon_{345}, \epsilon_{61}) - \frac{1}{2} \tilde{I}_3^{2m(2)}(\epsilon_2, \tilde{v}_6) \\ v_6 &= (\epsilon_3, \epsilon_4) \epsilon_{561} + ((\epsilon_5, \epsilon_{61}) + (\epsilon_{56}, \epsilon_1)) \epsilon_{34}, \\ \tilde{v}_6 &= ((\epsilon_3, \epsilon_{45}) + (\epsilon_{34}, \epsilon_5)) \epsilon_{61} + (\epsilon_6, \epsilon_1) \epsilon_{345}. \end{aligned} \quad (91)$$

$$\quad (92)$$

All the one-mass triangle diagrams combined with the corresponding bubble diagrams give vanishing contribution to the rational part, as all two neighboring gluons have opposite helicities for this helicity configuration. The remaining 3 bubble diagrams give the following contribution to the rational part:

$$R_8 = -\frac{1}{9} (s_{123} (\epsilon_{123}, \epsilon_{456}) + s_{234} (\epsilon_{234}, \epsilon_{561}) + s_{345} (\epsilon_{345}, \epsilon_{612})). \quad (93)$$

Adding all the above results with appropriate symmetry operations, the final result for the rational part is:

$$R = F \left[ R_0 + (R_0|_{i \rightarrow i+1})^* + R_1 + R_8 + \sum_{i=2}^6 \sum_{j=1}^6 R_i(j) \right], \quad (94)$$

where  $R_i(j) = \tau^{j-1}(R_i(1))$  and

$$F = \frac{1}{\langle 62 \rangle \langle 24 \rangle \langle 46 \rangle [15] [31] [53]}, \quad (95)$$

is an overall factor.

## 8 Factorization and the NMHV amplitudes

The results obtained in the last 2 sections for the rational parts of the 2 NMHV amplitude are quite complicated analytic expressions. To present

the results in a readable form we have used 3 level definitions. The first level is all scalar products (including all kinematic variables) in terms of spinor products. The second level is the composite polarization vectors  $\epsilon_{i(i+1)\dots}$ . The last level is the rational part of box and triangle integrals in terms of some elementary functions.

If we make a naive expansion of all the level 2 and 3 definitions, the resulting analytic expression is actually quite large. Although our results were derived rigorously from first principle, it would be best if we can make some independent checks for these results, especially to detect some human errors or typos. A trivial but quite useful check is that the rational part should have the correct spinor weight [18].

One stringent check is factorization. The factorization in one-loop gauge theory was discussed in detail by Bern and Chalmers in [43]. The 2-particle factorization properties were discussed in [4, 36]. As two momenta becomes collinear the one-loop amplitudes behave as [4, 15]

$$A_n^{1\text{-loop}} \longrightarrow \sum_{\lambda=\pm} \left( \text{Split}_{-\lambda}^{\text{tree}}(a^{\lambda_a}, b^{\lambda_b}) A_{n-1}^{1\text{-loop}}(\dots (a+b)^\lambda \dots) \right. \\ \left. + \text{Split}_{-\lambda}^{1\text{-loop}}(a^{\lambda_a}, b^{\lambda_b}) A_{n-1}^{\text{tree}}(\dots (a+b)^\lambda \dots) \right), \quad (96)$$

where  $k_a \rightarrow zK$  and  $k_b \rightarrow (1-z)K$ , with  $K = k_a + k_b$ ,  $K^2 = s_{ab} \rightarrow 0$ . Similar relation also holds for multi-particle factorization. The tree and 1-loop splitting amplitudes in massless QCD have been given in [15, 43, 20]. As we have a complete knowledge of the 4- and 5-gluon amplitudes at tree and one-loop level, it is straightforward to derive a set of relations which must be satisfied by the 6-gluon amplitude. These relations provide an extremely stringent check.

Berger, Bern, Dixon, Forde and Kosower have done this check by using their factorization check program and found that the results given in the last two sections in fact pass the check for all channels. The Mathematica codes used by them were input by us by using the formulas given in this paper. The codes are available up request and/or they will be stored somewhere for free use.

## Appendix: Formulas for the rational part of some Feynman integrals

For quick reference we list here the explicit formulas for the rational part of some Feynman integrals. The derivation can be found in [1].

Firstly, for the bubble integral we have:

$$\begin{aligned} I_2(\epsilon_1, \epsilon_2) &= \int \frac{d^D p}{i\pi^{D/2}} \frac{(\epsilon_1, p)(\epsilon_2, p)}{p^2(p+K)^2} \\ &= \frac{1}{18} ((\epsilon_1, K)(\epsilon_2, K) - 2K^2(\epsilon_1, \epsilon_2)). \end{aligned} \quad (97)$$

where  $K$  is the sum of momenta on one side of the bubble diagram.

For three mass triangle integral with external momenta  $\{K_1, K_2, K_3\}$  and arbitrary polarization vectors  $\epsilon_{1,2,3}$  we have

$$\begin{aligned} I_3(\epsilon_1, \epsilon_2, \epsilon_3) &\equiv \int \frac{d^D p}{i\pi^{D/2}} \frac{(\epsilon_1, p)(\epsilon_2, p - K_1)(\epsilon_3, p + K_3)}{p^2(p - K_1)^2(p + K_3)^2}, \\ &= -F_0(s_1, s_2, s_3)((\epsilon_1, K_1)(\epsilon_2, K_1)(\epsilon_3, K_2) \\ &\quad + (\epsilon_1, K_3)(\epsilon_2, K_2)(\epsilon_3, K_2) + (\epsilon_1, K_3)(\epsilon_2, K_1)(\epsilon_3, K_3) \\ &\quad + (\epsilon_1, K_3)(\epsilon_2, K_1)(\epsilon_3, K_2) - (\epsilon_1, K_1)(\epsilon_2, K_2)(\epsilon_3, K_3)) \\ &\quad - \sum_{i=1}^3 (\epsilon_1, K_i)(\epsilon_2, K_i)(\epsilon_3, K_i) F_i(s_1, s_2, s_3) \\ &\quad - \frac{1}{2\Delta} \left( (s_1 - s_2 - s_3)(\epsilon_1, K_1)(\epsilon_2, K_1)(\epsilon_3, K_3) \right. \\ &\quad \left. + (s_2 - s_3 - s_1)(\epsilon_1, K_1)(\epsilon_2, K_2)(\epsilon_3, K_2) \right. \\ &\quad \left. + (s_3 - s_1 - s_2)(\epsilon_1, K_3)(\epsilon_2, K_2)(\epsilon_3, K_3) \right) \\ &\quad + \frac{7}{36} \left( (\epsilon_1, \epsilon_2)(\epsilon_3, K_3 - K_2) + (\epsilon_2, \epsilon_3)(\epsilon_1, K_1 - K_3) \right. \\ &\quad \left. + (\epsilon_3, \epsilon_1)(\epsilon_2, K_2 - K_1) \right) \\ &\quad + \frac{1}{12\Delta} \left( (\epsilon_1, \epsilon_2)(\epsilon_3, K_3 - K_2)s_1(s_2 + s_3 - s_1) \right. \\ &\quad \left. + (\epsilon_2, \epsilon_3)(\epsilon_1, K_1 - K_3)s_2(s_3 + s_1 - s_2) \right. \\ &\quad \left. + (\epsilon_3, \epsilon_1)(\epsilon_2, K_2 - K_1)s_3(s_1 + s_2 - s_3) \right) \\ &\quad + \frac{1}{12\Delta} \left( (\epsilon_1, \epsilon_2)(\epsilon_3, K_1)(s_3 - s_2)(s_2 + s_3 - s_1) \right. \end{aligned}$$

$$\begin{aligned}
& +(\epsilon_2, \epsilon_3) (\epsilon_1, K_2) (s_1 - s_3)(s_3 + s_1 - s_2) \\
& +(\epsilon_3, \epsilon_1) (\epsilon_2, K_3) (s_2 - s_1)(s_1 + s_2 - s_3)), \tag{98}
\end{aligned}$$

where

$$F_0(s_1, s_2, s_3) = \frac{10 s_1 s_2 s_3}{3\Delta^2} + \frac{(s_1 + s_2 + s_3)}{6\Delta}, \tag{99}$$

$$F_1(s_1, s_2, s_3) = \frac{5(s_1 + s_2 - s_3) s_2 s_3}{3\Delta^2} + \frac{(s_1 - s_3)}{3\Delta}, \tag{100}$$

$$F_2(s_1, s_2, s_3) = \frac{5(s_2 + s_3 - s_1) s_3 s_1}{3\Delta^2} + \frac{(s_2 - s_1)}{3\Delta}, \tag{101}$$

$$F_3(s_1, s_2, s_3) = \frac{5(s_3 + s_1 - s_2) s_1 s_2}{3\Delta^2} + \frac{(s_3 - s_2)}{3\Delta}, \tag{102}$$

$$\Delta = s_1^2 + s_2^2 + s_3^2 - 2(s_1 s_2 + s_2 s_3 + s_3 s_1), \tag{103}$$

and  $s_i = K_i^2$ .

For degree 2 polynomial we have:

$$\begin{aligned}
I_3(\epsilon_i, \epsilon_j) & \equiv \int \frac{d^D p}{i\pi^{D/2}} \frac{(\epsilon_i, p) (\epsilon_j, p)}{p^2(p - K_1)^2(p + K_3)^2} \\
& = -\frac{1}{2\Delta} \left( s_1 ((\epsilon_i, K_2) (\epsilon_j, K_3) + (\epsilon_i, K_3) (\epsilon_j, K_2)) \right. \\
& + s_2 ((\epsilon_i, K_3) (\epsilon_j, K_1) + (\epsilon_i, K_1) (\epsilon_j, K_3)) \\
& + s_3 ((\epsilon_i, K_1) (\epsilon_j, K_2) + (\epsilon_i, K_2) (\epsilon_j, K_1)) \left. \right) + \frac{1}{2} (\epsilon_i, \epsilon_j). \tag{104}
\end{aligned}$$

For 2 mass triangle integrals the above formulas simplify greatly. They are:

$$\begin{aligned}
I_3(\epsilon_1, \epsilon_2) & \equiv \int \frac{d^D p}{i\pi^{D/2}} \frac{(\epsilon_1, p) (\epsilon_2, p)}{p^2(p - k_1)^2(p + K_3)^2} \\
& = \frac{1}{2} (\epsilon_1, \epsilon_2) + \frac{(K_2^2 + K_3^2)}{2(K_2^2 - K_3^2)^2} (\epsilon_1, k_1) (\epsilon_2, k_1) \\
& + \frac{((\epsilon_1, K_2) (\epsilon_2, k_1) - (\epsilon_1, k_1) (\epsilon_2, K_3))}{2(K_2^2 - K_3^2)}, \tag{105}
\end{aligned}$$

$$I_3(\epsilon_1, \epsilon_2) = \frac{1}{2} (\epsilon_1, \epsilon_2) + \frac{(\epsilon_1, K_2) (\epsilon_2, k_1)}{2(K_2^2 - K_3^2)}, \quad (\epsilon_1, k_1) = 0, \tag{106}$$

and

$$\begin{aligned}
I_3(\epsilon_i) &\equiv \int \frac{d^D p}{i\pi^{D/2}} \frac{(\epsilon_1, p)(\epsilon_2, p - k_1)(\epsilon_3, p)}{p^2 (p - k_1)^2 (p + K_3)^2} \\
&= \frac{1}{36} \left( (\epsilon_2, 4K_2 - 7k_1)(\epsilon_1, \epsilon_3) - (2 \leftrightarrow 3) + 4(\epsilon_1, K_2)(\epsilon_2, \epsilon_3) \right) \\
&\quad - \frac{(K_2^2 + K_3^2)}{6(K_2^2 - K_3^2)^2} (\epsilon_1, K_2)(\epsilon_2, k_1)(\epsilon_3, k_1) \\
&\quad - \frac{(\epsilon_1, K_2)((\epsilon_2, k_1)(\epsilon_3, K_3) - (\epsilon_2, K_2)(\epsilon_3, k_1))}{6(K_2^2 - K_3^2)} \\
&\quad - \frac{(K_2^2 + K_3^2)}{12(K_2^2 - K_3^2)} ((\epsilon_1, \epsilon_2)(\epsilon_3, k_1) + (\epsilon_1, \epsilon_3)(\epsilon_2, k_1)), \tag{107}
\end{aligned}$$

where  $\epsilon_1$  satisfies the physical condition  $(\epsilon_1, k_1) = 0$  and  $\epsilon_{2,3}$  are arbitrary 4-dimensional polarization vectors.

For degree 3 two mass easy box integrals we have:

$$I_4^{2me}(\lambda_3 \tilde{\lambda}_1, \epsilon_2, \epsilon_3) = \frac{\langle 3|K_2|1 \rangle}{2} \left[ \frac{(\epsilon_2, k_3)(\epsilon_3, k_3)}{(K_2^2 - t)(K_4^2 - s)} - (k_3 \rightarrow k_1, s \leftrightarrow t) \right]. \tag{108}$$

If two of the polarization vectors satisfy the physical condition, i.e.  $(\epsilon_1, k_1) = 0$  and  $(\epsilon_3, k_3) = 0$ , we have

$$\begin{aligned}
I_4^{2me}(\epsilon_1, \epsilon_2, \epsilon_3) &= -\frac{(\epsilon_1, k_3)(\epsilon_3, k_1)}{2(k_1, k_3)} \left[ \frac{(\epsilon_2, k_3)}{K_2^2 - t} + \frac{(\epsilon_2, k_1)}{K_4^2 - t} \right] \\
&\quad - \frac{(\epsilon_1, K_2)(\epsilon_2, k_1)(\epsilon_3, k_1)}{2(K_2^2 - s)(K_4^2 - t)} - \frac{(\epsilon_1, k_3)(\epsilon_2, k_3)(\epsilon_3, K_4)}{2(K_2^2 - t)(K_4^2 - s)}. \tag{109}
\end{aligned}$$

The formula for degree 3 two mass hard box integral is:

$$I_4^{2mh}(\lambda_1 \tilde{\lambda}_2, \epsilon_2, \epsilon_3) = \frac{\langle 1|K_3|2 \rangle}{4\delta} I_4(\epsilon_2, \epsilon_3), \tag{110}$$

$$I_4^{2mh}(\lambda_2 \tilde{\lambda}_1, \epsilon_2, \epsilon_3) = \frac{\langle 2|K_3|1 \rangle}{4\delta} I_4(\epsilon_2, \epsilon_3), \tag{111}$$

where

$$I_4(\epsilon_2, \epsilon_3) = (\epsilon_2, k_1)(\epsilon_3, K_4) + (\epsilon_2, K_4)(\epsilon_3, k_1) + (\epsilon_2, k_2)(\epsilon_3, K_3)$$

$$\begin{aligned}
& + (\epsilon_2, K_3)(\epsilon_3, k_2) - \frac{1}{\Delta} \left[ 2(K_3^2 K_4^2 - t^2 + \delta)(\epsilon_2, k_{12})(\epsilon_3, k_{12}) \right. \\
& + (K_3^2 + K_4^2 - s - 2t) \left( (K_3^2 - K_4^2 + s)(\epsilon_2, K_4)(\epsilon_3, K_4) \right. \\
& \quad \left. \left. + (K_4^2 - K_3^2 + s)(\epsilon_2, K_3)(\epsilon_3, K_3) \right) \right] \\
& + \frac{K_4^2 + t}{K_4^2 - t} (\epsilon_2, k_1)(\epsilon_3, k_1) + \frac{K_3^2 + t}{K_3^2 - t} (\epsilon_2, k_2)(\epsilon_3, k_2), \tag{112}
\end{aligned}$$

and

$$\delta = K_3^2 K_4^2 - (K_3^2 + K_4^2)t + (s+t)t, \tag{113}$$

$$\Delta = \Delta(k_{12}^2, K_3^2, K_4^2), \tag{114}$$

$$\Delta(s_1, s_2, s_3) = s_1^2 + s_2^2 + s_3^2 - 2(s_1 s_2 + s_2 s_3 + s_3 s_1), \tag{115}$$

are functions of the external momentum invariants.

In order to give the formulas for the rational part of the degree 4 polynomial, we define:

$$I_4(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4) \equiv \int \frac{d^D p}{i\pi^{D/2}} \frac{(\epsilon_1, p)(\epsilon_2, p - K_1)(\epsilon_3, p - K_{12})(\epsilon_4, p + K_4)}{p^2(p - K_1)^2(p - K_{12})^2(p + K_4)^2}, \tag{116}$$

where  $K_{12} = K_1 + K_2$ . For two mass easy case ( $K_{i,3} = k_{1,3}$  are massless external lines), we have

$$\begin{aligned}
I_4(\lambda_3 \tilde{\lambda}_1, \epsilon_2, \lambda_1 \tilde{\lambda}_3, \epsilon_4) &= -\frac{1}{4} \left( \frac{K_2^2 + s}{K_2^2 - s} + \frac{K_4^2 + t}{K_4^2 - t} \right) (\epsilon_2, k_1)(\epsilon_4, k_1) \\
&\quad - \frac{1}{4} \left( \frac{K_2^2 + t}{K_2^2 - t} + \frac{K_4^2 + s}{K_4^2 - s} \right) (\epsilon_2, k_3)(\epsilon_4, k_3) - \frac{5}{9} (k_1, k_3)(\epsilon_2, \epsilon_4) \\
&\quad + \frac{4}{9} \left( (\epsilon_2, k_1)(\epsilon_4, k_3) + (\epsilon_2, k_3)(\epsilon_4, k_1) \right), \tag{117}
\end{aligned}$$

$$\begin{aligned}
I_4(\lambda_1 \tilde{\lambda}_3, \epsilon_2, \lambda_1 \tilde{\lambda}_3, \epsilon_4) &= \frac{5}{9} \langle 1|\epsilon_2|3\rangle \langle 1|\epsilon_4|3\rangle \\
&\quad + \frac{\langle 1|K_2|3\rangle^2}{3} \left[ \frac{(\epsilon_2, k_1)(\epsilon_4, k_1)}{(K_2^2 - s)(K_4^2 - t)} + \frac{(\epsilon_2, k_3)(\epsilon_4, k_3)}{(K_2^2 - t)(K_4^2 - s)} \right]. \tag{118}
\end{aligned}$$

Other cases can be either obtained by conjugation or relabelling  $k_{1,3}$ .

For 2 mass hard box ( $K_{1,2} = k_{1,2}$  are two massless external lines), we define:

$$I_4^{2mh}(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4; c_3, c_4)$$

$$\begin{aligned}
&\equiv I_4[(\epsilon_1, p)(\epsilon_2, p - k_1)((\epsilon_3, p + K_4) + c_3)((\epsilon_4, p + K_4) + c_4)] \\
&= \int \frac{d^D p}{i\pi^{D/2}} \frac{(\epsilon_1, p)(\epsilon_2, p - k_1)((\epsilon_3, p + K_4) + c_3)((\epsilon_4, p + K_4) + c_4)}{p^2 (p - k_1)^2 (p - k_{12})^2 (p + K_4)^2},
\end{aligned} \tag{119}$$

and

$$\begin{aligned}
&I_4^{2mh}(\lambda_1 \tilde{\eta}_1, \eta_2 \tilde{\lambda}_2, \epsilon_3, \epsilon_4; c_3, c_4) \\
&= \frac{1}{\langle 2|K_3|1 \rangle} \left[ -\frac{1}{6} \langle \eta_2 | k_2 K_3 k_1 | \tilde{\eta}_1 \rangle (\epsilon_3, \epsilon_4) - t \langle \eta_2 2 \rangle [\tilde{\eta}_1 1] I_4^{2mh}(\lambda_1 \tilde{\lambda}_2, \epsilon_3, \epsilon_4) \right. \\
&+ t \langle \eta_2 | k_2 | \tilde{\eta}_1 \rangle I_3^{3m}(\epsilon_3, \epsilon_4) + \langle \eta_2 2 \rangle [\tilde{\eta}_1 1] \left( \frac{1}{2} (\langle 1|\epsilon_3|2 \rangle c_4 + \langle 1|\epsilon_4|2 \rangle c_3) \right. \\
&\quad \left. + \frac{1}{18} (\langle 1|\epsilon_3|2 \rangle \epsilon_4 + \langle 1|\epsilon_4|2 \rangle \epsilon_3, 7k_1 + 2k_2 + 9K_4) \right) \\
&+ \frac{1}{18} \langle \eta_2 2 \rangle [\tilde{\eta}_1 2] ((\epsilon_3, k_{12})(\epsilon_4, k_{12}) - 2s_{12}(\epsilon_3, \epsilon_4)) \\
&+ \frac{1}{18} \langle \eta_2 | (k_2 + K_3) | \tilde{\eta}_1 \rangle ((\epsilon_3, k_2 + K_3)(\epsilon_4, k_2 + K_3) - 2t(\epsilon_3, \epsilon_4)) \\
&- \frac{1}{18} \langle \eta_2 | K_3 | \tilde{\eta}_1 \rangle ((\epsilon_3, K_3)(\epsilon_4, K_3) - 2K_3^2(\epsilon_3, \epsilon_4)) \\
&+ \langle 2|K_3|\tilde{\eta}_1 \rangle (I_3^{2m}(\eta_2 \tilde{\lambda}_2, \epsilon_3, \epsilon_4) \\
&+ I_3^{2m}(\eta_2 \tilde{\lambda}_2, (c_3 - (\epsilon_3, K_3))\epsilon_4 + (c_4 + (\epsilon_4, K_4 + k_1))\epsilon_3)) \\
&+ I_3^{3m}(v, \epsilon_3, \epsilon_4) + I_3^{3m}(v, (c_3 - (\epsilon_3, K_3))\epsilon_4 + c_4\epsilon_3), \tag{120} \\
&+ \langle \eta_2 | K_4 | 1 \rangle (\tilde{I}_3^{2m}(\lambda_1 \tilde{\eta}_1, \epsilon_3, \epsilon_4) \\
&+ \tilde{I}_3^{2m}(\lambda_1 \tilde{\eta}_1, (c_3 - (\epsilon_3, k_2 + K_3))\epsilon_4 + (c_4 + (\epsilon_4, K_4))\epsilon_3) \Big], \tag{121}
\end{aligned}$$

where

$$v = \langle \eta_2 | K_3 | 1 \rangle \lambda_1 \tilde{\eta}_1 + \langle \eta_2 | K_3 | 2 \rangle \lambda_2 \tilde{\eta}_1 - (k_2, K_3) \eta_2 \tilde{\eta}_1. \tag{122}$$

The opposite helicity case is obtained by conjugation.

## Acknowledgments

We would like to thank Carola F. Berger, Zvi Bern, Lance J. Dixon, Darren Forde and David A. Kosower for sending us their results prior to publication [9], and for discussions, reading the paper, comments and assistance in comparing our six-gluon results with theirs. We also thank them for checking

the NMHV results by using their factorization check program. CJZ would like to thank J. -P. Ma for constant encouragements, helpful discussions and careful reading of the paper. His (financial) support (to buy a computer which was still in use today) actually goes back to the much earlier difficult times when I didn't have enough grants from other sources. What is more important is that there are no strings attached to his support and it is up to the last author to explore what he wants to. CJZ would also like to thank R. Iengo for encouragements and his interests in this work, helpful discussions and comments; to Z. Chang, B. Feng, E. Gava, H. -Y. Guo, K. S. Narain, K. Wu, Y. -S. Wu, Z. Xu and Z. -X. Zhang for discussions and comments; to Prof. X. -Q. Li and the hospitality at Nankai University where we can have good food; and finally to Prof. S. Randjbar-Daemi and the hospitality at Abdus Salam International Center for Theoretical Physics, Trieste, Italy. This work is supported in part by funds from the National Natural Science Foundation of China with grant number 10475104 and 10525522.

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