A SIMPLE PROOF OF WITTEN CONJECTURE THROUGH LOCALIZATION

YON-SEO KIM AND KEFENG LIU

Department of Mathematics, UCLA; Center of Math Sciences, Zhejiang University

Abstract. We obtain a system of relations between Hodge integrals with one λ-class. As an application, we show that its first non-trivial relation implies the Witten's Conjecture/Kontsevich Theorem [12, 6].

1. Introduction

In this paper, we obtain an alternate proof of the Witten's Conjecture [12] which claims that the tautological intersections on the moduli space of stable curves $\overline{\mathcal{M}}_{g,n}$ is governed by KdV hierarchy. It is first proved by M.Kontsevich [6] by constructing combinatorial model for the intersection theory of $\overline{\mathcal{M}}_{g,n}$ and interpreting the trivalent graph summation by a Feynman diagram expansion for a new matrix integral. Also, A.Okounkov and R.Pandharipande [11] used a connection between intersections in $\mathcal{M}_{g,n}$ and the enumeration of branched coverings of $\mathbb{P}^1$ and derived the key identity of Kontsevich, hence gave another approach to Witten's conjecture. Recently, M.Mirzakhani [10] derived a recursion formula by using the Weil-Petersen volume, which lead to a proof of Virasoro constraints.

Here we take an approach using virtual functorial localization on the moduli space of relative stable morphisms $\overline{\mathcal{M}}_g(\mathbb{P}^1,\mu)$ [8]. $\overline{\mathcal{M}}_g(\mathbb{P}^1,\mu)$ consists of maps from Riemann surfaces of genus g and $n = l(\mu)$ marked points to $\mathbb{P}^1$ which has prescribed ramification type μ at $\infty \in \mathbb{P}^1$. As the result, we obtain a system of relations between linear Hodge integrals. It recursively expresses each linear Hodge integral by lower-dimensional ones. The first non-trivial relation of this system is 'cut-and-join relation', and is of same recursion type as that of single Hurwitz numbers [7]. Moreover, as we increase the ramification degree, we can extract a relation between absolute Gromov-Witten invariants from this relation. And we show this relation implies the following recursion relation for the correlation functions of topological gravity [1]:

$$\langle\tilde\sigma_n \prod_{k\in S}\tilde\sigma_k\rangle_g = \sum_{k\in S}(2k+1)\langle\tilde\sigma_{n+k-1}\prod_{l\neq k}\tilde\sigma_l\rangle_g + \frac{1}{2}\sum_{a+b=n-2}\langle\tilde\sigma_a\tilde\sigma_b\prod_{l\neq a,b}\tilde\sigma_l\rangle_{g-1}$$
$$+\frac{1}{2}\sum_{S=X\cup Y, a+b=n-2, g_1+g_2=g}\langle\tilde\sigma_a\prod_{k\in X}\tilde\sigma_k\rangle_{g_1}\langle\tilde\sigma_b\prod_{l\in Y}\tilde\sigma_l\rangle_{g_2} \qquad \cdots(*)$$

which is equivalent to the Witten's Conjecture/Kontsevich Theorem. This recursion relation (*) is also equivalent to the Virasoro constraints; i.e. (*) can be expressed as linear, homogeneous differential equations for the τ-function [1]

$$\tau(\tilde{t}) = \exp \sum_{g=0}^{\infty} \langle \exp \sum_{n} \tilde{t}_n \tilde{\sigma}_n \rangle_g$$

$$L_n \cdot \tau = 0, \qquad (n \geq -1)$$

where L_n denote the differential operators

$$\begin{aligned}
L_{-1} &= -\frac{1}{2}\frac{\partial}{\partial \tilde{t}_0} + \sum_{k=1}^{\infty}(k+\frac{1}{2})\tilde{t}_k \frac{\partial}{\partial \tilde{t}_{k-1}} + \frac{1}{4}\tilde{t}_0^2 \\
L_0 &= -\frac{1}{2}\frac{\partial}{\partial \tilde{t}_1} + \sum_{k=0}^{\infty}(k+\frac{1}{2})\tilde{t}_k \frac{\partial}{\partial \tilde{t}_k} + \frac{1}{16} \\
L_n &= -\frac{1}{2}\frac{\partial}{\partial \tilde{t}_{n-1}} + \sum_{k=0}^{\infty}(k+\frac{1}{2})\tilde{t}_k \frac{\partial}{\partial \tilde{t}_{k+n}} + \frac{1}{4}\sum_{i=1}^{n} \frac{\partial^2}{\partial \tilde{t}_{i-1}\partial \tilde{t}_{n-i}}
\end{aligned}$$

As a remark, it is possible that the general recursion relation obtained from our approach implies the Virasoro conjecture for a general non-singular projective variety.

The rest of this paper is organized as follows: In section 2, we recall the recursion formula obtained in [5] and derive cut-and-join relation as its special case. In section 3, we prove asymptotic formulas for the coefficients in the cut-and-join relation. Then we derive first two relations of the system of relations between linear Hodge integrals, and show that the cut-and-join relation implies (*).

* *Please refer to* [5] *for miscellaneous notations.*

2. Recursion Formula

The following recursion formula was derived in [5].

Theorem 2.1. *For any partition μ and e with $|e| < |\mu| + l(\mu) - \chi$, we have*

$$\left[\lambda^{l(\mu)-\chi}\right] \sum_{|\nu|=|\mu|} \Phi^{\bullet}_{\mu,\nu}(-\lambda) z_\nu \mathcal{D}^{\bullet}_{\nu,e}(\lambda) = 0 \qquad (1)$$

where the sum is taken over all partitions ν of the same size as μ.

Here $\big[\lambda^a\big]$ means taking the coefficient of λ^a, and $\mathcal{D}^\bullet_{\nu,e}$ consists of linear Hodge integrals as follows;

$$\begin{aligned}
\mathcal{D}_{g,\nu,e} = \quad & \frac{\nu_1^{\nu_1-2}}{\nu_1!} && \text{, if } \big(g, l(\nu)+l(e)\big) = \big(0,1\big)\\
& \frac{1}{|\mathrm{Aut}\ \nu|}\frac{\nu_1^{\nu_1}\nu_2^{\nu_2}}{\nu_1!\nu_2!}\frac{1}{\nu_1+\nu_2} && \text{, if } \big(g, l(\nu), l(e)\big) = \big(0,2,0\big)\\
& \frac{\nu_1^{\nu_1}}{\nu_1!}\sum_{k=0}^{e_1}\frac{1}{\nu_1^{1+k}}\binom{e_1}{k} && \text{, if } \big(g, l(\nu), l(e)\big) = \big(0,1,1\big)\\
& \frac{1}{l(e)!\ |\ \mathrm{Aut}\ \nu\ |}\Big[\prod_{i=1}^{l(\nu)}\frac{\nu_i^{\nu_i}}{\nu_i!}\Big]\int_{\overline{\mathcal{M}}_{g,l(\nu)+l(e)}}\frac{\Lambda_g^\vee(1)\prod_{j=1}^{l(e)}(1-\psi_j)^{e_j}}{\prod_{i=1}^{l(\nu)}\big(1-\nu_i\psi_i\big)} && \text{, otherwise}\\
\mathcal{D}(\lambda,p,q) = \quad & \sum_{|\nu|\ge 1}\sum_{g\ge 0}\lambda^{2g-2+l(\nu)}p_\nu q_e\mathcal{D}_{g,\nu}\\
\mathcal{D}^\bullet(\lambda,p,q) = \quad & \exp\big(\mathcal{D}(\lambda,p,q)\big) =: \sum_{|\nu|\ge 0}\lambda^{-\chi+l(\nu)}p_\nu q_e\mathcal{D}^\bullet_{\chi,\nu,e} = \sum_{|\nu|\ge 0}p_\nu q_e\mathcal{D}^\bullet_\nu(\lambda)
\end{aligned}$$

where p_i, q_j's are formal variables with $p_\nu = p_{\nu_1}\times\cdots\times p_{\nu_{l(\nu)}}$, $q_e = q_{e_1}\times\cdots\times q_{e_{l(e)}}$, and $\Lambda_g^\vee(t)$ is the dual Hodge bundle;

$$\Lambda_g^\vee(t) = t^g - \lambda_1 t^{g-1} + \cdots + (-1)^g\lambda_g$$

The convoluted term $\Phi^\bullet_{\mu,\nu}(-\lambda)$ consists of double Hurwitz numbers as follows;

$$\Phi^\bullet_{\nu,\mu}(\lambda) = \sum_\chi H^\bullet_\chi(\nu,\mu)\frac{\lambda^{-\chi+l(\nu)+l(\mu)}}{(-\chi+l(\nu)+l(\mu))!} \qquad \Phi^\bullet(\lambda;p^0,p^\infty) = 1 + \sum_{\nu,\mu}\Phi^\bullet_{\nu,\mu}(\lambda)p^0_\nu p^\infty_\mu$$

Here, $H^\bullet_\chi(\nu,\mu)$ is the double Hurwitz number with ramification type ν,μ with Euler characteristic χ. The recursion formula (1) was derived by integrating point-classes over the relative moduli space $\overline{\mathcal{M}}_g(\mathbb{P}^1,\mu)$, and the 'cut-and-join relation' is only the first term in this much more general formula. This can also be seen as follows: Consider the following integral;

$$\int_{\overline{\mathcal{M}}_g(\mathbb{P}^1,\mu)}\mathrm{Br}^*\prod_{k=0}^{r-2}(H-k)$$

It is straightforward to show that preimages of p_r and p_{r-1} are the unique graph Γ_r and the 'cut-and-join graphs' of Γ_r, respectively. Hence we recover the 'cut-and-join relation' as the restriction of (1) to the first two fixed points $\{p_r, p_{r-1}\}$;

$$(2) \qquad r\Gamma_r = \sum_{i=1}^n\Big[\sum_{j\neq i}\frac{\mu_i+\mu_j}{1+\delta^{\mu_i}_{\mu_j}}\Gamma^{ij}_J + \sum_{p=1}^{\mu_i-1}\frac{p(\mu_i-p)}{1+\delta^p_{\mu_i-p}}\Big(\Gamma^{i,p}_{C1} + \sum_{g_1+g_2=g,\nu_1\cup\nu_2=\nu}\Gamma^{i,p}_{C2}\Big)\Big]$$

where I denote, by abuse of notation, the contributions from 'cut-and-join' graphs as follows;

$$\Gamma_r = \frac{1}{|\text{Aut } \mu|} \prod_{i=1}^{n} \frac{\mu_i^{\mu_i}}{\mu_i!} \int_{\overline{\mathcal{M}}_{g,n}} \frac{\Lambda_g^\vee(1)}{\prod(1-\mu_i\psi_i)}$$

$$\Gamma_J^{ij} = \frac{1}{|\text{Aut } \eta|} \prod_{k=1}^{n-1} \frac{\eta_k^{\eta_k}}{\eta_k!} \int_{\overline{\mathcal{M}}_{g,n-1}} \frac{\Lambda_g^\vee(1)}{\prod(1-\eta_k\psi_k)} \qquad \text{for } \eta \in J_{ij}(\mu)$$

$$\Gamma_{C1}^i = \frac{1}{|\text{Aut } \nu|} \prod_{k=1}^{n+1} \frac{\nu_k^{\nu_k}}{\nu_k!} \int_{\overline{\mathcal{M}}_{g-1,n+1}} \frac{\Lambda_{g-1}^\vee(1)}{\prod(1-\nu_k\psi_k)} \qquad \text{for } \nu \in C_i(\mu)$$

$$\Gamma_{C2}^i = \frac{1}{|\text{Aut } \nu_1|} \frac{1}{|\text{Aut } \nu_2|} \prod_{k=1}^{n+1} \frac{\nu_k^{\nu_k}}{\nu_k!} \int_{\overline{\mathcal{M}}_{g_1,n_1}} \frac{\Lambda_{g_1}^\vee(1)}{\prod(1-\nu_{1,k}\psi_k)} \int_{\overline{\mathcal{M}}_{g_2,n_2}} \frac{\Lambda_{g_2}^\vee(1)}{\prod(1-\nu_{2,k}\psi_k)}$$

Here $J_{ij}(\mu)$ and $C_i(\mu)$ are cut-and-join partitions:

$$J_{ij}(\mu) = \{ \ \eta^{ij} = (\mu_1, \cdots, \hat{\mu}_i, \cdots, \hat{\mu}_j, \cdots, \mu_n, \mu_i + \mu_j)\}$$
$$C_i(\mu) = \{ \ \nu^{i,p} = (\mu_1, \cdots, \hat{\mu}_i, \cdots, \mu_n, p, q) \mid p + q = \mu_i, \ \ p, q \geq 1\}$$

When there's no confusion, we will denote by $\eta = \eta^{ij}$ for the join-partition and $\nu = \nu^{i,p}$ for the cut-partition of splitting $\mu_i = p + (\mu_i - p)$ for some $1 \leq p < \mu_i$. Also denote by ν_1 and ν_2 for the splitting of cut-partition ν such that $\nu_1 \cup \nu_2 = \nu$ with $p \in \nu_1, \mu_i - p \in \nu_2$. Note that in the Γ_{C2}-type contribution, unstable vertices (i.e. $g = 0$ and n=1,2) are included. As mentioned in [9], this 'cut-and-join relation' (2) reduces to the recursion formula for single Hurwitz numbers [7] if the Hodge integral terms in the graph contributions are identified with single Hurwitz numbers via ELSV formula [2].

We can also use any set $\{p_{k_0}, \cdots, p_{k_n}\}$, $n > 0$ of fixed points and obtain relations between linear Hodge integrals. And these can be applied to derive deeper relations.

3. Degree Analysis

In this section, we study asymptotic behaviour of the 'cut-and-join relation' and obtain a system of relations between linear Hodge integrals. The Hodge integral terms in the graph contributions can be expanded as follows:

$$(3) \qquad \int_{\overline{\mathcal{M}}_{g,n}} \frac{\Lambda_g^\vee(1)}{\prod(1-\mu_i\psi_i)} = \sum_{k} \prod \mu_i^{k_i} \int_{\overline{\mathcal{M}}_{g,n}} \prod \psi_i^{k_i} + \text{lower degree terms}$$

where $\tilde{k} = (k_1, \cdots, k_n)$ are multi-indices running over condition $\sum k_i = 3g - 3 + n$. Hence the top-degree terms consist of Hodge-integral of ψ-classes and lower degree terms involve λ-classes. This will give a system of relations between Hodge integrals involving one λ-class. More precisely, integrals will be determined recursively by either lower-dimensional or lower-degree λ-class integrals. The following asymptotic formula is crucial in degree analysis.

Proposition 3.1. *As $n \longrightarrow \infty$, we have for $k, l \geq 0$*

$$e^{-n}\sum_{p+q=n}\frac{p^{p+k+1}q^{q+l+1}}{p!q!} \longrightarrow \frac{1}{2}\Big[\frac{(2k+1)!!(2l+1)!!}{2^{k+l+2}(k+l+2)!}\Big]n^{k+l+2}+o(n^{k+l+2})$$

$$e^{-n}\sum_{p+q=n}\frac{p^{p+k+1}q^{q-1}}{p!q!} \longrightarrow \frac{n^{k+\frac{1}{2}}}{\sqrt{2\pi}}-\Big[\frac{(2k+1)!!}{2^{k+1}k!}\Big]n^{k}+o(n^{k})$$

Proof. Let m be an integer such that $1 < m < n$ and consider three ranges of p, q as follows:

$$\begin{aligned} R_l &= \{\ (p,q)\mid p>n-m \text{ and } q<m\} \\ R_c &= \{\ (p,q)\mid m\leq p,q\leq n-m\ \} \\ R_r &= \{\ (p,q)\mid p<m \text{ and } q>n-m\} \end{aligned}$$

Recall the Stirling's formula;

$$n! = \frac{\sqrt{2\pi}n^{n+1/2}}{e^n}\Big(1+\frac{1}{12n}+\cdots\Big)$$

For the summation over R_c, let $m = n\epsilon$ and $p = nx$ for some $\epsilon, x \in \mathbb{R}_{>0}$ so that $m, p \in \mathbb{N}$, then we have

$$\begin{aligned} e^{-n}\sum_{p=m}^{n-m}\frac{p^{p+k+1}}{p!}\frac{q^{q+l+1}}{q!} &= \sum_{p=m}^{n-m}\frac{1}{2\pi}p^{k+\frac{1}{2}}q^{l+\frac{1}{2}}\big[1+o(1)\big] \\ &= \frac{n^{k+l+2}}{2\pi}\sum_{p=m}^{n-m}x^{k+\frac{1}{2}}(1-x)^{l+\frac{1}{2}}\frac{1}{n}+o(n^{k+l+2}) \\ &\longrightarrow \frac{n^{k+l+2}}{2\pi}\int_{\epsilon}^{1-\epsilon}x^{k+\frac{1}{2}}(1-x)^{l+\frac{1}{2}}dx+o(n^{k+l+2}) \qquad \text{as } n \text{ goes to } \infty \\ &= \frac{n^{k+l+2}}{2\pi}\frac{(2k+1)!!(2l+1)!!}{(2(k+l)+3)!!}\int_{\epsilon}^{1-\epsilon}\frac{(1-x)^{k+l+\frac{3}{2}}}{\sqrt{x}}dx+o(n^{k+l+2})+O(\sqrt{\epsilon}) \\ &= \frac{1}{2}\Big[\frac{(2k+1)!!(2l+1)!!}{2^{k+l+2}(k+l+2)!}\Big]n^{k+l+2}+o(n^{k+l+2})+O(\sqrt{\epsilon}) \end{aligned}$$

As $n \longrightarrow \infty$, we can send $\epsilon \longrightarrow 0$. For the summation over R_l and R_r, the top-degree terms belong to $O(n^{k+1/2})$ and $O(n^{l+1/2})$, respectively. Since we assume $k, l \geq 0$, both cases belong to $o(n^{k+l+2})$, and this proves the first formula. For the second formula, R_l has highest order of $n^{k+1/2}$ and one can show that the leading term in the asymptotic behaviour is $n^{k+1/2}/\sqrt{2\pi}$. After integration by parts, R_c gives the second highest term in the asymptotic behaviour

$$\begin{aligned}
e^{-n}\sum_{p=m}^{n-1}\frac{p^{p+k+1}}{p!}\frac{q^{q-1}}{q!} &= \sum_{p=m}^{n-1}\frac{1}{2\pi}p^{k+\frac{1}{2}}q^{l-\frac{3}{2}}\big[1+o(1)\big]\\
&= \frac{n^k}{2\pi}\sum_{p=m}^{n-1}x^{k+\frac{1}{2}}(1-x)^{-3/2}\frac{1}{n}+o(n^k)\\
&\longrightarrow \frac{n^k}{2\pi}\int_{\epsilon}^{1}x^{k+\frac{1}{2}}(1-x)^{-3/2}dx+o(n^k) \qquad \text{as } n \text{ goes to } \infty\\
&= \frac{n^{k+1/2}}{\sqrt{2\pi}}-\frac{n^k}{2\pi}(2k+1)\int_{\epsilon}^{\delta}\frac{x^{k-\frac{1}{2}}}{\sqrt{1-x}}dx+o(n^k)\\
&= \frac{n^{k+1/2}}{\sqrt{2\pi}}-\Big[\frac{(2k+1)!!}{2^{k+1}k!}\Big]n^k+o(n^k)+O(\sqrt{\epsilon})
\end{aligned}$$

This proves the second formula. □

Let $\mu_i = Nx_i$ for some $x_i \in \mathbb{R}$ and $N \in \mathbb{N}$. By taking general values of x_i, we can assume, without loss of generality, that $|\text{Aut }\mu| = 1$. As the ramification degree tends to infinity, i.e. as $N \longrightarrow \infty$, the Hodge integral expansion (3) tends to

$$\prod_{i=1}^{n}\frac{\mu_i^{\mu_i+k_i}}{\mu_i!}\int_{\overline{\mathcal{M}}_{g,n}}\prod\psi_i^{k_i}+O(e^N N^{m-1}) \longrightarrow e^{|\mu|}\prod_{i=1}^{n}\frac{\mu_i^{k_i-1/2}}{\sqrt{2\pi}}\int_{\overline{\mathcal{M}}_{g,n}}\prod\psi_i^{k_i}+O(e^N N^{m-1})$$

where $m = 3g-3+n-(n/2)$ is the highest degree of N in (3). Same expansion applies to each term in (2). By taking out the common factor $e^{|\mu|}$ and applying the asymptotic formula (3.1), we find that

$$\begin{aligned}
r\Gamma_r &= N^{m+1}\Big[(x_1+\cdots+x_n)\prod\frac{x_i^{k_i-1/2}}{\sqrt{2\pi}}\int_{\overline{\mathcal{M}}_{g,n}}\prod\psi_i^{k_i}\Big]+O(N^m)\\
\Gamma_{C1}^i &= \frac{N^{m+1/2}}{2}\sum_{k+l=k_i-2}\frac{(2k+1)!!(2l+1)!!}{2^{k+l+2}(k+l+2)!}x_i^{k+l+2}\prod_{j\neq i}\frac{x_j^{k_j-1/2}}{\sqrt{2\pi}}\Big[\int_{\overline{\mathcal{M}}_{g-1,n+1}}\psi_1^k\psi_2^l\prod\psi_j^{k_j}\\
&\quad+\sum_{g_1+g_2=g,\nu_1\cup\nu_2=\nu}\int_{\overline{\mathcal{M}}_{g_1,n_1}}\psi_1^k\prod\psi_j^{k_j}\int_{\overline{\mathcal{M}}_{g_2,n_2}}\psi_1^l\prod\psi_j^{k_j}\Big]+O(N^m)\\
\Gamma_{C2}^i &= N^{m+1}\Big[(x_1+\cdots+x_n)\prod\frac{x_i^{k_i-1/2}}{\sqrt{2\pi}}\int_{\overline{\mathcal{M}}_{g,n}}\prod\psi_i^{k_i}\Big]\\
&\quad-N^{m+1/2}\sum_{i=1}^{n}\Big[\frac{(2k_i+1)!!}{2^{k_i+1}k_i!}x_i^{k_i}\prod_{j\neq i}\frac{x_j^{k_j-1/2}}{\sqrt{2\pi}}\int_{\overline{\mathcal{M}}_{g,n}}\prod\psi_j^{k_j}\Big]+O(N^m)\\
\Gamma_J^{ij} &= N^{m+1/2}\frac{(x_i+x_j)^{k_i+k_j-1/2}}{\sqrt{2\pi}}\prod_{l\neq i,j}\frac{x_l^{k_l-1/2}}{\sqrt{2\pi}}\int_{\overline{\mathcal{M}}_{g,n-1}}\psi^{k_i+k_j-1}\prod\psi_l^{k_l}+O(N^m)
\end{aligned}$$

Putting them together in the 'cut-and-join relation' (2) yields a system of relations between Hodge integrals with one λ-class as follows:

- For N^{m+1}, we have trivial identity:

$$(x_1+\cdots+x_n)\prod\frac{x_i^{k_i-1/2}}{\sqrt{2\pi}}\int_{\overline{\mathcal{M}}_{g,n}}\prod\psi_i^{k_i}-(x_1+\cdots+x_n)\prod\frac{x_i^{k_i-1/2}}{\sqrt{2\pi}}\int_{\overline{\mathcal{M}}_{g,n}}\prod\psi_i^{k_i}=0$$

- For $N^{m+1/2}$, we have a relation between cut-and-join graphs:

$$\begin{aligned}
&\sum_{i=1}^{n}\Big[\frac{(2k_i+1)!!}{2^{k_i+1}k_i!}x_i^{k_i}\prod_{j\neq i}\frac{x_j^{k_j-1/2}}{\sqrt{2\pi}}\int_{\overline{\mathcal{M}}_{g,n}}\prod\psi_j^{k_j}\\
&-\sum_{j\neq i}\frac{(x_i+x_j)^{k_i+k_j-1/2}}{\sqrt{2\pi}}\prod_{l\neq i,j}\frac{x_l^{k_l-1/2}}{\sqrt{2\pi}}\int_{\overline{\mathcal{M}}_{g,n-1}}\psi^{k_i+k_j-1}\prod\psi_l^{k_l}\\
&-\frac{1}{2}\sum_{k+l=k_i-2}\frac{(2k+1)!!(2l+1)!!}{2^{k+l+2}(k+l+2)!}x_i^{k_i}\prod_{j\neq i}\frac{x_j^{k_j-1/2}}{\sqrt{2\pi}}\Big[\int_{\overline{\mathcal{M}}_{g-1,n+1}}\psi_1^{k}\psi_2^{l}\prod\psi_j^{k_j}\\
&+\sum_{g_1+g_2=g,\nu_1\cup\nu_2=\nu}\int_{\overline{\mathcal{M}}_{g_1,n_1}}\psi_1^{k}\prod\psi_j^{k_j}\int_{\overline{\mathcal{M}}_{g_2,n_2}}\psi_1^{l}\prod\psi_j^{k_j}\Big]\Big]=0\qquad\cdots(**)
\end{aligned}$$

- Lower degree strata will give relations for Hodge integrals involving non-trivial λ-class in terms of lower-dimensional ones.

In particular,the first non-trivial relation (**) implies the Witten's Conjecture (*):

Theorem 1. *The relation (**) implies (*).*

Proof. Introduce formal variables $s_i\in\mathbb{R}_{>0}$ and recall the Laplace Transformation:

$$\int_0^\infty\frac{x^{k-1/2}}{\sqrt{2\pi}}e^{-x/2s}dx=(2k-1)!!\ s^{k+1/2},\qquad\int_0^\infty x^ke^{-x/2s}dx=k!\ (2s)^{k+1}$$

Applying Laplace Transformation to the $N^{m+1/2}$-stratum gives the following relation:

$$\sum_{i=1}^{n}\Big[\quad s_i^{k_i+1}(2k_i+1)!!\prod_{j\neq i}s_j^{k_j+1/2}(2k_j-1)!!\int_{\overline{\mathcal{M}}_{g,n}}\prod\psi_l^{k_l}$$
$$-\sum_{a+b=k_i-2}s_i^{k_i+1}(2a+1)!!(2b+1)!!\prod_{j\neq i}s_j^{k_j+1/2}(2k_j-1)!!$$
$$\times\Big(\int_{\overline{\mathcal{M}}_{g-1,n+1}}\psi_1^a\psi_2^b\prod\psi_l^{k_l}+\sum_{g_1+g_2=g,\cdots}\int_{\overline{\mathcal{M}}_{g_1,n_1}}\psi^a\prod\psi_l^{k_l}\int_{\overline{\mathcal{M}}_{g_2,n_2}}\psi^b\prod\psi_l^{k_l}\Big)$$
$$-\sum_{j\neq i}\frac{(2w+1)!!}{\sqrt{s_i}+\sqrt{s_j}}\big(s_is_j^{w+2}+s_i^{3/2}s_j^{w+3/2}+\cdots+s_i^{w+2}s_j\big)$$
$$\times\prod_{l\neq i,j}s_l^{k_l+1/2}(2k_l-1)!!\int_{\overline{\mathcal{M}}_{g,n-1}}\psi^w\prod\psi_l^{k_l}\quad\Big]=0$$

where $w=k_i+k_j-1$. The last term is derived from direct integration;

$$\frac{N^{k+\frac{1}{2}}}{\sqrt{2\pi}}\int_0^\infty\int_0^\infty(x_i+x_j)^{k+\frac{1}{2}}e^{-x_iy_i}e^{-x_jy_j}dx_idx_j=\frac{N^{k+\frac{1}{2}}}{2\sqrt{2\pi}}\int_0^\infty\int_{-r}^{r}r^{k+\frac{1}{2}}e^{-\frac{r+s}{2}y_i}e^{-\frac{r-s}{2}y_j}dsdr$$
$$=\frac{N^{k+\frac{1}{2}}}{2\sqrt{2\pi}}\int_0^\infty\Big[\int_{-r}^{r}e^{\frac{y_j-y_i}{2}s}ds\Big]r^{k+\frac{1}{2}}e^{-\frac{y_i+y_j}{2}r}dr=\frac{N^{k+\frac{1}{2}}}{\sqrt{y_i}+\sqrt{y_j}}\frac{(2k+1)!!}{(2y_iy_j)^{k+\frac{3}{2}}}\Big[y_i^{k+1}+y_i^{k+\frac{1}{2}}y_j^{\frac{1}{2}}+\cdots+y_j^{k+1}\Big]$$

under change of variable $r=x_i+x_j$ and $s=x_i-x_j$. Considering this as a polynomial in s_i's, we can isolate out coefficients to obtain

$$(\#)\cdots(2k_i+1)!!\prod_{j\neq i}(2k_j-1)!!\int_{\overline{\mathcal{M}}_{g,n}}\prod\psi_l^{k_l}=\sum_{j\neq i}(2w+1)!!\prod_{l\neq i,j}(2k_l-1)!!\int_{\overline{\mathcal{M}}_{g,n-1}}\psi^w\prod_{l\neq i,j}\psi_l^{k_l}+$$
$$\sum_{a+b=k_i-2}(2a+1)!!(2b+1)!!\Big[\int_{\overline{\mathcal{M}}_{g-1,n+1}}\psi^a\psi^b\prod_{l\neq i}\psi_l^{k_l}+\sum\int_{\overline{\mathcal{M}}_{g_1,n_1}}\psi^a\prod\psi_l^{k_l}\int_{\overline{\mathcal{M}}_{g_2,n_2}}\psi^b\prod\psi_l^{k_l}\quad\Big]$$

The reason for getting 1 as coefficient in the Join-case is due to the following expansion

$$\frac{1}{\sqrt{s_i}+\sqrt{s_j}}(s_is_j^{w+2}+s_i^{3/2}s_j^{w+3/2}+\cdots+s_i^{w+2}s_j)$$
$$=\frac{1}{\sqrt{s_j}}(1-\sqrt{\frac{s_i}{s_j}}+\frac{s_i}{s_j}-(\frac{s_i}{s_j})^{3/2}+\cdots)(s_is_j^{w+2}+s_i^{3/2}s_j^{w+3/2}+\cdots+s_i^{w+2}s_j)$$
$$=\cdots+1\cdot s_i^{k_i+1}s_j^{k_j+1/2}+\cdots$$

In the notations of (*), we have $\tilde{\sigma}_n=(2n+1)!!\sigma_n=(2n+1)!!\psi^n$ and

$$\langle\tilde{\sigma}_{k_1}\cdots\tilde{\sigma}_{k_n}\rangle_g=\Big[\prod_{i=1}^{n}(2k_i+1)!!\Big]\int_{\overline{\mathcal{M}}_{g,n}}\psi_1^{k_1}\cdots\psi_n^{k_n}$$

After multiplying a common factor $\prod_{l\neq i}(2k_l+1)$ on both sides of (#), we obtain

$$\langle\tilde{\sigma}_n\prod_{k\in S}\tilde{\sigma}_k\rangle_g=\sum_{k\in S}(2k+1)\langle\tilde{\sigma}_{n+k-1}\prod_{l\neq k}\tilde{\sigma}_l\rangle_g+\frac{1}{2}\sum_{a+b=n-2}\langle\tilde{\sigma}_a\tilde{\sigma}_b\prod_{l\neq a,b}\tilde{\sigma}_l\rangle_{g-1}$$
$$+\frac{1}{2}\sum_{S=X\cup Y,a+b=n-2,g_1+g_2=g}\langle\tilde{\sigma}_a\prod_{k\in X}\tilde{\sigma}_k\rangle_{g_1}\langle\tilde{\sigma}_b\prod_{l\in Y}\tilde{\sigma}_l\rangle_{g_2}$$

which is the desired recursion relation (*). The factor $2k+1$ comes from missing j-th marked point in the Join-graph contribution, and the extra 1/2-factor on Cut-graph contributions is due to graph counting conventions. Hence we derived Witten's Conjecture / Kontsevich Theorem through localization on the relative moduli space. □

References

[1] R. Dijkgraaf, *Intersection Theory, Integrable Hierarchies and Topological Field Theory*, New symmetry principles in quantum field theory (Cargse, 1991), 95–158, NATO Adv. Sci. Inst. Ser. B Phys., 295, Plenum, New York, 1992.
[2] T. Ekedahl, S. Lando, M. Shapiro, A. Vainshtein, *Hurwitz numbers and intersections on moduli spaces of curves*, Invent. Math. **146** (2001), 297–327.
[3] I.P. Goulden, D.M. Jackson, A. Vainshtein, *The number of ramified coverings of the sphere by the torus and surfaces of higher genera*, Ann. of Comb. **4** (2000), 27-46.
[4] T. Graber, R. Vakil, *Relative virtual localization and vanishing of tautological classes on moduli spaces of curves*, preprint, math.AG/0309227.
[5] Y.-S. Kim, *Computing Hodge integrals with one λ-class*, preprint, math-ph/0501018
[6] M. Kontsevich, *Intersection theory on the moduli space of curves and the matrix Airy function*, Comm. Math. Phys. **147** (1992), no. 1, 1–23.
[7] A.M. Li, G. Zhao, Q. Zheng, *The number of ramified coverings of a Riemann surface by Riemann surface*, Comm. Math. Phys. **213** (2000), no. 3, 685–696.
[8] J. Li, *Stable Morphisms to singular schemes and relative stable morphisms*, J. Diff. Geom. **57** (2001), 509-578.
[9] C.-C. Liu, K. Liu, J. Zhou, *A proof of a conjecture of Mariño-Vafa on Hodge Integrals*, J. Differential Geom. **65** (2003), no. 2, 289–340.
[10] M. Mirzakhani, *Simple geodesics and Weil-Petersson volumes of moduli spaces of bordered Riemann surfaces*, preprint, 2003.
[11] A. Okounkov, R. Pandharipande, *Gromov-Witten theory, Hurwitz numbers, and Matrix models, I*, preprint, math.AG/0101147
[12] E. Witten, *Two-dimensional gravity and intersection theory on moduli space*, Surveys in differential geometry (Cambridge, MA, 1990), 243–310, Lehigh Univ., Bethlehem, PA, 1991.
[13] J. Zhou, *Hodge integrals, Hurwitz numbers, and symmetric groups*, preprint, math.AG/0308024.

Department of Mathematics,University of California at Los Angeles, Los Angeles, CA 90095-1555, USA
E-mail address: yskim@math.ucla.edu

Center of Math Sciences, Zhejiang University, Hangzhou, Zhejiang 310027, China; Department of Mathematics,University of California at Los Angeles, Los Angeles, CA 90095-1555, USA
E-mail address: liu@math.ucla.edu, liu@cms.zju.edu.cn