THE YAMABE PROBLEM FOR HIGHER ORDER CURVATURES

Weimin Sheng, Neil S. Trudinger, Xu-Jia Wang

Abstract. Let $\mathcal{M}$ be a compact Riemannian manifold of dimension n. The k-curvature, for $k = 1, 2, \cdots, n$, is defined as the k-th elementary symmetric polynomial of the eigenvalues of the Schouten tenser. The k-Yamabe problem is to prove the existence of a conformal metric whose k-curvature is a constant. When $k = 1$, it reduces to the well-known Yamabe problem. Under the assumption that the metric is admissible, the existence of solutions to the k-Yamabe problem was recently proved by Gursky and Viaclovsky for $k > \frac{n}{2}$. In this paper we prove the existence of solutions for the remaining cases $2 \leq k \leq \frac{n}{2}$, assuming that the equation is variational.

1. Introduction

In recent years the Yamabe problem for the k-curvature of the Schouten tensor, or simply the k-Yamabe problem, has been extensively studied. Let $(\mathcal{M}, g_0)$ be a compact Riemannian manifold of dimension n. Denote by $Riem$, Ric, and R the Riemannian curvature tensor, the Ricci tensor, and the scalar curvature, respectively. Then one has the standard decomposition $Riem = W + A \odot g_0$, where W is the Weyl tensor, A is the Schouten tensor given in (1.2) below, and $\odot$ denotes the Kulkarni-Nomizu product [B]. As the Weyl curvature tensor is conformally invariant, the transformation of the Riemannian curvature tensor under conformal changes of metrics is determined by that of the Schouten tensor. Therefore it is of interest to study curvature functions of the Schouten tensor under conformal deformation. A fundamental problem is the k-Yamabe problem, to prove the existence of a conformal metric $g = g_v = v^{\frac{4}{n-4}} g_0$ whose k-curvature is equal to a constant, that is

$$\sigma_k(\lambda(A_g)) = 1, \tag{1.1}$$

where $1 \leq k \leq n$ is an integer, $\lambda = (\lambda_1, \cdots, \lambda_n)$ are the eigenvalues of A_g with respect to the metric g, and

$$A_g = \frac{1}{n-2}(Ric_g - \frac{R_g}{2(n-1)} g) \tag{1.2}$$

The first and third authors were supported by the Natural Science Foundation of China. The second and third authors were supported by the Australian Research Council

is the Schouten tensor. As usual we denote by

$$\sigma_k(\lambda) = \sum_{i_1<\cdots<i_k} \lambda_{i_1}\cdots\lambda_{i_k} \tag{1.3}$$

the k-th elementary symmetric polynomial. When $k=1$, we arrive at the well known Yamabe problem.

When $k \geq 2$, the k-Yamabe problem was initiated by Viaclovsky [V1] and also arose in the study of the Paneitz operator [Br, CGY1]. Equation (1.1) is a fully nonlinear partial differential equation. To work in the realm of elliptic operators, one assumes that the eigenvalues $\lambda(A_g)$ lie in the convex cone Γ_k [CNS], where

$$\Gamma_k = \{\lambda \in \mathbf{R}^n \mid \sigma_j(\lambda) > 0 \text{ for } j = 1, \cdots, k\}. \tag{1.4}$$

Under this assumption, the k-Yamabe problem has been solved in the cases when $n=4$ and $k=2$ [CGY1, CGY2], or when the manifold is locally conformally flat [LL1, GW2]. Very recently Gursky and Viaclovsky [GV2] solved the problem for $k > \frac{n}{2}$, using the positivity of the Ricci curvature in this case.

In this paper we employ a variational method to treat the problem for the cases $2 \leq k \leq \frac{n}{2}$. We prove that equation (1.1) has a solution as long as it is variational, namely it is the Euler equation of a functional, which includes the cases when $k=2$ and when $\mathcal{M}$ is locally conformally flat. In Section 2 we give a sufficient and necessary condition for equation (1.1) to be variational.

When (1.1) is variational and $k \neq \frac{n}{2}$, its solutions correspond to critical points of the functional

$$J(g) = \frac{n-2}{2(n-2k)} \int_{\mathcal{M}} \sigma_k(\lambda(g))d\,vol_g - \frac{n-2}{2n}\int_{\mathcal{M}} d\,vol_g \tag{1.5}$$

in the conformal class $[g_0] = \{g \mid g = v^{\frac{4}{n-2}} g_0,\ v > 0\}$. When $k = \frac{n}{2}$, the first integral in (1.5) is a constant and we need to replace it by (2.32) below. We will find a min-max (Mountain Pass Lemma) solution, as in the case $k=1$. Note that in the case $k > \frac{n}{2}$, the coefficients are negative and the functional is negative.

The progressive resolution of the Yamabe problem ($k=1$) by the second author, Aubin and Schoen [Ya, Tr1, Au1, S1] was a milestone in differential geometry. Roughly speaking, the overall proof consists of two parts. The first one is to show that the Yamabe problem is solvable if the Yamabe constant Y_1 satisfies the condition

$$Y_1(\mathcal{M}) < Y_1(S^n), \tag{1.6}$$

and the second one is to verify the condition (1.6) for manifolds not conformally diffeomorphic to the unit sphere S^n with standard metric. When $\mathcal{M}$ is locally conformally flat, different proofs were found later [SY1, Ye].

For the k-Yamabe problem, $2 \le k \le \frac{n}{2}$, our variational approach basically comprises the same two steps. Namely one first shows that (1.1) has a solution when the k-Yamabe constant Y_k satisfies

$$Y_k(\mathcal{M}) < Y_k(S^n), \tag{1.7}$$

and then verify the condition (1.7) for manifolds not conformal to the unit sphere S^n. But since equation (1.1) is fully nonlinear, our treatment is technically different and more complicated. For the first step, we cannot apply the variational method directly, since we need to restrict the functional (1.5) to a subset of the conformal class $[g_0]$, given by

$$[g_0]_k = \{g \mid g = v^{\frac{4}{n-2}} g_0, v > 0, \lambda(A_g) \in \Gamma_k\}, \tag{1.8}$$

and the set of functions v with $v^{\frac{4}{n-2}} g_0 \in [g_0]_k$ may not be convex. Through the functional (1.5), we introduce a descent gradient flow, establish appropriate a priori estimates, and prove the convergence of solutions to the flow under assumption (1.7). We need to choose a particular gradient flow to obtain the a priori estimates, locally in time.

For the second step, it seems impossible to find an explicit k-admissible test function. The function

$$v_\varepsilon(x) = \Big(\frac{\varepsilon}{\varepsilon^2 + r^2}\Big)^{(n-2)/2}, \tag{1.9}$$

which is the unique solution of (1.1) on the Euclidean space $\mathbf{R}^n$ for all $1 \le k \le n$, is k-admissible only when $r \le C\varepsilon^{1/2}$ on a general manifold, where r is the geodesic distance. Fortunately we found a simple way to deduce (1.7) directly from (1.6).

This paper is arranged as follows. In Section 2, we state the main results, specifically in §2.1, while in §2.2 we outline the proof. In §2.3 we collect some related results on the k-Hessian equation. In §2.4 we give a sufficient and necessary condition for a partial differential equation to be variational. In Section 3 we study the regularity for the gradient flow of the functional (1.5) for solutions with $\lambda(A_{g_v}) \in \Gamma_k$, and give counterexamples to interior regularity for solutions with $\lambda(A_{g_v}) \in (-\Gamma_k)$. In Section 4 we investigate the asymptotic behavior of a descent gradient flow and prove the convergence of the flow under condition (1.7). We then prove (1.7) for manifolds not conformal to S^n in Section 5. The final Section 6 contains some remarks.

The authors are grateful to Kaiseng Chou for useful discussions. This research was largely carried out in the winter of 2004-05 while the third author was at the Nankai Institute of Mathematics in China under a Yangtze River Fellowship. The other authors are also grateful for the Nankai Institute for hospitality when we were all there together in November 2004.

2. The main results

2.1. The main results. Let $(\mathcal{M}, g_0)$ be a Riemannian manifold. If $g = v^{\frac{4}{n-2}} g_0$ is a solution of (1.1), then the Schouten tensor is given by $A_g = \frac{2}{(n-2)v} V$, and v satisfies the equation

$$L[v] := v^{(1-k)\frac{n+2}{n-2}} \sigma_k(\lambda(V)) = v^{\frac{n+2}{n-2}}, \tag{2.1}$$

where

$$V = -\nabla^2 v + \frac{n}{n-2} \frac{\nabla v \otimes \nabla v}{v} - \frac{1}{n-2} \frac{|\nabla v|^2}{v} g_0 + \frac{n-2}{2} v A_{g_0}. \tag{2.2}$$

Equation (2.1) is a fully nonlinear equation of similar type to the k-Hessian equations [CNS, CW2, I1, TW2]. For the operator L to be elliptic, we need to restrict to metrics with eigenvalues $\lambda(A_g) \in \cup(\pm\Gamma_k)$, which we will simply denote as $g \in \pm\Gamma_k$. Therefore equation (2.1) has two elliptic branches, one is when the eigenvalues $\lambda \in \Gamma_k$ and the other one is when $\lambda \in (-\Gamma_k)$. In this paper we will mainly consider solutions with eigenvalues in Γ_k. Accordingly we say v is *k-admissible* (that is v is strictly subharmonic with respect to L) if $g = v^{\frac{4}{n-2}} g_0 \in \Gamma_k$. The set of all k-admissible functions will be denoted by $\Phi_k = \Phi_k(\mathcal{M}, g_0)$. In this paper we will always assume, unless otherwise indicated, that $2 \le k \le \frac{n}{2}$ and the following two conditions hold,

(C1) The set $\Phi_k(\mathcal{M}, g_0) \neq \emptyset$;

(C2) The operator L is variational.

The condition (C1) ensures that the operator L is elliptic, and may be replaced by $Y_j(\mathcal{M}) > 0$ for $j = 1, \cdots, k$, as in the case when $k = 2$ and $n = 4$ [CGY1, GV1]. Note that in condition (C1), we do not assume directly that the metric $g_0 \in \Gamma_k$, rather we assume that there exists a positive function v such that $v^{\frac{4}{n-2}} g_0 \in \Gamma_k$. Conditions (C1) (C2) are automatically satisfied when $k = 1$.

As for the Yamabe problem, we introduce the k-Yamabe constant for $2 \le k \le \frac{n}{2}$,

$$Y_k(\mathcal{M}) = \inf\{\mathcal{F}_k(g) \mid g \in [g_0]_k, \mathrm{Vol}(\mathcal{M}_g) = 1\}, \tag{2.3}$$

where $[g_0]_k$ is defined in (1.8), and

$$\begin{aligned} \mathcal{F}_k(g) &= \int_{\mathcal{M}} \sigma_k(\lambda(A_g)) d\,vol_g \\ &= \int_{\mathcal{M}} v^{\frac{2n}{n-2} - k\frac{n+2}{n-2}} \sigma_k(\lambda(V))\, d\,vol_{g_0}. \end{aligned} \tag{2.4}$$

Note that we have ignored a coefficient $(\frac{2}{n-2})^k$ in the second equality. The main result of the paper is the following.

Theorem 2.1. *Assume $2 \leq k \leq \frac{n}{2}$ and the conditions (C1) (C2) hold. Then the k-Yamabe problem (1.1) is solvable.*

As indicated in the introduction, the proof of Theorem 2.1 is divided into two parts. The first part is the following lemma.

Lemma 2.1. *If the critical inequality (1.7) holds, then the k-Yamabe problem (1.1) is solvable.*

The second part provides the condition for (1.7).

Lemma 2.2. *The critical inequality (1.7) holds for any compact manifold which is not conformal to the unit sphere S^n.*

When $k = \frac{n}{2}$, we prove that $\mathcal{F}_{n/2}(g) \equiv Y_{n/2}(\mathcal{M})$, that is it is a constant for any $g \in [g_0]_k$ (Lemma 4.8). Hence (1.7) implies that $\mathcal{F}_{n/2}(g) < Y_{n/2}(S^n)$ provided $\mathcal{M}$ is not conformal to the unit sphere.

2.2. Strategy of the proof. A solution of the k-Yamabe problem is a min-max type critical point of the corresponding functional. As we need to restrict ourselves to k-admissible functions, we cannot directly use variational theory (such as the Ekeland variational principle). Instead we study a descent gradient flow of the functional and investigate its convergence. We need to choose a special gradient flow (similar to [CW2]) for which the necessary a priori estimates can be established.

As with the original Yamabe paper [Ya], we first study the approximating problems

$$L(v) = v^p, \tag{2.5}$$

where $1 < p \leq \frac{n+2}{n-2}$. Equation (2.5) is the Euler equation of the functional

$$J_p(v) = J_p(v; \mathcal{M}) = \frac{n-2}{2n-4k} \int_{(\mathcal{M},g_0)} v^{\frac{2n}{n-2} - k\frac{n+2}{n-2}} \sigma_k(\lambda(V)) - \frac{1}{p+1} \int_{(\mathcal{M},g_0)} v^{p+1}. \tag{2.6}$$

Let $\varphi_1 = \varepsilon$ and $\varphi_2 = \varepsilon^{-1}$, where $\varepsilon > 0$ is a small constant. Then $J_p(\varphi_1) \to 0$ (when $k < \frac{n}{2}$) and $J_p(\varphi_2) \to -\infty$ as $\varepsilon \to 0$. Let P denote the set of paths in Φ_k connecting φ_1 and φ_2, namely

$$P = \{\gamma \in C([0,1], \Phi_k) \mid \gamma(0) = \varphi_1, \gamma(1) = \varphi_2\}. \tag{2.7}$$

Obviously $\Phi_k \neq \emptyset$. Denote

$$c_p[\mathcal{M}] = \inf_{\gamma \in P} \sup_{s \in [0,1]} J_p(\gamma(s); \mathcal{M}). \tag{2.8}$$

Then (1.7) is equivalent to

$$c_p[\mathcal{M}] < c_p[S^n] \tag{2.9}$$

with $p = \frac{n+2}{n-2}$. We will prove that J_p has a min-max critical point v_p with $J_p(v_p) = c_p[\mathcal{M}]$, in the sub-critical case $p < \frac{n+2}{n-2}$. By a blow-up argument, we prove furthermore that v_p converges to a solution of (2.1) under the assumption (2.9).

The descent gradient flow will be chosen so that appropriate a priori estimates can be established. To simplify the computations, we will also use the conformal transformations $g = u^{-2}g_0$ or $g = e^{-2w}g_0$. That is

$$u = e^w = v^{-\frac{2}{n-2}}. \tag{2.10}$$

We say u or w is k-admissible if v is, and also denote $u, w \in \Phi_k$ if $v \in \Phi_k$.

Our gradient flow is given by

$$F[w] - w_t = \mu(f(x, w)), \tag{2.11}$$

where

$$F[w] := \mu(\sigma_k(\lambda(A_g))) \tag{2.12}$$

and $g = e^{-2w}g_0$. When $f(x, w) = e^{-2kw}$, a stationary solution of (2.11) is a solution to the k-Yamabe problem. The function μ is monotone increasing and satisfies

$$\lim_{t \to 0^+} \mu(t) = -\infty. \tag{2.13}$$

Condition (2.13) ensures the solution is k-admissible at any time t. For if $u(\cdot, t)$ is a smooth solution, then (2.13) implies $\sigma_k(\lambda) > 0$ at any time $t > 0$. A natural candidate for the choice of μ is the logarithm function $\mu(t) = \log t$ [Ch1, W1, TW4]. However for the flow (2.11), we need to choose a different μ to ensure appropriate a priori estimates.

In the case $k = \frac{n}{2}$, $\mathcal{F}_{n/2}(g)$ is a constant less than $Y_{n/2}(S^n)$. Hence by the Liouville theorem in [LL1], it is easy to prove that the set of solutions of (2.1) is compact. Hence the existence of solutions can be obtained by a degree argument. When $2 \le k < \frac{n}{2}$, by the Liouville theorem in [LL2], one can also prove the set of solutions of (2.5) is compact when $p < \frac{n+2}{n-2}$. But to use the condition (1.7) in the blow-up argument, we need a solution v_p of (2.5) satisfying $J_p(v_p) = c_p$. This is the reason for us to employ the gradient flow.

For the verification of (1.7), one cannot mimic the argument in the case $k = 1$, as the test function (1.9) is in general not k-admissible in a geodesic ball B_{ρ_0}. Instead we let

v_1 be the solution to the Yamabe problem ($k=1$), and v be the k-admissible solution of the equation

$$\sigma_k(\lambda(V)) = v_1^{k\frac{n+2}{n-2}}.$$

We will verify (1.7) by using the solution v as the test function.

The idea of using a gradient flow was inspired by [CW2], where a similar problem for the k-Hessian problem (see (2.22) below) was studied. However technically the argument in this paper is different. For the k-Yamabe problem, the corresponding Sobolev type inequality was not available, and the a priori estimates only allow us to get a local (in time) solution. The argument in this paper is also self-contained, except we will use the Liouville theorem in [LL1, LL2], proved by the moving plane method; see also [CGY3].

2.3. The k-Hessian equation. Equations (2.1) is closely related to the k-Hessian equation

$$\sigma_k(\lambda(D^2 v)) = f(x) \quad x \in \Omega, \tag{2.15}$$

where $1 \le k \le n$, $\lambda = (\lambda_1, \cdots, \lambda_n)$ denote the eigenvalues of the Hessian matrix $(D^2 v)$, Ω is a bounded domain in the Euclidean n-space $\mathbf{R}^n$. For later applications we collect here some elementary properties of the polynomial σ_k, and give a very brief summary of related results for the equation (2.15).

We write $\sigma_0(\lambda) = 1$, $\sigma_k(\lambda) = 0$ for $k > n$, and denote $\sigma_{k;i}(\lambda) = \sigma_k(\lambda)\big|_{\lambda_i = 0}$.

Lemma 2.3. *Let $\lambda \in \Gamma_k$ with $\lambda_1 \ge \cdots \ge \lambda_n$. Then*

$$\lambda_k \ge 0 \tag{i}$$

$$\sigma_k(\lambda) = \sigma_{k;i}(\lambda) + \lambda_i \sigma_{k-1;i}(\lambda), \tag{ii}$$

$$\Sigma_{i=1}^n \sigma_{k-1;i}(\lambda) = (n-k+1)\sigma_{k-1}(\lambda), \tag{iii}$$

$$\sigma_{k-1;n}(\lambda) \ge \cdots \ge \sigma_{k-1;1}(\lambda) > 0, \tag{iv}$$

$$\sigma_{k-1;k}(\lambda) \ge C_{n,k} \Sigma_{i=1}^n \sigma_{k-1;i}(\lambda), \tag{v}$$

$$\sigma_{k-1}(\lambda) \ge \frac{k}{n-k+1} \tbinom{n}{k}^{1/k} [\sigma_k(\lambda)]^{(k-1)/k}. \tag{vi}$$

Moreover, the function $[\sigma_k]^{1/k}$ is concave on Γ_k.

We just listed a few basic formulae, there are many other useful ones, see for example [CNS, CW2, LT]. For our investigation of equation (2.1) and its parabolic counterpart, Lemma 2.1 will be sufficient. These formulas can be extended to $\sigma_k(\lambda(r))$, regarded as functions of $n \times n$ symmetric matrices r. In particular $[\sigma_k(\lambda(r))]^{1/k}$ is concave in r [CNS].

We say a function $v \in C^2(\Omega)$ is k-admissible (relative to equation (2.15)) if the eigenvalues $\lambda(D^2 v) \in \Gamma_k$. Equation (2.15) is elliptic if v is k-admissible. The existence of k-admissible solutions to the Dirichlet problem for (2.15) was proved by Caffarelli-Nirenberg-Spruck [CNS], see also Ivochkina [I].

Relevant to the k-Yamabe problem is the variational property of the k-Hessian equation (2.15), investigated in [CW2, TW4, W1]. It is well known that the k-Hessian equation is the Euler equation of the functional

$$I_k(v) = \frac{1}{k+1} \int_\Omega (-v)\sigma_k(\lambda(D^2 v)). \tag{2.16}$$

The Sobolev-Poincaré type inequality, for k-admissible functions vanishing on the boundary,

$$I_l^{1/(l+1)}(v) \le C I_k^{1/(k+1)}(v), \tag{2.17}$$

was established in [W1] for the case $l = 0$ and $k \ge 1$, and in [TW4] for the case $k > l \ge 1$, where $0 \le l \le k \le n$,

$$I_0(v) = \Big[\int_\Omega |v|^{k^*} dx \Big]^{1/k^*}, \tag{2.18}$$

and

$$\begin{aligned} & k^* = n(k+1)/(n-2k) \quad \text{if} \;\; k < n/2, \\ & k^* < \infty \quad \text{if} \;\; k = n/2, \\ & k^* = \infty \quad \text{if} \;\; k > n/2. \end{aligned} \tag{2.19}$$

The best constant in the inequality (2.17) is attained by

$$v(x) = (1 + |x|^2)^{(2k-n)/2k} \tag{2.20}$$

when $l = 0$, $k < \frac{n}{2}$, and $\Omega = \mathbf{R}^n$; and by the unique solution of

$$\frac{\sigma_k}{\sigma_l}(\lambda(D^2 v)) = 1 \;\; \text{in} \;\; \Omega \tag{2.21}$$

for $1 \le l < k \le n$.

From the inequalities (2.17), it was proved in [CW2] that the Dirichlet problem

$$\begin{aligned} \sigma_k(\lambda(D^2 v)) = |v|^p + f(v) \quad & \text{in} \;\; \Omega, \\ v = 0 \quad & \text{on} \;\; \partial\Omega, \end{aligned} \tag{2.22}$$

admits a nonzero k-admissible solution, where $1 \le k \le \frac{n}{2}$, $1 < p < k^* - 1$, f is a lower order term of $|v|^p$. The existence result was proved for the problem with more general right hand side.

Remark 2.1. The 1996 preprint [CW1] also contains the existence of solutions to the Dirichlet problem (2.22) in the critical growth case $p = k^* - 1$. The result was obtained by a blow-up argument and the symmetrization of functions, see Theorem 9.1 in [CW1].

2.4. A necessary and sufficient condition for an equation to be variational.

The following proposition was communicated to the authors by Kaiseng Chou several years ago.

Proposition 2.1. *Let $\mathcal{M}$ be a compact manifold without boundary, $v \in C^4(\mathcal{M})$. An operator $F[v] = F[\nabla^2 v, \nabla v, v, x]$ is variational if and only if its linearized operator is self-adjoint. The functional is given by*

$$I[v] = \int G[v], \tag{2.23}$$

except when F is homogeneous of degree -1, where

$$G[v] = \int_0^1 vF[\lambda v]. \tag{2.24}$$

This proposition can be found in [O]. We give a proof of the "if" part, as we need some related formulae.

Proof. The linearized operator of $F[v]$ is given by

$$L(\varphi) = F^{ij}\varphi_{ij} + F_{p_j}\varphi_j + F_v\varphi. \tag{2.25}$$

We have

$$\begin{aligned}
\int_{\mathcal{M}} v\, L(\varphi) &= \int_{\mathcal{M}} [v\nabla_i(F^{ij}\nabla_j\varphi) + v\varphi F_v] - A \\
&= \int_{M} [-v_i\varphi_j F^{ij} + v\varphi F_v] - A \\
&= \int_{\mathcal{M}} \varphi[F^{ij}v_{ij} + F_{p_i}v_i + F_v v] - A + B \\
&= \int_{\mathcal{M}} \varphi\, L(v) - A + B,
\end{aligned}$$

where

$$\begin{aligned}
A &= \int_{\mathcal{M}} v\varphi_j(\nabla_i F^{ij} - F_{p_j}), \\
B &= \int_{\mathcal{M}} v_i\varphi(\nabla_j F^{ij} - F_{p_i}), \\
-A + B &= -\int_{\mathcal{M}} (\frac{\varphi}{v})_i v^2(\nabla_j F^{ij} - F_{p_i}) \\
&= \int_{\mathcal{M}} \frac{\varphi}{v}\nabla_i[v^2(\nabla_j F^{ij} - F_{p_i})]
\end{aligned}$$

Hence L is self-adjoint if and only if

$$\sum_{i,j=1}^{n} \nabla_i [v^2(\nabla_j F^{ij} - F_{p_i})] = 0. \tag{2.26}$$

If L is self-adjoint,

$$\begin{aligned}
\langle I'[v], \varphi\rangle &= \int_{\mathcal{M}} \varphi \int_0^1 F[\lambda v] + \int_0^1 \int_{\mathcal{M}} \lambda v [F^{ij}[\lambda v]\varphi_{ij} + F_{p_i}\varphi_i + F_v \varphi] \\
&= \int_{\mathcal{M}} \varphi \int_0^1 F[\lambda v] + \int_0^1 \int_{\mathcal{M}} \lambda \varphi [F^{ij}[\lambda v] v_{ij} + F_{p_i} v_i + F_v v] \\
&= \int_{\mathcal{M}} \varphi \int_0^1 F[\lambda v] + \int_{\mathcal{M}} \varphi \int_0^1 \lambda \frac{d}{d\lambda} F[\lambda v] d\lambda \\
&= \int_{\mathcal{M}} \varphi F[v].
\end{aligned}$$

Hence F is the Euler equation of the functional I. □

Conversely if the operator F is the Euler operator of the functional I, from the above argument we must have $-A + B = 0$, namely (2.26) holds. In other words, F is the Euler operator of I if and only if (2.26) holds. Observe that if

$$\sum_i \nabla_i F^{ij} = F_{p_j} \quad \forall\ j, \tag{2.27}$$

then (2.26) holds.

From Proposition (2.5) we can recover the results on the variational structure of (2.1) in [V1]. First, if locally $\mathcal{M}$ is Euclidean, one verifies directly that (2.26) holds, as it is a pointwise condition. The locally conformally flat case is equivalent to the Euclidean case by a conformal deformation to the Euclidean metric. Finally if $k = 2$, we note that to verify (2.26) for arbitrary v with a fixed background metric g_0 is equivalent to verify it for $v \equiv 1$ with respect to an arbitrary conformal metric $g = \hat{v}^{\frac{4}{n-2}} g_0$. However when $v \equiv 1$, condition (2.26) becomes $\sum_{i,j=1}^{n} \nabla_i \nabla_j F^{ij} = 0$, where $F^{ij} = \frac{\partial}{\partial r_{ij}} \sigma_k(\lambda(r))$ at $r = A_g$. But we have

$$\nabla_i F^{ij} = \frac{1}{2(n-2)} (R_{,j} - 2R_{ij,i}) = 0 \tag{2.28}$$

by the second Bianchi identity.

When $k \geq 3$, it is easy to find metrics for which (2.26) does not hold (at $v \equiv 1$). So equation (2.1) need not be variational. As an example, let $k = n = 3$, and in a local coordinate system, let the metric $g = \{g_{ij}\}$ be given by

$$g_{11} = 1, \quad g_{22} = 1 + x, \quad g_{33} = 1 + y^2 + z^2, \quad g_{ij} = \delta_{ij} \ \text{ for } \ i \neq j. \tag{2.29}$$

Then $\nabla_i \nabla_j F^{ij} \neq 0$.

By (2.23) we also see that (2.6) is the functional of (2.5). When $k = \frac{n}{2}$, the integral (2.24) may not exist. We may consider v as a composite function $v = \varphi(w)$ and write equation (2.5) in the form

$$F[w] =: \varphi'(w)L(\varphi(w)) = \varphi^p(w)\varphi'(w). \tag{2.30}$$

If the operator L in (2.1) satisfies (2.26) with respect to v, the operator F in (2.30) satisfies (2.26) with respect to w. Hence the corresponding functional is given by

$$\mathcal{E}_{n/2}(w) = \int_{(\mathcal{M},g_0)} \int_0^1 wF[tw]. \tag{2.31}$$

In particular if $v = e^{-\frac{n-2}{2}w}$, then we obtain the functional in [BV],

$$\mathcal{E}_{n/2}(w) = -\int_{(\mathcal{M},g_0)} \int_0^1 w\sigma_{n/2}(\lambda(A_{g_t})), \tag{2.32}$$

where $g_t = e^{-2tw}g_0$.

3. The a priori estimates

In this section we study the regularity of k-admissible solutions ($2 \leq k \leq n$) to equation (2.1) and its parabolic counterpart (2.11). The global a priori estimates for the elliptic equation (2.1) (for solutions with eigenvalues in Γ_k) have already been established by J. Viaclovsky [V2], with interior estimates by P. Guan and G. Wang [GW1]. We will provide a simpler proof for the elliptic equation (2.1), and extend the estimates to the parabolic equation (2.11) on general manifolds, which is necessary for our proof of Theorem 2.1. Previously the estimates for the parabolic equation were proved on locally conformally flat manifolds [Ye, GW2, GW3]. Regularity has also been studied in many other papers [CGY1,LL1].

We will also present an example showing that the interior a priori estimates do not hold for solutions with eigenvalues in the negative cone $-\Gamma_k$.

3.1. A priori estimates for equation (2.1). For the regularity of (2.1), we will use the conformal changes $g = u^{-2}g_0$. For function u, equation (2.1) becomes

$$\sigma_k(\lambda(U)) = u^{-k}, \tag{3.1}$$

where

$$U = \nabla^2 u - \frac{|\nabla u|^2}{2u} g_0 + u A_{g_0}.$$

Lemma 3.1. *[GW1] Let $u \in C^3$ be a k-admissible positive solution of (3.1) in a geodesic ball $B_r(0) \subset \mathcal{M}$. Suppose $A_{g_0} = (a_{ij}) \in C^1(B_r(0))$. Then we have*

$$\frac{|\nabla u|}{u}(0) \leq C, \tag{3.2}$$

where C depends only on n, k, r, $\inf u$, and $\|A_{g_0}\|_{C^1}$, ∇ denotes the covariant derivative with respect to the initial metric g_0.

Proof. Let μ be a smooth, monotone increasing function. Write equation (3.1) in the form

$$F[u] = \mu[f(x,u)], \tag{3.3}$$

where $F[u] = \mu[\sigma_k(\lambda(U))]$. We will prove (3.2) for more general function f. Moreover the constant C is independent of $\sup_{B_r} u$ if $f = \kappa u^{-p}$ for some constant $p > 0$ and smooth, positive function κ.

Let $z = |\nabla u|^2 \varphi^2(u)\rho^2$, where $\varphi(u) = \frac{1}{u}$, and $\rho(x) = (1 - \frac{|x|^2}{r^2})^+$ is a cut-off function, $|x|$ denotes the geodesic distance from 0. Suppose z attains maximum at $x_0 \in B_1(0)$, and $|\nabla u(x_0)| = u_1(x_0)$. Then at x_0, in an orthonormal frame,

$$\frac{1}{2}(\log z)_i = \frac{u_{1i}}{u_1} + \frac{\varphi'}{\varphi} u_i + \frac{\rho_i}{\rho} = 0, \tag{3.4}$$

$$\frac{1}{2}(\log z)_{ij} = \frac{u_{1ij}}{u_1} + \sum_{\alpha>1} \frac{u_{\alpha i} u_{\alpha j}}{u_1^2} - \frac{u_{1i} u_{1j}}{u_1^2} + \frac{\varphi'}{\varphi} u_{ij} + (\frac{\varphi''}{\varphi} - \frac{\varphi'^2}{\varphi^2}) u_i u_j + (\frac{\rho_{ij}}{\rho} - \frac{\rho_i \rho_j}{\rho^2}). \tag{3.5}$$

Differentiating equation (3.3) gives

$$F^{ij}[u_{ij1} + (\frac{u_1^3}{2u^2} - \frac{u_1 u_{11}}{u})\delta_{ij}] = \Delta, \tag{3.6}$$

where for a matrix $r = (r_{ij})$, $F^{ij}(r) = \frac{\partial}{\partial r_{ij}} \mu[\sigma_k(\lambda(r))] = \mu' \frac{\partial}{\partial r_{ij}} \sigma_k(\lambda(r))$,

$$\Delta = \nabla_1 \mu(f) - F^{ij} \nabla_1 (a_{ij} u).$$

By (3.4)-(3.6) we have, at x_0,

$$0 \geq \frac{1}{2} F^{ij} (\log z)_{ij} = \frac{1}{u_1}[\frac{u_1 u_{11}}{u} - \frac{u_1^3}{2u^2}]\mathcal{F} + \sum_{\alpha>1} F^{ij} \frac{u_{\alpha i} u_{\alpha j}}{u_1^2} - F^{ij}(\frac{\varphi'}{\varphi} u_i + \frac{\rho_i}{\rho})(\frac{\varphi'}{\varphi} u_j + \frac{\rho_j}{\rho})$$
$$+ \frac{\varphi'}{\varphi} F^{ij}(u_{ij} - \frac{|\nabla u|^2}{2u}\delta_{ij}) + \frac{u_1^2}{2u}\frac{\varphi'}{\varphi}\mathcal{F} + (\frac{\varphi''}{\varphi} - \frac{\varphi'^2}{\varphi^2}) F^{11} u_1^2 + F^{ij}(\frac{\rho_{ij}}{\rho} - \frac{\rho_i \rho_j}{\rho^2}) + \frac{\Delta}{u_1} + \Delta',$$

where $\mathcal{F} = \sum F^{ii}$, Δ' arises in the exchange of derivatives, with $|\Delta'| \leq C\mathcal{F}$. Note that

$$F^{ij}(u_{ij} - \frac{|\nabla u|^2}{2u}\delta_{ij}) = k\mu' \sigma_k(\lambda) - u F^{ij} a_{ij} \geq -C_a u \mathcal{F},$$

where $C_a = 0$ if $A_{g_0} = (a_{ij}) = 0$. By (3.4) and since $\varphi(u) = \frac{1}{u}$,

$$[\frac{u_{11}}{u} - \frac{u_1^2}{2u^2}] + \frac{u_1^2}{2u}\frac{\varphi'}{\varphi} = -\frac{u_1 \rho_1}{u\rho},$$
$$-F^{ij}(\frac{\varphi'}{\varphi} u_i + \frac{\rho_i}{\rho})(\frac{\varphi'}{\varphi} u_j + \frac{\rho_j}{\rho}) + (\frac{\varphi''}{\varphi} - \frac{\varphi'^2}{\varphi^2}) F^{11} u_1^2 = -F^{ij}(\frac{2\varphi' u_i \rho_j}{\varphi\rho} + \frac{\rho_i \rho_j}{\rho^2})$$

Hence we obtain

$$0 \geq \sum_{\alpha>1} F^{ij} \frac{u_{\alpha i} u_{\alpha j}}{u_1^2} - C(\frac{1}{r^2 \rho^2} + \frac{u_1}{u}\frac{1}{r\rho} + C_a)\mathcal{F} + \frac{\Delta}{u_1} + \Delta'. \tag{3.7}$$

Denote $b = \frac{|\nabla u|^2}{2u}(x_0)$. We claim

$$\sum_{\alpha>1} F^{ij} u_{\alpha i} u_{\alpha j} \geq C b^2 \mathcal{F} - C' u^2 \mathcal{F} \tag{3.8}$$

for some positive constant C, C' ($C' = 0$ if $a_{ij} = 0$). Note that by Lemma 2.3 (iii) (vi), $\mathcal{F} \ge C_{n,k}\mu' \sigma_k^{(k-1)/k}$. From (3.8) we have

$$\frac{|\nabla u|}{u}\rho \le \frac{C_1}{r} + C_2 \tag{3.9}$$

at x_0, where C_1 is independent of f and C_2 independent of r. Hence $z(0) \le z(x_0) \le C$, namely (3.2) holds.

Denote $\tilde{u}_{ij} = u_{ij} + ua_{ij}$. For any two unit vectors ξ, η, we denote formally $\tilde{u}_{\xi\eta} = \sum \xi_i \eta_j \tilde{u}_{ij}$. Then to prove (3.8) it suffices to prove

$$A =: \sum_{\alpha > 1} F^{ij} \tilde{u}_{\alpha i} \tilde{u}_{\alpha j} \ge C b^2 \mathcal{F}. \tag{3.10}$$

By a rotation of the coordinates we suppose $\{\tilde{u}_{ij}\}$ is diagonal at x_0. Then

$$\lambda_1 = \tilde{u}_{11} - b, \cdots, \lambda_n = \tilde{u}_{nn} - b$$

are the eigenvalues of the matrix $\{\tilde{u}_{ij} - \frac{|Du|^2}{2u}\delta_{ij}\}$. Suppose $\lambda_1 \ge \cdots \ge \lambda_n$. At x_0 we have $|Du(x_0)| = u_\xi(x_0)$ for some unit vector ξ. In the new coordinates we have

$$A = \sum_i (F^{ii}\tilde{u}_{ii}^2 - F^{ii}\tilde{u}_{\xi i}^2).$$

If there exists a small $\delta_0 > 0$ such that $\langle e_i, \xi\rangle < 1 - \delta_0$ for all unit axial vectors e_i, then $A \ge \delta_0 F^{ii}\tilde{u}_{ii}^2$. Since $\lambda = (\lambda_1, \cdots, \lambda_n) \in \Gamma_k^+$, we have $\lambda_k > 0$ and so $\tilde{u}_{kk} > b$. Hence by Lemma 2.3(v), $A \ge \delta_0 b^2 F^{kk} \ge \delta_1 b^2 \mathcal{F}$. We obtain (3.8).

If there is i^* such that $\langle e_{i^*}, \xi\rangle \ge 1 - \delta_0$, then we have $A \ge \frac{1}{2}\sum_{i \ne i^*} F^{ii}\tilde{u}_{ii}^2$. If there exists $j \ge k$, $j \ne i^*$ such that $\tilde{u}_{jj} \ge \alpha b$ for some $\alpha > 0$, then by Lemma 2.3(iv)(v), $A \ge \frac{1}{2}F^{jj}(\alpha b)^2 \ge \delta_2 b^2 \mathcal{F}$ and the claim holds. Otherwise we have $i^* = k$ since $\tilde{u}_{kk} = \lambda_k + b \ge b$.

Case 1: $k \le n-2$. Observing that $\frac{\partial}{\partial\lambda_1}\cdots\frac{\partial}{\partial\lambda_{k-1}}\sigma_k(\lambda) = \lambda_k + \cdots + \lambda_n \ge 0$, we have $\lambda_k \ge -(\lambda_{k+1} + \cdots + \lambda_n)$. Since $\tilde{u}_{jj} \le \alpha b$ for $j \ge k+1$, we have $\lambda_j \le -(1-\alpha)b$. Hence $\lambda_k \ge (n-k)(1-\alpha)b \ge 2(1-\alpha)b$.

On the other hand, by (3.4), we may suppose at x_0, $|\frac{\rho_\xi}{\rho}| \le \alpha\frac{u_\xi}{u}$, for otherwise we have the required estimate (3.2). Hence $\tilde{u}_{\xi\xi} \le (2+\alpha)b$ for a different small $\alpha > 0$. By the relation $\tilde{u}_{\xi\xi} = \sum_i \xi_i^2 \tilde{u}_{ii} \ge \sum_{i \le k}\xi_i^2\tilde{u}_{ii} - n\alpha b$ where $\xi = (\xi_1, \cdots, \xi_n)$, we have $\tilde{u}_{kk} \le (1+\alpha)\tilde{u}_{\xi\xi} \le (2+2\alpha)b$. Hence $\lambda_k = \tilde{u}_{kk} - b \le (1+2\alpha)b$. We reach a contradiction when α is sufficiently small.

Case 2: $k = n-1$. We have

$$\frac{\partial\sigma_k}{\partial\lambda_{k-1}}\lambda_{k-1} = \sigma_k(\lambda) - \frac{\lambda_1\cdots\lambda_n}{\lambda_{k-1}} \ge -\frac{\lambda_1\cdots\lambda_n}{\lambda_{k-1}}.$$

Since $\lambda_n = \tilde{u}_{nn} - b \le -(1-\alpha)b$ and by $\frac{\partial}{\partial \lambda_1} \cdots \frac{\partial}{\partial \lambda_{n-2}} \sigma_k(\lambda) = \lambda_{n-1} + \lambda_n \ge 0$, we have $\lambda_{n-1} \ge (1-\alpha)b$ and so $\lambda_i \ge (1-\alpha)b$ for any $1 \le i \le n-1$. Hence $\frac{\partial \sigma_k}{\partial \lambda_{k-1}} \lambda_{k-1} \ge (1-\alpha)^2 b^2 \lambda_1 \cdots \lambda_{n-3}$. Note that $\mu' \frac{\partial \sigma_k}{\partial \lambda_i}(\lambda) = F^{ii}$. It follows that

$$\begin{aligned} A &\ge \frac{1}{2} \mu' \frac{\partial \sigma_k}{\partial \lambda_{k-1}} \tilde{u}^2_{k-1\, k-1} \ge \frac{1}{2} \mu' \frac{\partial \sigma_k}{\partial \lambda_{k-1}} \lambda^2_{k-1} \\ &\ge \frac{1}{2} \mu' (1-\alpha)^2 b^2 \lambda_1 \cdots \lambda_{n-2} \ge C b^2 \mathcal{F}. \end{aligned} \tag{3.11}$$

Case 3: $k = n$. As in Case 1, we assume that $|\frac{\rho_\xi}{\rho}| \le \alpha \frac{u_\xi}{u}$. Then by (3.4), $\tilde{u}_{\xi\xi} \ge (2-\alpha)b$. Note that when $k = n$, $\lambda_i > 0$ for all i. Hence $\tilde{u}_{ii} = \lambda_i + b > b$. Recall that when $k = n$, we have $i^* = n$. It follows that $\tilde{u}_{nn} \ge (2-\alpha)b$ for a different small $\alpha > 0$. Hence $\lambda_n \ge (1-\alpha)b$ and

$$A \ge \frac{1}{2} \Sigma_{i \ne i^*} F^{ii} \tilde{u}^2_{ii} \ge \frac{1}{2} F^{ii} \lambda_i^2 = \frac{1}{2} \lambda_i \lambda_n F^{nn} \ge C b^2 \mathcal{F}. \tag{3.12}$$

This completes the proof. □

We remark that in our proof of (3.10), we didn't use the equation (3.3). Hence we can also use (3.10) for the corresponding equation (3.21) below. We also note that the gradient estimate is independent of the choice of μ. From Lemma 3.1, we obtain the following Liouville theorem.

Corollary 3.1. *Let $u \in C^3$ be an entire k-admissible positive solution of*

$$\sigma_k\big(\lambda(\nabla^2 u - \frac{|\nabla u|^2}{2u} I)\big) = 0. \tag{3.13}$$

Then $u \equiv$ constant.

Proof. For equation (3.13), the constant C_2 in (3.9) vanishes. Letting $r \to \infty$, by (3.9), we see that either $\frac{|\nabla u|}{u} \equiv 0$, or $\mathcal{F} = 0$. In the former case, u is a constant. In the latter case, u satisfies $\sigma_{k-1}(\lambda) = 0$ and so it is also a constant by induction. □

By approximation as in [MTW,W2] one can show that Corollary 3.1 holds for continuous positive viscosity solutions. The proof of the interior gradient estimate (3.2) can be simplified if one allows the estimate to depend on both $\inf_{B(0,r)} u$ and $\sup_{B(0,r)} u$. Indeed, let $\varphi(u) = \frac{1}{u-\delta}$ in the auxiliary function z, where $\delta = \frac{1}{2} \inf_{B(0,r)} u$. Then one obtains the extra good term $\frac{\delta u_1^2}{(u-\delta)u^2} \mathcal{F}$ on the right hand side of (3.7). The proof after (3.8) is not needed.

For the k-Yamabe problem, $f(u) = u^{-k}$. The constant C in (3.2) is independent of $\sup u$. Therefore we have the Harnack inequality [GW1].

Corollary 3.2. *Let $u \in C^3$ be a positive solution of (3.1). If $\inf u \geq C_0 > 0$, then* $\sup u \leq C_1$.

Next we prove the second order derivative estimate.

Lemma 3.2. *Let $u \in C^4$ be a k-admissible positive solution of (3.1) in a geodesic ball $B_r(0) \subset \mathcal{M}$. Suppose $A \in C^2(B_r(0))$. Then we have*

$$|\nabla^2 u|(0) \leq C, \tag{3.14}$$

where C depends only on n, k, r, $\inf u$, $\sup u$, and $\|A_{g_0}\|_{C^2}$.

Proof. Again we will consider the more general equation (3.3). Choose $\mu(t) = t^{1/k}$ such that equation (3.3) is concave in U_{ij}. Differentiating (3.3) we get

$$F^{ij} U_{ij,kk} = -\frac{\partial^2 \mu(\sigma_k(\lambda(U)))}{\partial U_{ij} \partial U_{rs}} U_{ij,k} U_{rs,k} + \nabla_k^2 \mu(f) \geq \nabla_k^2 \mu(f), \tag{3.15}$$

where $U_{ij,k} = \nabla_k U_{ij}$. As above denote $\tilde{u}_{ij} = u_{ij} + u a_{ij}$. Let T denote the unit tangent bundle of $B_r(0)$ with respect to g_0. Assume the auxiliary function z on T, $z(x, e_p) = \rho^2 \nabla^2 \tilde{u}(e_p, e_p)$, attains its maximum at x_0 and in direction $e_1 = (1, 0, \cdots, 0)$, where $\rho(x) = (1 - \frac{|x|^2}{r^2})^+$. In an orthonormal frame at x_0, we may assume by a rotation of axes that $\{U_{ij}\}$ is diagonal at x_0. Then at x_0, F^{ij} is diagonal and

$$0 = (\log z)_i = \frac{2\rho_i}{\rho} + \frac{\tilde{u}_{11,i}}{\tilde{u}_{11}}, \tag{3.16}$$

$$0 \geq (\log z)_{ii} = (\frac{2\rho_{ii}}{\rho} - \frac{6\rho_i^2}{\rho^2}) + \frac{\tilde{u}_{11,ii}}{\tilde{u}_{11}}. \tag{3.17}$$

By (3.16), the gradient estimate, and the Ricci identities,

$$U_{ij,11} = u_{ij11} - \frac{u_{k1}^2}{u}\delta_{ij} + O(\frac{1+u_{11}}{\rho}) = u_{11ij} - \frac{u_{k1}^2}{u}\delta_{ij} + O(\frac{1+u_{11}}{\rho}). \tag{3.18}$$

Hence we obtain

$$\begin{aligned} 0 \geq \sum_i F^{ii} (\log z)_{ii} &\geq -\frac{C}{\rho^2}\mathcal{F} + F^{ii}\frac{\tilde{u}_{11,ii}}{\tilde{u}_{11}} \\ &\geq -\frac{C}{\rho^2}\mathcal{F} + \frac{u_{11}^2}{2u\tilde{u}_{11}}\mathcal{F} + \frac{1}{\tilde{u}_{11}}\nabla_k^2 \mu(f). \end{aligned}$$

Since $\mu(t) = t^{1/k}$, we have $\mathcal{F} \geq C > 0$. Hence (3.14) holds. □

The second order derivative estimate (3.14) was established in [GW1]. As the proof is straightforward, we included it here for completeness. The estimate is also similar to that in [GW4] for the equation

$$\det\big(\nabla^2 u - \frac{|\nabla u|^2}{2u} I + \frac{u}{2} I\big) = f(x, u, \nabla u) \quad \text{in} \quad \Omega \subset S^n \tag{3.19}$$

which arises in the design of a reflector antenna, where I is the unit matrix. See also [W2] for $n = 2$.

By Lemma 3.2, equation (3.1) becomes a uniformly elliptic equation. By the Evans-Krylov estimates and linear theory [GT], we have the following interior estimates.

Theorem 3.1. *Let $u \in C^{3,1}$ be a positive solution of (3.1) in a geodesic ball $B_r(0) \subset \mathcal{M}$. Suppose $f > 0, \in C^{1,1}$. Then for any $\alpha \in (0,1)$,*

$$\|u\|_{C^{3,\alpha}(B_{r/2}(0))} \leq C, \tag{3.20}$$

where C depends only on n, k, r, $\inf_{\mathcal{M}} u$, and g_0.

Theorem 3.1 also holds for equation (3.3) with $f = \kappa u^{-p}$ for a constant $p > 0$ and a smooth, positive function κ.

3.2. The parabolic equation. It is more convenient to study the parabolic equation for the function $w = \log u$. In this section we will extend the a priori estimates in §3.1 to the parabolic equation

$$F[w] - w_t = \mu(f), \tag{3.21}$$

where $F[w] = \mu[\sigma_k(\lambda(W))]$, and

$$W = \nabla^2 w + \nabla w \otimes \nabla w - \frac{1}{2}|\nabla w|^2 g_0 + A_{g_0}.$$

When $f = e^{-2kw}$, a stationary solution of (3.21) satisfies the equation

$$\sigma_k(\lambda(W)) = e^{-2kw},$$

which is equivalent to (3.1).

We choose a monotone increasing function μ such that F is concave in $D^2 w$ and

$$\mu(t) = \begin{cases} t^{1/k} & t \geq 10, \\ \log t & t \in (0, \frac{1}{10}), \end{cases}$$

and furthermore

$$(t-s)(\mu(t) - \mu(s)) \geq c_0 (t-s)(t^{1/k} - s^{1/k}) \tag{3.22}$$

for some constant $c_0 > 0$ independent of t. Condition (3.22) will be used in the next section.

We say w is k-admissible if for any fixed t, w is k-admissible as a function of x. Denote $Q_r = B_r(0) \times (0, r^2]$. In the following lemmas we establish interior (in both time and spatial variables) a priori estimates for w,

Lemma 3.3. *Let w be a k-admissible solution of (3.21) on Q_r. Then we have the estimates*

$$|\nabla_x w(0, r^2)| \leq C, \tag{3.23}$$

where C is independent of $\sup w$, if $f = \kappa(x)e^{-pw}$ for some constant $p > 0$ and smooth, positive function κ.

Proof. The proof is similar to that of Lemma 3.1. We outline the proof here. Let $u = e^w$. Then u satisfies the equation

$$\tilde{F}[u] - \frac{u_t}{u} = \mu(f), \tag{3.24}$$

where

$$\tilde{F}[u] = \mu\big[\frac{1}{u^k}\sigma_k\big(\lambda(\nabla^2 u - \frac{|\nabla u|^2}{2u}g_0 + uA_{g_0})\big)\big].$$

Let $z = \big(\frac{|\nabla u|}{u}\big)^2\rho^2$ be the auxiliary function as in the proof of Lemma 3.1. Here we choose

$$\rho(x, t) = \frac{t}{r^2}(1 - \frac{|x|^2}{r^2})^+.$$

Suppose z attains its maximum at (x_0, t_0). Then $t_0 > 0$. By a rotation of axes we assume $|\nabla u| = u_1$. Then at (x_0, t_0), $z_i = 0$, $\{z_{ij}\} \leq 0$, and $z_t \geq 0$. Hence we have (3.4), (3.5) and

$$\frac{u_{1t}}{u} - \frac{u_1 u_t}{u^2} + \frac{\rho_t}{\rho} \geq 0. \tag{3.25}$$

Differentiating equation (3.24) we obtain (3.6) with F^{ij} and Δ replaced by

$$\begin{aligned}
\tilde{F}^{ij}(r) &= \frac{\partial}{\partial r_{ij}}\mu[\frac{1}{u^k}\sigma_k(\lambda(r))] = \frac{\mu'}{u^k}\frac{\partial}{\partial r_{ij}}\sigma_k(\lambda(r)),\\
\Delta &= [\frac{u_{1t}}{u} - \frac{u_1 u_t}{u^2}] + \frac{ku_1\mu'}{u^{k+1}}\sigma_k(\lambda) + [\nabla_1\mu(f) - \tilde{F}^{ij}\nabla_1(a_{ij}u)]\\
&\geq -\frac{\rho_t}{\rho} + \nabla_1\mu(f) - \tilde{F}^{ij}\nabla_1(a_{ij}u).
\end{aligned}$$

By Lemma 2.3 and our choice of μ, $\tilde{\mathcal{F}} = \sum \tilde{F}^{ii}$ has a positive lower bound,

$$\tilde{\mathcal{F}} \geq \frac{\mu'(u^{-k}\sigma_k(\lambda))}{u^k}\sigma_k^{(k-1)/k}(\lambda(U)) \geq \frac{C}{u}$$

for some $C > 0$ depending only on n, k. Hence similarly as the proof of Lemma 3.1 we have (3.7). From (3.8), we obtain estimate (3.23). □

Since the constant C in (3.23) is independent of $\sup w$, hence we have a similar Harnack type inequality as Corollary 3.2.

Lemma 3.4. *Let w be a k-admissible solution of (3.21) on Q_r. Then we have the estimate*

$$|\nabla_x^2 w(0,r^2)| \le C, \tag{3.26}$$

where C depends only on n, k, r, μ, $\inf w$, $\sup w$, and $\|A_{g_0}\|_{C^2}$.

Proof. Differentiating equation (3.21) twice, we get

$$F^{ij} W_{ij,k} = w_{tk} + \nabla_k \mu(f), \tag{3.27}$$

$$F^{ij} W_{ij,kk} = -F^{ij,rs} W_{ij,k} W_{rs,k} + w_{tkk} + \nabla_k^2 \mu(f) \ge w_{tkk} + \nabla_k^2 \mu(f). \tag{3.28}$$

where $F^{ij} = \frac{\partial F}{\partial W_{ij}}$, $W_{ij,k} = \nabla_k W_{ij}$, and $F^{ij,rs} = \frac{\partial^2 \mu(\sigma_k(\lambda(W)))}{\partial W_{ij} \partial W_{rs}}$. Denote $\tilde{w}_{ij} = w_{ij} + a_{ij}$, $a_{ij} = (A_{g_0})_{ij}$. Let T denote the unit tangent bundle of $\mathcal{M}$ with respect to g_0. Consider the auxiliary function z defined on $T \times [0, r^2]$, given by $z = \rho^2 \big(\nabla^2 \tilde{w} + (\nabla w)^2\big)(e_p, e_p)$, where ρ is the cut-off function in the proof of Lemma 3.3. Assume that z attains its maximum at (x_0, t_0) and in direction $e_1 = (1, 0, \cdots, 0)$. We choose an orthonormal frame at (x_0, t_0), such that after a rotation of axes, $\{W_{ij}\}$ is diagonal. Then F^{ij} is diagonal and at (x_0, t_0),

$$0 = (\log z)_i = \frac{2\rho_i}{\rho} + \frac{\tilde{w}_{11,i} + 2w_1 w_{1i}}{\tilde{w}_{11} + w_1^2}, \tag{3.29}$$

$$0 \le (\log z)_t = \frac{2\rho_t}{\rho} + \frac{w_{11t} + 2w_1 w_{1t}}{\tilde{w}_{11} + w_1^2}, \tag{3.30}$$

$$0 \ge (\log z)_{ii} = (\frac{2\rho_{ii}}{\rho} - \frac{6\rho_i^2}{\rho^2}) + \frac{\tilde{w}_{11,ii} + 2w_1 w_{1ii} + 2w_{1i}^2}{\tilde{w}_{11} + w_1^2}. \tag{3.31}$$

We have, by (3.29) and the Ricci identities,

$$\begin{aligned} W_{ij,11} &= w_{ij11} + w_{i11} w_j + w_{j11} w_i + 2w_{i1} w_{j1} - w_{k1}^2 \delta_{ij} + O(\frac{1}{\rho}(\tilde{w}_{11} + w_1^2)) \\ &= w_{11ij} + 2w_{i1} w_{j1} - w_{k1}^2 \delta_{ij} + O(\frac{1}{\rho}(\tilde{w}_{11} + w_1^2)). \end{aligned}$$

Hence we obtain

$$\begin{aligned} 0 &\ge \sum_i F^{ii} (\log z)_{ii} - (\log z)_t \\ &\ge -\frac{C}{\rho^2}\mathcal{F} + \frac{1}{\tilde{w}_{11} + w_1^2} F^{ii} (\tilde{w}_{11,ii} + 2w_1 w_{ii1} + 2w_{i1}^2) - \frac{w_{11t} + 2w_1 w_{1t}}{\tilde{w}_{11} + w_1^2} - \frac{2\rho_t}{\rho} \\ &\ge -\frac{C}{\rho^2}\mathcal{F} + \frac{1}{\tilde{w}_{11} + w_1^2} F^{ii} [(W_{ii,11} + w_{k1}^2) + 2w_1 w_{1ii}] - \frac{w_{11t} + 2w_1 w_{1t}}{\tilde{w}_{11} + w_1^2} - \frac{2\rho_t}{\rho} \\ &\ge -\frac{C}{\rho^2}\mathcal{F} + \frac{1}{\tilde{w}_{11} + w_1^2} (F^{ii} W_{ii,11} - w_{11t}) + w_{11}\mathcal{F} + \frac{2w_1}{\tilde{w}_{11} + w_1^2} (F^{ii} w_{ii1} - w_{t1}) - \frac{2\rho_t}{\rho} \\ &\ge -\frac{C}{\rho^2}\mathcal{F} + \frac{1}{\tilde{w}_{11} + w_1^2} \nabla_1^2 \mu(f) + w_{11}\mathcal{F} + \frac{2w_1}{\tilde{w}_{11} + w_1^2} \nabla_1 \mu(f) - \frac{2\rho_t}{\rho}. \end{aligned} \tag{3.32}$$

By our choice of μ, $\mathcal{F} \geq C$ for some C depending only on n, k. We obtain $w_{11}\rho^2 \leq C$ at (x_0, t_0). Whence $z(0, r^2) \leq z(x_0, t_0) \leq C$. □

Remark 3.1. The a priori estimates (3.23) and (3.26) also hold for the equation

$$\frac{1}{a}\mu(a^k\sigma(\lambda(W)) - w_t = \frac{1}{a}\mu(a^k f), \tag{3.33}$$

where $a > 0$ is a constant. We claim that the constants C in (3.23) and (3.26) are independent of $a \geq 1$.

First we note that this is obvious for (3.23) as the gradient estimate is independent of the choice of μ.

For the estimate (3.26), we have, by Lemma 2.3,

$$\begin{aligned}\mathcal{F} &= \sum_i \frac{\partial}{\partial W_{ii}}\left[\frac{1}{a}\mu(a^k\sigma_k(\lambda(W))\right] \\ &= (n-k+1)a^{k-1}\sigma_{k-1}(\lambda(W))\mu' \\ &\geq Ca^{k-1}\sigma_k^{(k-1)/k}\mu'(a^k\sigma_k) \\ &\geq C\inf_{t>0} t^{(k-1)/k}\mu'(t),\end{aligned}$$

By our choice of μ, $\inf_{t>0} t^{(k-1)/k}\mu'(t) \geq C > 0$. Hence $\mathcal{F} > C > 0$.

Therefore by (3.32) it suffices to show that

$$|\nabla_1 g| + |\nabla_1^2 g| \leq C$$

for some $C > 0$ independent of $a \geq 1$, where $g = \frac{1}{a}\mu(a^k f)$. By our choice of μ,

$$g = \begin{cases} \mu(f) & \text{if } a^k f > 10 \\ \frac{1}{a}(k\log a + \log f) & \text{if } a^k f < \frac{1}{10}, \end{cases}$$

Hence $\sup(|\nabla_1 g| + |\nabla_1^2 g|)$ is independent of $a \geq 1$ if $a^k f > 10$ or $a^k f < \frac{1}{10}$. When $a^k f \in (\frac{1}{10}, 10)$, we can choose μ properly such that $\sup|\nabla_1^2 g|$ is independent of $a \geq 1$. Alternatively one can compute directly

$$\begin{aligned}|\nabla_1 g| &= |a^{k-1}f^{(k-1)/k}\mu'(a^k f)|\,|k\nabla_1 f^{1/k}| \\ &\leq C|\nabla_1 f^{1/k}|, \\ |\nabla_1^2 g| &= |a^{k-1}\mu'(a^k f)\nabla_1^2 f| + \big|a^{k-1}\mu'(a^k f)\frac{(\nabla_1 f)^2}{f}\big|\,\big|\frac{a^k f\mu''(a^k f)}{\mu'(a^k f)}\big| \\ &\leq C\frac{|\nabla_1^2 f|}{f^{1-\frac{1}{k}}}| + C\frac{|\nabla_1 f|^2}{f^{2-\frac{1}{k}}},\end{aligned}$$

where C depends on $\sup_{t>0} t^{1-1/k}\mu'(t)$ and $\sup_{t>0}\frac{t\mu''(t)}{\mu'(t)} < \infty$. Hence the estimate (3.26) is independent of $a \geq 1$.

Remark 3.1 will be used in the next section. The choice of function μ in the parabolic equation (3.21) is critical.

Lemma 3.5. *Let w be a k-admissible solution of (3.21) on Q_r. Then we have the estimates*

$$|w_t(0,r^2)| \leq C, \tag{3.34}$$

where C depends only on $n, k, r, \mu, \inf w, \sup w$, and $\|A_{g_0}\|_{C^2}$.

Proof. From the equation (3.21) and by the estimate (3.26) we have an upper bound for w_t. It suffices to show that w_t is bounded from below. Let $z = \frac{w_t}{(M-w)^\alpha}\rho^\beta$, where $M = 2\sup_{Q_r}|w|$, ρ is the cut-off function as above. Suppose $\min_{Q_r} z$ attains its minimum at (x_0, t_0), $t_0 > 0$. Then at the point we have $z_t \leq 0$, $z_i = 0$ and the matrix $\{z_{ij}\} \geq 0$, namely

$$\frac{w_{tt}}{w_t} + \alpha\frac{w_t}{M-w} + \beta\frac{\rho_t}{\rho} \geq 0, \tag{3.35}$$

$$\frac{w_{ti}}{w_t} + \alpha\frac{w_i}{M-w} + \beta\frac{\rho_i}{\rho} = 0 \quad i = 1, \cdots, n, \tag{3.36}$$

$$\{\frac{w_{ijt}}{w_t} - \frac{w_{it}w_{jt}}{w_t^2} + \alpha\frac{w_{ij}}{M-w} + \alpha\frac{w_iw_j}{(M-w)^2} + \beta\frac{\rho_{ij}}{\rho} - \beta\frac{\rho_i\rho_i}{\rho^2}\} \leq 0, \tag{3.37}$$

where we have changed the direction of the inequalities as we assume that $w_t < 0$. Differentiating equation (3.21) gives

$$F^{ij}W_{ijt} - w_{tt} = \frac{\partial}{\partial t}\mu(f). \tag{3.38}$$

Hence by (3.35),

$$\begin{aligned}\alpha\frac{w_t}{M-w} &\geq -\frac{w_{tt}}{w_t} - \beta\frac{\rho_t}{\rho}\\ &= \frac{-1}{w_t}F^{ij}W_{ijt} + \frac{1}{w_t}\frac{\partial}{\partial t}\mu(f) - \beta\frac{\rho_t}{\rho}\end{aligned}$$

By (3.36), the matrix in (3.37) is equal to

$$\{\frac{w_{ijt}}{w_t} + \frac{\alpha w_{ij}}{M-w} + \frac{\alpha(1-\alpha)w_iw_j}{(M-w)^2} - \frac{2\alpha\beta w_i\rho_j}{(M-w)\rho} + \beta\frac{\rho_{ij}}{\rho} - \beta(1+\beta)\frac{\rho_i\rho_i}{\rho^2}\} \leq 0.$$

We have

$$\begin{aligned}\frac{-1}{w_t}F^{ij}W_{ijt} &= \frac{-1}{w_t}F^{ij}(w_{ijt} + w_{it}w_j + w_{jt}w_i - w_kw_{kt}\delta_{ij})\\ &\geq F^{ij}(\frac{\alpha w_{ij}}{M-w} + \frac{\alpha(1-\alpha)w_iw_j}{(M-w)^2} - \frac{2\alpha\beta w_i\rho_j}{(M-w)\rho} + \beta\frac{\rho_{ij}}{\rho} - \beta(1+\beta)\frac{\rho_i\rho_i}{\rho^2})\\ &\quad + F^{ij}(2\alpha\frac{w_iw_j}{M-w} + 2\beta\frac{\rho_iw_j}{\rho} - \alpha\frac{|\nabla w|^2}{M-w}\delta_{ij} - \beta\frac{w_k\rho_k}{\rho}\delta_{ij}\}\\ &\geq \frac{\alpha}{M-w}F^{ij}(w_{ij} + 2w_iw_j - |\nabla w|^2\delta_{ij})\\ &\quad + \frac{\alpha}{M-w}F^{ij}(\frac{(1-\alpha)w_iw_j}{M-w} - 2\beta\frac{w_i\rho_j}{\rho}) - \frac{C}{\rho^2},\end{aligned}$$

where the constant C depends on the gradient estimate (3.23) and the second derivative estimate (3.26). Choose $\alpha = \frac{1}{2}$. By the Holder inequality,

$$F^{ij}(\frac{w_i w_j}{2(M-w)} - 2\beta \frac{w_i \rho_j}{\rho}) \geq -\frac{C}{\rho^2}.$$

By the k-admissibility, $F^{ij}W_{ij} \geq 0$. Hence we obtain

$$\frac{-1}{w_t} F^{ij} W_{ijt} \geq \frac{\alpha}{M-w} F^{ij}(W_{ij} + w_i w_j - \frac{1}{2}|\nabla w|^2 \delta_{ij} - a_{ij}) - \frac{C}{\rho^2} \geq -\frac{C}{\rho^2}.$$

It follows that

$$\alpha \frac{w_t}{M-w} \geq -\frac{C}{\rho^2} + \frac{1}{w_t}\frac{\partial}{\partial t}\mu(f) - \beta \frac{\rho_t}{\rho}.$$

Now we choose $\beta = 2$. Then we obtain

$$z(x_0, t_0) = \frac{w_t}{(M-w)^{1/2}} \rho^2(x_0, t_0) \geq -C.$$

It follows that $z(0, r^2) \geq z(x_0, t_0) \geq -C$. Hence w_t is bounded from below. □

Theorem 3.2. *For any $w_0 \in \Phi_k$, there is a smooth k-admissible solution $w \in C^{3,2}(\mathcal{M} \times [0, T))$ of (3.21) with $w(\cdot, 0) = w_0$ on a maximal time interval $[0, T)$. If $T < \infty$, we have $\inf_{\mathcal{M}} w(\cdot, t) \to -\infty$ as $t \nearrow T$.*

Proof. First we point out that a k-admissible solution of (3.21) is locally bounded. Indeed, at the minimum point of w, by equation (3.21) we have

$$w_t = F[w] - \mu(f) \geq \mu\big(\sigma_k(\lambda(A_{g_0}))\big) - \mu(f).$$

Hence locally in time the solution is bounded from below. By the interior gradient estimate (3.23), the solution is also bounded from above. Therefore by Lemmas 3.3-3.5, equation (3.21) is uniformly parabolic. By Krylov's regularity theory, we obtain the $C^{3,2}$ a priori estimate for (3.21), and so the local existence follows. Let $[0, T)$ be the maximal time interval for the solution. If $T < \infty$, we must have $\inf_{\mathcal{M}} w(\cdot, t) \to -\infty$ as $t \nearrow T$. □

Remark 3.2. The a priori estimates in §3.1 and §3.2 can be extended to the quotient equation

$$\sigma_{k,l}\big(\lambda(\nabla^2 u - \frac{|\nabla u|^2}{2u} g_0 + uA)\big) = u^{l-k} \quad (1 \leq l < k \leq n) \tag{3.39}$$

and its parabolic counterpart, where $\sigma_{k,l}(\lambda) = \frac{\sigma_k}{\sigma_l}(\lambda)$. Indeed, let μ be a monotone increasing function such that $\mu[\sigma_{k,l}(\lambda)]$ is concave. Write equation (3.39) in the form $F[u] = \mu(f)$ as (3.3). Then the proof for the second derivative estimates (3.14) and (3.26) can be extended to the quotient equation without change. For the gradient estimates (3.2) and (3.23), denote $\sigma_{k-1;i}(\lambda) = \frac{\partial}{\partial\lambda_i}\sigma_k(\lambda)$. By Newton's inequality [LT],

$$\begin{aligned}\frac{\partial\sigma_{k,l}}{\partial\lambda_i}(\lambda) &= \frac{\sigma_l\sigma_{k-1;i} - \sigma_k\sigma_{l-1;i}}{\sigma_l^2} \\ &= \frac{\sigma_{l;i}\sigma_{k-1;i} - \sigma_{k;i}\sigma_{l-1;i}}{\sigma_l^2} \geq \frac{n(k-l)}{k(n-l)}\frac{\sigma_{l;i}\sigma_{k-1;i}}{\sigma_l^2}.\end{aligned} \tag{3.40}$$

As before we arrange the eigenvalues in the descending order $\lambda_1 \geq \cdots \geq \lambda_n$. Then by Lemma 2.3(v), $\sigma_{l;i}(\lambda) \geq C\sigma_l(\lambda)$ when $i \geq l+1$. Hence $\frac{\partial\sigma_{k,l}}{\partial\lambda_k}(\lambda) \geq C\sum_i \frac{\partial\sigma_{k,l}}{\partial\lambda_i}(\lambda)$, namely $F^{kk} \geq C\mathcal{F}$. Hence the proof of (3.2) and (3.23) can also be extended to the quotient equation. But we need to replace (3.11) in Case 2 by

$$\begin{aligned}A &\geq \frac{1}{2}\mu'\frac{\partial\sigma_{k,l}}{\partial\lambda_{k-1}}\tilde{u}^2_{k-1\,k-1} \geq \frac{1}{2}\mu'\frac{n(k-l)}{k(n-l)}\frac{\sigma_{l;k-1}\sigma_{k-1;k-1}}{\sigma_l^2}\tilde{u}^2_{k-1\,k-1} \\ &\geq C\frac{\sigma_{l;k-1}\sigma_{k-1;k-1}}{\sigma_l^2}\lambda_{k-1}^2 \geq C\frac{\sigma_{k-1;k-1}}{\sigma_l}\lambda_{k-1}^2 \geq Cb^2\mathcal{F},\end{aligned}$$

and (3.12) in Case 3 by

$$A \geq \frac{1}{2}\Sigma_{i\neq i^*}F^{ii}\tilde{u}_{ii}^2 \geq \frac{1}{2}F^{ll}\tilde{u}_{ll}^2 \geq C\frac{\sigma_{n-1,l}}{\sigma_l}\tilde{u}_{ll}^2 \geq C\lambda_l\lambda_n\frac{\sigma_{n-1,n}}{\sigma_l} \geq Cb^2\mathcal{F},$$

where we have used $F^{ll} \geq C\sigma_{k-1,l}/\sigma_l$ by (3.40).

For the corresponding parabolic equation (3.21), where $F[w] = \mu\big(\sigma_{k,l}(\lambda(W))\big)$, choose a monotone increasing function μ such that F is concave in λ, and $\mu(t) = t^{1/(k-l)}$ when $t \geq 10$, $\mu(t) = \log t$ for $t > 0$ small. Then we can prove Theorem 3.2 for the quotient equation in the same way as before.

We remark that the a priori estimates for (3.39), and for its parabolic counterpart on locally conformally flat manifolds, were obtained in [GW3].

We also note that the a priori estimates in §3.1 can be extended to the more general equation

$$s_k = \sum_{l=0}^{k-1}\beta_l s_l, \tag{3.41}$$

where β_l are nonnegative constants, $\sum_l \beta_l > 0$, and $s_k = u^k\sigma_k(\lambda(\nabla^2 u - \frac{|\nabla u|^2}{2u} + uA))$ is the k-curvature.

3.3. Counterexamples. Theorem 3.1 applies to solutions of (3.1) with eigenvalues in the positive cone Γ_k. The a priori estimate (3.14) relies critically on the negative sign of the term $\frac{|\nabla u|^2}{2u}$, which yields the dominating term u_{k1}^2 in (3.18). Equation (3.1) has another elliptic branch, namely when the eigenvalues λ lie in the negative cone $-\Gamma_k$. An open problem is whether the a priori estimate (3.14) holds for solutions with eigenvalues in the negative cone $-\Gamma_k$. This is also an open problem for equations from optimal transportation [MTW], in particular the reflector antenna design problem (3.19). Here we give a counter example to the regularity. Our example is a modification of the Heinz-Levy counterexample in [Sc].

We will consider the two dimensional case. By making the change $u \to -u$, we consider equation

$$\det(u_{ij} + |\nabla u|^2 I + a_{ij}) = f \tag{3.42}$$

with positive sign before the term $|\nabla u|^2$, where f is a $C^{1,1}$ positive function to be determined. We want to show that equation (3.42) has no interior a priori estimates for solutions with eigenvalues in the positive cone.

Set

$$u(x) = \frac{b}{2}x_2^2 + \varphi(x_1), \tag{3.43}$$

where b is constant, φ is an even function. Let

$$a_{11} = -b^2x_2^2, \quad a_{12} = 0, \quad a_{22} = -b - b^2x_2^2. \tag{3.44}$$

Then equation (3.42) becomes

$$(\varphi'' + \varphi'^2)\varphi'^2 = f. \tag{3.45}$$

Let $\psi = (\varphi')^3$. Then ψ satisfies the equation

$$\frac{1}{3}\psi' + \psi^{4/3} = f. \tag{3.46}$$

Let

$$\psi(x_1) = x_1 - \frac{9}{7}x_1^{7/3}. \tag{3.47}$$

Then

$$f(x) = \frac{1}{3} - x_1^{4/3} + (x_1 - \frac{9}{7}x_1^{7/3})^{4/3} \tag{3.48}$$

is a positive C^2 function, but the solution $u \notin C^2$.

If instead of (3.44), we choose

$$a_{11} = c_0 - b^2x_2^2, \quad a_{12} = 0, \quad a_{22} = \varepsilon - b - b^2x_2^2, \tag{3.49}$$

where c_0, ε are constants, $\varepsilon > 0$ small. Then we have the equation

$$(\varphi'' + \varphi'^2 + c_0)(\varepsilon + \varphi'^2) = f. \tag{3.50}$$

Let $f \equiv 1$ and denote $g = \varphi'$. Then $g(0) = 0$ and g satisfies

$$g' = \frac{1}{\varepsilon + g^2} - g^2 - c_0. \tag{3.51}$$

This equation has a unique solution g_ε. Obviously the gradient of g_ε is not uniformly bounded. Hence there is no interior $C^{1,1}$ a priori estimate for equation (3.43). Note that the matrix $A = (a_{ij})$ can either be in the positive cone or in negative cone by choosing proper constants b, c_0.

Write equation (3.1) in the form

$$\sigma_k(\lambda(\nabla^2 w - \nabla w \otimes \nabla w + \frac{1}{2}|\nabla w|^2 I + A)) = f. \tag{3.52}$$

Then similarly as above we can construct a sequence of functions satisfying equation (3.52) with $f = 1$ whose second derivatives are not uniformly bounded.

Remark 3.3. In many situations [MTW, W2] there arise equations of the form

$$\sigma_k(\lambda(D^2 u + A(x, u, Du)) = f, \tag{3.53}$$

where A is a matrix. From the discussions in this section, we see that the interior a priori estimates hold in general when A is negative definite with respect to Du, and do not hold if A is positive definite. When $A = 0$, there is no interior regularity in general, but if the solution vanishes on the boundary, interior a priori estimates have been established in [CW2].

4. Proof of Lemma 2.1

4.1. Existence of solutions in the sub-critical growth case. In this subsection we first study the existence of k-admissible solutions, for $2 \le k < \frac{n}{2}$, to equation (2.5) in the subcritical growth case $1 < p < \frac{n+2}{n-2}$. We then extend the existence result to the critical case $p = \frac{n+2}{n-2}$ in §4.2 by the blow-up argument. In §4.3 we consider the case $k = \frac{n}{2}$.

Theorem 4.1. *Suppose $2 \le k < \frac{n}{2}$. Then for any given $1 < p < \frac{n+2}{n-2}$, there is a solution v_p of (2.5) with $J_p(v_p) = c_p > 0$, where J_p, c_p are defined respectively in (2.6) and (2.8). Moreover, the set of solutions of (2.5) is compact.*

A solution of (2.5) is a critical point of the functional $J = J_p$. To study the critical points of the functional J, we will employ the parabolic equation (3.21), which can also be written in terms of v as (ignoring a coefficient $\frac{2}{n-2}$ before v_t)

$$F[v] + \frac{v_t}{v} = \mu(f(v)), \tag{4.1}$$

where $f(v) = v^{\frac{4k}{n-2}-\varepsilon}$, $F[v] = \mu(\sigma_k(\lambda(\frac{V}{v})))$, μ is the function in (3.21), and

$$\varepsilon = \frac{n+2}{n-2} - p. \tag{4.2}$$

Write functional (2.6) in the form

$$J(v) = \frac{n-2}{2n-4k}\int_{(\mathcal{M},g_0)} v^{\frac{2n-4k}{n-2}}\sigma_k(\lambda(\frac{V}{v})) - \frac{1}{p+1}\int_{(\mathcal{M},g_0)} v^{\frac{2n-4k}{n-2}} v^{\frac{4k}{n-2}-\varepsilon}. \tag{4.3}$$

Equation (4.1) is a descent gradient flow of the functional J,

$$\begin{aligned} \frac{d}{dt}J(v) &= \int_{(\mathcal{M},g_0)} v^{\frac{2n-4k}{n-2}-1}\big[\sigma_k(\lambda(\frac{V}{v})) - v^{\frac{4k}{n-2}-\varepsilon}\big]v_t \\ &= -\int_{(\mathcal{M},g_0)} v^{\frac{2n-4k}{n-2}}\big[\sigma_k(\lambda(\frac{V}{v})) - v^{\frac{4k}{n-2}-\varepsilon}\big]\big[\mu\big(\sigma_k(\lambda(\frac{V}{v}))\big) - \mu\big(v^{\frac{4k}{n-2}-\varepsilon}\big)\big] \le 0. \end{aligned} \tag{4.4}$$

Given an initial k-admissible function v_0, by Theorem 3.2, the flow (4.1) has a unique smooth positive solution v on a maximal time interval $[0, T)$, where $T \le \infty$.

Lemma 4.1. *Suppose $J(v(\cdot,t))$ is bounded from below for all $t \in (0,T)$. If $v(\cdot,t)$ is uniformly bounded, then either $v(\cdot,t) \to 0$ or there is a sequence $t_j \to \infty$ such that $v(\cdot,t_j)$ converges to a solution of (2.5).*

Proof. By the assumption that $v(\cdot,t)$ is uniformly bounded, we have $T = \infty$. At the maximum point of $v(\cdot,t)$, by equation (4.1) we have

$$v_t \le v[\mu(f(v)) - \mu(\sigma_k(\lambda(A_{g_0})))]. \tag{4.5}$$

Hence if $\sup v(\cdot,t_0)$ is sufficiently small at some t_0, by the assumptions $g_0 \in \Gamma_k$ and $v > 0$, we have $v(\cdot,t) \to 0$ uniformly. Therefore if v does not converges to zero uniformly, by the gradient estimate (3.23), we have $v \geq c$ for some constant $c > 0$. In the latter case, by Theorem 3.2 and the assumption that v is uniformly bounded, we have $\|v(\cdot,t)\|_{C^3(\mathcal{M})} \leq C$ for any $t \geq 0$.

Choose a sequence $t_j \to \infty$ such that

$$\frac{d}{dt}J(v(\cdot,t_j)) \to 0. \tag{4.6}$$

By the above C^3 a priori estimate, we may abstract a subsequence, still denoted as t_j, such that $v(\cdot,t_j)$ converges in $C^{2,\alpha}$. By (4.4) we concludes that $v(x,t_j)$ converges as $j \to \infty$ to a solution of (2.5). □

Lemma 4.2. *Suppose $J(v(\cdot,t))$ is bounded from below for all $t \in (0,T)$. Then $T = \infty$ and $v(\cdot,t)$ is uniformly bounded.*

Proof. Suppose to the contrary that there exists a sequence $t_j \nearrow T$ such that $m_j = \sup v(\cdot,t_j) \to \infty$. Assume the maximum is attained at $z_j \in \mathcal{M}$. By choosing a normal coordinate centered at z_j, we may identify a neighbourhood of z_j in $\mathcal{M}$ with the unit ball in $\mathbf{R}^n$ such that z_j becomes the origin. We make the local transformation

$$\begin{aligned} v_j(y,s) &= m_j^{-1}v(x,t), \\ y &= m_j^{\frac{2}{n-2}-\frac{\varepsilon}{2k}}x, \\ s &= m_j^{\frac{4}{n-2}-\frac{\varepsilon}{k}}(t-t_j). \end{aligned} \tag{4.7}$$

For the transformation $x \to y$, more precisely it should be understood as a dilation of $\mathcal{M}$, regarded as a submanifold in $\mathbf{R}^N$ for some $N > n$ with induced metric. Denote $\mathcal{M}_j = \{Y m_j^{\frac{2}{n-2}-\frac{\varepsilon}{2k}} X \mid X \in \mathcal{M} \subset \mathbf{R}^N\}$, with induced metric from $\mathbf{R}^N$. Then we have $0 < v_j(y,0) \leq m_j^{-1}v(0,t_j) = 1$, v_j is defined for $y \in \mathcal{M}_j$ and $s \leq s_0$, where by (4.5), $s_0 > 0$ is a positive constant independent of j. Moreover v_j satisfies the equation

$$m_j^{-\frac{4}{n-2}+\frac{\varepsilon}{k}}\mu\big[m_j^{\frac{4k}{n-2}-\varepsilon}\sigma_k(\lambda(\frac{V_j}{v_j}))\big] + \frac{(v_j)_s}{v_j} = m_j^{-\frac{4}{n-2}+\frac{\varepsilon}{k}}\mu(m_j^{\frac{4k}{n-2}-\varepsilon}f(v_j)). \tag{4.8}$$

By direct computation,

$$\begin{aligned} \int_{\mathcal{M}_j} v_j^{\frac{2n-4k}{n-2}}\sigma_k(\lambda(\frac{V_j}{v_j}))dy &= m_j^{\varepsilon(1-\frac{n}{2k})}\int_{\mathcal{M}} v^{\frac{2n-4k}{n-2}}\sigma_k(\lambda(\frac{V}{v}))dx \\ \int_{\mathcal{M}_j} v_j^{\frac{2n}{n-2}-\varepsilon}dy &= m_j^{\varepsilon(1-\frac{n}{2k})}\int_{\mathcal{M}} v^{\frac{2n}{n-2}-\varepsilon}dx \end{aligned} \tag{4.9}$$

Hence

$$J(v_j,\mathcal{M}_j) =: \frac{n-2}{2n-4k}\int_{\mathcal{M}_j} v_j^{\frac{2n-4k}{n-2}}\sigma_k(\lambda(\frac{V_j}{v_j}))dy - \frac{1}{p+1}\int_{\mathcal{M}_j} v_j^{\frac{2n}{n-2}-\varepsilon}dy$$
$$= m_j^{\varepsilon(1-\frac{n}{2k})}J(v,\mathcal{M}) \le C. \tag{4.10}$$

We may choose $s_j \in (0,\frac{1}{2}s_0)$ such that

$$\frac{d}{ds}J(v_j(\cdot,s_j)) \to 0. \tag{4.11}$$

By (4.4), (4.11) is equivalent to

$$\int_{\mathcal{M}_j} v_j^{\frac{2n-4k}{n-2}}\Big\{\sigma_k(\lambda(\frac{V_j}{v_j})) - v_j^{\frac{4k}{n-2}-\varepsilon}\Big\}\cdot$$
$$\Big\{m_j^{-\frac{4}{n-2}+\frac{\varepsilon}{k}}\Big[\mu\big(m_j^{\frac{4k}{n-2}-\varepsilon}\sigma_k(\lambda(\frac{V_j}{v_j}))\big) - \mu\big(m_j^{\frac{4k}{n-2}-\varepsilon}v_j^{\frac{4k}{n-2}-\varepsilon}\big)\Big]\Big\} \to 0.$$

By (3.22), we obtain

$$\int_{\mathcal{M}_j} v_j^{\frac{2n-4k}{n-2}}\Big\{\sigma_k(\lambda(\frac{V_j}{v_j})) - v_j^{\frac{4k}{n-2}-\varepsilon}\Big\}\cdot\Big\{\big(\sigma_k(\lambda(\frac{V_j}{v_j}))\big)^{1/k} - \big(v_j^{\frac{4k}{n-2}-\varepsilon}\big)^{1/k}\Big\} \to 0. \tag{4.12}$$

By the gradient estimate (3.23), $v_j + \frac{1}{v_j}$ (at $s = s_j$) is locally uniformly bounded. Hence

$$\sigma_k(\lambda(\frac{V_j}{v_j})) - v_j^{\frac{4k}{n-2}-\varepsilon} \to 0 \ \text{ in } \ L^{(k+1)/k}. \tag{4.13}$$

Note that by Remark 3.1, v_j are locally uniformly bounded in $C^{1,1}$ and the convergence in (4.13) is locally uniform.

By extracting a subsequence we can assume that $v_j(\cdot,s_j)$ converges to a function $v_0 \in C^{1,1}(\mathbf{R}^n)$ with $v_0(0) = 1$. We claim that v_0 is a smooth solution of the equation

$$F_0[v] := \sigma_k^{1/k}(\lambda(V)) = v^p \tag{4.14}$$

in $\mathbf{R}^n$. On the other hand, by the Liouville Theorem in [LL2], there is no entire positive solution to (4.14) when $\varepsilon > 0$. This is a contradiction. Hence Lemma 4.2 holds.

To prove that v_0 is a smooth solution of (4.14), we note that since $v_0 \in C^{1,1}(\mathbf{R}^n)$, v_0 is twice differentiable almost everywhere. Suppose now that $F_0[v_0] > v_0^p$ at some point x_0 where v_0 is twice differentiable. Without loss of generality we assume that $x = 0$. Let

$$\varphi(x) = v_0(0) + Dv_0(0)x + \frac{1}{2}D_{ij}v_0(0)x_ix_j + \frac{\varepsilon}{2}|x|^2 - \delta,$$

where ε, δ are positive constants. By choosing δ sufficiently small we have

$$\varphi > v_0 \quad \text{on} \quad \partial B_r(0) \quad \text{and} \quad \varphi(0) < v_0(0).$$

Since $v_j \to v_0$ uniformly, we have $\varphi > v_j$ on $\partial B_r(0)$ and $\varphi(0) < v_j(0)$ when j is sufficiently large. Since v_0 is locally uniformly bounded in $C^{1,1}$, by the inequality $F_0[v_0] > v_0^p$ we have $\lambda(V_{v_0} - \varepsilon I) \in \Gamma_k$ and

$$F_j[\varphi] := \sigma_k^{1/k}[\lambda(-\nabla^2\varphi + \frac{n}{n-2}\frac{\nabla v_j \times \nabla v_j}{v_j} - \frac{1}{n-2}\frac{|\nabla v_j|^2}{v_j} g_0 + \frac{n-2}{2} v_j A_{g_0}] \geq v_j^p$$

when $\varepsilon > 0$ is sufficiently small, where V_{v_0} is the matrix relative to v_0, given in (2.2). Hence by the concavity of $\sigma_k^{1/k}$,

$$F^{ab}[v_j] D_{ab}(\varphi - v_j) \leq F(v_j) - v_j^p \to 0 \tag{4.15}$$

in $L^{\tilde{p}}(\Omega)$ for any $\tilde{p} < \infty$, where $F^{ab}[v_j] = \frac{\partial}{\partial r_{ab}} \sigma_k^{1/k}(\lambda(r))$ at $r = V_{v_j}$ $(a, b = 1, \cdots, n)$, which satisfy $\det F^{ab} \geq C > 0$ for some $C > 0$ depending only on n, k. Applying the Aleksandrov-Bakelman maximum principle [GT] to (4.15) in $\{\varphi < v_j\}$ and sending $j \to \infty$, we conclude that $\varphi \geq v_0$ near 0, which is a contradiction so that $F_0[v_0] \leq v_0^p$ at x_0. By a similar argument we obtain the reverse inequality and hence we conclude (4.14) for v_0. Since the limit equation (4.14) is locally uniformly elliptic with respect to v_0, we then conclude further regularity by the Evans-Krylov estimates and linear theory [GT]. In particular we obtain $v_0 \in C^\infty$. □

A more general approach to the approximation argument to obtain (4.14) can be obtained by extending the theory of Hessian measures in [TW1, TW3] to the operators of the type F_0.

Lemma 4.3. *There exists a function $v_0 \in \Phi_k$ such that the solution v of (4.1) satisfies $J(v(\cdot, t)) \geq -C$ and $\sup v(\cdot, t) \geq c_0 > 0$ for all $t \geq 0$.*

Proof. Let P be the set of paths introduced in §2.2. For $\gamma \in P$, let v_s $(s \in [0, 1])$ be the solution of (4.1) with initial condition $v_s(\cdot, 0) = \gamma(s)$. Then by (4.5) and the comparison principle, there is an $s_0 > 0$ such that $v_s(\cdot, t) \to 0$ uniformly for $s \leq s_0$. Denote by I_γ the set of $s \in [0, 1]$ such that $J(v_s(\cdot, t)) \geq 0$ for all $t > 0$. Then $(0, s_0) \subset I_\gamma$. Let $s^* = \sup\{s \mid s \in I_\gamma\}$.

Obviously $s^* \in I_\gamma$. For if there exists t such that $J(v_{s^*}(\cdot, t)) < 0$, then $J(v_s(\cdot, t)) < 0$ for $s < s^*$ sufficiently close to s^*, which implies $s^* \neq \sup\{s \mid s \in I_\gamma\}$. It is also easy to see that $v_{s^*}(\cdot, t)$ does not converges to zero uniformly, for otherwise $v_s(\cdot, t) \to 0$ uniformly for $s > s^*$ and near s^*. Finally by our definition of the set P, we have $1 \notin I_\gamma$, namely $s^* < 1$. Hence $v_0 = \gamma(s^*)$ satisfies Lemma 4.3. □

From the above three Lemmas, one sees that there is a sequence $t_j \to \infty$ such that $v_{s^*}(\cdot, t_j)$ converges to a solution of (2.5) for $1 < p < \frac{n+2}{n-2}$. Next we prove

Lemma 4.4. *For any given $1 < p < \frac{n+2}{n-2}$, the set of solutions of (2.5) is compact.*

Proof. By the a priori estimates it suffices to show that the set of solutions is uniformly bounded. If on the contrary that there is a sequence of solutions $v_j \in \Phi_k$ such that $\sup v_j \to \infty$, denote $m_j = \sup v_j$ and assume that the sup is attained at z_j. Similar to (4.7) we make a translation and a dilation of coordinates and a scaling for solution, namely

$$\begin{aligned} \tilde{v}_j(y) &= m_j^{-1} v_j(x), \\ y = R_j x \quad R_j &= m_j^{\frac{2}{n-2} - \frac{\varepsilon}{2k}}. \end{aligned}$$

Then $0 < \tilde{v}_j \le 1$, and $\tilde{v}_j$ satisfies

$$\sigma_k(\lambda(\tilde{V})) = \tilde{v}^{k\frac{n+2}{n-2} - \varepsilon}.$$

By the a priori estimates in §3.1, $\tilde{v}$ is locally uniformly bounded in C^3. Hence $\tilde{v}_j$ converges by a subsequence to a positive solution $\tilde{v}$ of

$$\sigma_k(\lambda(V)) = v^{k\frac{n+2}{n-2} - \varepsilon} \quad \text{in} \quad \mathbf{R}^n. \tag{4.16}$$

By the Liouville Theorem [LL2], there is no nonzero solution to the above equation. We reach a contradiction. □

Let v be a k-admissible solution of (2.5). Then we have

$$\int_{(\mathcal{M}, g_0)} v^{\frac{2n-4k}{n-2}} \sigma_k(\lambda(\frac{V}{v})) - \int_{(\mathcal{M}, g_0)} v^{\frac{2n-4k}{n-2}} v^{\frac{4k}{n-2} - \varepsilon} = 0.$$

Hence

$$J(v) = \sup_{t>0} J(tv).$$

By (4.3) we have

$$\begin{aligned} J(v) &= (\frac{n-2}{2n-4k} - \frac{1}{p+1}) \int_{(\mathcal{M}, g_0)} v^{\frac{2n-4k}{n-2}} v^{\frac{4k}{n-2} - \varepsilon} \\ &\ge C > 0 \end{aligned} \tag{4.17}$$

By the compactness in Lemma 4.4, the constant C is bounded away from zero.

Lemma 4.5. *There exists a solution v_p of (2.5) such that $J(v_p) = c_p$.*

Proof. For any given constant $\delta > 0$, choose a path $\gamma \in P$ such that $\sup_{s\in(0,1)} J(\gamma(s)) \le c_p + \delta$. By the proof of Lemma 4.3, there exists $s^* \in (0, 1)$ such that the solution of (4.1) with initial condition $v(\cdot,t)=\gamma_{s^*}$ converges to a solution v^*_δ of (2.5). Since (4.1) is a descent gradient flow, we have $J(v^*_\delta)<c_p+\delta$. Letting $\delta\to 0$, by the compactness in Lemma 4.4, v^*_δ converges along a subsequence to a solution v of (2.5) with $J(v)\le c_p$. Note that $J(v)=\sup_{s>0}J(sv)\ge c_p$. Hence $J(v)=c_p$. □

From (4.17) we also have

$$c_p\ge C>0. \tag{4.18}$$

We have thus proved Theorem 4.1.

4.2. Proof of Lemma 2.1 (for $2\le k<\frac{n}{2}$). Let v_p be a solution of (2.5) with $J_p(v_p)=c_p$. If there is a sequence $p_j\nearrow\frac{n+2}{n-2}$ such that $\sup v_{p_j}$ is uniformly bounded, by the a priori estimate in §3.1, v_{p_j} sub-converges to a solution of (2.1) and Lemma 2.1 is proved.

If $\sup v_p\to\infty$ as $p\nearrow\frac{n+2}{n-2}$, noting that $c_p\le\sup_{s>0}J(sv_0)$ for any given $v_0\in\Phi_k$, we see that c_p is uniformly bounded from above for $p\in[1,\frac{n+2}{n-2}]$. By (4.17),

$$\int_{(\mathcal{M},g_0)}v_p^{p+1}\le C, \tag{4.19}$$

where C is independent of $p\le\frac{n+2}{n-2}$. Denote $m_p=\sup v_p$ and assume that the sup is attained at $z_p=0$. As before we make a dilation of coordinates and a scaling for solution, namely

$$\begin{aligned}\tilde v_p(y)&=m_p^{-1}v_p(x),\\ y=R_px\quad R_p&=m_p^{\frac{2}{n-2}-\frac{\varepsilon}{2k}}.\end{aligned}$$

Then $0<\tilde v_p\le 1$, and $\tilde v_p$ satisfies

$$\sigma_k(\lambda(\tilde V))=\tilde v^{k\frac{n+2}{n-2}-\varepsilon}$$

in B_{cR_p} for some constant $c>0$ independent of p. Note that in the present case, $\varepsilon=\frac{n+2}{n-2}-p\to 0$. By the a priori estimates in §3.1, $\tilde v$ is locally uniformly bounded in C^3. Hence $\tilde v_p$ converges by a subsequence to a positive solution $\tilde v$ of

$$\sigma_k(\lambda(V)=v^{k\frac{n+2}{n-2}}\ \text{ in }\ \mathbf{R}^n.$$

By the Liouville Theorem [LL1],

$$\tilde v(y)=\overline{c}(1+|y|^2)^{\frac{2-n}{2}}, \tag{4.20}$$

where $\overline{c}=[n(n-2)]^{(n-2)/4}$. Moreover

$$\sup_{s>0}J_{p^*}(s\tilde v;\mathbf{R}^n)=c_{p^*}[S^n], \tag{4.21}$$

with $p^* = \frac{n+2}{n-2}$, where c_p was defined in (2.9).

The above argument implies that the metric $g = v_p^{\frac{4}{n-2}} g_0$ is a bubble near the maximum point z_p. By (4.20), v_p has the asymptotical behavior

$$v_p(x) = \bar{c}\,(\frac{\delta}{\delta^2 + r^2})^{\frac{n-2}{2}}(1 + o(1)) \qquad \delta = m_p^{-\frac{2}{n-2}+\frac{\varepsilon}{2k}}. \tag{4.22}$$

For a sufficiently small $\theta > 0$, let $\Omega_p = \{x \in \mathcal{M} \mid v_p(x) > \theta m_p\}$, and let

$$\hat{v}_p(x) = \begin{cases} v_p(x) & x \in \mathcal{M} - \Omega_p, \\ \theta m_p & x \in \Omega_p. \end{cases}$$

Note that by assumption (1.7),

$$\sup_{s>0} J(sv_p) = c_p < c_{p^*}[S^n] \tag{4.23}$$

when $p < \frac{n+2}{n-2}$ and is close to $\frac{n+2}{n-2}$.

Combining (4.21), (4.22) and (4.23), we see that

$$\int_{(\mathcal{M},g_0)} \hat{v}_p^{p+1} \geq C > 0$$

for some C independent of θ, provided θ is sufficiently small and m_p is sufficiently large, and

$$\sup_{s>0} J(s\hat{v}_p) < \sup_{s>0} J(sv_p).$$

Namely $\sup_{s>0} J(s\hat{v}_p) < c_p$, which is in contradiction of our definition of c_p. Note that $\hat{v}_p$ is not smooth, but can be approximated by smooth, k-admissible functions. This completes the proof of Lemma 2.1. □

4.3. The case $k = \frac{n}{2}$. In this case, the proof of Lemma 4.3 does not apply, due to that $J(v) \to -\infty$ as $v \to 0$, and also we don't know if $\mathcal{E}_{n/2}(v)$ is bounded from below for any admissible function v with $\mathrm{Vol}\mathcal{M}_{g_v} = 1$, where

$$J(v) = \mathcal{E}_{n/2}(v) - \frac{1}{p+1}\int_{(\mathcal{M},g_0)} v^{\frac{2n}{n-2}-\varepsilon}$$

is the corresponding functional and $\mathcal{E}_{n/2}$ is given in (2.32). However when $k = \frac{n}{2}$, we have the following

Lemma 4.6. *Assume that equation (2.1) is variational. Then $\mathcal{F}_{n/2}(v)$ is a constant.*

Proof. When $k = \frac{n}{2}$, we write the equation (2.1) and the functional $\mathcal{F}_{n/2}$ in the form

$$\sigma_{n/2}(\lambda(W)) = e^{-nw},$$
$$\mathcal{F}_{n/2}(w) = \int_{(\mathcal{M},g_0)} \sigma_{n/2}(\lambda(W)), \tag{4.24}$$

To prove that $\mathcal{F}_{n/2}$ is equal to a constant, we have

$$\begin{aligned}\mathcal{F}_{n/2}(w) - \mathcal{F}_{n/2}(w_0) &= \int_{(\mathcal{M},g_0)} \int_0^1 \frac{d}{dt}\sigma_{n/2}(\lambda(W_t)) \\ &= \int_0^1 \int_{(\mathcal{M},g_0)} L_{w_t}(w)\end{aligned}$$

where $w_t = tw$, $w_0 = 0$, L_{w_t} is the linearized operator of $\sigma_{n/2}(\lambda(W))$ at w_t. By the assumption that equation (2.1) is variational, we have (see §2.4)

$$\int_{(\mathcal{M},g_0)} L_{w_t}(w) = \int_{(\mathcal{M},g_0)} wL_{w_t}(1) = 0.$$

This completes the proof of Lemma 4.6. □

By assumption (1.7), we have

$$\mathcal{F}_{n/2}(v) = c_0 < Y_{n/2}(S^n) \tag{4.25}$$

for some constant c_0 depending on $(\mathcal{M}, g_0)$. Lemma 4.6 enables us to prove the following

Lemma 4.7. *For $1 < p \le \frac{n+2}{n-2}$, the set of solutions of (2.5) is compact.*

Proof. When $1 < p < \frac{n+2}{n-2}$, the proof is the same as that of Lemma 4.4.

When $p = \frac{n+2}{n-2}$, we use the same argument of Lemma 4.4. Instead of (4.16), we have the equation

$$\sigma_{n/2}(\lambda(V)) = v^{\frac{n}{2}\frac{n+2}{n-2}} \quad \text{in} \quad \mathbf{R}^n. \tag{4.26}$$

By the Liouville theorem [LL1], v must be the function given in (4.20). Hence we have

$$\int_{\mathbf{R}^n} \sigma_{n/2}(\lambda(V)) = Y_{n/2}(S^n).$$

By (4.9), we obtain that

$$\underline{\lim}_{j\to\infty} \mathcal{F}_{n/2}(v_j) \ge Y_{n/2}(S^n).$$

This is in contradiction with (4.25). □

By Lemma 4.7, we can prove the existence of solutions of (2.1) by a degree argument, see [CGY2, LL1]. We omit the details here.

4.4. A Sobolev type inequality. As a consequence of our argument above, we have the following Sobolev type inequality.

Corollary 4.1. *Let $2 \leq k < \frac{n}{2}$. Then there exists a constant $C > 0$ such that the inequality*

$$\big[Vol(\mathcal{M}_g)\big]^{\frac{n-2}{2n}} \leq C\big[\int_{\mathcal{M}} \sigma_k(\lambda(A_g))d\,vol_g\big]^{\frac{n-2}{2n-4k}} \tag{4.27}$$

holds for any conformal metric $g = v^{\frac{4}{n-2}} g_0$ with $v \in \Phi_k$.

Proof. Note that (4.27) is equivalent to

$$\big[\int_{(\mathcal{M},g_0)} v^{\frac{2n}{n-2}}\big]^{\frac{n-2}{2n}} \leq C\big[\int_{(\mathcal{M},g_0)} v^{\frac{2n-4k}{n-2}} \sigma_k(\lambda(\frac{V}{v}))\big]^{\frac{n-2}{2n-4k}}, \tag{4.28}$$

which is equivalent to (4.18). □

Remark 4.1. The Sobolev type inequality (4.27) is similar to (2.17) of $l = 0$, $k \geq 1$, and was proved in [GW2] for locally conformally flat manifolds. If $\mathcal{M}$ is locally conformally flat and $f = \mu(e^{-2kw})$, the flow (3.21) has a remarkable property. That is by the moving plane argument of Ye [Ye] and the conformal invariance of the equation, one obtains the gradient estimate (3.23) at all time t, with the upper bound C depending only on the initial function $w(\cdot, 0)$, for any monotone increasing μ satisfying (2.13). Therefore by the second derivative estimate (3.26), the solution of (3.21) converges to a solution of the k-Yamabe equation (2.1), and accordingly one also obtains the Sobolav type inequality (4.27). Theorem 4.2 shows that the Sobolev type inequality also holds on general manifolds provided equation (1.1) is variational.

5. Proof of Lemma 2.2

We let v_ε be the function given by

$$v_\varepsilon(x)\Big(\frac{\varepsilon}{\varepsilon^2+r^2}\Big)^{\frac{n-2}{2}}, \tag{5.1}$$

where $r=|x|$, $x\in\mathbf{R}^n$, and $\varepsilon>0$ is a small constant. Let V_ε be the matrix relative to v_ε, see (2.2). Then we have

$$\frac{V_\varepsilon}{v_\varepsilon}=(n-2)v_\varepsilon^{\frac{4}{n-2}}I.$$

Hence v_ε is k-admissible on $\mathbf{R}^n$ and

$$\sigma_k(\lambda(\frac{V_\varepsilon}{v_\varepsilon})=C_{n,k}v_\varepsilon^{\frac{4k}{n-2}},$$

where $C_{n,k}=\frac{n!(n-2)^k}{k!(n-k)!}$. It follows that

$$\int_{\mathbf{R}^n}v^{\frac{2n-4k}{n-2}}\sigma_k(\lambda(\frac{V}{v}))=C_{n,k}\int_{R^n}v^{\frac{2n}{n-2}}.$$

So we have

$$Y_k(S^n)=\frac{\int_{\mathbf{R}^n}v^{\frac{2n-4k}{n-2}}\sigma_k(\lambda(\frac{V}{v}))}{\big[\int_{\mathbf{R}^n}v^{\frac{2n}{n-2}}\big]^{(n-2k)/n}}=C_{n,k}\Big[\int_{\mathbf{R}^n}v^{\frac{2n}{n-2}}\Big]^{2k/n}. \tag{5.2}$$

In particular we have

$$Y_k(S^n)=\frac{C_{n,k}}{(n(n-2))^k}[Y_1(S^n)]^k. \tag{5.3}$$

In the above $v=v_\varepsilon$ and the integrations are independent of ε.

To verify (1.7), it would be natural to use the function (5.1) as a test function, as in the case $k=1$ [Au1, S1]. However on a general manifold, the function v_ε, where r denotes the geodesic distance from a given point, is k-admissible only when $r\le C\varepsilon^{1/2}$. It seems impossible to find an explicit test function.

Instead we shall deduce (1.7) directly from (1.6). First note that by assumption, there exists a function $v>0$ such that $\tilde g=v^{4/(n-2)}g\in\Gamma_k$. Hence $\sigma_1(\lambda(A_{\tilde g}))>0$. That is the scalar curvature of $(\mathcal{M},\tilde g)$ is positive. Hence the comparison principle for the operator $\sigma_1(\lambda(A_g))$ holds on $\mathcal{M}$.

Let v_1 be a solution of the Yamabe problem (with $k=1$) such that $Q_1(v_1)<Y_1(S^n)$, where

$$Q_1(v)=\frac{\int_{\mathcal{M}}v\sigma_1(\lambda(V))}{[\int_{\mathcal{M}}v^{2n/(n-2)}]^{(n-2)/n}}.$$

Let v_k be the solution of

$$\sigma_k(\lambda(V)) = C_{n,k} v_1^{k\frac{n+2}{n-2}} \quad \text{in } \mathcal{M}. \tag{5.4}$$

By Lemma 2.3(vi), we have

$$-\Delta v_k + \frac{n-2}{4(n-1)} R v_k = \sigma_1(\lambda(V_k)) \geq n(n-2) v_1^{\frac{n+2}{n-2}}.$$

Since v_1 satisfies

$$-\Delta v + \frac{n-2}{4(n-1)} R v = n(n-2) v_1^{\frac{n+2}{n-2}},$$

by the comparison principle,

$$v_k \geq v_1 \tag{5.5}$$

Now writing

$$Q_k(v) = \frac{\int_{\mathcal{M}} v^{\frac{2n}{n-2} - k\frac{n+2}{n-2}} \sigma_k(\lambda(V))}{\left[\int_{\mathcal{M}} v^{2n/(n-2)}\, dvol_g\right]^{(n-2k)/n}},$$

we claim that

$$Q_k(v_k) < Y_k(S^n), \tag{5.6}$$

namely (1.7) holds. Indeed, when $k \geq 2$, we have $\frac{2n}{n-2} - k\frac{n+2}{n-2} < 0$. Hence by (5.5),

$$v_1^{\frac{2n}{n-2} - k\frac{n+2}{n-2}} \geq v_k^{\frac{2n}{n-2} - k\frac{n+2}{n-2}}.$$

Hence

$$\begin{aligned}
\int_{\mathcal{M}} v_k^{\frac{2n}{n-2} - k\frac{n+2}{n-2}} \sigma_k(\lambda(V_k))\, dvol_g &\leq C_{n,k} \int_{B_\rho} v_1^{\frac{2n}{n-2} - k\frac{n+2}{n-2}} v_1^{k\frac{n+2}{n-2}}\, dvol_g \\
&\leq C_{n,k} \int_{B_\rho} v_1^{\frac{2n}{n-2}}\, dvol_g
\end{aligned}$$

and

$$\int_{\mathcal{M}} v_k^{\frac{2n}{n-2}}\, dvol_g \geq \int_{B_\rho} v_1^{\frac{2n}{n-2}}\, dvol_g.$$

Therefore we obtain

$$Q_k(v_k) \leq C_{n,k} \left[\int_{\mathcal{M}} v_1^{\frac{2n}{n-2}}\, dvol_g\right]^{2k/n}, \tag{5.7}$$

so that (5.6) follows from (5.3).

We remark that a similar argument can be used to prove the inequality $Y_1(\mathcal{M}) < Y_1(S^n)$ for some manifolds.

6. Some remarks

6.1. Compactness of the solution set. For the Yamabe problem ($k = 1$), Schoen [S2] has shown furthermore that the set of solutions is compact if the manifold is locally conformally flat and not conformally equivalent to the sphere. Schoen's result was extended to general compact manifolds for which the positive mass theorem holds, such as in low dimensions $3 \le n \le 7$ [LZ, M].

When $k > \frac{n}{2}$, the compactness of solutions has also been established in [GV2], for the more general equation

$$\sigma_k(\lambda(A_{g_v})) = f\, v^{\frac{n+2}{n-2}}, \tag{6.1}$$

where f is any positive, smooth function f. Their proof relies crucially on the fact that the Ricci curvature $Ric_{g_v} \ge C g_v$ if $g_v = v^{\frac{4}{n-2}} g_0$ is a solution to (6.1). For the convenience of the reader we indicate the idea of their proof here. From the positivity of the Ricci curvature, the volume of $(\mathcal{M}, g_v)$ is uniformly bounded. Hence if there exists a sequence of solutions $\{v_k\}$ with $\sup v_k \to \infty$, there are at most finitely many blow-up points $P =: \{p_0, p_1, \cdots, p_s\}$. By the interior first and second derivative estimates (3.23) and (3.26), $v_k / \inf v_k$ converges locally uniformly on $\mathcal{M} - P$ to a $C^{1,1}$ function v with $\sigma_k(\lambda(A_{g_v})) = 0$. By a technical analysis one has $v(x) = C(1 + o(1)) r^{2-n}$ near the singularity set P. Since $Ric_{g_v} > 0$, the ratio $h(r) =: \frac{|B_r(p_0)|}{r^n}$ is non-increasing, where $|B_r(p_0)|$ denotes the volume of the geodesic ball on $(\mathcal{M}, g_v)$. On the other hand, since $v(x) = C(1 + o(1)) r^{2-n}$, we have $\lim_{r\to\infty} h(r) = (s+1)\omega_n$, where ω_n is the volume of the Euclidean unit ball. Hence $s = 0$ and $(\mathcal{M}, g_v)$ is isometric to the Euclidean space $\mathbf{R}^n$, and so $\mathcal{M}$ is conformal to the unit sphere. Note that when $2 \le k \le \frac{n}{2}$, the Ricci curvature of $(\mathcal{M}, g_v)$ may not be positive anymore.

6.2. Conditions (C1) and (C2). As indicated earlier, we impose condition (C1) so that equation (1.1) is elliptic. If a fully nonlinear partial differential equation is not elliptic, little is known about the existence and regularity of solutions. For example it is unknown whether there is a local solution to the Monge-Ampere equation $\det D^2 u = f$ when the right hand side f changes sign, even in dimension two. But possibly condition (C1) may be replaced by the positivity of the Yamabe constant $Y_k(\mathcal{M})$, as in the case $k = 2$, $n = 4$ [CGY1, GV1].

As for the condition (C2), the variational approach to the k-Yamabe problem is natural, as in the case $k = 1$. Indeed this approach has already been employed in [CGY1, CGY2, GW2, GW3], and Theorem 2.1 was proved in [CGY1, CGY2] when $n = 4$, $k = 2$, and in [GW2, GW3] when $\mathcal{M}$ is locally conformally flat. At the moment we are not aware of any other possible ways to remove the variational structure condition (C2) for the case $2 \le k \le \frac{n}{2}$, even in low dimensional cases.

6.3. The full k-Yamabe problem [K, La]. We bring to the attention of the readers the full k-Yamabe problem. On a Riemannian manifold (M^n, g), one can define a series of scalar curvatures

$$s_k = s_k(Riem) = s_k(W + A \odot g), \tag{6.2}$$

for $k = 1, 2, \cdots, [\frac{n}{2}]$, where $Riem, W, A$ are introduced at the beginning in the introduction. The k-scalar curvature can also be expressed simply as

$$s_k = \frac{1}{(2k)!} c^{2k} Riem^k, \tag{6.3}$$

where c is the standard contraction operator, and $Riem^k = Riem \circ \cdots \circ Riem$ is the product introduced in [K].

When $k = 1$, s_1 is the usual scalar curvature. When n is even, $s_{n/2}$ is the Lipschitz-Killing curvature. Furthermore if $\mathcal{M}$ is a hypersurface, the k-scalar curvature s_k is the $2k^{th}$ mean curvature H_{2k}, which is equal to the $2k^{th}$ elementary symmetric polynomial of the principal curvatures of the hypersurface, which is intrinsic quantity [Sp]. When $\mathcal{M}$ is locally conformally flat, then the Weyl curvature in (6.2) vanishes, and s_k turns out to be the k-curvature given in (1.1).

The full k-Yamabe problem concerns the existence of a conformal metric such that the k-scalar curvature is a constant. This problem coincides with the k-Yamabe problem for locally conformally flat manifolds. The corresponding equation of the k-Yamabe problem is always variational, as in the case $k = 1$ [La]. However the equation is of mixed type in general.

Note. After this paper was completed, we were informed by Guofang Wang that he and Yuxin Ge had recently solved the case $k = 2, n > 8$ of Theorem 2.1.

References

[Au1] T. Aubin, Equations différentielles non linéaires et problme de Yamabe concernant la courbure scalaire. J. Math. Pures Appl. (9) 55 (1976), 269–296.

[Au2] T. Aubin, Some nonlinear problems in Riemannian geometry, Springer, 1998.

[B] A.L. Besse, Einstein manifolds, Springer-Verlag, Berlin, 1987.

[Br] T. Branson, Differential operators canonically associated to a conformal structure, Math. Scand., 57(1985), 293–345.

[BV] S. Brendle and J. Viaclovsky, A variational characterization for $\sigma_{n/2}$, Calc. Var. PDE. 20(2004), 399–402.

[CNS] L.A. Caffarelli, L. Nirenberg, and J. Spruck, Dirichlet problem for nonlinear second order elliptic equations III. Functions of the eigenvalues of the Hessian, Acta Math. **1985**, *155*, 261–301.

[CGY1] A. Chang, M. Gursky, P. Yang, An equation of Monge-Amère type in conformal geometry, and four-manifolds of positive Ricci curvature, Ann. of Math. (2) 155(2002), 709–787.

[CGY2] A. Chang, M. Gursky, P. Yang, An a priori estimate for a fully nonlinear equation on four-manifolds, J. Anal. Math. 87 (2002), 151–186.

[CGY3] A. Chang, M. Gursky, and P. Yang, Entire solutions of a fully nonlinear equation, Lectures on partial differential equations, 43–60, Inter. Press, 2003.

[Ch1] K.S. Chou (K. Tso), On a real Monge-Ampere functional, Invent. Math. 101(1990), 425–448.

[CW1] K.S. Chou and X-J. Wang, Variational solutions to Hessian equations, preprint, 1996. Available at wwwmaths.anu.edu.au/research.reports/96mrr.html.

[CW2] K.S. Chou and X-J. Wang, A variational theory of the Hessian equation, Comm. Pure Appl. Math. 54 (2001), 1029–1064.

[GV1] M. Gursky and J. Viaclovsky, A fully nonlinear equation on four-manifolds with positive scalar curvature, J. Differential Geom. 63 (2003), 131–154.

[GV2] M. Gursky and J.Viaclovsky, Prescribing symmetric functions of the eigenvalues of the Ricci tensor, arXiv:math.DG/0409187.

[GW1] P. Guan and G. Wang, Local estimates for a class of fully nonlinear equations arising from conformal geometry, Int. Math. Res. Not. (2003), 1413–1432.

[GW2] P. Guan and G. Wang, A fully nonlinear conformal flow on locally conformally flat manifolds, J. Reine Angew. Math. 557 (2003), 219–238.

[GW3] P. Guan and G. Wang, Geometric inequalities on locally conformally flat manifolds, Duke Math. J. 124 (2004), 177–212.

[GW4] P. Guan and Xu-Jia Wang, On a Monge-Ampère equation arising in geometric optics, J. Diff. Geom., 48(1998), 205–223.

[I1] N. Ivochkina, Solution of the Dirichlet problem for certain equations of Monge-Ampère type (Russian), Mat. Sb. (N.S.) 128(170) (1985), 403–415.

[K] R.S. Kulkarni, On the Bianchi identity, Math. Ann., 199(1972), 175-204.

[La] M.-L. Labbi, On a variational formula for the H. Weyl curvature invariants, arXiv:math.DG/0406548.

[Lb] D. Labutin, Potential estimates for a class of fully nonlinear elliptic equations. Duke Math. J. 111 (2002), 1–49.

[LP] J.M. Lee and T.H. Parker, The Yamabe problem, Bull. AMS 17(1987), 37–91.

[LL1] A. Li and Y.Y. Li, On some conformally invariant fully nonlinear equations. Comm. Pure Appl. Math. 56 (2003), 1416–1464.

[LL2] A. Li and Y.Y. Li, On some conformally invariant fully nonlinear equations II, Liouville, Harnack, and Yamabe, preprint.

[LS] J. Li and W.M. Sheng, Deforming metrics with negative curvature by a fully nonlinear flow, Calc. Var. PDE, to appear.

[LZ] Y.Y. Li and L. Zhang, Compactness of solutions to the Yamabe problem, C. R. Math. Acad. Sci. Paris 338 (2004), 693–695.

[LT] M. Lin and N.S. Trudinger, On some inequalities for elementary symmetric functions Bull. Aust. Math. Soc., 50(1994), 317–326.

[MTW] X.N. Ma, N.S. Trudinger, and X-J. Wang, Regularity of potential functions to the optimal transportation problem, Arch. Rat. Mech. Anal., to appear.

[M] F.C. Marques, A priori estimates for the Yamabe problem in non-locally conformal flat cases,

arXiv:math.DG/0408063.
[O] P.J. Olver, Applications of Lie groups to differential equations, Springer, 1993.
[S1] R. Schoen, Conformal deformation of a Riemannian metric to constant scalar curvature, J. Diff. Geom. 20(1984), 479–495.
[S2] R. Schoen, Variational theory for the total scalar curvature functional for Riemannian metrics and related topics. Topics in Calculus of Variations. Lectures Notes in Math. 1365. Springer, Berlin (1989), pp. 120–154.
[SY1] R. Schoen and S.T. Yau, Conformally flat manifolds, Kleinian groups and scalar curvature, Invent. Math. 92 (1988), 47–71.
[SY2] R. Schoen and S.T. Yau, Lectures on Differential geometry. International Press, 1994.
[Sc] F. Schulz, Regularity theory for quasi-linear elliptic systems and Monge-Ampère equations in two dimensions, Lecture Notes in Math. 1445 (1990).
[Sp] M. Spivak, A comprehensive introduction to differential geometry, Vol.4, Publish or Perish Inc. 1979.
[Tr1] N.S. Trudinger, Remarks concerning the conformal deformation of Riemannian structures on compact manifolds, Ann. Scuola Norm. Sup. Pisa (3) 22(1968), 265–274.
[TW1] N.S. Trudinger and X-J. Wang, Hessian measures I, Topol. Methods Nonlinear Anal. 10 (1997), 225–239.
[TW2] N.S. Trudinger and X-J. Wang, Hessian measures II, Ann. of Math. (2) 150 (1999), 579–604.
[TW3] N.S. Trudinger and X-J. Wang, Hessian measures III, J. Funct. Anal. 193 (2002), 1–23.
[TW4] N.S. Trudinger and X-J. Wang, A Poincaré type inequality for Hessian integrals, Calc. Var. PDE 6(1998), 315–328.
[TW5] N.S. Trudinger and X-J. Wang, On the weak continuity of elliptic operators and applications to potential theory, Amer. J. Math. 124 (2002), 369–410.
[V1] J. Viaclovsky, Conformal geometry, contact geometry, and the calculus of variations. Duke Math. J. 101 (2000), no. 2, 283–316.
[V2] J. Viaclovsky, Estimates and existence results for some fully nonlinear elliptic equations on Riemannian manifolds, Comm. Anal. Geom. 10 (2002), 815–846.
[W1] X-J. Wang, A class of fully nonlinear elliptic equations and related functionals, Indiana Univ. Math. J. 43 (1994), 25–54.
[W2] X-J. Wang, On the design of a reflector antenna, Inverse Problems, 12(1996), 351–375.
[Ya] H. Yamabe, On a deformation of Riemannian structures on compact manifolds, Osaka Math. J. 12(1960), 21–37.
[Ye] R. Ye, Global existence and convergence of Yamabe flow. J. Differential Geom. 39 (1994), 35–50.

Weimin Sheng, Department of Mathematics, Zhejiang University, Hangzhou 310028, China
E-mail address: weimins@zju.edu.cn

Neil S. Trudinger, Centre for Mathematics and Its Applications, Australian National University, Canberra, ACT 0200, Australia
E-mail address: neil.trudinger@maths.anu.edu.au

Xu-Jia Wang, Centre for Mathematics and its Applications, Australian National University, Canberra ACT 0200, Australia
Nankai Institute of Mathematics, Nankai University, Tianjin 300071, China
E-mail address: wang@maths.anu.edu.au