Invariant properties of representations under cleft extensions*

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September 7, 2005

Abstract

The main aim of this paper is to give invariant properties of representations of algebras under cleft extensions over a semisimple Hopf algebra. Firstly, we explain the concept of cleft extension and give the relation between cleft extension and crossed product which is the approach we depend upon. Then, by using of them, we prove that for a finite dimensional Hopf algebra H which is semisimple as well as its dual H^*, the representation type of an algebra is an invariant property under a finite dimensional H-cleft extension over an algebraically closed field. In the other part, we still show that the Nakayama property of an artin algebra is also an invariant property under an H-cleft extension when the radical of the algebra is H-stable.

2000 Mathematics Subject Classifications: 16G60; 16G10; 16W30; 16S35

1 Introduction

Assume k is always a field. A module, which is not explained, is always left.

Let B be a k-algebra, A a subalgebra of B, and H a Hopf k-algebra. The following concepts are well-known:

1) $A \subset B$ is called a *(right) H-extension* if B is a right H-comodule algebra with structure map ρ satisfying $B^{coH} = A$, where B^{coH} is defined as the subcomodule $\{b \in B : \rho(b) = b \otimes 1\}$;

2) An H-extension $A \subset B$ is called *H-cleft* if there exists a right H-comodule map

*Project supported by the Program for New Century Excellent Talents in University (No.04-0522), the Natural Science Foundation of Zhejiang Province of China (No.102028) and partially by the Cultivation Fund of the Key Scientific and Technical Innovation Project, Ministry of Education of China (No.704004)

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$\gamma: H \to B$ which is (convolution) invertible;

3) An H-extension $A \subset B$ is called *right H-Galois* if the map $\beta: B \otimes_A B \to B \otimes_k H$ given by $\beta(a \otimes b) = (a \otimes 1)\rho(b)$, is bijective;

4) An H-extension $A \subset B$ is said to have the *(right) normal basis property* if $B \cong A \otimes H$ as left A-modules and right H-comodules.

The following theorem of Doi and Takeuchi [8] characterizes Galois extensions with the normal basis property.

Theorem 1.1 *[8] Let $A \subset B$ be an H-extension. The the following are equivalent:*
(i) $A \subset B$ is H-cleft;
(ii) $A \subset B$ is H-Galois and has the normal basis property.

From this conclusion, we know that cleft extension is a special kind of Galois extensions and generalizes the classical theorem in the Galois theory of groups which says that if $F \subset E$ is a finite Galois extension of fields with Galois group G, then E/F has a normal basis. So, cleft extension is important in the Galois theory of Hopf algebras.

Our main aim is to give invariant properties of representations of algebras under cleft extensions over a semisimple Hopf algebra.

In Section 1, firstly, we explain the concept of cleft extension and give the relation between cleft extension and crossed product which is the method we depend upon. In Section 2, by using of them, we prove that for a finite dimensional Hopf algebra H which is semisimple as well as its dual H^*, the representation type of an algebra is an invariant property under a finite dimensional H-cleft extension over an algebraically closed field. In Section 3, we still show that the Nakayama property of an artin algebra is also an invariant property under an H-cleft extension when the radical of the algebra is H-stable. The major approach we will use in Section 2 and 3 is based on the correspondent relation between cleft extension and crossed product.

First, we recall some notions on crossed product. A Hopf algebra H is said to *measure* an algebra A if there is a k-linear map $H \otimes A \to A$ given by $h \otimes a \mapsto h \cdot a$ such that $h \cdot 1 = \varepsilon(h)1$ and $h \cdot (ab) = \sum (h_1 \cdot a)(h_2 \cdot b)$ for all $h \in H$, $a, b \in A$.

Assume H measures A and σ is a convolution invertible map in $Hom(H \otimes H, A)$. The *crossed product* $A\#_\sigma H$ of A with H is the set $A \otimes H$ as a vector space with multiplication

$$(a\#h)(b\#k) = \sum a(h_1 \cdot b)\sigma(h_2, k_1)\#h_3k_2$$

for all $h, k \in H$, $a, b \in A$. Here write $a\#h$ for the tensor product $a \otimes h$.

Theorem 1.2 *[8][4] $A\#_\sigma H$ is an associative algebra with identity element $1\#1$ if and only if the following two conditions are satisfied:*

(i) A is a twisted H-module with action $\cdot$, that is, $1 \cdot a = a$ and

$$h \cdot (k \cdot a) = \sum \sigma(h_1, k_1)(h_2 k_2 \cdot a)\sigma^{-1}(h_3, k_3)$$

for all $h, k \in H$, $a \in A$;

(ii) σ is a cocycle, that is, $\sigma(h, 1) = \sigma(1, h) = \varepsilon(h)1$ and

$$\sum (h_1 \cdot \sigma(k_1, m_1))\sigma(h_2, k_2 m_2) = \sum \sigma(h_1, k_1)\sigma(h_2 k_2, m)$$

for all $h, k, m \in H$.

Trivially, for any $a \in A$, $h \in H$, we have $(a\#1)(1\#h) = a\#h$; and $A \cong A\#_\sigma 1$ and $H \cong 1\#_\sigma H$ can always be said to be subalgebras of $A\#H$ respectively. Thus, $(a\#1)$ and $(1\#h)$ often be written briefly as a and h respectively. So, $ah = a\#h$ and etc.

Now we can give the following relation:

Theorem 1.3 *[12] An H-extension $A \subset B$ is H-cleft with right convolution invertible H-comodule map $\gamma: H \to B$ if and only if $B \cong A\#_\sigma H$ as algebras with a convolution invertible k-map $\sigma: H \otimes H \to A$, where the twisted H-module action on A of $A\#_\sigma H$ is given by*

$$h \cdot a = \sum_{(h)} \gamma(h')a\gamma^{-1}(h'') \tag{1}$$

moreover, γ and σ are constructed each other by

$$\sigma(h, k) = \sum_{(h)(k)} \gamma(h')\gamma(k')\gamma^{-1}(h''k'') \tag{2}$$

and γ, γ^{-1} from H to $A\#_\sigma H$ by

$$\gamma(h) = 1\#h \qquad \gamma^{-1}(h) = \sum_{(h)} \sigma^{-1}(Sh'', h''')\#Sh' \tag{3}$$

for all $a \in A$, $h, k \in H$ and the antipode S of H.

2 Representation Type

Note that in this section, representations of any considered algebra Λ, or say, its modules, always are assumed to lie in the subcategory modΛ of finite generated modules of the algebra.

As well known [9][6], an algebra A is said to be of *finite representation type* provided there are only finitely many non-isomorphic finitely generated indecomposable A-modules.

Moreover, an algebra A is said to be of *tame type* or a *tame* algebra if A is not of finite representation type, whereas for any dimension $d > 0$, there are a finite number of

A-$k[T]$-bimodules M_i which are free as right $k[T]$-modules such that all but a finite number of indecomposable A-modules of dimension d are isomorphic to $M_i \otimes_{k[T]} k[T]/(T-\lambda)$ for $\lambda \in k$.

And, an algebra A is said to be of *wild type* or a *wild* algebra if there is a finitely generated A-$k\langle X,Y\rangle$-bimodule B which is free as a right $k\langle X,Y\rangle$-module such that the functor $B \otimes_{k\langle X,Y\rangle} -$ from mod-$k\langle X,Y\rangle$, the category of finitely generated $k\langle X,Y\rangle$-modules, to mod-A, the category of finitely generated A-modules, preserves indecomposability and reflects isomorphisms.

The famous tame-and-wild theorem of Drozd's in [9][6] states that a finite dimensional algebra over an algebraically closed field k , which is not of finite representation type, is either of tame representation type or of wild representation type, and not both. Therefore, it gives the classification of finite dimensional algebras over an algebraically closed field due to representation type. Now, we use this conclusion to discuss the representation type of a crossed product.

In the sequel, for a Hopf algebra H and a k-algebra A, when we write $A\#_\sigma H$ it is always assumed the crossed product structure $A\#_\sigma H$ exists as an associative algebra for a cocycle σ.

The fundamental facts on the right integral space $\int_H^r$ of a Hopf algebra H are that when $\dim H < +\infty$, $\int_H^r$ is always one-dimensional over k, and H is semisimple if and only if $\varepsilon(\int_H^r) \neq 0$.

Note that for a ring R and two R-modules M, N, we will write $M|N$ if M is a direct summand of N as a R-module.

Lemma 2.1 *For a finite dimensional semisimple Hopf algebra H, a twisted H-module algebra A and a cocycle $\sigma \in Hom(H\otimes H, A)$, suppose $A\#_\sigma H$ is a crossed product algebra. Then, for any $A\#_\sigma H$-module X, it holds that $X|(A\#_\sigma H)\otimes_A X$.*

Proof: Define $\varphi: (A\#_\sigma H)\otimes_A X \to X$ by $(a\#h)\otimes x \mapsto (a\#h)x$ for $a \in A, h \in H$ and $x \in X$. Clearly, φ is an $A\#_\sigma H$-epimorphism.

Let $0 \neq t \in \int_H^r$ and assume $\varepsilon(t) = 1$. Define $\psi: X \to (A\#_\sigma H)\otimes_A X$ by

$$\psi(x) = \sum_{(t)} \gamma^{-1}(t') \otimes_A \gamma(t'')x$$

for $x \in X$, where γ and γ^{-1} are given by (3). We have, for $a \in A,\ h \in H,\ x \in X$, firstly,

$$\begin{aligned} \gamma(h)(ax) &= (1\#h)((a\#1)x) = \sum_{(h)} (h' \cdot a\#h'')x \\ &= \sum_{(h)} ((h' \cdot a\#1)(1\#h''))x = \sum_{(h)} (h' \cdot a\#1)(\gamma(h'')x). \end{aligned}$$

Then it follows that

$$\begin{aligned}
\psi(ax) &= \sum_{(t)}\gamma^{-1}(t')\otimes_A \gamma(t'')ax = \sum_{(t)}\gamma^{-1}(t')\otimes_A (t''\cdot a\#1)(\gamma(t''')x)\\
&= \sum_{(t)}\gamma^{-1}(t')(t''\cdot a)\otimes_A \gamma(t''')x = \sum_{(t)}\gamma^{-1}(t')\gamma(t'')a\gamma^{-1}(t''')\otimes_A \gamma(t^{(4)})x \qquad (by\ \ (1))\\
&= \sum_{(t)}a\gamma^{-1}(t')\otimes_A \gamma(t'')x = \sum_{(t)}(a\#1)\gamma^{-1}(t')\otimes_A \gamma(t'')x = \sum_{(t)}a(\gamma^{-1}(t')\otimes_A \gamma(t'')x)\\
&= a\psi(x).
\end{aligned}$$

And, for any $g,h\in H$, $(1\#g)(1\#h) = (g'\cdot 1)\sigma(g'',h')\#g'''h'' = \sigma(g',h')\#g''h'' = \sigma(g',h')(g''h'')$. It can also be written as $\gamma(g)\gamma(h) = \sigma(g',h')\gamma(g''h'')$. From this, it can be proved easily that $\sigma(g,h) = \gamma(g')\gamma(h')\gamma^{-1}(g''h'')$. Thus, we have

$$\begin{aligned}
\psi(hx) &= \psi(\gamma(h)x) = \sum_{(t)}\gamma^{-1}(t')\otimes_A \gamma(t'')(\gamma(h)x) = \sum_{(t)}\gamma^{-1}(t')\otimes_A (\gamma(t'')\gamma(h))x\\
&= \sum_{(t)}\gamma^{-1}(t')\otimes_A \sigma(t'',h')\gamma(t'''h'')x = \sum_{(t)}\gamma^{-1}(t')\sigma(t'',h')\otimes_A \gamma(t'''h'')x\\
&= \sum_{(t)}\gamma^{-1}(t')\gamma(t'')\gamma(h')\gamma^{-1}(t'''h'')\otimes_A \gamma(t^{(4)}h''')x = \sum_{(t)}\gamma(h')\gamma^{-1}(t'h'')\otimes_A \gamma(t''h''')x\\
&= \sum_{(t)}\gamma(h)\gamma^{-1}(t')\otimes_A \gamma(t'')x \qquad (by\ \ the\ \ fact:\ \ h\otimes\Delta(t) = \sum_{(h)(t)}h'\otimes t'h''\otimes t''h''')\\
&= h\psi(x)
\end{aligned}$$

Therefore, ψ is an $A\#_\sigma H$-morphism. Finally,

$$\begin{aligned}
\varphi\psi(x) &= \varphi(\sum_{(t)}\gamma^{-1}(t')\otimes\gamma(t'')x) = \sum_{(t)}\gamma^{-1}(t')(\gamma(t'')x)\\
&= \sum_{(t)}(\gamma^{-1}(t')\gamma(t''))x = \varepsilon(t)x\\
&= x.
\end{aligned}$$

Hence, $\varphi\psi = id_X$. It means that $X|(A\#_\sigma H)\otimes_A X$. $\square$

Although this lemma is a generalization of the similar result[11] in the case for smash product, its proof is different with that of the latter since the definitions of their ψ's are distinct.

In Hopf theory, the Blattner-Montgomery Duality Theorem [12] is very valid which says that for a finite dimensional Hopf algebra H and a crossed product algebra $A\#_\sigma H$, one has $(A\#_\sigma H)\#H^* \cong M_n(A)$, where $n=$dim$H$. On the other hand, Drozd's theorem holds also only for finite dimensional algebras. Due to them, we will always suppose the Hopf algebra H and the algebra A are finite-dimensional in the sequel.

Proposition 2.2 *Let H be a finite dimensional semisimple Hopf algebra and H^* be semisimple simultaneously over a field k. Then the following hold:*
(i) *If A is a finite dimensional twisted H-module algebra such that $A\#_\sigma H$ exists as a crossed product algebra, then A is of finite representation type if and only if $A\#_\sigma H$ is so;*
(ii) *If A and B are finite dimensional k-algebras and $A \subset B$ is an H-cleft extension, then A is of finite representation type if and only if B is so.*

Proof: "Only if" of (i): Suppose X is a finitely generated indecomposable $A\#_\sigma H$-module. Let $\{B_1, \ldots, B_t\}$ be a complete set of non-isomorphic finitely generated indecomposable A-modules. Viewing X as an A-module, then $X \cong \oplus_{j=1}^{t} n_j B_j$ for some non-negative integers n_j and so $(A\#_\sigma H) \otimes_A X \cong \oplus_{j=1}^{t}(n_j(A\#_\sigma H) \otimes_A B_j)$. Since, by Lemma 2.1, X is an $A\#_\sigma H$-summand of $(A\#_\sigma H) \otimes_A X$. Since X is finitely generated over $A\#_\sigma H$ and $\dim A\#_\sigma H < \infty$, X is of finite length. Due to the Krull-Schmidt theorem, we can know that X is an $A\#_\sigma H$-summand of $(A\#_\sigma H) \otimes_A B_i$ for some i. Therefore, the non-isomorphic finitely generated indecomposable $A\#_\sigma H$-summands of all the $(A\#_\sigma H) \otimes_A B_i$ give a complete set of non-isomorphic finitely generated indecomposable $A\#_\sigma H$-modules. It can be seen that this set is finite. In fact, for a fixed i, B_i is finitely generated as A-module, so $(A\#_\sigma H) \otimes_A B_i$ is also finitely generated as $(A\#_\sigma H)$-module, then $(A\#_\sigma H) \otimes_A B_i$ is an $(A\#_\sigma H)$-module of finite length. Thus, by the Krull-schmidt theorem, $(A\#_\sigma H) \otimes_A B_i$ has a unique finite indecomposable decomposition. Therefore, $A\#_\sigma H$ is of finite representation type.

"If" of (i): Since H^* is also semisimple and smash product is a crossed product in case σ is trivial, we can use the the necessity as above to know that $(A\#_\sigma H)\#H^*$ is of finite representation type due to $A\#_\sigma H$ is so. By the Blattner-Montgomery Duality Theorem, $(A\#_\sigma H)\#H^* \cong M_n(A)$ which is Morita equivalent to A. Thus A is of finite representation type.

(ii): By (i) and Theorem 1.3. □

For a finite dimensional algebra Λ, the so-called *generic category*[11] from Λ, denoted as $GC(\Lambda)$, is defined as that whose objects are all Λ-$k[T]$-bimoduls which are free as right $k[T]$-modules and whose morphisms are all Λ-$k[T]$-morphisms. It is closely related with tame algebras and thus with *generic modules*[7].

The following lemma and its proof is from [11]:

Lemma 2.3 *Let $X \in GC(\Lambda)$. X is indecomposable in $GC(\Lambda)$ if and only if $X \otimes_{k[T]} k[T]/(T-\lambda)$ is indecomposable as a Λ-$k[T]/(T-\lambda)$-bimodule for $\lambda \in k$.*

Proof: "If": If X is decomposed as $X = X_1 \oplus X_2$ in $GC(\Lambda)$, then $X \otimes_{k[T]} k[T]/(T-\lambda) =$

$X_1 \otimes_{k[T]} k[T]/(T-\lambda) \oplus X_2 \otimes_{k[T]} k[T]/(T-\lambda)$ as Λ-$k[T]/(T-\lambda)$-bimodules.

"Only if": Assume $X \otimes_{k[T]} k[T]/(T-\lambda) = N_1 \oplus N_2$ for Λ-$k[T]/(T-\lambda)$-bimodules N_1 and N_2.

It is easy to see $k[T]/(T-\lambda) \cong k$ as algebras. So, we can say $k[T]$ is a $k[T]/(T-\lambda)$-module with suitable module structure. Then

$$\begin{aligned} X &\cong X \otimes_{k[T]} k[T] \cong X \otimes_{k[T]} k[T]/(T-\lambda) \otimes_{k[T]/(T-\lambda)} k[T] \\ &= (N_1 \otimes_{k[T]/(T-\lambda)} k[T]) \oplus (N_2 \otimes_{k[T]/(T-\lambda)} k[T]) \end{aligned}$$

It follows that $N_1 = 0$ or $N_2 = 0$ from the indecomposability of X. That is, $X \otimes_{k[T]} k[T]/(T-\lambda)$ is indecomposable. $\square$

Proposition 2.4 *Let H and H^* be finite dimensional semisimple Hopf algebras over a field k. Then the following hold:*

(i) *If a finite dimensional twisted H-module algebra A such that $A\#_\sigma H$ exists as a crossed product algebra, then A is tame if and only if $A\#_\sigma H$ is tame;*

(ii) *If A and B are finite dimensional k-algebras and $A \subset B$ is an H-cleft extension, then A is tame if and only if B is tame.*

Proof: "Only if" of (i): Our proof is based on the definition of tame algebra. By Proposition 2.2, $A\#_\sigma H$ is not of finite representation type. Let d be a positive integer and X an indecomposable left $(A\#_\sigma H)$-module with dimension d over k.

By Lemma 2.1, $X|(A\#_\sigma H) \otimes_A X$. Denote X by ${}_AX$ when X is considered as a left A-module in the canonical way.

Since A is a tame algebra, there are a finite number of A-$k[T]$-bimodules M_j ($j = 1, \ldots, n$) which are free as right $k[T]$-modules such that all, but a finite number, of indecomposable A-modules of dimension d are isomorphic to $M_j \otimes_{k[T]} k[T]/(T-\lambda)$ for some j and some $\lambda \in k$. No loss of generality, for each M_j, suppose there is at least one indecomposable A-module Y of dimension d such that $Y \cong M_j \otimes_{k[T]} k[T]/(T-\lambda)$. Equivalently, let $\{M_j : j = 1, \ldots, n\}$ be the minimal such set. It is easy to see that $(A\#_\sigma H) \otimes_A M_j$ is free as a right $k[T]$-module from the same property of M_j. Hence, $(A\#_\sigma H) \otimes_A M_j \in GC(A\#_\sigma H)$.

Since M_j is finitely generated over A, $(A\#_\sigma H) \otimes_A M_j$ is finitely generated over $(A\#_\sigma H)$. And, since $\dim(A\#_\sigma H) < \infty$, one can decompose $(A\#_\sigma H) \otimes_A M_j$ into a direct sum of a finite number of indecomposable objects M_{jl} in $GC(A\#_\sigma H)$, i.e. $(A\#_\sigma H) \otimes_A M_j = \oplus_{l \in I_j} M_{jl}$ for a finite index set I_j. Then $\{M_{jl} : j = 1, \ldots, n, \ l \in I_j\}$ is a finite set of $(A\#_\sigma H)$-$k[T]$-bimodules which are free as right $k[T]$-modules.

By Lemma 2.3, $M_{jl} \otimes_{k[T]} k[T]/(T-\lambda)$ is indecomposable as an $(A\#_\sigma H)$-$(k[T]/(T-\lambda))$-bimodule. This is equivalent to say that $M_{jl} \otimes_{k[T]} k[T]/(T-\lambda)$ is indecomposable as an $A\#_\sigma H$-module since $k[T]/(T-\lambda) \cong k$ as algebras.

Let ${}_AX = X_1 + \ldots + X_m$ be a direct sum of indecomposable A-modules. Now, we claim for all, but a finite number, of X, there exists an indecomposable A-submodule X_t of ${}_AX$ such that $X|(A\#_\sigma H) \otimes_A X_t$ and $X_t \cong M_{j_t} \otimes_{k[T]} k[T]/(T-\lambda)$ for some M_{j_t} where $j_t \in \{1, \ldots, n\}$. In fact, by Lemma 2.1, $X|(A\#_\sigma H) \otimes_A X$. But, $(A\#_\sigma H) \otimes_A X = ((A\#_\sigma H) \otimes_A X_1) \oplus \ldots \oplus ((A\#_\sigma H) \otimes_A X_m)$. So, by Krull-Schmidt theorem, there exists t such that $X|(A\#_\sigma H) \otimes_A X_t$. However, since A is tame, for all, but a finite number, of such X and X_t, it satisfies that $X_t \cong M_{j_t} \otimes_{k[T]} k[T]/(T-\lambda)$ for some M_{j_t} and λ.

Moreover, $(A\#_\sigma H) \otimes_A X_t \cong (A\#_\sigma H) \otimes_A M_{j_t} \otimes_{k[T]} k[T]/(T-\lambda) = \oplus_{l \in I_{j_t}} M_{j_t l} \otimes_{k[T]} k[T]/(T-\lambda)$. We have known above each $M_{j_t l} \otimes_{k[T]} k[T]/(T-\lambda)$ is indecomposable as left $A\#_\sigma H$-module. Therefore, there exists $s \in I_{j_t}$ such that $X \cong M_{j_t s} \otimes_{k[T]} k[T]/(T-\lambda)$.

We have known $\{M_{jl} : \ j = 1, \ldots, n, \ l \in I_j\}$ is a finite set of $(A\#_\sigma H)$-$k[T]$-bimodules which are free as right $k[T]$-modules and such X are of almost all. Therefore, $A\#_\sigma H$ is tame.

"If" of (i): In similar to the "if" part of Proposition 2.2, we can use the the necessity as above to know that $(A\#_\sigma H)\#H^*$ is of tame representation type due to $A\#_\sigma H$ is so. By the Blattner-Montgomery Duality Theorem, $(A\#_\sigma H)\#H^* \cong M_n(A)$ which is Morita equivalent to A. Thus A is of tame representation type.

(ii): By (i) and Theorem 1.3. $\square$

In order to make the major conclusions in this paper to be more self-contained, it would be better to remark that the tame representation type property of algebras is a Morita invariant which has been used in the "if" part of the proof of Proposition 2.4. In fact, let two algebras A and B be Morita equivalent, denote their basic algebras as I_A and I_B respectively, then I_A and I_B are Morita equivalent each other, thus by Lemma I.2.6 of [10], I_A and I_B are isomorphic. It follows that I_A is of tame representation type if and only if I_B is so. But, note that an algebra is tame if and only if its basic algebra is tame. Therefore, A is tame if and only if B is tame.

With the additional condition for the ground field k to be algebraically closed, according to the Drozd's tame-and-wild theorem and Proposition 2.2 and 2.4, we obtain the following:

Corollary 2.5 *Let H and H^* be finite dimensional semisimple Hopf algebras over an algebraically closed field k. Then the following hold:*

(i) *If A is a finite dimensional twisted H-module algebra such that $A\#_\sigma H$ exists as a crossed product algebra, then A is wild if and only if $A\#_\sigma H$ is wild;*

(ii) *If A and B are finite dimensional k-algebras and $A \subset B$ is an H-cleft extension, then A is wild if and only if B is wild.*

Finally, as a summary, we get the main results in this section as follows:

Theorem 2.6 *Let H and H^* be finite dimensional semisimple Hopf algebras over an algebraically closed field k. Then the following hold:*
(i) *If A is a finite dimensional twisted H-module algebra such that $A\#_\sigma H$ exists as a crossed product algebra, then A and $A\#_\sigma H$ have the same representation type;*
(ii) *If A and B are finite dimensional k-algebras and $A \subset B$ is an H-cleft extension, then A and B have the same representation type.*

Only due to the Drozd's tame-and-wild theorem, the condition for the ground field k to be algebraically closed is added to Corollary 2.5 and Theorem 2.6. It is the reason that this condition is not necessary for Proposition 2.2 and 2.4.

3 Nakayama Property

This section is devoted to discuss when a crossed product is a Nakayama algebra and equivalently, to characterize the Nakayama property of cleft extensions.

In this section, we always assume A is an artin R-algebra over a commutative ring R unless in some special cases we will explain. And, for any ring Γ, denote its (Jacobson) radical as $J(\Gamma)$.

It is known that a module M is called a *uniserial module* if the set of submodules is totally ordered by inclusion, an aritn algebra A is said to be a *Nakayama algebra* if its both indecomposable projective and indecomposable injective modules are uniserial. Nakayama algebras are of considerable interest because next to semisimple algebras they are the best understood artin algebras. For examples, the following valid conclusions [1] hold for them.

Lemma 3.1 *Let A be an artin algebra and $J(A)$ the Jacobson radical of A. Then A is a Nakayama algebra if and only if $A/J(A)^2$ is Nakayama.*

Lemma 3.2 *Let A be an artin algebra with $J(A)^2 = 0$. Then A is a Nakayama algebra if and only if the injective envelope $I(A)$ of A is a projective module.*

Lemma 3.3 *Let $f : \Lambda \to \Gamma$ be a morphism of artin algebras. Consider Γ as the Λ-module induced naturally by f. Then the following are equivalent:*

(i) Γ *is a projective right Λ-module;*

(ii) *Every injective left Γ-module is an injective left Λ-module.*

The major result in this section is as follows:

Theorem 3.4 *Let H be a finite dimensional Hopf algebra such that H and H^* are both semisimple and A a twisted H-module artin algebra such that $A\#_\sigma H$ exists as a crossed product algebra. If the radical $J(A)$ of A is H-stable, then $A\#_\sigma H$ is a Nakayama algebra if and only if A is Nakayama.*

The following lemmas are needed for our discussion.

Lemma 3.5 *Let H be a semisimple Hopf algebra and A an artinian twisted H-module algebra such that $A\#_\sigma H$ exists as a crossed product algebra. If the radical $J(A)$ of A is H-stable, then $J(A\#_\sigma H) = J(A)\#_\sigma H$.*

Proof: Let $J = J(A)$. Since J is H-stable, we have the crossed product $J\#_\sigma H$. Denote the canonical projection $A \to A/J$ by p which induces the canonical algebra morphism $\pi : A\#_\sigma H \to (A/J)\#_\sigma H$ by $\pi(\sum a\#h) = \sum p(a)\#h$ for all $a \in A,\ h \in H$. Then $\mathrm{Ker}\pi = J\#_\sigma H$. Therefore, $(A\#_\sigma H)/(J\#_\sigma H) \cong (A/J)\#_\sigma H$. Since A/J and H are semisimple, $(A/J)\#_\sigma H$ is a semisimple algebra by [3]. It implies $J(A\#_\sigma H) \subseteq J\#_\sigma H$. And, since $(J\#_\sigma H)^i = J^i\#_\sigma H$ for any positive integer i and A is artinian, $J\#_\sigma H$ is a nilpotent ideal of $A\#_\sigma H$ and therefore $J\#_\sigma H \subseteq J(A\#_\sigma H)$. Thus $J(A\#_\sigma H) = J\#_\sigma H$. □

Note that (i) In this lemma, A is supposed to be artinian, i.e. satisfying the descending chain condition. This condition is more general than that A is an artin R-algebra. (ii) Since $ab\#g = (a\#1)(b\#g)$ for $a, b \in A,\ g \in H$, we have $(I\#_\sigma H)^i = I^i\#_\sigma H$ for any ideal I of A and positive integer i.

For an artin R-algebra and its radical $J = J(A)$, $A\#_\sigma H$ is also an artin R-algebra. Then by Lemma 3.1, $A\#_\sigma H$ is Nakayama if and only if $(A\#_\sigma H)/(J(A\#_\sigma H))^2$ is Nakayama. By Lemma 3.5, $J(A\#_\sigma H) = J\#_\sigma H$. So, $(J(A\#_\sigma H))^2 = J^2\#_\sigma H$. Then, using of the similar discussion as in the proof of Lemma 3.5, $(A\#_\sigma H)/(J(A\#_\sigma H))^2 = (A\#_\sigma H)/(J^2\#_\sigma H) \cong (A/J^2)\#_\sigma H$. Hence, $A\#_\sigma H$ is Nakayama if and only if $(A/J^2)\#_\sigma H$ is so. It implies that in order to prove Theorem 3.4, no loss of generality, we can assume that $J^2 = 0$ for $J = J(A)$ from now on.

Proof of Theorem 3.4:

"Only if": Since $J^2 = 0$, $(J(A\#_\sigma H))^2 = J^2\#_\sigma H = 0$. By Lemma 3.2, the $A\#_\sigma H$-injective envelope $I(A\#_\sigma H)$ of $A\#_\sigma H$ is $A\#_\sigma H$-projective. Then it suffices to prove the A-injective envelope $I(A)$ of A is A-projective.

In fact, $A\#_\sigma H$ is both a right and left free A-module by means of the inclusion $A \hookrightarrow A\#_\sigma H$(see [13]). Of course, $A\#_\sigma H$ is a projective right A-module, then by Lemma 3.3, every left injective $A\#_\sigma H$-module is a left injective A-module. So, $I(A\#_\sigma H)$ is a left injective A-module. Since $I(A\#_\sigma H)$ is a projective $A\#_\sigma H$-module and $A\#_\sigma H$ is a projective A-module, it is easy to see that $I(A\#_\sigma H)$ is a projective A-module. And, $A \hookrightarrow A\#_\sigma H \hookrightarrow I(A\#H)$ as left A-modules. Hence by The Fundamental Lemma for Injective Envelopes, $I(A)$ is a summand of $I(A\#H)$ as a left A-module. As a result, $I(A)$ is also a projective A-module.

Before proving "if", we first give the following lemma:

Lemma 3.6 *For any positive integer n, if an algebra A is Nakayama, then the $n \times n$ full matrix algebra $M_n(A)$ is also Nakayama.*

Proof: We know from [1] that an algebra Λ is Nakayama if and only if its all indecomposable projective Λ-modules and Λ^{op}-modules are uniserial. It implies that Λ is Nakayama if and only if Λ^{op} is Nakayama. So, we only need to verify any indecomposable projective $M_n(A)$-modules are uniserial.

Since A is artin, there are a finite number of primitive idempotents in A, which are denoted as $e_1, \cdots, e_m$. Then $P_i = Ae_i$ $(i = 1, \cdots, m)$ is the complete set of non-isomorphic indecomposable projective A-modules. From this, we get the complete set of primitive idempotents in $M_n(A)$ as follows:

$$\begin{pmatrix} e_i & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix}, \begin{pmatrix} 0 & & & \\ & e_i & & \\ & & \ddots & \\ & & & 0 \end{pmatrix}, \cdots, \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & \ddots & \\ & & & e_i \end{pmatrix}$$

for all $i = 1, \cdots, m$. Moreover, the complete set of non-isomorphic indecomposable projective $M_n(A)$-modules consists of the following:

$$\begin{pmatrix} P_i & 0 & \cdots & 0 \\ P_i & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ P_i & 0 & \cdots & 0 \end{pmatrix}, \begin{pmatrix} 0 & P_i & \cdots & 0 \\ 0 & P_i & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & P_i & \cdots & 0 \end{pmatrix}, \cdots, \begin{pmatrix} 0 & 0 & \cdots & P_i \\ 0 & 0 & \cdots & P_i \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & P_i \end{pmatrix}$$

for all $i = 1, \cdots, m$. It is easy to see that any $M_n(A)$-submodule of $\begin{pmatrix} 0 & \cdots & P_i & \cdots & 0 \\ 0 & \cdots & P_i & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & P_i & \cdots & 0 \end{pmatrix}$

is of the form $\begin{pmatrix} 0 & \cdots & Q & \cdots & 0 \\ 0 & \cdots & Q & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & Q & \cdots & 0 \end{pmatrix}$ for a submodule Q of P_i.

Because A is Nakayama, P_i is uniserial, i.e. all A-submodule Q of P_i compose a totally ordered set by inclusion. From this, all $M_n(A)$-submodules $\begin{pmatrix} 0 & \cdots & Q & \cdots & 0 \\ 0 & \cdots & Q & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & Q & \cdots & 0 \end{pmatrix}$ of $\begin{pmatrix} 0 & \cdots & P_i & \cdots & 0 \\ 0 & \cdots & P_i & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & P_i & \cdots & 0 \end{pmatrix}$ compose a totally ordered set by inclusion. It means that $\begin{pmatrix} 0 & \cdots & P_i & \cdots & 0 \\ 0 & \cdots & P_i & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & P_i & \cdots & 0 \end{pmatrix}$ is uniserial. Finally, $M_n(A)$ is Nakayama. $\square$

Now, we return to prove Theorem 3.4:

"If": Assume that A is a Nakayama algebra. Let $\dim H = n$. By the Blattner-Montgomery Duality Theorem and Lemma 3.6, $(A\#_\sigma H)\#H^* \cong M_n(A)$ is a Nakayama algebra.

By Lemma 3.5, $J(A\#_\sigma H) = J\#_\sigma H$. Due to the definition of $(A\#_\sigma H)\#H^*$ in [2], H^* acts trivially on A and on H by the usual $\rightarrow$ action. Obviously, $J\#_\sigma H$ is H^*-stable. Thus according to the "only if" part of Theorem 3.4, $A\#_\sigma H$ is Nakayama. $\square$

It is known that for a finite dimensional Hopf algebra H with antipode S, H and H^* are both semisimple if and only if $Tr(S^2) \neq 0$. Therefore, in Proposition 2.2, 2.4, Corollary 2.5, Theorem 2.6 and Theorem 3.4, we can replace the condition for H and H^* to be semisimple by $Tr(S^2) \neq 0$. In particular, when $\mathrm{char}k = 0$, this condition is satisfied naturally. Hence all major results in this paper hold when H is a semisimple Hopf algebra over a field k of characteristic 0. Of course, for Corollary 2.5 and Theorem 2.6, the field k is required to be algebraically closed at the same time.

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