The Rational Part of QCD Amplitude I: the General Formalism

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Abstract

A general formalism for computing only the rational part of the one-loop QCD amplitude is developed. Starting from the Feynman integral representation of the one-loop amplitude, we use tensor reduction and recursive relations to compute the rational part directly. Explicit formulas for the rational part are given for all bubble and triangle integrals. Formulas are also given for box integrals up to two mass hard boxes which are the needed ingredients to compute up to 6-gluon QCD amplitudes. We use this method to compute explicitly the rational part of the 5- and 6-gluon QCD amplitudes in two accompanying papers.

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1 Introduction

The forthcoming experimental program at CERN's Large Hadron Collider (LHC) requires many computations at the next-leading-order (NLO) or one-loop with many particles as final states [1]. However the analytic computation of the matrix elements is very difficult. Only for special helicity [2, 3, 4] or special models [5], some analytic results are known for higher point amplitudes. The current state of the art for NLO computation is 5-point for QCD processes and 6-point for electroweak processes [6, 7, 8]. The recent development for tackling the multi-leg amplitudes by semi-numerical/analytic methods shows promise for improving traditional capabilities [9, 10, 11, 12, 13, 14, 15, 16]. All helicity configurations for the 6-gluon amplitude are evaluated for a single-space point [9]. These results are used to check the available analytic results [17].

Following Witten's twistor string theory [18] and the CSW approach [19] and the use of maximally-helicity-violating (MHV)[1] vertices [21, 22], there has been spectacular progress [23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47] in the perturbative QCD computations in the last two years or so, by using the unitarity cut method of Bern, Dunbar, Dixon and Kosower [3, 48, 50] and the spinor-helicity formalism [51, 52] (see [53] for a review). In particular, Bedford, Brandhuber, Spence and Travaglini [28, 31] applied the the MHV vertices to one-loop calculations. Britto, Buchbinder, Cachazo, Feng and Mastrolia [38, 39, 40] developed efficient technique for evaluating the rational coefficients in an expansion of the one-loop amplitude in terms of scalar box, triangle and bubble integrals (the cut-constructible part, see below and Sect. 4). By using their technique, it is much easier to calculate the coefficient of box integrals without doing any integration. Recently, Britto, Feng and Mastrolia completed the computation of the cut-constructible terms for all the 6-gluon helicity amplitudes [40].

In order to complete the QCD calculation for the 6-gluon amplitude, the remaining challenge is to compute the rational part of the amplitude with scalar circulating in the loop, commonly called the $N = 0$ case in a

[1]The 2 dimensional origin of the MHV amplitudes in gauge theory was first given in [20]

supersymmetric decomposition of the QCD amplitude:

$$A^{QCD} = A^{N=4} - 4A^{N=1 \text{ chiral}} + A^{N=0 \text{ or scalar}}. \quad (1)$$

The above strategy of splitting the computation of QCD amplitude into various supersymmetric parts plus a scalar part is quite fruitful. By using a theorem of Bern, Dunbar, Dixon and Kosower [50], the supersymmetric parts are cut-constructible, meaning that these amplitudes can be determined completely by using 4-dimensional unitarity. Even for the scalar part which is not cut-constructible, we can still split it into two parts: a cut-constructible part and a rational part. As we said before, the recent development inspired by twistor string theory has lead to very efficient techniques to compute the cut-constructible part [38, 39, 40]. To complete the program it is quite important to have efficient and powerful method to compute the rational part.

There are various ways to compute the rational part. The first approach [48] is to use the factorization properties by trial and error. This is a quite effective method if the end result is simple enough. The difficulty with this approach is that we don't know how to automate the method to make effective use of the advances in computer industry. The correctness of the obtained result is almost guaranteed by checking factorization for all channels. For higher point amplitudes, the complexity of the analytic results makes this method impractical.

The second approach used the unitarity relation. In principle the rational part can be constructed by using the D-dimensional unitarity method [49, 50, 54, 42]. The problem with this approach is that too much information is kept and tree amplitudes in D-dimension are even more difficult [55]. In fact this approach loses the simplicity of 4-dimensional helicity amplitudes as given by the MHV formula.

The third approach is the bootstrap recursive approach of Bern, Dixon and Kosower [34]. This approach is quite promising and powerful. It is a streamlined approach of the first one by adding the insights of recent tree-level recursive method of Britto, Cachazo, Feng and Witten [32]. This approach has already produced a wealth of general results for special helicity configurations, notably the "split-helicity" configuration [34, 46, 17]. It can also be used to compute one-loop QCD amplitudes with general helicities as outlined in [17].

Given the complexity of the results for the cut-constructible part of the 6-gluon amplitude [40] and its important application to LHC related experiments, it is quite worthy and even mandatory to have other methods to compute the rational part of the QCD amplitude. In particular one would like to bypass the need of using the cut-constructible part and have an independent method to compute the rational part. Of course, the testing ground for any method is a complete computation of the 6-gluon QCD amplitude where only partial results for some helicity configurations exist. The present status for the marching to one loop 6-gluon QCD amplitude were summarized in [55, 56]. For more recent developments, we refer the reader to [57, 58].

In this paper we will study the problem of computing the rational part of the one-loop amplitude directly from the Feynman integral representations. These integrals can be written down directly by drawing all Feynman diagrams and by using the Feynman rules. With the present technology these can be done quite effectively by using the various packages like GRACE [62], FeynArts [63] and Qgraf [64] et. al. (See [65, 62] for reviews.) Fortunately these powerful methods are not needed to compute up to the 6-gluon amplitude. For computing higher point amplitudes or 6-parton amplitudes they may be a necessity.

It is easy to imagine that the rational part is already contained in the integral representation of the amplitude. If one could obtain the complete rational coefficients by doing tensor reduction to scalar box, triangle and bubble integrals, one can simply get the rational part by making an expansion with the dimension D around 4 ($D = 4 - 2\epsilon$ is the parameter of dimensional regularization). This is extraordinarily difficult because of the complexity of tensor reduction for $N \geq 5$. However if one only needs to compute the rational part, it is not necessary to know the complete coefficients from tensor reductions. By the BDDK theorem [3], we know that many terms simply do not contribute to the rational part. Following this path of thought, the remaining problem is: is there an efficient way to compute these rational parts in one-loop QCD amplitudes directly from Feynman integrals?

In this paper we show that there is actually a quite efficient and powerful method to compute the rational part directly from Feynman integrals. Because we are concerned only with the rational part of the amplitude, there is no need for tensor reduction all the way down to scalar integrals. We only need tensor reduction to reduce the degree of the numerator by 2. So the original complexity of tensor reduction is bypassed in the computation of rational part.

In our approach of computing the rational part, we will exploit the theorem of Bern, Dunbar, Dixon and Kosower [3] and directly extract the rational part from the one-loop Feynman integrals. We also use the simple tensor reduction by using spinors as developed in [48, 66, 67]. We point out that the tensor reduction formulas used in our computations are actually quite simple, as one can see from eqs. (11) and (13) in Sect. 3.

As we will demonstrate in this paper, the computation of the rational part is reduced to tree-level like calculations. As our method also applies to massive theory and theories with fermions, we envisage more widely application of our method in the computation of one-loop amplitudes, in combination with the $D = 4$ unitarity method [3, 48] and the efficient technique for computing generic unitarity cuts [38]. The once most difficult part of the one-loop amplitude can actually be attacked by the traditional technique.

In this paper and the accompanying two papers [68, 69], we will develop our method and apply it to the computation of the rational part of the one-loop 5- and 6-gluon QCD amplitudes. This paper mainly deals with the general theoretical formalism of the method. In [68], we show the efficiency of the method by computing the rational part for the 5-gluon amplitude for the two MHV helicity configurations by giving most of the intermediate steps. In [69], we will present the results for the rational part of the 6-gluon amplitude for the two MHV and two NMHV helicity configurations. The rational parts of the 6-gluon amplitude for the "split helicity" configurations are known already [34, 17]. Recently all one-loop maximally helicity violating gluonic amplitudes were computed by Berger, Bern, Dixon, Forde and Kosower [70]. We refer the reader to [70, 69] for details about the explicit analytic results and comparison.

This paper is organized as follows: in Sect. 2 we set our notation for spinor products and composite currents for sewing trees to one-loop. Some simple tensor reduction formulas are given in Sect. 3. Starting from Sect. 4, we begin to develop the method of extracting the rational part of Feynman integral. We use the recursive approach to compute any Feynman integral as developed in [60]. In Sect. 6 and 7 we give explicit results for triangle and box integrals. In Sect. 8 we compute the correction terms from using the naive $D = 4$ tensor reduction of box and triangle integrals which arises from the ultra-violet divergent part of the box and triangle amplitude.

2 Notation

We mainly follow the notation of BDK [34] and the QCD-literature convention for the square bracket $[i\,j]$. By abusing of notation the product between 2 holomorphic spinors or 2 anti-holomorphic spinors is formed by a round bracket:

$$(\lambda_i, \lambda_j) = \langle i\,j\rangle, \qquad (\tilde\lambda_i, \tilde\lambda_j) = [i\,j]. \tag{2}$$

The scalar product between 2 vectors (written either in 4d vector notation or in 2 spinor notation) is also denoted by a round bracket. We have

$$\begin{aligned} (\lambda_{i_1}\tilde\lambda_{j_1}, \lambda_{i_2}\tilde\lambda_{j_2}) &= (\lambda_{i_1}, \lambda_{i_2})\,(\tilde\lambda_{j_2}, \tilde\lambda_{j_1}) = \langle i_1\, i_2\rangle[j_2\, j_1], & (3)\\ 2\,k_i\cdot k_j &= (\lambda_i\tilde\lambda_i, \lambda_j\tilde\lambda_j) = \langle i\,j\rangle\,[j\,i]. & (4) \end{aligned}$$

For spinor strings, we simply use $\langle i|(k_a+k_b)|\lambda_j\rangle$ or $\langle i|(k_a+k_b)|j]$ to denote $\langle i^-|(a+b)|j^-\rangle$:

$$\begin{aligned} \langle\lambda_i|(k_a+k_b)|\tilde\lambda_j\rangle &= \langle i|(k_a+k_b)|j] = \langle i|(a+b)|j] \\ &= \langle i^-|(a+b)|j^-\rangle = \langle i\,a\rangle\,[a\,j] + \langle i\,b\rangle\,[b\,j]. \end{aligned} \tag{5}$$

We don't use gamma matrix traces. Instead we use bra and ket notation with multiple insertion of momenta:

$$\langle i|k_1\,k_2\cdots k_n|j\rangle = \begin{cases} \langle i\,1\rangle[1\,2]\cdots[n\,j], & n = \text{odd},\\ \langle i\,1\rangle[1\,2]\cdots\langle n\,j\rangle, & n = \text{even}. \end{cases} \tag{6}$$

Sometimes we also write $\langle i|K|j\rangle$ for $\langle i|K|j]$, with the understanding that sometimes the last j should actually stand for $\tilde\lambda_j$ and with the bracket]. For example we have

$$\langle i|k_1\,k_2\,k_3|j] = \langle i\,1\rangle\,[1\,2]\,\langle 2\,3\rangle\,[3\,j]. \tag{7}$$

For $i = j$ the above notation is just the gamma matrix trace. For simplicity we will not write the slash:

$$\langle i|k_1\,k_2\,k_3|i] = \mathrm{tr}_-(k_i\,k_1\,k_2\,k_3). \tag{8}$$

We note that the above notation only happens for an odd number of momenta inserted between 2 spinors (one holomorphic and one anti-holomorphic). Of course the momentum can be either massless or a sum of several massless momenta.

The sums of cyclicly consecutive external momenta is denoted generically by K_i in a Feynman diagram. In our explicit computation we use $k_{12} = k_1 + k_2$ and $k_{234} = k_2 + k_3 + k_4$, etc. The kinematic variables are denoted as $s_{12} = (k_1 + k_2)^2$ and $s_{123} = (k_1 + k_2 + k_3)^2$ in a self explaining notation. For 6-gluon case we also have $s_{123} = (k_4 + k_5 + k_6)^2$ by momentum conservation.

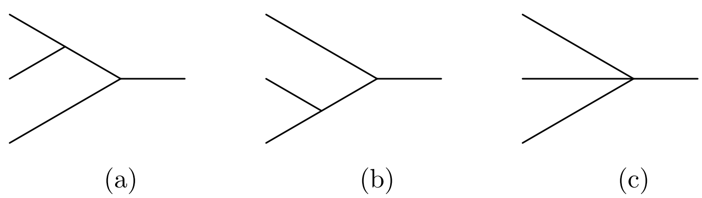

Figure 1: The composition of 3 external particles in tree amplitude.

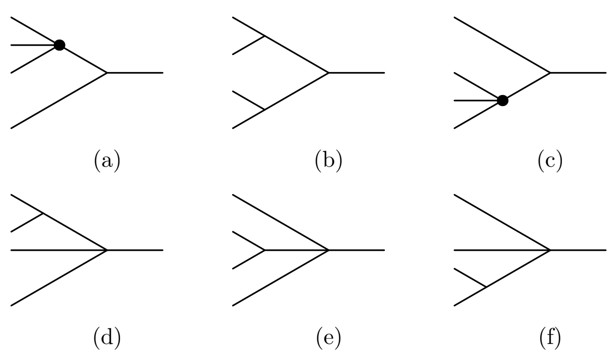

Figure 2: The composition of 4 external particles in tree amplitudes. The blob denotes an expansion as given in Fig. 1. The explicit expression of $\epsilon_{i(i+1)(i+2)(i+3)}$ will not be given.

For sewing trees to the loop, we define the following composite currents or polarization vectors:

$$\begin{aligned} \epsilon_{i(i+1)} &= P(\epsilon_i, k_i; \epsilon_{i+1}, k_{i+1}) \equiv \frac{1}{(k_i + k_{i+1})^2}\Big((\epsilon_i, k_{i+1})\,\epsilon_{i+1} \\ &- (\epsilon_{i+1}, k_i)\,\epsilon_i + \frac{1}{2}(\epsilon_i, \epsilon_{i+1})\,(k_i - k_{i+1})\Big), \end{aligned} \tag{9}$$

$$\epsilon_{i(i+1)(i+2)} = P(\epsilon_{i(i+1)}, k_{i(i+1)}; \epsilon_{i+2}, k_{i+2}) + P(\epsilon_i, k_i; \epsilon_{(i+1)(i+2)}, k_{(i+1)(i+2)})$$
$$+\frac{1}{s_{i(i+1)(i+2)}} \left((\epsilon_i, \epsilon_{i+2})\, \epsilon_{i+1} - \frac{1}{2} (\epsilon_i, \epsilon_{i+1})\, \epsilon_{i+2} - \frac{1}{2} (\epsilon_{i+1}, \epsilon_{i+2})\, \epsilon_i \right), \quad (10)$$

where $s_{i(i+1)(i+2)} = (k_i + k_{i+1} + k_{i+2})^2$. The above procedure is a simplified version of the general recursive calculation of the tree-level n-gluon amplitudes [71]. We note that $\epsilon_{i(i+1)}$ is anti-symmetric and $\epsilon_{i(i+1)(i+2)}$ is symmetric under the reversing of the order of the particles. This generalizes to higher composite currents which we haven't written down explicitly. The diagrammatic representations of $\epsilon_{i(i+1)(i+2)}$ and $\epsilon_{i(i+1)(i+2)(i+3)}$ are given in Figs. 1 and 2.

3 Tensor reduction of the one-loop amplitude

There is a vast literature on this subject. The original Passrino-Veltman approach [59] is quite general but it is not quite practical to obtain compact analytic results. In fact the tensor reduction relations we will use for our calculation of the 5- and 6-gluon amplitudes are quite simple. It was based on the BDK trick [48] of multiplying and dividing by spinor square roots. To make more effective use of this trick, we have purposely chosen the reference momenta to make the tensor reduction simple. See [69] for further details about the specific choice of the reference momenta and the tensor reduction involved in the computation of (the rational part of) the 6-gluon amplitude.

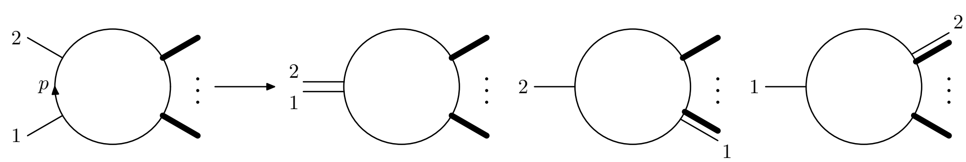

Figure 3: The tensor reduction for two adjacent same helicity generally gives three terms.

There are basically only two different cases to consider. As shown in Fig. 3, the two polarization vectors have the same helicities. If we choose the same reference momentum (denoted by the spinor η), we have

$$(\eta\tilde{\lambda}_1, p)\,(\eta\tilde{\lambda}_2, p) = -\frac{(\eta\tilde{\lambda}^{(\eta)}_{k_{12}}, p + k_1)}{\langle 1\,2\rangle}\, I^{(2)}$$

$$+ \quad \frac{\langle \eta\, 1\rangle}{\langle 1\, 2\rangle}\,(\eta\tilde{\lambda}_1, p)\, I^{(3)} + \frac{\langle \eta\, 2\rangle}{\langle 1\, 2\rangle}\,(\eta\tilde{\lambda}_2, p)\, I^{(1)}, \tag{11}$$

$$\tilde{\lambda}^{(\eta)}_{k_{12}} \;=\; \langle \eta\, 1\rangle\tilde{\lambda}_1 + \langle \eta\, 2\rangle\tilde{\lambda}_2, \tag{12}$$

where $I^{(1)} = (p+k_1)^2$, $I^{(2)} = p^2$ and $I^{(3)} = (p-k_2)^2$ are various inverse propagators. The above tensor reduction formula is shown diagrammatically in Fig. 3, omitting the relevant factors.

In deriving eq. (11), we assumed that p is a four-dimensional vector. Because pentagon and higher point one-loop amplitudes are ultra-violet convergent, the use of the above formula in tensor reduction is correct in dimensional regularization up to infinitesimal terms. However one must be quite careful to apply the above formula to the tensor reduction of the box and triangle tensor integrals because the difference is a finite rational part. Some correction terms must be included for tensor reduction with box and triangle tensor integrals. This also applies to the following tensor reduction formulas given later in this section. We will compute these correction terms later in Sect. 8.

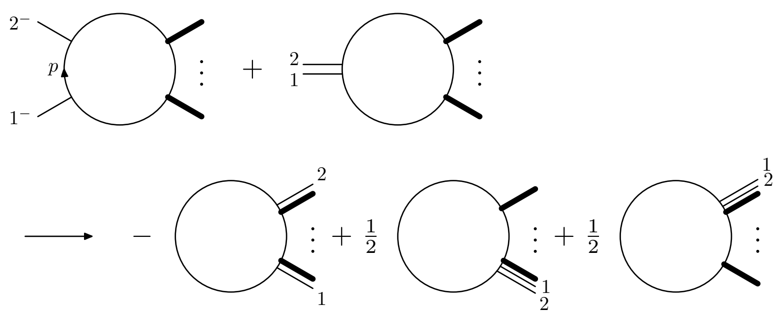

Figure 4: For two adjacent same helicity, the tensor reduction for the combination of two diagrams is even simpler by a judicious choice of the reference momenta.

An even simpler version of the above tensor reduction relation is to consider a combination of two diagrams together as shown in Fig. 4. The reduction formulas is:

$$\frac{(\epsilon_1, p+k_1)(\epsilon_2, p)}{(p+k_1)^2 p^2 (p-k_2)^2} \;+\; \frac{(\epsilon_{12}, p+k_1) - (\epsilon_1, \epsilon_2)/2}{(p+k_1)^2 (p-k_2)^2}$$

$$= \; -\frac{1}{p^2} + \frac{1/2}{(p+k_1)^2} + \frac{1/2}{(p-k_2)^2}, \tag{13}$$

for $\epsilon_1 = \lambda_1\tilde{\lambda}_2$ and $\epsilon_2 = \lambda_2\tilde{\lambda}_1$. All the factors appearing in the left-hand side of the above equations are read off directly from Feynman rules. These tensor reduction relations show quite clearly the simplicity of the diagrams when there are adjacent particles with the same helicity.

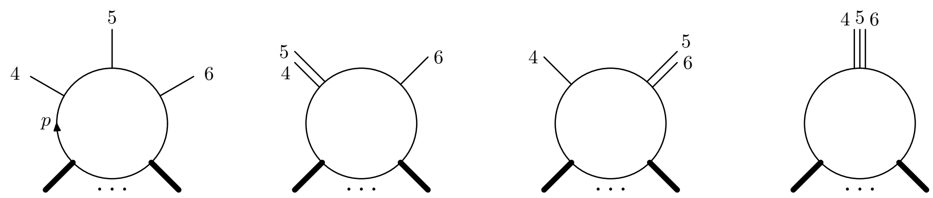

Figure 5: For three adjacent same helicity, the tensor reduction for the combination of these four diagrams is also quite simple if we choose the reference momenta appropriately.

For three adjacent particles with the same helicity gluons, we choose the following polarization vectors (omitting an overall factor for each polarization vector):

$$\epsilon_4 = \lambda_5\tilde{\lambda}_4, \qquad \epsilon_5 = \eta\tilde{\lambda}_5, \qquad \epsilon_6 = \lambda_5\tilde{\lambda}_6. \tag{14}$$

Then we have

$$\epsilon_{45} = -\eta\tilde{\lambda}_4 + \frac{1}{2}\frac{\langle\eta\, 5\rangle}{\langle 4\, 5\rangle}\, k_{45}, \tag{15}$$

$$\epsilon_{56} = \eta\tilde{\lambda}_6 - \frac{1}{2}\frac{\langle\eta\, 5\rangle}{\langle 6\, 5\rangle}\, k_{56}, \tag{16}$$

and

$$(\epsilon_{45}, p - k_{45}) - \frac{1}{2}(\epsilon_4, \epsilon_5) = -(\eta\tilde{\lambda}_4, p - k_{45}) + \frac{1}{2}\frac{\langle\eta\, 5\rangle}{\langle 4\, 5\rangle}\,(I^{(4)} - I^{(6)}), \tag{17}$$

$$(\epsilon_{56}, p - k_4) - \frac{1}{2}(\epsilon_5, \epsilon_6) = (\eta\tilde{\lambda}_6, p - k_4) - \frac{1}{2}\frac{\langle\eta\, 5\rangle}{\langle 6\, 5\rangle}\,(I^{(5)} - I^{(*)}), \tag{18}$$

where $I^{(4)} = p^2$, $I^{(5)} = (p - k_4)^2$, $I^{(6)} = (p - k_{45})^2$ and $I^{(*)} = (p - k_{456})^2$ are inverse propagators.

Now considering the three terms coming from the first 3 Feynman diagrams shown in Fig. 5, we have:

$$\begin{aligned}
A_{456} &= (\epsilon_4, p-k_4)(\epsilon_5, p-k_4)(\epsilon_6, p-k_4) \\
&+ ((\epsilon_{45}, p-k_{45}) - \frac{1}{2}(\epsilon_4,\epsilon_5))(\epsilon_6, p-k_4)\, I^{(5)} \\
&+ (\epsilon_4, p-k_4)((\epsilon_{56}, p-k_4) - \frac{1}{2}(\epsilon_5,\epsilon_6))\, I^{(6)} \\
&= ((\lambda_5\tilde\lambda_4, p-k_4)(\eta\tilde\lambda_6, p-k_4) - (\eta\tilde\lambda_4, p-k_4)(\lambda_5\tilde\lambda_6, p-k_4))\, I^{(6)} \\
&+ \frac{1}{2}\frac{\langle\eta\, 5\rangle}{\langle 4\, 5\rangle}\,(\lambda_5\tilde\lambda_6, p-k_4)\,(I^{(4)} - I^{(6)})\, I^{(5)} \\
&- \frac{1}{2}\frac{\langle\eta\, 5\rangle}{\langle 6\, 5\rangle}\,(\lambda_5\tilde\lambda_4, p-k_4)\,(I^{(5)} - I^{(*)})\, I^{(6)}, \qquad (19)
\end{aligned}$$

by doing tensor reduction with k_{45} for the first term. This can be further simplified by expressing η in terms of a linear combination of $\lambda_{4,5}$. The final result is:

$$\begin{aligned}
A_{456} &= \frac{1}{2}\frac{\langle\eta\, 5\rangle}{\langle 4\, 5\rangle}\,(\lambda_5\tilde\lambda_6, p-k_4)\, I^{(4)}\, I^{(5)} + \frac{1}{2}\frac{\langle\eta\, 5\rangle}{\langle 6\, 5\rangle}\,(\lambda_5\tilde\lambda_4, p-k_4)\, I^{(*)}\, I^{(6)} \\
&+ I^{(5)}\, I^{(6)}\,\langle\eta\, 5\rangle\,\left[[4\, 6] - \frac{1}{2}(\lambda_5(\frac{\tilde\lambda_4}{\langle 6\, 5\rangle} + \frac{\tilde\lambda_6}{\langle 4\, 5\rangle}), p-k_4)\right] \\
&= \frac{1}{2}\frac{\langle\eta\, 5\rangle}{\langle 4\, 5\rangle}\,(\lambda_5\tilde\lambda_6, p-k_4)\, I^{(4)}\, I^{(5)} + \frac{1}{2}\frac{\langle\eta\, 5\rangle}{\langle 6\, 5\rangle}\,(\lambda_5\tilde\lambda_4, p-k_4)\, I^{(*)}\, I^{(6)} \\
&+ \frac{1}{2}\, I^{(5)}\, I^{(6)}\,\langle\eta\, 5\rangle\,\left[[4\, 6] + \frac{1}{\langle 4\, 5\rangle\langle 6\, 5\rangle}(\lambda_5\tilde\lambda^{(5)}_{k_{456}}, p)\right], \qquad (20)
\end{aligned}$$

which has nice symmetric property under the flipping operation $4 \leftrightarrow 6$. The last term in eq. (20) actually cancels the contribution from the last Feynman diagram in Fig. 5.

For the case of different neighboring helicities, we can use the following reduction formula:

$$\begin{aligned}
(\lambda_1\tilde\eta, p)\,(\lambda_1\tilde\lambda_2, p) &= \frac{\langle\lambda_1|p\, k_2\, K\, k_1\, p|\tilde\eta\rangle}{\langle 2|K|1\rangle} \\
&= \frac{1}{\langle 2|K_4|1\rangle}\Big(I^{(1)}\,(\lambda_1\tilde\lambda_2, p)\,\langle 2|(K_4+k_1)|\tilde\eta\rangle
\end{aligned}$$

$$+I^{(2)}\,\langle 1|(p-k_2)(K_4k_1-k_2K_4)|\tilde\eta\rangle - I^{(3)}\,\langle 1|p\,K_4|1\rangle[1\,\tilde\eta]$$
$$-I^{(4)}\,(\lambda_1\tilde\lambda_2,p)\,\langle 2\,1\rangle[1\,\tilde\eta] + (K_4+k_1)^2\,(\lambda_1\tilde\lambda_2,p)\,\langle 2\,1\rangle[1\,\tilde\eta]\Big), \quad (21)$$

$$\begin{aligned}(\lambda_1\tilde\lambda_2,p)\,(\eta\tilde\lambda_2,p) &= \frac{\langle\eta|p\,k_2\,K\,k_1\,p|\tilde\lambda_2\rangle}{\langle 2|K|1\rangle}\\ &= \frac{1}{\langle 2|K_3|1\rangle}\Big(I^{(1)}\,\langle\eta|k_2\,K_3\,p|2\rangle\\ &+I^{(2)}\,\langle\eta|(K_3k_1-k_2K_3)(p+k_1)|2\rangle - I^{(3)}\,\langle\eta|(k_2+K_3)|1\rangle(\lambda_1\tilde\lambda_2,p)\\ &+I^{(*)}\,(\lambda_1\tilde\lambda_2,p)\,\langle\eta\,2\rangle[2\,1] - (k_2+K_3)^2\,(\lambda_1\tilde\lambda_2,p)\,\langle\eta\,2\rangle[2\,1]\Big), \quad (22)\end{aligned}$$

where I's are the various inverse propagators:

$$I^{(1)} = (p+k_1)^2, \qquad I^{(2)} = p^2, \qquad I^{(3)} = (p-k_2)^2, \quad (23)$$
$$I^{(4)} = (p+k_1+K_4)^2, \qquad I^{(*)} = (p-k_2-K_3). \quad (24)$$

In eqs. (21) and (22), the momentum K can be chosen as one of the nearby momentum to avoid the spurious pole associated with a composite momenta. For 2 mass hard box, K can only be chosen as one of the composite momentum.

Because of the complexity of the above tensor reduction formula for differing helicity, it is better not to use them directly. Luckily we are able to avoid using them directly for tensor reduction with 5- and 6-point diagrams by a judicious choice of reference momenta for all the polarization vectors. The details will be given in [69]. For 2 mass hard box integrals, we must use the above general tensor reduction in order to obtain comparatively compact analytic expressions. We will use a slightly different reduction formula in Sect. 7.4 to compute the rational part of the two mass hard box integral.

The above is just the first step for the tensor reduction. Of course this procedure can be applied recursively. By a rough examination of this recursive method, one immediately finds two problems: 1) the association of the resulting polarization vector with an external (composite) momentum may not satisfy the physical conditions; 2) the above formula is no-longer applicable if one has a massive external momentum (a composite one arising from the reduction or a pinched line by sewing tree to loop). There are other methods to do further tensor reduction, some involving Gram determinant. Exactly because of these problems, tensor reduction is usually the most difficult part and the bottleneck for directly computing the one loop amplitude.

Quite elaborate methods are developed to tackle these problems. See, for example, [8, 61].

Because of the complexity of doing further tensor reduction, we immediately see the problem why directly computing the amplitude by using Feynman rules is an extraordinarily difficult task for higher point amplitudes. There are mainly two difficulties to overcome: too many diagrams and the complexity for tensor reduction (especially at a later stage of tensor reduction as mentioned in the above). By using computer it is not too difficult to manage the many Feynman diagrams (of the order 1000). But the complexity of tensor reduction with the appearance of spurious poles is actually the bottleneck for analytic computation. Tensor reduction is also the bottleneck for doing numerical calculation. See, for example, [62].

In contrast we also see that why it is possible to compute the rational part by using the conventional Feynman integrals. First by using supersymmetric decomposition, the number of Feynman diagrams is about 50 (1 hexagon, 6 pentagons, 15 boxes, 20 triangles and 15 bubbles) by only computing the scalar loop contributions. Second by computing only the rational part, it is not necessary to do tensor reduction all the way down to scalar integrals. One needs only to do tensor reduction to reduce the degree by 2 (see next section). So tensor reduction doesn't complicate the analytic expressions significantly. In fact for special helicity assignments, there is an almost mutual cancellation between higher point diagrams and lower point diagrams, as we demonstrated in eqs. (11) and (13). This is a manifestation of gauge invariance. This property can be used to check and to organize the results of our calculation. It is very important to have some "local" cancellation before adding all the results together in order to obtain relatively compact analytic results for the rational part of QCD amplitude.

4 The BDDK theorem and the structure of one-loop amplitude

A generic m-point one-loop Feynman diagram shown in Fig. 6 and its integral is given as follows (by using Feynman rules):

$$I_m^D[f(p)] = \int \frac{\mathrm{d}^D p}{i\,\pi^{D/2}} \frac{f(p)}{p^2(p-K_1)^2\cdots(p+K_m)^2}, \tag{25}$$

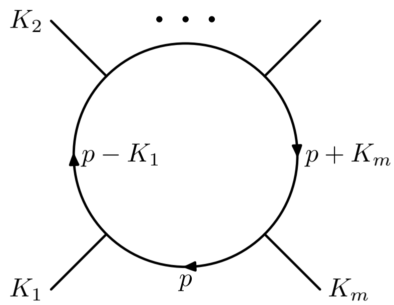

Figure 6: A generic one-loop diagram with external momenta $K_1, \cdots, K_m$. p is the internal momentum between the external lines K_1 and K_m.

where $f(p)$ is a polynomial function of the internal momentum p. For phenomenologically interesting models and by choosing a suitable gauge, the degree of $f(p)$ is always not greater than m. $f(p)$ also depends on the external momenta k (K_i's are sums of cyclicly consecutive external momenta k's) and the polarization vectors ϵ_i. For $f(p) = 1$ it is called the scalar integral. The strategy of computing $I_m^D[f(p)]$ is to reduce it recursively into lower degree polynomial and/or lower point integral. It is a well-known result that $I_n^D[f]$ can be generically written as:

$$\begin{aligned} I_m^D[f] &= \sum_i c_{4,i}(\epsilon, k; D)\, I_4^{D(i)}[1] \\ &+ \sum_i c_{3,i}(\epsilon, k; D)\, I_3^{D(i)}[1] + \sum_i c_{2,i}(\epsilon, k; D)\, I_2^{D(i)}[1], \end{aligned} \tag{26}$$

up to infinitesimal terms arising from tensor reduction from pentagon or higher point diagrams. Here $c_{j,i}$'s are rational functions of the external momenta when all polarization vectors are written in terms of spinor products. We note that these coefficients also depends on the (arbitrary) space-time dimension D in dimensional regularization (in the FDH scheme [72]).

A brute-force computation of these coefficients from Feynman integrals is an impossible task for 6 or higher point amplitude. The 5-point case was computed by string-inspired method by using a table for all Feynman parameter integrals (see below) [7]. However the string-inspired method is still not powerful enough to compute even the 6-gluon amplitude due to the complexity of the Feynman integrals and the intermediate expressions.

For physical application what we actually need is an expansion in ϵ of the above formula for $D = 4 - 2\epsilon$, up to finite terms. If we forget the infrared divergence for the moment, there are only simple pole ($\frac{1}{\epsilon}$) terms in the scalar integrals $I_{4,3,2}^D[1]$ with rational coefficients. So we can write the n-gluon amplitude (addition of all contributing $I_m^D[f(p)]$ from all Feynman diagrams) as follows:

$$\begin{aligned} \mathcal{A}_n &= \sum_i c_{4,i}(\epsilon, k; 4)\, I_4^{D(i)}[1] + \sum_i c_{3,i}(\epsilon, k; 4)\, I_3^{D(i)}[1] \\ &+ \sum_i c_{2,i}(\epsilon, k; 4)\, I_2^{D(i)}[1] + (\text{rational function}) + O(\epsilon). \end{aligned} \quad (27)$$

For supersymmetric theory, Bern, Dixon, Dunbar and Kosower [3] proved a theorem which states that the rational function is exactly zero. This is due to the better ultra-violet behaviour of one-loop amplitude in supersymmetric theory. What they proved is actually a more general theorem: if $f(p)$ is a polynomial (in p) of degree $m-2$ or less, the rational part for $I_m^D[f(p)]$ arising by expanding in ϵ is exactly zero. Generally speaking, the rational part is non-vanishing for degree m and $m-1$ polynomials.

So we need only to compute the rational coefficients (called the cut-constructible part hereafter) exactly at $D = 4$. For non-supersymmetric theory like QCD we also need to compute the rational function (called the rational part hereafter). In a series of papers [3, 48, 50], Bern, Dunbar, Dixon and Kosower have developed the method of computing the cut-constructible part, i.e., the coefficients $c_{j,i}(\epsilon, k; 4)$, from 4-dimensional unitarity. The nicety of 4-dimensional unitarity is that all the ingredients in the unitarity relation are on-shell quantity. However 4-dimensional unitarity loses all information about the rational function part in eq. (27). One must use some other methods to compute the rational part. In [48], they use the factorization properties with trials and errors. The difficulty of computing the rational function prevents the wider application of unitarity method to calculate more general amplitudes. As we mentioned in the introduction, the rational function part could be computed by going to D-dimensional unitarity [50, 42].

In the following sections we will exploit the BDDK theorem to compute the rational part directly from the Feynman integrals. By using the recursive relations satisfied by the tensor integrals we will derive the recursive relations for the rational part by making an expansion in ϵ. Our integration method of computing the rational part may also be used to compute the rational part by using D-dimensional unitarity. We use then recursive relations to derive explicit formulas for the rational part of all bubble and triangle integrals in Sect. 6. In Sect. 7, we derive the formulas for box integrals up to 2 mass hard boxes. Formulas for 3 mass and 4 mass box integrals can also be derived and will be given elsewhere. They are much more complicated than the 2 mass box formulas. Fortunately they are not needed in the computation of 6-gluon amplitude. We note that the recursive relations for tensor integrals and the rational parts can also be derived for massive internal loop and/or external fermion lines. For simplicity all formulas are given only for vanishing internal mass.

5 The recursive relations of one-loop amplitudes

In this section we study the recursive relations of one-loop amplitude [60, 61, 8]. By using Feynman parametrization we have

$$\begin{aligned} I_n^D[1] &\equiv \int \frac{\mathrm{d}^D p}{i\pi^{D/2}} \frac{1}{p^2(p-k_1)^2\cdots(p+k_n)^2} \\ &= (-1)^n\,\Gamma(n-D/2)\int \mathrm{d}^n a\,\frac{\delta(1-\sum_i a_i)}{(a\cdot S\cdot a)^{n-\frac{D}{2}}}, \end{aligned} \tag{28}$$

where

$$a\cdot S\cdot a = \sum_{i,j=1}^{n} a_i\,a_j\,S_{ij}$$

and the matrix S,

$$S = -\frac{1}{2}\begin{pmatrix} 0 & k_1^2 & (k_1+k_2)^2 & \cdots & (k_1+k_2+\cdots k_{n-1})^2 \\ * & 0 & k_2^2 & \cdots & (k_2+k_3+\cdots k_{n-1})^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ * & * & * & 0 & k_{n-1}^2 \\ * & * & * & * & 0 \end{pmatrix}, \tag{29}$$

is an $n\times n$ symmetric matrix of external kinematic variables (extension to massive loop is straightforward). For tensor integral $I_n^D[f(p)]$ it is given by the same integral with an extra polynomial of a in the denominator[2]:

$$I_n^D[g(a)] = \Gamma(n - D/2) \int \mathrm{d}^n a \, \frac{\delta(1 - \sum_i a_i)\, g(a)}{(a \cdot S \cdot a)^{n - \frac{D}{2}}}. \tag{30}$$

The degree of $g(a)$ is the same as the degree of $f(p)$ in p.

As explained in [60], a set of recursive relations for these tensor integrals can be derived by performing the following integration:

$$\begin{aligned} F &= \Gamma(\alpha) \int_0^1 \mathrm{d}a_{n-1} \int_0^{1-a_{n-1}} \mathrm{d}a_{n-2} \cdots \int_0^{1-a_1-a_2-\cdots-\hat{a}_m-\cdots-a_{n-1}} \mathrm{d}a_m \cdots \\ &\quad \times \frac{\partial}{\partial a_m} \left[\left. \frac{f(a)}{(a \cdot S \cdot a)^\alpha} \right|_{a_n = 1 - a_1 - \cdots - a_{n-1}} \right], \end{aligned} \tag{31}$$

in two ways: one by partial integration and one by direct differentiation. Then we obtain the following recursive relations:

$$\begin{aligned} & -2\Gamma(\alpha+1) \int_0^1 \mathrm{d}^n a \delta(1-a) \, \frac{f(a)((S \cdot a)_m - (S \cdot a)_n)}{(a \cdot S \cdot a)^{\alpha+1}} \\ &= \Gamma(\alpha) \int_0^1 \mathrm{d}^n a \delta(1-a) \left. \frac{f(a)}{(a \cdot S \cdot a)^\alpha} \right|_{a_n=0} \\ &\quad -\Gamma(\alpha) \int_0^1 \mathrm{d}^n a \delta(1-a) \left. \frac{f(a)}{(a \cdot S \cdot a)^\alpha} \right|_{a_m=0} \\ &\quad -\Gamma(\alpha) \int_0^1 \mathrm{d}^n a \delta(1-a) \, \frac{\partial_m f(a) - \partial_n f(a)}{(a \cdot S \cdot a)^\alpha}. \end{aligned} \tag{32}$$

By carefully examining the composition of $a \cdot S \cdot a$ for $a_m = 0$ one recognizes that this corresponds to an pinching limit of the original Feynman diagram: $k_{m-1},\ k_m \to k_{m-1} + k_m$. By setting $\alpha = n - 1 - \frac{D}{2}$ and using the definition for $I_n^D[f]$, the above recursive relation translates into the following form:

$$\begin{aligned} & I_n^D[-2f(a)((S \cdot a)_m - (S \cdot a)_n)] \\ &= I_{n-1}^{D(n)}[f(a)] - I_{n-1}^{D(m)}[f(a)] - I_n^{D+2}[\partial_m f(a)] + I_n^{D+2}[\partial_n f(a)]. \end{aligned} \tag{33}$$

[2]We note that $I_n[f]$ can stand for either $I_n[f(p)]$ (momenta integral) or $I_n[f(a)]$ (Feynman parameter integral), depending on the context. Reader should be careful that there is a $(-1)^n$ difference between the definitions of these two kinds of integral.

Here $I_{n-1}^{D(i)}[f(a)]$ denotes the $(n-1)$-point one-loop Feynman integral obtained by pinching k_{i-1} and k_i. Coupled with one more equation from the delta function:

$$\sum_{i=1}^{n} I_n^D[f(a)a_i] = I_N^D[f(a)], \tag{34}$$

we can solve $I_n^D[f(a)a_i]$ in terms of $I_{n-1}^{D(m)}[f(a)]$, $I_n^{D+2}[\partial_m f(a)]$ and $I_n^D[f(a)]$:

$$I_n^D[f(a)a_i] \;=\; c_{ij}^{(n)} I_{n-1}^{D(j)}[f(a)] + d_{ij}^{(n)} I_n^{D+2}[\partial_j f(a)] + c_i^{(n)} I_n^D[f(a)]. \tag{35}$$

The above reasoning goes through so long as we can invert the relevant matrix. This matrix is

$$\begin{bmatrix} & -2(S_{ij}-S_{nj}) & \\ 1 & \cdots & 1 \end{bmatrix} \tag{36}$$

and it is singular for $n \geq 6$. For our purpose of computing the rational part we will only need these recursive relations for $n \leq 4$. Higher point tensor integrals are reduced directly in $D=4$ as we have done in the last section. For further discussion about tensor reduction and its close relation with the above recursive relation, we refer the reader to [61, 8].

The above recursive relation is not symmetric for all a_i. The symmetric recursive relation can be obtained by firstly solving $I_n^D[f(a)(S\cdot a)_n]$ and substituting it back to the system of equations. By taking $f = g_l(a)\,a_m$ ($g_l(a)$ is a homogeneous polynomial of degree l in a), we can solve $I_n^D[g_l(a)\,(S\cdot a)_n]$ to get:

$$2I_n^D[g_l(a)\,(S\cdot a)_n] = I_{n-1}^{D(n)}[g_l(a)] + (n-1-l-D)\,I_n^{D+2}[g_l(a)] + I_n^{D+2}[\partial_n g_l(a)]. \tag{37}$$

Substituting this back into eq. (33) and multiplying both sides of the equation by S_{im}^{-1} and then summing over m, we have:

$$\begin{aligned} I_n^D[g_l(a)\,a_i] \;&=\; \frac{1}{2}(n-1-l-D)\,\gamma_i\,I_n^{D+2}[g_l(a)] \\ &+\; \frac{1}{2}\sum_j S_{ij}^{-1}\,I_{n-1}^{D(j)}[g_l(a)] + \frac{1}{2}\sum_j S_{ij}^{-1}\,I_n^{D+2}[\partial_j g_l(a)], \end{aligned} \tag{38}$$

where

$$\gamma_i = \sum_j S_{ij}^{-1}. \tag{39}$$

One can check that the above recursive relation is equivalent to the following recursive relation:

$$I_n^D[a_i f(a)] = P_{ij}\left(I_n^{D(j)}[f(a)] + I_n^{D+2}[\partial_j f(a)]\right) + \frac{\gamma_i}{\Delta} I_n^D[f(a)], \tag{40}$$

$$P_{ij} = \frac{1}{2}\left(S_{ij}^{-1} - \frac{\gamma_i\,\gamma_j}{\Delta}\right), \qquad \Delta = \sum_i \gamma_i. \tag{41}$$

The good point of this recursive relation is that all coefficients have no explicit dependence on the space-time dimension D and so it is well suited to compute the rational part.

We note that the above reasoning would go through if S is invertible. In the 2 mass and 1 mass triangle cases, S is not invertible. Nevertheless we can still use the above formulas by taking a limit from the general 3 mass triangle case. The point is that all we need is to have a well-behaved limit for the quantity P_{ij} and $\frac{\gamma_i}{\Delta}$. We have verified that directly taking the massless limit gives the same result as one would obtained by solving the non-singular system of equations (33) and (34).

From eq. (40) we see that higher dimensional integrals also appeared in the recursive relations. These higher dimensional tensor integrals can be reduced to even higher and/or lower point tensor integrals by using these recursive relations repeatedly. At the end only scalar integrals are left. These higher dimensional scalar integrals can be reduced to lower dimensional and lower point scalar integrals by using the following recursive relation:

$$I_n^{D+2}[1] = \frac{1}{(n-1-D)\,\Delta}\left[2\,I_n^D[1] - \sum_j \gamma_j\, I_{n-1}^{D(j)}[1]\right]. \tag{42}$$

This equation was derived from eq. (38) by setting $g_l(a) = 1$ and summing over i. The explicit dependence on the space-time dimension D in eq. (42) is very important. Otherwise all the rational coefficients in eq. (26) have no explicit dependence on D and there would be no rational part.

To be more specific, we give some explicit examples for bubble and triangle integrals in what follows.

For bubble integral we have

$$I_2^D[f(p)] \equiv \int \frac{\mathrm{d}^D p}{i\,\pi^{D/2}} \frac{f(p)}{p^2\,(p+K)^2}, \tag{43}$$

where K is the sum of momenta on one side of the bubble diagram. For $K^2 = 0$ this integral is 0 in dimensional regularization. So we will assume $K^2 \neq 0$ hereafter. By direct computation we have the following results ($D = 4 - 2\epsilon$):

$$I_2^D[1] = \frac{\gamma_\Gamma}{\epsilon(1-2\epsilon)}\,(-K^2)^{-\epsilon}, \tag{44}$$

$$I_2^D[p^\mu] = -\frac{K^\mu}{2}\,I_2^D[1], \tag{45}$$

$$I_2^D[a_1^2] = I_2^D[a_2^2] = \frac{2-\epsilon}{2(3-2\epsilon)}\,I_2^D[1]. \tag{46}$$

The recursive relation (42) becomes:

$$I_2^{D+2}[1] = \frac{K^2}{2(D-1)}\,I_2^D[1]. \tag{47}$$

This can be applied recursively to compute arbitrarily higher dimensional bubble integrals.

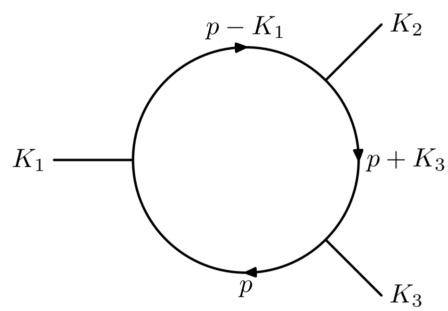

Figure 7: A generic three mass triangle diagram. The 2 mass triangle diagram is obtained by setting one of momentum to be massless, for example $K_1 = k_1$ and $k_i^1 = 0$.

A generic triangle diagram is shown in Fig. 7 and the integral is[3]

$$I_3^D[f(p)] \equiv \int \frac{\mathrm{d}^D p}{i\,\pi^{D/2}}\,\frac{f(p)}{p^2\,(p-K_1)^2\,(p+K_3)^2}. \tag{48}$$

[3]A minus sign is not included in the definition of $I_3^D[f(p)]$.

This integral is finite (free of ultra-violet (for degree 3 or less polynomial $f(p)$) and infrared divergences) for generic external momenta (i.e., $K_i^2 \neq 0$, $i = 1, 2, 3$). The explicit formula can be found in [3]. For degenerate case we have

$$I_3^D[1] = -\frac{1}{\epsilon^2} \frac{\Gamma(1-\epsilon)^2\Gamma(1+\epsilon)}{\Gamma(1-2\epsilon)} \frac{(-K_2^2)^{-\epsilon} - (-K_3^2)^{-\epsilon}}{(-K_2^2) - (-K_3^2)}, \quad K_1^2 = 0, \tag{49}$$

$$I_3^D[1] = -\frac{1}{\epsilon^2} \frac{\Gamma(1-\epsilon)^2\Gamma(1+\epsilon)}{\Gamma(1-2\epsilon)} (-K_3^2)^{-1-\epsilon}, \qquad K_1^2 = K_2^2 = 0. \tag{50}$$

In the next section we will use these results and the more general recursive relations to derive the rational part of the bubble, triangle and box integrals by making an expansion in the space-time parameter ϵ.

6 The rational part of the triangle integral

6.1 ϵ-expansion and the rational part: the bubble integral

The ϵ-expansion of the scalar bubble integral is:

$$I_2^D[1] = \frac{1}{\epsilon} + O(1). \tag{51}$$

By using this result in eq. (47) we have ($D = 4 - 2\epsilon$):

$$I_2^{D+2}[1] = \frac{K^2}{6} I_2^D[1] + \frac{K^2}{9} + O(\epsilon), \tag{52}$$

$$I_2^{D+4}[1] = \frac{(K^2)^2}{60} I_2^D[1] + \frac{4\,(K^2)^2}{225} + O(\epsilon). \tag{53}$$

We note that the first term on the right-hand of the above two equations still depends on D through the scalar integral $I_2^D[1]$. What is important is the second term which is a pure rational function and doesn't depend on the space-time dimension D. This is the rational function part we need to keep track of later.

By making an expansion in ϵ for eq. (46), we have:

$$I_2^D[a_1^2] = I_2^D[a_2^2] = \frac{1}{3} I_2^D[1] + \frac{1}{18} + O(\epsilon). \tag{54}$$

From this equation we read off the rational part of $I_2^D[a_1^2]$ and $I_2^D[a_2^2]$ as $\frac{1}{18}$. Translating back to the momentum integral the result is:

$$\begin{aligned} I_2^D[(\epsilon_1,p)\,(\epsilon_2,p)] &= \left(\frac{(\epsilon_1,K)\,(\epsilon_2,K)}{3} - \frac{K^2\,(\epsilon_1,\epsilon_2)}{6}\right) I_2^D[1] \\ &+ \frac{1}{18}\,((\epsilon_1,K)\,(\epsilon_2,K) - 2\,K^2\,(\epsilon_1,\epsilon_2)). \end{aligned} \tag{55}$$

This result has already appeared in [50] (eq. (31) on p. 133). We interpret the second term in the above equation as the rational part. If we discard the first term we can simply write:

$$I_2[(\epsilon_1,p)\,(\epsilon_2,p)] = \frac{1}{18}\,((\epsilon_1,K)\,(\epsilon_2,K) - 2\,K^2\,(\epsilon_1,\epsilon_2)), \tag{56}$$

by dropping also the explicit dependence of I_2 on D. However we still retain this dependence for higher dimensional Feynman integral and simply drop all the cut-constructible part. Explicitly we have:

$$I_2[1] = 0, \tag{57}$$

$$I_2[a_1^2] = I_2[a_2^2] = -I_2[a_1\,a_2] = \frac{1}{18}, \tag{58}$$

$$I_2^{D+2}[1] = \frac{K^2}{9}. \tag{59}$$

6.2 The rational part of the higher-dimensional scalar integral

For higher dimensional scalar integrals we can use the recursive relation eq. (42) to derive their rational parts.

For three mass triangle the explicit recursive relation is:

$$\begin{aligned} I_3^{3m\,(D+2)} &= \frac{1}{(2-D)\,\Delta}\Big[2\,s_1\,s_2\,s_3\,I_3^{3m} + s_2(s_1+s_3-s_2)\,I_2^{(1)} \\ &+ s_3\,(s_1+s_2-s_3)\,I_2^{(2)} + s_1\,(s_2+s_3-s_1)\,I_2^{(3)}\Big], \end{aligned} \tag{60}$$

$$\Delta = s_1^2 + s_2^2 + s_3^2 - 2(s_1\,s_2 + s_1\,s_3 + s_2\,s_3), \tag{61}$$

where $s_i = K_i^2$, $i = 1,2,3$.

For 3 mass triangle integral $I_3^D[1]$ is regular as $D \to 4$. By using the ϵ-expansion of I_2^D we have:

$$I_3^{3m\,(D+2)}[1] \;=\; -\frac{1}{2\,\Delta}\,\Big[2\,s_1\,s_2\,s_3\,I_3^{3m} + s_2(s_1+s_3-s_2)\,I_2^{(1)} \\ +s_3\,(s_1+s_2-s_3)\,I_2^{(2)} + s_1\,(s_2+s_3-s_1)\,I_2^{(3)}\Big] + \frac{1}{2} + O(\epsilon). \quad (62)$$

So the rational part of (the Feynman parameter integral) $I_3^{D+2}[1]$ is $\frac{1}{2}$. This result also applies to the 2 mass and 1 mass triangle, although there are intricacies of infrared divergences. This can be explicitly checked by using the following explicit recursive relations for the 2 mass and 1 mass triangle integrals:

$$I_3^{2m\,(D+2)} \;=\; \frac{1}{D-2}\left(\frac{s_1}{s_1-s_2}\,I_2^{(3)} - \frac{s_2}{s_1-s_2}\,I_2^{(1)}\right), \quad (63)$$

$$I_3^{1m\,(D+2)} \;=\; \frac{1}{D-2}\;I_2^{(3)}, \quad (64)$$

and the explicit formulas for $I_2^{D(i)}$'s. We note that the above recursive relations can be derived either by taking the limit $s_1 = K_1^2 \to 0$ (2 mass) or $s_{1,2} = K_{1,2}^2 \to 0$ (1 mass) or by solving eqs. (33) and (34).

The end result of the above analysis is

$$I_3^{D+2}[1] = \frac{1}{2} + c(s)\,I_3^D[1] + \sum_i c_i(s) I_2^{D(i)}[1] + O(\epsilon), \quad (65)$$

where $c(s)$ and $c_i(s)$'s are (rational) functions of the external kinematic variables s_i and the polarization vectors. This formula applies to all possible triangle integrals. This shows explicitly that the rational part of the higher dimensional scalar integral $I_3^{D+2}[1]$ is a purely ultra-violet effect.

Setting the cut-constructible part to 0, we can effectively write the following formula for the rational part:

$$I_3^{D+2}[1] = \frac{1}{2}. \quad (66)$$

We have also studied in detail the box integrals by using the explicit formulas of [3]. We checked all the degenerate cases. The end result can be simply stated as follows:

$$\begin{aligned} I_4^{D+2}[1] &= I_4^{D+2}[a_i] = 0, &(67)\\ I_4^{D+4}[1] &= \frac{1}{6} + \frac{1}{9} = \frac{5}{18}, &(68)\end{aligned}$$

by dropping all the cut-constructible part. For 5-point integral we have

$$\begin{aligned} I_5^{D+2}[1] &= I_5^{D+2}[a_i] = I_5^{D+2}[a_i\, a_j] = 0, &(69)\\ I_5^{D+4}[1] &= I_5^{D+4}[a_i] = 0, &(70)\\ I_5^{D+6}[1] &= \frac{13}{144}, &(71)\end{aligned}$$

although they were not used in the computation of the QCD amplitude.

The rational part for the tensor integral is computed by first transforming it into Feynman parameter integrals and then use the recursive relations. The recursive relations for the rational part is exactly the same as for the complete Feynman integral. However all lower degree ($m-2$ or less for m-point) Feynman integral can be set to zero by BDDK theorem. Effectively the recursive relations are truncated. In the next few sections we compute explicitly the rational part for triangle and box integrals. From hereafter all Feynman integrals are meant the rational part of the Feyman integral, except explicitly stated otherwise.

6.3 The triangle integrals: the general case

For triangle integrals we will consider 2 cases: the 2 mass triangle and the 3 mass triangle, although the 2 mass case can be obtained simply by setting one of the mass to be 0. The reason for doing this is that the formulas simplified greatly for the 2 mass triangle integral. For some computations one may only need the simplified formulas (for example the 5-gluon amplitude in [68]). We will first give the formulas for the general 3 mass triangle integrals.

The 3 mass triangle diagram is shown in Fig. 7 and the external momenta are denoted by K_i ($i = 1, 2, 3$). We use the convention of denoting a light-like momentum by a lower case k, i.e. $k^2 = 0$.

We will give the rational part for both the degree 3 and degree 2 polynomials[4].

[4]It is also possible to derive the rational part for higher degree polynomials but they have no practical usage in application to computations in the electroweak theory and QCD.

First we make the following definitions:

$$I_3(\epsilon_1,\epsilon_2,\epsilon_3) \equiv \int \frac{\mathrm{d}^D p}{i\pi^{D/2}} \frac{(\epsilon_1,p)\,(\epsilon_2,p-K_1)\,(\epsilon_3,p+K_3)}{p^2(p-K_1)^2(p+K_3)^2}, \tag{72}$$

$$I_3(\epsilon_1,\epsilon_2) \equiv \int \frac{\mathrm{d}^D p}{i\pi^{D/2}} \frac{(\epsilon_1,p)\,(\epsilon_2,p)}{p^2(p-K_1)^2(p+K_3)^2}. \tag{73}$$

The actual computation of the rational part is done by using Feynman parametrization and we have:

$$\begin{aligned} I_3(\epsilon_1,\epsilon_2,\epsilon_3) &= -I_3[(\epsilon_1,a)\,(\epsilon_2,a)(\epsilon_3,a)] \\ &+ \sum_{i=1}^{3}(\epsilon_i,\epsilon_{i+1}) I_3^{D+2}[(\epsilon_{i+2},a)], \end{aligned} \tag{74}$$

$$I_3(\epsilon_1,\epsilon_2) = -I_3[(\epsilon_1,a)\,(\epsilon_2,a)] + \frac{1}{2}\,(\epsilon_1,\epsilon_2), \tag{75}$$

by using the previous result for $I_3^{D+2}[1]$. In the above we have used the following shortened notation:

$$(\epsilon_1,a) = (\epsilon_1,K_1)\,a_2 - (\epsilon_1,K_3)\,a_3, \tag{76}$$

$$(\epsilon_2,a) = (\epsilon_2,K_2)\,a_3 - (\epsilon_2,K_1)\,a_1, \tag{77}$$

$$(\epsilon_3,a) = (\epsilon_3,K_3)\,a_1 - (\epsilon_3,K_2)\,a_2, \tag{78}$$

where a_i's are Feynman parameters as used in Sect. 5. In order to give compact formulas for the various quantities appearing in eqs. (74) and (75), we first define the following functions:

$$\begin{aligned} F_0(s_1,s_2,s_3) &\equiv -I_3[a_1a_2a_3] \\ &= \frac{10\,s_1s_2s_3}{3\Delta^2} + \frac{(s_1+s_2+s_3)}{6\Delta}, \end{aligned} \tag{79}$$

$$\begin{aligned} F_1(s_1,s_2,s_3) &\equiv -I_3[a_1a_2(a_2+a_3)] \\ &= \frac{5\,(s_1+s_2-s_3)\,s_2s_3}{3\Delta^2} + \frac{(s_1-s_3)}{3\Delta}, \end{aligned} \tag{80}$$

$$\begin{aligned} F_2(s_1,s_2,s_3) &\equiv -I_3[a_2a_3(a_3+a_1)] \\ &= \frac{5\,(s_2+s_3-s_1)\,s_3s_1}{3\Delta^2} + \frac{(s_2-s_1)}{3\Delta}, \end{aligned} \tag{81}$$

$$\begin{aligned} F_3(s_1,s_2,s_3) &\equiv -I_3[a_3a_1(a_1+a_2)] \\ &= \frac{5\,(s_3+s_1-s_2)\,s_1s_2}{3\Delta^2} + \frac{(s_3-s_2)}{3\Delta}. \end{aligned} \tag{82}$$

where $s_i = K_i^2$ and $\Delta = s_1^2 + s_2^2 + s_3^2 - 2(s_1 s_2 + s_2 s_3 + s_3 s_1)$ is the "Gram determinant" for the triangle diagram.

By using these function we have

$$
\begin{aligned}
I_3[(\epsilon_1, a)\,(\epsilon_2, a)\,(\epsilon_3, a)] = {} & F_0(s_1, s_2, s_3)((\epsilon_1, K_1)\,(\epsilon_2, K_1)\,(\epsilon_3, K_2) \\
& +(\epsilon_1, K_3)\,(\epsilon_2, K_2)\,(\epsilon_3, K_2) + (\epsilon_1, K_3)\,(\epsilon_2, K_1)\,(\epsilon_3, K_3) \\
& +(\epsilon_1, K_3)\,(\epsilon_2, K_1)\,(\epsilon_3, K_2) - (\epsilon_1, K_1)\,(\epsilon_2, K_2)\,(\epsilon_3, K_3)) \\
& + \sum_{i=1}^{3} (\epsilon_1, K_i)\,(\epsilon_2, K_i)\,(\epsilon_3, K_i)\, F_i(s_1, s_2, s_3) \\
& + \frac{1}{2\,\Delta}\Big((s_1 - s_2 - s_3)\,(\epsilon_1, K_1)\,(\epsilon_2, K_1)\,(\epsilon_3, K_3) \\
& + (s_2 - s_3 - s_1)\,(\epsilon_1, K_1)\,(\epsilon_2, K_2)\,(\epsilon_3, K_2) \\
& + (s_3 - s_1 - s_2)\,(\epsilon_1, K_3)\,(\epsilon_2, K_2)\,(\epsilon_3, K_3)\Big), \qquad (83)
\end{aligned}
$$

$$
\begin{aligned}
I_3^{D+2}[(\epsilon_1, a)] &= (\epsilon_1, K_1 - K_3)\left(\frac{7}{36} + \frac{s_2(s_3 + s_1 - s_2)}{12\Delta}\right) \\
& \quad +(\epsilon_1, K_2)\,\frac{(s_1 - s_3)(s_3 + s_1 - s_2)}{12\Delta}, \qquad (84) \\
I_3^{D+2}[(\epsilon_2, a)] &= (\epsilon_2, K_2 - K_1)\left(\frac{7}{36} + \frac{s_3(s_1 + s_2 - s_3)}{12\Delta}\right) \\
& \quad +(\epsilon_2, K_3)\,\frac{(s_2 - s_1)(s_1 + s_2 - s_3)}{12\Delta}, \qquad (85) \\
I_3^{D+2}[(\epsilon_3, a)] &= (\epsilon_3, K_3 - K_2)\left(\frac{7}{36} + \frac{s_1(s_2 + s_3 - s_1)}{12\Delta}\right) \\
& \quad +(\epsilon_3, K_1)\,\frac{(s_3 - s_2)(s_2 + s_3 - s_1)}{12\Delta}, \qquad (86)
\end{aligned}
$$

and

$$
\begin{aligned}
I_3[(\epsilon_i, a)\,(\epsilon_j, a)] &= \frac{1}{2\,\Delta}\Big(s_1\,((\epsilon_i, K_2)\,(\epsilon_j, K_3) + (\epsilon_i, K_3)\,(\epsilon_j, K_2)) \\
& + \quad s_2\,((\epsilon_i, K_3)\,(\epsilon_j, K_1) + (\epsilon_i, K_1)\,(\epsilon_j, K_3)) \\
& +s_3\,((\epsilon_i, K_1)\,(\epsilon_j, K_2) + (\epsilon_i, K_2)\,(\epsilon_j, K_1))\Big) - \frac{1}{2}\,(\epsilon_i, \epsilon_j). \qquad (87)
\end{aligned}
$$

We note that some formulas (and also the function F_i) in the above are related by permutations. We purposely wrote down the complete formulas in order to see the pattern.

6.4 The triangle integrals: the two mass case

For 2 mass triangle we set $K_1 = k_1$. Then we can derive the simplified formulas in the 2 mass triangle case by setting $s_1 = 0$ in the above formulas. Also we redefine (ϵ_3, a) to be $(\epsilon_3, k_1)\, a_2 - (\epsilon_3, K_3)\, a_3$. This corresponds to the change of $(\epsilon_3, p + K_3)$ to (ϵ_3, p). This gives a more symmetric form for the result, as one can see from the Feynman integral representation by doing the transformation $2 \leftrightarrow 3$. We list all the explicit formulas here because they are used heavily in the the actual computation of the 6 particle amplitude. This is necessary to obtain compact analytic formulas for the rational part of the amplitude. We have:

$$
\begin{aligned}
I_3[(\epsilon_1, a)\,(\epsilon_2, a)\,(\epsilon_3, a)] &= \frac{(s_2 + s_3)}{6(s_2 - s_3)^2}\,(\epsilon_1, K_2)\,(\epsilon_2, k_1)\,(\epsilon_3, k_1) \\
&\quad + \frac{(\epsilon_1, K_2)}{6(s_2 - s_3)}\,((\epsilon_2, k_1)\,(\epsilon_3, K_3) - (\epsilon_2, K_2)\,(\epsilon_3, k_1)). \qquad (88)
\end{aligned}
$$

$$
\begin{aligned}
I_3^{D+2}[(\epsilon_1, a)] &= \frac{1}{9}\,(\epsilon_1, K_2), \qquad (89) \\
I_3^{D+2}[(\epsilon_2, a)] &= -\frac{7}{36}\,(\epsilon_2, k_1) + \frac{1}{9}\,(\epsilon_2, K_2) - \frac{(s_2 + s_3)\,(\epsilon_2, k_1)}{12(s_2 - s_3)}, \qquad (90) \\
I_3^{D+2}[(\epsilon_3, a)] &= \frac{7}{36}\,(\epsilon_3, k_1) - \frac{1}{9}\,(\epsilon_3, K_3) - \frac{(s_2 + s_3)\,(\epsilon_3, k_1)}{12(s_2 - s_3)}. \qquad (91)
\end{aligned}
$$

$$
\begin{aligned}
I_3[(\epsilon_i, a)\,(\epsilon_j, a)] &= -\frac{(s_2 + s_3)}{2(s_2 - s_3)^2}\,(\epsilon_i, k_1)\,(\epsilon_j, k_1) \\
&\quad - \frac{((\epsilon_i, k_1)\,(\epsilon_j, K_2 - K_3) + (\epsilon_j, k_1)\,(\epsilon_i, K_2 - K_3))}{4\,(s_2 - s_3)}. \qquad (92)
\end{aligned}
$$

We note that in the last formula the double pole term is absent if one of the polarization vector is associated with the first momentum k_1 and satisfies the physical condition $(\epsilon, k_1) = 0$.

By using the above results we have

$$
I_3(\epsilon_1, \epsilon_2) \equiv \int \frac{\mathrm{d}^D p}{i\pi^{D/2}}\, \frac{(\epsilon_1, p)\,(\epsilon_2, p)}{p^2 (p - k_1)^2 (p + K_3)^2}
$$

$$
\begin{aligned}
&= \frac{1}{2}(\epsilon_1,\epsilon_2) + \frac{(K_2^2+K_3^2)}{2(K_2^2-K_3^2)^2}(\epsilon_1,k_1)(\epsilon_2,k_1) \\
&+ \frac{((\epsilon_1,K_2)(\epsilon_2,k_1)-(\epsilon_1,k_1)(\epsilon_2,K_3))}{2(K_2^2-K_3^2)}, \qquad (93)\\
I_3(\epsilon_1,\epsilon_2) &= \frac{1}{2}(\epsilon_1,\epsilon_2) + \frac{(\epsilon_1,K_2)(\epsilon_2,k_1)}{2(K_2^2-K_3^2)}, \qquad (\epsilon_1,k_1)=0, \qquad (94)
\end{aligned}
$$

and

$$
\begin{aligned}
I_3(\epsilon_i) &\equiv \int \frac{\mathrm{d}^D p}{i\pi^{D/2}} \frac{(\epsilon_1,p)(\epsilon_2,p-k_1)(\epsilon_3,p)}{p^2(p-k_1)^2(p+K_3)^2} \\
&= \frac{1}{36}\Big((\epsilon_2,4\,K_2-7\,k_1)(\epsilon_1,\epsilon_3)-(2\leftrightarrow 3)+4(\epsilon_1,K_2)(\epsilon_2,\epsilon_3)\Big) \\
&- \frac{(K_2^2+K_3^2)}{6\,(K_2^2-K_3^2)^2}(\epsilon_1,K_2)(\epsilon_2,k_1)(\epsilon_3,k_1) \\
&- \frac{(\epsilon_1,K_2)((\epsilon_2,k_1)(\epsilon_3,K_3)-(\epsilon_2,K_2)(\epsilon_3,k_1))}{6\,(K_2^2-K_3^2)} \\
&- \frac{(K_2^2+K_3^2)}{12\,(K_2^2-K_3^2)}((\epsilon_1,\epsilon_2)(\epsilon_3,k_1)+(\epsilon_1,\epsilon_3)(\epsilon_2,k_1)). \qquad (95)
\end{aligned}
$$

The above formula is anti-symmetric under the exchange $2 \leftrightarrow 3$ by noting $(\epsilon_1,K_2)=-(\epsilon_1,K_3)$. For later application we denote the 2 mass rational part of I_3 by $I_3^{2m(i)}$ where i denote the massless external line. To distinguish the two possible 2 mass triangle diagrams with the same massless external line in the 6-gluon amplitude case, we put a tilde on I_3^{2m} for one of the 2 mass triangle integral with 1, 3 and 2 external momenta in a clockwise direction. Referring to Fig. 7, $I_3^{2m(i)}$ has external momenta $\{k_i, k_{i+1}+k_{i+2}, k_{i+3}+k_{i+4}+k_{i+5}\}$, whereas $\tilde{I}_3^{2m(i)}$ has external momenta $\{k_i, k_{i+1}+k_{i+2}+k_{i+3}, k_{i+4}+k_{i+5}\}$.

7 The rational part of the box integrals

A generic box diagram is shown in Fig. 8. The kinematic variables s and t are defined by following the standard notation:

$$
s=(K_1+K_2)^2=(K_3+K_4)^2, \qquad t=(K_2+K_3)^2=(K_4+K_1)^2. \qquad (96)
$$

For our purpose of computing up to 6-gluon amplitude, we need to consider up to two mass boxes. There are two kinds of 2 mass boxes: the 2 mass hard box and 2 mass easy box. By a judicious choice of reference momenta the 2 mass hard box doesn't show up in the computation of MHV amplitudes. So we will discuss only the 2 mass case here and set $K_1 = k_1$, etc. by following the convention of writing the light-like momenta in lower case k. The one mass box case is obtained as a special case of the 2 mass easy box case by setting further $K_2 = k_2$. We will list the explicit formula for 1 mass box for quick reference and the easy of use.

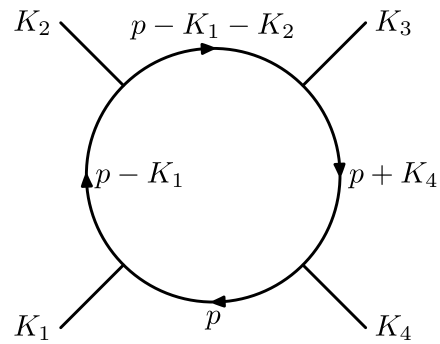

Figure 8: A generic box diagram. p is the internal momenta between K_4 and K_1. Other internal momenta are also shown explicitly.

7.1 The box integrals of degree 3 polynomial: the 2 mass easy case

Generally we need to compute the rational part of the following box integral:

$$I_4(\epsilon_1, \epsilon_2, \epsilon_3) \equiv \int \frac{\mathrm{d}^D}{i\pi^{D/2}} \frac{(\epsilon_1, p)\,(\epsilon_2, p)\,(\epsilon_3, p)}{p^2(p-K_1)^2(p-K_1-K_2)^2(p+K_4)^2}. \tag{97}$$

The complete rational part in the general case is quite complicated and should be avoided. From our experience it is always the case that at least one of the polarization vector satisfies the physical condition for one of the massless external momenta k_1 or k_3. In fact one can always expand an arbitrary 4-dimensional vector in terms of the 2 independent spinors of the two external massless external momenta. So we can assume that ϵ_1 satisfies the physical condition: $(\epsilon_1, k_1) = 0$. To be specific we can take $\epsilon_1 = \eta\lambda_1$. The negative helicity case can be obtained from this case (the positive helicity one) by conjugation. If one of polarization satisfies the physical condition for k_3 we can rotate the 2 mass easy box diagram by π and relabel k_3 as k_1.

By explicit computation, we found that if the reference momentum of ϵ_1 is k_3, the rational part becomes quite simple and is given as follows:

$$I_4(\lambda_3\tilde{\lambda}_1, \epsilon_2, \epsilon_3) = \frac{\langle 3|K_2|1]}{2}\left[\frac{(\epsilon_2, k_3)\,(\epsilon_3, k_3)}{(K_2^2 - t)(K_4^2 - s)} - \frac{(\epsilon_2, k_1)\,(\epsilon_3, k_1)}{(K_2^2 - s)(K_4^2 - t)}\right]. \quad (98)$$

By using this result, the computation of the rational part of the degree 3 polynomial can be proceeded by changing the reference momentum of ϵ_1 to k_3. This is equivalent to expanding the spinor in terms of $\lambda_{1,3}$:

$$\eta = \frac{\langle \eta\, 3\rangle}{\langle 1\, 3\rangle}\,\lambda_1 + \frac{\langle \eta\, 1\rangle}{\langle 3\, 1\rangle}\,\lambda_3. \quad (99)$$

In so doing we also generate 2 triangle diagrams which have been computed in the last subsection. Explicitly we have:

$$\begin{aligned} I_4(\eta\tilde{\lambda}_1, \epsilon_2, \epsilon_3) &= \frac{\langle \eta\, 1\rangle}{\langle 3\, 1\rangle}\frac{\langle 3|K_2|1]}{2}\left[\frac{(\epsilon_2, k_3)\,(\epsilon_3, k_3)}{(K_2^2 - t)(K_4^2 - s)} - \frac{(\epsilon_2, k_1)\,(\epsilon_3, k_1)}{(K_2^2 - s)(K_4^2 - t)}\right] \\ &+ \frac{\langle \eta\, 3\rangle}{\langle 1\, 3\rangle}\left[\tilde{I}_3^{2m}(\epsilon_2, \epsilon_3) - I_3^{2m}(\epsilon_2, \epsilon_3)\right], \quad (100) \\ \tilde{I}_3^{2m}(\epsilon_2, \epsilon_3) &= \frac{1}{2}\,(\epsilon_2, \epsilon_3) + \frac{(K_2^2 + t)}{2(K_2^2 - t)^2}\,(\epsilon_2, k_3)\,(\epsilon_3, k_3) \\ &+ \frac{((\epsilon_2, k_3)\,(\epsilon_3, K_2) - (\epsilon_3, k_3)\,(\epsilon_2, K_4 + k_1))}{2(K_2^2 - t)}, \\ I_3^{2m}(\epsilon_2, \epsilon_3) &= \frac{1}{2}\,(\epsilon_2, \epsilon_3) + \frac{(K_4^2 + s)}{2(K_4^2 - s)^2}\,(\epsilon_2, k_3)\,(\epsilon_3, k_3) \\ &+ \frac{((\epsilon_2, k_3)\,(\epsilon_3, K_4) - (\epsilon_3, k_3)\,(\epsilon_2, k_1 + K_2))}{2(K_4^2 - s)}. \quad (101) \end{aligned}$$

We note that no 3 mass triangle integrals are generated in this reduction. Taking into account the anti-symmetric property of the product $(\lambda_3\tilde{\lambda}_1, K_2) = -(\lambda_3\tilde{\lambda}_1, K_4)$, the above formula is actually symmetric (which must be the case) under the interchange $1 \leftrightarrow 3$ and $2 \leftrightarrow 4$ (the polarization vectors are kept fixed because they should be invariant under permutation by themselves).

7.2 The box integrals of degree 3 polynomial: the 2 mass hard case

For the 2 mass hard box case, we follow the same strategy. For $\epsilon_1 = \lambda_1\tilde{\lambda}_2$ or $\lambda_2\tilde{\lambda}_1$, we have

$$
\begin{aligned}
I_4^{2mh}(\lambda_1\tilde{\lambda}_2, \epsilon_2, \epsilon_3) &= \frac{\langle 1|K_3|2]}{4\,\delta}\, I_4(\epsilon_2, \epsilon_3), && (102)\\
I_4^{2mh}(\lambda_2\tilde{\lambda}_1, \epsilon_2, \epsilon_3) &= \frac{\langle 2|K_3|1]}{4\,\delta}\, I_4(\epsilon_2, \epsilon_3), && (103)
\end{aligned}
$$

where

$$
\begin{aligned}
I_4(\epsilon_2, \epsilon_3) &= (\epsilon_2, k_1)(\epsilon_3, K_4) + (\epsilon_2, K_4)(\epsilon_3, k_1) + (\epsilon_2, k_2)(\epsilon_3, K_3)\\
&+ (\epsilon_2, K_3)(\epsilon_3, k_2) - \frac{1}{\Delta}\Big[2\,(K_3^2 K_4^2 - t^2 + \delta)(\epsilon_2, k_{12})(\epsilon_3, k_{12})\\
&+ (K_3^2 + K_4^2 - s - 2\,t)\,((K_3^2 - K_4^2 + s)(\epsilon_2, K_4)(\epsilon_3, K_4)\\
&\quad +(K_4^2 - K_3^2 + s)\,(\epsilon_2, K_3)(\epsilon_3, K_3))\Big]\\
&+ \frac{K_4^2 + t}{K_4^2 - t}\,(\epsilon_2, k_1)(\epsilon_3, k_1) + \frac{K_3^2 + t}{K_3^2 - t}\,(\epsilon_2, k_2)(\epsilon_3, k_2), && (104)
\end{aligned}
$$

and

$$
\begin{aligned}
\delta &= K_3^2\, K_4^2 - (K_3^2 + K_4^2)\, t + (s + t)\, t, && (105)\\
\Delta &= \Delta(k_{12}^2, K_3^2, K_4^2), && (106)\\
\Delta(s_1, s_2, s_3) &= s_1^2 + s_2^2 + s_3^2 - 2(s_1 s_2 + s_2 s_3 + s_3 s_1), && (107)
\end{aligned}
$$

are function of the external momentum invariants. In particular Δ is the Gram determinant of the three mass triangle integral arising from the tensor reduction of the two mass hard box integral.

The general case is obtained by changing the reference momentum:

$$
\begin{aligned}
I_4(\eta\tilde{\lambda}_1, \epsilon_2, \epsilon_3) &= \frac{\langle \eta\, 1\rangle}{\langle 2\, 1\rangle}\, I_4(\lambda_2\tilde{\lambda}_1, \epsilon_2, \epsilon_3)\\
&+ \frac{\langle \eta\, 2\rangle}{\langle 1\, 2\rangle}\left[I_3^{2m}(\epsilon_2, \epsilon_3) - I_3^{3m}(\epsilon_2, \epsilon_3)\right], && (108)\\
I_3^{2m}(\epsilon_2, \epsilon_3) &= \frac{1}{2}\,(\epsilon_2, \epsilon_3) + \frac{(K_3^2 + t)}{2(K_3^2 - t)^2}\,(\epsilon_2, k_2)\,(\epsilon_3, k_2)\\
&+ \frac{((\epsilon_2, k_2)\,(\epsilon_3, K_3) - (\epsilon_3, k_2)\,(\epsilon_2, K_4 + k_1))}{2(K_3^2 - t)}, && (109)
\end{aligned}
$$

and $I_3^{3m}(\epsilon_2, \epsilon_3)$ is the three mass triangle integral with external momenta $\{k_{12}, K_3, K_4\}$. There are two triangle diagrams from this reduction and one is a three mass triangle diagram.

7.3 The box integrals of degree 4 polynomial: the 2 mass easy case

Now we discuss the computation of the rational part for the degree 4 polynomial. First we define

$$I_4(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4) \equiv \int \frac{\mathrm{d}^D p}{i\pi^{D/2}} \frac{(\epsilon_1, p)\,(\epsilon_2, p-K_1)\,(\epsilon_3, p-K_{12})\,(\epsilon_4, p+K_4)}{p^2(p-K_1)^2(p-K_{12})^2(p+K_4)^2}, \tag{110}$$

where $K_{12} = K_1 + K_2$. By using Feynman parametrization we have

$$\begin{aligned} I_4(\epsilon_i) &= I_4[(\epsilon_1, a)(\epsilon_2, a)(\epsilon_3, a)(\epsilon_4, a)] - \sum_{i<j}(\epsilon_i, \epsilon_j) I_4^{D+2}[(\epsilon_k, a)(\epsilon_l, a)] \\ &+ ((\epsilon_1, \epsilon_2)(\epsilon_3, \epsilon_4) + (\epsilon_1, \epsilon_3)(\epsilon_2, \epsilon_4) + (\epsilon_1, \epsilon_4)(\epsilon_2, \epsilon_3))\, I_4^{D+4}[1]. \end{aligned} \tag{111}$$

We also assume that $\epsilon_{1,3}$ satisfy the physical conditions in the 2 mass easy box integral and $\epsilon_{1,2}$ satisfy the physical conditions for the 2 mass hard box integral. As we said before this can always be done by expanding all polarization vector in terms of the 2 spinors from the 2 massless momenta.

The direct calculation of the rational part of the two mass easy box integral gives rather complicated formulas for generic polarization vectors (even after using the physical condition for $\epsilon_{1,3}$). By choosing appropriate reference momenta for $\epsilon_{1,3}$, the result simplifies greatly. In particular the reference momentum of ϵ_1 should be k_3 and the reference momentum of ϵ_3 should be k_1. In some sense this is equivalent to the tensor reduction with the two factors $(\epsilon_1, p)(\epsilon_3, p - k_1 - K_2)$. The explicit results are given as follows:

$$\begin{aligned} I_4(\lambda_3\tilde{\lambda}_1, \epsilon_2, \lambda_1\tilde{\lambda}_3, \epsilon_4) &= -\frac{1}{4}\left(\frac{K_2^2+s}{K_2^2-s} + \frac{K_4^2+t}{K_4^2-t}\right)(\epsilon_2, k_1)(\epsilon_4, k_1) \\ &-\frac{1}{4}\left(\frac{K_2^2+t}{K_2^2-t} + \frac{K_4^2+s}{K_4^2-s}\right)(\epsilon_2, k_3)(\epsilon_4, k_3) - \frac{5}{9}(k_1, k_3)(\epsilon_2, \epsilon_4) \\ &+ \frac{4}{9}\Big((\epsilon_2, k_1)(\epsilon_4, k_3) + (\epsilon_2, k_3)(\epsilon_4, k_1)\Big), \end{aligned} \tag{112}$$

$$I_4(\lambda_1\tilde{\lambda}_3, \epsilon_2, \lambda_1\tilde{\lambda}_3, \epsilon_4) = \frac{5}{9}\langle 1|\epsilon_2|3]\,\langle 1|\epsilon_4|3] + \frac{\langle 1|K_2|3]^2}{3}\left[\frac{(\epsilon_2, k_1)\,(\epsilon_4, k_1)}{(K_2^2 - s)\,(K_4^2 - t)} + \frac{(\epsilon_2, k_3)\,(\epsilon_4, k_3)}{(K_2^2 - t)\,(K_4^2 - s)}\right]. \quad (113)$$

Other cases can be either obtained by conjugation or relabelling $k_{1,3}$. In fact $I_4(\lambda_1\tilde{\lambda}_3, \epsilon_2, \lambda_3\tilde{\lambda}_1, \epsilon_4) = I_4(\lambda_3\tilde{\lambda}_1, \epsilon_2, \lambda_1\tilde{\lambda}_3, \epsilon_4)$ as it is conjugation invariant.

For the general case, we use the reduction formula by changing the reference appropriately. For example we have:

$$\begin{aligned} I_4(\eta_1\tilde{\lambda}_1, \epsilon_2, \eta_3\tilde{\lambda}_3, \epsilon_4) &= \frac{\langle \eta_1\, 3\rangle}{\langle 1\, 3\rangle}\, I_4(k_1, \epsilon_2, \epsilon_3, \epsilon_4) + \frac{\langle \eta_3\, 1\rangle}{\langle 3\, 1\rangle}\, I_4(\epsilon_1, \epsilon_2, k_3, \epsilon_4) \\ &- \frac{\langle \eta_1\, 1\rangle\,\langle \eta_3\, 3\rangle}{\langle 1\, 3\rangle^2}\, I_4(\lambda_3\tilde{\lambda}_1, \epsilon_2, \lambda_1\tilde{\lambda}_3, \epsilon_4) \\ &+ \frac{\langle \eta_1\, 3\rangle\,\langle \eta_3\, 1\rangle}{\langle 1\, 3\rangle^2}\, I_4(k_1, \epsilon_2, k_3, \epsilon_4). \end{aligned} \quad (114)$$

By using the explicit result of the 2 mass easy triangle integral we have

$$\begin{aligned} I_4(\epsilon_1, \epsilon_2, k_3, \epsilon_4) &= \frac{K_2^2 + s}{6(K_2^2 - s)^2}\,(\epsilon_1, K_2)(\epsilon_2, k_1)(\epsilon_4, k_1) \\ &+ \frac{K_4^2 + t}{6(K_4^2 - t)^2}\,(\epsilon_1, K_4)(\epsilon_2, k_1)(\epsilon_4, k_1) \\ &+ \frac{1}{12}\left(\frac{K_2^2 + s}{K_2^2 - s} + \frac{K_4^2 + t}{K_4^2 - t}\right)((\epsilon_1, \epsilon_2)(\epsilon_4, k_1) + (\epsilon_1, \epsilon_4)(\epsilon_2, k_1) \\ &+ \frac{(\epsilon_1, K_2)}{6(K_2^2 - s)}\,(\epsilon_2, k_1)(\epsilon_4, k_3) + \frac{(\epsilon_1, K_4)}{6(K_4^2 - t)}\,(\epsilon_2, k_3)(\epsilon_4, k_1) \\ &+ \left[\frac{(\epsilon_1, K_2)}{6\,(K_2^2 - s)} - \frac{(\epsilon_1, K_4)}{6\,(K_4^2 - t)}\right]((\epsilon_2, k_1)\,(\epsilon_4, K_4) - (\epsilon_2, K_2)\,(\epsilon_4, k_1)) \\ &+ \frac{1}{9}((\epsilon_1, \epsilon_2)(\epsilon_4, k_3) + (\epsilon_1, \epsilon_4)(\epsilon_2, k_3) + (\epsilon_2, \epsilon_4)(\epsilon_1, k_3)) \\ &+ \frac{1}{2}\,(\epsilon_2, k_1)\,(\epsilon_4, K_4)\left[\frac{(\epsilon_1, K_4)}{K_4^2 - t} - \frac{(\epsilon_1, K_2)}{K_2^2 - s}\right]. \end{aligned} \quad (115)$$

Except for the last term, this formula is invariant under the interchange

$2 \leftrightarrow 4$ $(s \leftrightarrow t)$. By setting $\epsilon_1 = k_1$ we get

$$I_4(k_1, \epsilon_2, k_3, \epsilon_4) = \frac{1}{18}(2(k_1, k_3)(\epsilon_2, \epsilon_4) - ((\epsilon_2, k_1)(\epsilon_2, k_3) + (\epsilon_2, k_3)(\epsilon_2, k_1))). \tag{116}$$

This agrees with the result by direct computation by first doing the tensor reduction and then using the result for bubble integrals. These formulas are quite useful for obtaining compact analytic formulas for QCD amplitude.

For easy reference we also gave here the relevant formulas in terms of Feynman parameters:

$$\begin{aligned} & I_4[(\tilde{\epsilon}_1, a)(\epsilon_2, a)(\tilde{\epsilon}_3, a)(\epsilon_4, a)] \\ & = \frac{(\tilde{\epsilon}_1, K_2)\,(\tilde{\epsilon}_3, K_2)}{3}\left(\frac{(\epsilon_2, k_1)\,(\epsilon_4, k_1)}{(K_2^2 - s)(K_4^2 - t)} + \frac{(\epsilon_2, k_3)\,(\epsilon_4, k_3)}{(K_2^2 - t)(K_4^2 - s)}\right) \\ & I_4^{D+2}[(\epsilon_2, a)(\epsilon_4, a)] = \frac{1}{6(K_2^2 + K_4^2 - s - t)}\Big[(\epsilon_2, k_1)\,(\epsilon_4, k_3) + (\epsilon_2, k_3)\,(\epsilon_4, k_1) \\ & - \;(K_2^2\, K_4^2 - s\, t)\Big(\frac{(\epsilon_2, k_1)\,(\epsilon_4, k_1)}{(K_2^2 - s)(K_4^2 - t)} + \frac{(\epsilon_2, k_3)\,(\epsilon_4, k_3)}{(K_2^2 - t)(K_4^2 - s)}\Big)\Big]. \end{aligned} \tag{117}$$

All other $I_4^{D+2}[(\epsilon_i, a)(\tilde{\epsilon}_j, a)]$'s are identically zero. In the above formulas $\tilde{\epsilon}_{1,3}$ satisfy 2 conditions: $(\tilde{\epsilon}_i, k_{1,3}) = 0$, i.e. the reference momentum of $\tilde{\epsilon}_1$ is k_3 and the reference momentum of $\tilde{\epsilon}_3$ is k_1. We also note when $(\tilde{\epsilon}_1, \tilde{\epsilon}_3)$ is not equal to 0, it cancels with the factor $(K_2^2 + K_4^2 - s - t) = (k_1, k_3) = -(\tilde{\epsilon}_1, \tilde{\epsilon}_3)$.

7.4 The box integrals of degree 4 polynomial: the 2 mass hard case

As before, the direct computation of the rational part of the two mass hard box gives very complicated algebraic expressions. In fact it gives the most complicated formula up to now. Changing the reference momenta simplifies a little bit, but the resulting formula is still not intelligible. The main complication comes from the presence of the 3 mass triangle integrals. To organize the final result, we proceed to do tensor reduction one more time. To begin with let's define the integral we want to compute:

$$\begin{aligned} & I_4^{2mh}(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4; c_3, c_4) \\ & \quad \equiv I_4[(\epsilon_1, p)(\epsilon_2, p - k_1)((\epsilon_3, p + K_4) + c_3)((\epsilon_4, p + K_4) + c_4)] \end{aligned}$$

$$= \int \frac{\mathrm{d}^D p}{i\pi^{D/2}} \frac{(\epsilon_1, p)(\epsilon_2, p-k_1)((\epsilon_3, p+K_4)+c_3)((\epsilon_4, p+K_4)+c_4)}{p^2\,(p-k_1)^2\,(p-k_{12})^2\,(p+K_4)^2}. \tag{118}$$

The external momenta are k_1, k_2, K_3 and K_4 as shown in Fig. 8 by setting $K_1 = k_1$ and $K_2 = k_2$. $k_{1,2}$ are the two massless external legs and $K_{3,4}$ are the two massive external legs. We set $t = (k_2 + K_3)^2 = (K_4 + k_1)^2$. We also require that $\epsilon_{1,2}$ satisfy the physical condition, i.e. $(\epsilon_1, k_1) = 0$ and $(\epsilon_2, k_2) = 0$. The formula for generic $\epsilon_{1,2}$ is beyond the scope of this paper. ($\epsilon_{3,4}$ are arbitrary 4-dimensional polarization vectors.)

There are 4 possible cases for $\epsilon_{1,2}$. The same helicity cases are quite easy. So we will only consider the difficult cases where $\epsilon_{1,2}$ have different helicities. To be definite we set $\epsilon_1 = \lambda_1\tilde{\eta}_1$ and $\epsilon_2 = \eta_2\tilde{\lambda}_2$. The opposite case can be obtained from this one simply by conjugation. We will give the explicit formula for this case at the end of this subsection.

To do tensor reduction for factors associated with the two massless external legs, we have:

$$T^{-+} \;=\; (\epsilon_1, p)\,(\epsilon_2, p-k_1) = \frac{\langle\eta_2|(p-k_1)\,k_2\,K_3\,k_1\,(p-k_1)|\tilde{\eta}_1\rangle}{\langle 2|K_3|1]}. \tag{119}$$

The middle factor in the above can be decomposed by moving the first factor of $(p-k_1)$ towards the second $(p-k_1)$. Explicitly we have:

$$(p-k_1)\,k_2\,K_3\,k_1\,(p-k_1) = T_1 + T_2, \tag{120}$$

$$T_1 \;=\; I^{(1)}\,(p-k_1)k_2K_3 + I^{(2)}\,(K_3k_1 - k_2K_3)p + I^{(3)}K_4k_1p, \tag{121}$$

$$\begin{aligned} T_2 \;&=\; p^2_{-2\epsilon}k_2K_3k_1 + (I^{(4)}-t)k_2k_1p \\ &+\; I^{(1)}(-I^{(2)}K_3 + I^{(3)}(k_2+K_3) - (I^{(4)}-t)k_2). \end{aligned} \tag{122}$$

Omitting the factor $\langle 2|K_3|1]$, we now compute the various terms. We have:

$$\begin{aligned} \langle\eta_2|T_2|\tilde{\eta}_1\rangle \;&=\; p^2_{-2\epsilon}\langle\eta_2|k_2K_3k_1|\tilde{\eta}_1\rangle \\ &+\; (I^{(4)}-t)(\langle\eta_2\,2\rangle\,[\tilde{\eta}_1\,1](\lambda_1\tilde{\lambda}_2, p) - I^{(2)}\,\langle\eta_2|k_2|\tilde{\eta}_1\rangle) \\ &+\; I^{(1)}\,(I^{(3)}\,\langle\eta_2|(k_2+K_3)|\tilde{\eta}_1\rangle - I^{(2)}\,\langle\eta_2|K_3|\tilde{\eta}_1\rangle). \end{aligned} \tag{123}$$

This gives the following rational terms:

$$A_2 \;=\; -\frac{1}{6}\,\langle\eta_2|k_2K_3k_1|\tilde{\eta}_1\rangle\,(\epsilon_3,\epsilon_4) - t\,\langle\eta_2\,2\rangle\,[\tilde{\eta}_1\,1]\,I_4^{2mh}(\lambda_1\tilde{\lambda}_2, \epsilon_3, \epsilon_4)$$

$$
\begin{aligned}
&+ \quad t\,\langle\eta_2|k_2|\tilde{\eta}_1]\,I_3^{3m}(\epsilon_3,\epsilon_4) + \langle\eta_2\,2\rangle\,[\tilde{\eta}_1\,1]\left(\frac{1}{2}(\langle 1|\epsilon_3|2]\,c_4 + \langle 1|\epsilon_4|2]\,c_3)\right.\\
&\quad \left.+\frac{1}{18}(\langle 1|\epsilon_3|2]\,\epsilon_4 + \langle 1|\epsilon_4|2]\,\epsilon_3, 7\,k_1 + 2\,k_2 + 9\,K_4)\right)\\
&+ \quad \frac{1}{18}\,\langle\eta_2\,2\rangle\,[\tilde{\eta}_1\,2]\,((\epsilon_3,k_{12})\,(\epsilon_4,k_{12}) - 2\,s_{12}\,(\epsilon_3,\epsilon_4))\\
&+ \quad \frac{1}{18}\,\langle\eta_2|(k_2+K_3)|\tilde{\eta}_1]((\epsilon_3,k_2+K_3)(\epsilon_4,k_2+K_3) - 2\,t\,(\epsilon_3,\epsilon_4))\\
&- \quad \frac{1}{18}\,\langle\eta_2|K_3|\tilde{\eta}_1]\,((\epsilon_3,K_3)(\epsilon_4,K_3) - 2K_3^2(\epsilon_3,\epsilon_4))
\end{aligned} \tag{124}
$$

The explicit formulas for the other 3 terms in T_1 are not quite illuminating and we refrain to write the explicit results here. We can write the result in terms of the rational part for the 3 mass and 2 mass triangle integrals arising from the tensor reduction. We have:

$$
\begin{aligned}
A_1 &= \langle 2|K_3|\tilde{\eta}_1\rangle(I_3^{2m}(\eta_2\tilde{\lambda}_2,\epsilon_3,\epsilon_4)\\
&+ I_3^{2m}(\eta_2\tilde{\lambda}_2,(c_3-(\epsilon_3,K_3))\epsilon_4 + (c_4+(\epsilon_4,K_4+k_1))\epsilon_3))\\
&+ I_3^{3m}(v,\epsilon_3,\epsilon_4) + I_3^{3m}(v,(c_3-(\epsilon_3,K_3))\epsilon_4 + c_4\epsilon_3), \qquad (125)\\
&+ \langle\eta_2|K_4|1]\,(\tilde{I}_3^{2m}(\lambda_1\tilde{\eta}_1,\epsilon_3,\epsilon_4)\\
&+ \tilde{I}_3^{2m}(\lambda_1\tilde{\eta}_1,(c_3-(\epsilon_3,k_2+K_3))\epsilon_4 + (c_4+(\epsilon_4,K_4))\epsilon_3), \qquad (126)
\end{aligned}
$$

where

$$
v = \langle\eta_2|K_3|1]\lambda_1\tilde{\eta}_1 + \langle\eta_2|K_3|2]\lambda_2\tilde{\eta}_1 - (k_2,K_3)\eta_2\tilde{\eta}_1. \tag{127}
$$

Combining the above results together we have:

$$
I_4^{2mh}(\lambda_1\tilde{\eta}_1,\eta_2\tilde{\lambda}_2,\epsilon_3,\epsilon_4;c_3,c_4) = \frac{A_1+A_2}{\langle 2|K_3|1]}. \tag{128}
$$

For the opposite helicity case the formula is:

$$
\begin{aligned}
&I_4^{2mh}(\eta_1\tilde{\lambda}_1,\lambda_2\tilde{\eta}_2,\epsilon_3,\epsilon_4;c_3,c_4) = \frac{1}{\langle 1|K_3|2]}\left[-\frac{1}{6}\,\langle\eta_1|k_1K_3k_2|\tilde{\eta}_2\rangle\,(\epsilon_3,\epsilon_4)\right.\\
&- \quad t\,\langle\eta_1\,1\rangle\,[\tilde{\eta}_2\,2]\,I_4^{2mh}(\lambda_2\tilde{\lambda}_1,\epsilon_3,\epsilon_4) + t\,\langle\eta_1|k_2|\tilde{\eta}_2]\,I_3^{3m}(\epsilon_3,\epsilon_4)\\
&+ \quad \langle\eta_1\,1\rangle\,[\tilde{\eta}_2\,2]\left(\frac{1}{2}(\langle 2|\epsilon_3|1]\,c_4 + \langle 2|\epsilon_4|1]\,c_3)\right.\\
&\quad \left.+\frac{1}{18}(\langle 2|\epsilon_3|1]\,\epsilon_4 + \langle 2|\epsilon_4|1]\,\epsilon_3, 7k_1 + 2k_2 + 9K_4)\right)
\end{aligned}
$$

$$
\begin{aligned}
+ &\quad \frac{1}{18}\,\langle \eta_1\, 2\rangle\,[\tilde\eta_2\, 2]\,((\epsilon_3, k_{12})\,(\epsilon_4, k_{12}) - 2\, s_{12}\,(\epsilon_3, \epsilon_4))\\
+ &\quad \frac{1}{18}\,\langle \eta_1|(k_2+K_3)|\tilde\eta_2]((\epsilon_3, k_2+K_3)(\epsilon_4, k_2+K_3) - 2\, t\,(\epsilon_3, \epsilon_4))\\
- &\quad \frac{1}{18}\,\langle \eta_1|K_3|\tilde\eta_2]\,((\epsilon_3, K_3)(\epsilon_4, K_3) - 2K_3^2(\epsilon_3, \epsilon_4))\\
+ &\quad \langle \eta_1|K_3|2](I_3^{2m}(\lambda_2\tilde\eta_2, \epsilon_3, \epsilon_4)\\
&\quad + I_3^{2m}(\lambda_2\tilde\eta_2, (c_3 - (\epsilon_3, K_3))\epsilon_4 + (c_4 + (\epsilon_4, K_4 + k_1))\epsilon_3))\\
+ &\quad I_3^{3m}(\tilde v, \epsilon_3, \epsilon_4) + I_3^{3m}(\tilde v, (c_3 - (\epsilon_3, K_3))\epsilon_4 + c_4\epsilon_3), \qquad (129)\\
+ &\quad \langle 1|K_4|\tilde\eta_2\rangle\,(\tilde I_3^{2m}(\eta_1\tilde\lambda_1, \epsilon_3, \epsilon_4)\\
&\quad + \tilde I_3^{2m}(\eta_1\tilde\lambda_1, (c_3 - (\epsilon_3, k_2+K_3))\epsilon_4 + (c_4 + (\epsilon_4, K_4))\epsilon_3)\Big], \qquad (130)
\end{aligned}
$$

where

$$
\tilde v = \langle 1|K_3|\tilde\eta_2\rangle\,\eta_1\tilde\lambda_1 + \langle 2|K_3|\tilde\eta_2\rangle\,\eta_1\tilde\lambda_2 - (k_2, K_3)\eta_1\tilde\eta_2. \qquad (131)
$$

8 Extra terms for box and triangle tensor reduction

8.1 Extra terms for box tensor reduction

In order to get a comparatively compact analytic formula for the 2 mass hard box integral, it is necessary to do tensor reduction because a direct computation of the rational part by recursive method gives a very complicated formula. It would be a better idea to organize it into lower degree and lower point integral. A naive tensor reduction directly in $D = 4$ would give an incorrect result because the box integral is ultra-violet divergent for degree 4 polynomial of momenta in the numerator. It turns out that the difference between the $D = 4$ tensor reduction and the correct tensor reduction is just a rational function. In this subsection we will compute this extra rational function explicitly.

For a degree 2 polynomial $g(\epsilon, k, p)$ in the internal momentum p, the general form of the $D = 4$ tensor reduction we used is as follows:

$$
g(\epsilon, k, p) = \tilde g(\epsilon, k, p) - p^2\, f(\epsilon, k). \qquad (132)
$$

We assume that all polarization vector ϵ and momenta k are 4-dimensional and the above relation is derived by assuming p is also a 4-dimensional momentum. We also assume that g and $\tilde{g}$ depends on the momentum p through scalar product (ϵ, p) and/or (k, p). Because we use FDH regularization [72], p is actually promoted to be in $D = 4-2\epsilon$ dimension in our later calculation of the rational part. This affects only the last term in eq. (132). For arbitrary dimensional internal momentum p, the above formula is still valid if we make the substitution:

$$p^2 \to p^2 - p_{D-4}^2. \tag{133}$$

That is, the extra dimensional part $(D - 4 = -2\epsilon)$ of the momentum p must be subtracted from p^2 which is actually stand for the scalar product in D dimension in the subsequent computation in four-dimensional helicity regularization scheme. Because pentagon and hexagon diagrams are ultraviolet convergent we can safely discard this term in the tensor reduction for 5- or 6-point diagrams. This term does give a non-vanishing contribution to the rational part for box and triangle integrals.

For box integral we have

$$\begin{aligned} A_{\mathrm{box}} &= I_4^{D=4-2\epsilon}[p_{-2\epsilon}^2\,(\epsilon_3, p)\,(\epsilon_4, p)] \\ &= \int_0^1 \mathrm{d}^4 a_i \delta(1-\sum_i a_i) \int_0^\infty \mathrm{d}T\, T^3 \\ &\quad \times \int \frac{\mathrm{d}^D p}{i\pi^{D/2}}\, p_{-2\epsilon}^2\,(\epsilon_3, p)\,(\epsilon_4, p)\,\mathrm{e}^{p^2 T - T\,a\cdot S\cdot a}, \end{aligned} \tag{134}$$

by transforming it into Feynman parameter integral and omitting terms which are vanishing in the limit $\epsilon \to 0$.

The integration over the momentum p can be done easily by splitting it into a $D = 4$ part and a (-2ϵ)-dimensional part. We have then

$$A_{\mathrm{box}} = \epsilon\,(-(\epsilon_3, \epsilon_4))\, I_4^{D+4}[1] = -\frac{1}{6}\,(\epsilon_3, \epsilon_4) + O(\epsilon). \tag{135}$$

By using this result the correct formula for computing the rational part of the box tensor integral is:

$$\begin{aligned} I_4[g(\epsilon, k, p)\,((\epsilon_3, p) + c_3)\,((\epsilon_4, p) + c_4))] &= I_4[(\tilde{g}(\epsilon, k, p) - p^2\, f(\epsilon, k)) \\ &\quad \times((\epsilon_3, p) + c_3)\,((\epsilon_3, p) + c_4))] - \frac{1}{6}\, f(\epsilon, k)\,(\epsilon_3, \epsilon_4). \end{aligned} \tag{136}$$

8.2 Extra terms in triangle tensor reduction

For triangle integral, using the same reduction formula as given in eq. (132) also gives incorrect results. The extra terms are given by the following formulas (p is the internal momentum between the external legs K_1 and K_2):

$$\begin{aligned} I_3[g(\epsilon, K, p)\,((\epsilon_3, p) + c_3)] &= I_3[(\tilde{g}(\epsilon, K, p) - p^2\, f(\epsilon, k)) \\ &\quad \times((\epsilon_3, p) + c_3)] - f(\epsilon, k)\left[\frac{1}{2}\,c_3 + \frac{1}{6}\,(\epsilon_3, K_1 - K_3)\right]. \end{aligned} \tag{137}$$

We note that the rational part $I_3[g(\epsilon, K, p)\,((\epsilon_3, p) + c_3)]$ is defined by the Feynman integral without the usual minus sign, just as we did before in eq. (48).

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