CALABI-YAU VARIETIES WITH FIBRE STRUCTURES I.

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Abstract. Motivated by the Strominger-Yau-Zaslow conjecture, we study fibre spaces whose total space has trivial canonical bundle. Especially, we are interest in Calabi-Yau varieties with fibre structures. In this paper, we only consider semi-stable families. We use Hodge theory and the generalized Donaldson-Simpson-Uhlenbeck-Yau correspondence to study the parabolic structure of higher direct images over higher dimensional quasi-projective base, and obtain some results on parabolic-semi-positivity. We then apply these results to study nonisotrivial Calabi-Yau varieties fibred by Abelian varieties (or fibred by hyperkähler varieties), we obtain that the base manifold for such a family is rationally connected and the dimension of a general fibre depends only on the base manifold.

Contents

1. The Generalized Donaldson-Simpson-Uhlenbeck-Yau Correspondence

An introduction to the correspondence. Let the base M be a quasi-projective manifold such that there is a smooth projective completion $\overline{M}$ with a reduced normal crossing divisor $D_\infty = \overline{M} - M$.

Definition 1.1. Let M be a quasi-projective manifold as above. A smooth projective curve $C \subset \overline{M}$ is *sufficiently general* if it satisfies that

(1) C intersects D_∞ transversely;
(2) $\pi_1(C_0) \twoheadrightarrow \pi_1(M) \to 0$ is surjectve where $C_0 = C \cap M$.

Remark. It is obvious that there are many *sufficiently general* curves: Let C be a complete intersection of very ample divisors such that it is a smooth projective curve in $\overline{M}$ intersecting D_∞ transversally. The quasi-projective version of the *Lefschetz hyperplane theorem* guarantees the subjectivity of $\pi_1(C_0) \twoheadrightarrow \pi_1(M) \to 0$ (cf. [6]).

1. Let (V, ∇) be a flat $\mathrm{GL}(n, \mathbb{C})$ vector bundle on M. (V, ∇) one to one corresponds to a fundamental representation $\rho : \pi_1(M) \to \mathrm{GL}(n, \mathbb{C})$. A Hermitian metric H on V leads

to a decomposition $\nabla = D_H + \vartheta$ corresponding to the Cartan decomposition of Lie algebra $gl(n, \mathbb{C}) = \mathfrak{u}(n) \oplus \mathfrak{p}$. D_H is a unitary connection preserving the metric H. With respect to the complex structure of M, one has the decomposition

$$D_H = D_H^{1,0} + D_H^{0,1},\ \vartheta = \vartheta^{1,0} + \vartheta^{0,1}.$$

The following two conditions are equivalent:

- $\nabla_H^*(\vartheta) = 0$ (∇_H^* is defined by $(e, \nabla_H^*(f))_H := (\nabla(e), f)_H$);
- $(D_H^{0,1})^2 = 0,\ D_H^{0,1}(\vartheta^{1,0}) = 0,\ \vartheta^{1,0} \wedge \vartheta^{1,0} = 0.$

If one of the above conditions holds, the metric H is called *harmonic* (or V is called *harmonic*). Altogether, if H is *harmonic* one has:

a) $(E, \overline{\partial}_E, \theta)$ is a Higgs bundle with respect to the holomorphic structure $\overline{\partial}_E := D_H^{0,1}$ where E takes the underlying bundle as V and $\theta := \vartheta^{1,0}$.
b) D_H is the unique metric connection with respect to $\overline{\partial}_E$.
c) H is the *Hermitian-Yang-Mills metric* of (E, θ), i.e.,

$$D_H^2 = -(\theta_H^* \wedge \theta + \theta \wedge \theta_H^*).$$

The existence of the *harmonic metric* was proven by Simpson in case that $\dim M = 1$ (cf. [21]), by Jost-Zuo in case that M is of higher dimension (cf. [8]). If M is a projective manifold, the *harmonic metric* on V is unique and depends only on the fundamental representation ρ, but the uniqueness does not hold if M is not compact. One can extend the induced Higgs bundle E over $\overline{M}$ to get a coherent sheaf $\overline{E}$, also extend θ to $\overline{\theta} \in \Gamma(\overline{M}, \mathcal{E}nd(\overline{E}) \otimes \Omega^1_{\overline{M}}(\log D_\infty))$. Though the extension of (E, θ) is not unique, one can treat this nonuniqueness by taking filtered extensions $(E, \theta)_\alpha$, and obtains a filtered Higgs bundle $\{(E, \theta)_\alpha\}$.

Conversely, let $(E, \overline{\partial}_E, \theta)$ be a Higgs bundle equipped with a Hermitian metric H. there is a unique metric connection D_H on (E, θ) with respect to the holomorphic structure $\overline{\partial}_E$, and θ has an H-adjoint $(0,1)$-form θ_H^*. Denote

$$\partial_E := D_H - \overline{\partial}_E,\ \nabla' := \partial_E + \theta_H^*,\ \text{and } \nabla'' := \overline{\partial}_E + \theta.$$

Then, $\nabla'' \circ \nabla'' = 0$ and $(\partial_E)^2 = 0$ as $D_H^2 = \pi^{1,1}(D_H^2)$. With respect to ∇'', there is another holomorphic vector bundle (V, ∇'') where V takes the underlying bundle as E. The metric H on (E, θ) is called *Hermitian-Yang-Mills* if $\nabla := \nabla' + \nabla''$ is integrable, and then H is said to be a *harmonic metric* on V (cf. [7]).

If M is a compact Kähler manifold or a quasi-projective curve, one has the *Donaldson-Simpson-Uhlenbeck-Yau correspondence* (DSUY correspondence), i.e., the *Hermitian-Yang-Mills metric* exists (cf. [5],[20],[21],[23]). Suppose that (V, ∇) is flat, one has

$$\partial_E(\theta) = \overline{\partial}_E(\theta_H^*) = 0.$$

Hence $\nabla' \circ \nabla' = 0$ and ∇' then is the Gauss-Manin connection of (V, ∇'').

2. An algebraic vector bundle E over M is said to have a *parabolic structure* if there is a collection of algebraic bundles E_α extending E over $\overline{M}$ whose extensions form a decreasing left continuous filtration such that $E_{\alpha+1} = E_\alpha \otimes \mathcal{O}_M(-D_\infty)$ and $E_\alpha \subset E_\beta$ for $\alpha \geq \beta$ as sheaves with $E_{\alpha-\epsilon} = E_\alpha$ for small ϵ.

One only needs to consider the index $0 \leq \alpha < 1$. The set of values where the filtration jumps is discrete, so it is finite and that the filtration is actually a proper filtration. Over a punctured curve $C_0 = C - S$, the *parabolic degree* of E is defined by

$$\text{par.deg}(E) = \deg \overline{E} + \sum_{s \in S} \sum_{0 \leq \alpha < 1} \alpha \dim(\text{Gr}_\alpha \overline{E}(s)) \tag{1.1.1}$$

where $\overline{E} := E_0 = \cup E_\alpha$. The *parabolic degree* of a filtered bundle $\{E_\alpha\}$ on a higher dimensional M is defined by taking the *parabolic degree* of the restriction $\{(E_\alpha)|_C\}$ over a *sufficiently general* curve C in $\overline{M}$. Since any subsheaf of a *parabolic* vector bundle E has a *parabolic* structure induced from E, one then has the definition of the *stability* for *parabolic* bundles (cf.[21]).

3. A *harmonic bundle* (V, H, ∇) is called *tame* if the metric H has at most polynomial growth near the infinity. In other words, a *harmonic bundle* (V, H, ∇) is *tame* if and only if (V, ∇) has only regular singularity at D_∞. Hence any *tame harmonic bundle* and its induced Higgs bundle over M are algebraic.

   Simpson and Jost-Zuo proved that any $\mathbb{C}$-local system on M has a *tame harmonic metric*. Moreover, if (E, θ) is a Higgs bundle induced from a *tame harmonic bundle*, E has a *parabolic structure* which is compatible with the extensive Higgs field, i.e., there is a filtered regular Higgs bundle $\{(E, \theta)_\alpha\}$. If local monodromies are all quasi-unipotent, the extension $\overline{E} = E_0$ can be chosen to be the Deligne quasi-unipotent extension.

   If M is a punctured curve, Simpson proved that a filtered Higgs bundle $\{(E, \theta)_\alpha\}$ is ploy-stable of *parabolic degree* zero if and only if it corresponds to a ploy-stable local system of *degree* zero (cf.[21]). For higher dimensional base, we have the following *generalized Donaldson-Simpson-Uhlenbeck-Yau correspondence* obtained by Simpson and Jost-Zuo:

Theorem 1.2 (cf. [8],[21],[23], [29])**.** *Let M be a quasi-projective manifold such that it has a smooth projective completion $\overline{M}$ and $D_\infty = \overline{M} \setminus M$ is a normal crossing divisor. Let (V, H, ∇) be a tame harmonic bundle on M and $\{(E, \theta)_\alpha\}$ be the induced filtered Higgs bundle. Then, one has:*

1. (V, H, ∇) is a direct sum of irreducible ones and $\{(E, \theta)_\alpha\}$ is a poly-stable filtered Higgs bundle of parabolic degree zero.

2. If (V, H, ∇) is irreducible, $\{(E, \theta)_\alpha\}$ is a stable filtered Higgs bundle of parabolic degree zero.

Remark. The parabolic stability of Higgs sheafs which induced from of *harmonic bundles* is independent of the choice of *sufficiently general* curve.

Chern classes of Hodge bundles. Let the base M be a quasi-projective manifold such that there is a smooth projective completion $\overline{M}$ with a reduced normal crossing divisor $D_\infty = \overline{M} - M$. Let $\mathbb{V}$ be a $\mathbb{Q}$-local system over M. Assume that $\mathbb{V}$ underlies a polarizable variation of Hodge structure of weight k such that all local monodromies around D are quasi-unipotent. Let $\overline{\mathcal{V}}$ be the Deligne quasi-canonical extension of the holomorphic vector bundle $\mathcal{V} = \mathbb{V} \otimes \mathcal{O}_M$ to $\overline{M}$ with the regular Gauss-Manin connection

$$\overline{\nabla} : \overline{\mathcal{V}} \to \overline{\mathcal{V}} \otimes \Omega^1_{\overline{M}}(\log D).$$

The Hodge filtration $F^\bullet$ on $\mathcal{V}$ can also extend to $\overline{\mathcal{V}}$. Denote $\overline{\mathcal{F}}^p = F^p\overline{\mathcal{V}}$. For any polarized VHS over M, the Hodge metrics are polynomial growth near D_∞ by Schmid's nilpotent orbit theorem (cf. [18]), thus $\mathcal{V}$ is is a *tame harmonic bundle* and it gives rise to a Higgs bundle $(E = \oplus E^{p,q}, \theta = \oplus \theta^{p,q})$ with the Higgs structure $\theta^{p,q} : E^{p,q} \to E^{p-1,q+1} \otimes \Omega^1_M$. The Higgs bundle (E, θ) has a *parabolic structure* with only regular singularity at D_∞, and it has Deligne canonical extension:

$$(\overline{E} = \bigoplus \overline{E}^{p,q}, \overline{\theta} = \bigoplus \overline{\theta}_{p,q}),$$

where $\overline{\theta}^{p,q} : \overline{E}^{p,q} \to \overline{E}^{p-1,q+1} \otimes \Omega^1_{\overline{M}}(\log D)$, and so $\overline{\theta}$ is an $\mathcal{O}_{\overline{M}}$-liner map and $\overline{\theta} \wedge \overline{\theta} = 0$. As this extensive Higgs bundle comes from VHS, it is called*Hodge bundle.*

The most important fact is that (E, θ) is a poly-stable *parabolic* Higgs bundle of *parabolic degree* 0 (cf. [18],[20, 21]). The following results related to the Chern classes of Hodge bundles are well-known:

1. Let (V, H, ∇) is a *tame harmonic bundle* over a quasi-projective smooth curve C_0 and (E, θ) be the induced *parabolic* Higgs bundle. Let F be a holomorphic subbundle of E (it takes the *parabolic* structure of E) and H_F be the restricted metric on F. Simpson showed that $\int_{C_0} c_1(F, H_F)$ is convergent (cf. [21]), moreover
$$\text{par.deg}(F) = \int_{C_0} c_1(F, H_F) = \int_{C_0} \text{Trace}(\Theta(F, H_F)).$$
2. Suppose that all monodromies are unipotent. Cattani-Kaplan-Schmid proved that the Chern form of the Hodge metric on the various $E^{p,q}$ defines *current* on $\overline{M}$ (cf.[2]). Moreover, the first Chern form computes the first Chern class of the Deligne canonical extension $\overline{E}^{p,q}$ on $\overline{M}$.

2. The Parabolic-Semi-Positivity of Bottom Filtrations of VHSs

Definition 2.1. Let $\pi : X \to Y$ be an algebraic fibre space with $d = \dim X - \dim Y$. We say π has *unipotent reduction condition* (*URC*) if the following conditions are satisfied:

(1) there is a Zariski open dense subset Y_0 of Y such that $D = Y \setminus Y_0$ is a *divisor of normal crossing on* Y, i.e., D is a reduced effective divisor and if $D = \sum_{i=1}^N D_i$ is the decomposition to irreducible components, then all D_i are non-singular and cross normally;
(2) $\pi : X_0 \to Y_0$ is smooth where $X_0 = \pi^{-1}(Y_0)$;
(3) all local monodromies of $R^d\pi_*\mathbb{Q}_{X_0}$ around D are unipotent.

The *URC* holds automatically for a semistable family, and one always has the semistable reduction if the base is a curve. But for any higher dimensional base, the semistable reduction theorem is still an enigma. Fortunately, one always has the unipotent reduction.

Proposition 2.2 (Fujita-Kawamata's positivity cf. [9])**.** *Let $\pi : X \to Y$ be a proper algebraic family with connected fibre and $\omega_{X/Y} := \omega_X \otimes \pi^*\omega_Y^{-1}$ be the relative dualizing sheaf. Assume that π satisfies URC as in the definition 2.1. Let $\mathcal{F}$ be the bottom filtration of the VHS $R^d\pi_*\mathbb{Q}_{X_0}$ where $X_0 = f^{-1}(Y_0)$ and $d = \dim X - \dim Y$. Then, one has:*

1. *$\pi_*\omega_{X/Y} = \overline{\mathcal{F}}$, where $\overline{\mathcal{F}}$ is the Deligne canonical extension. Thus, $\pi_*\omega_{X/Y}$ is locally free.*
2. *$\pi_*\omega_{X/Y}$ is semi-positive, i.e., for every projective curve T and morphism $g : T \to Y$ every quotient line bundle of $g^*(\pi_*\omega_{X/Y})$ has non-negative degree.*

Viehweg obtained more advanced results on the *weak positivity* without the assumption of *URC*. Studying VHSs with quasi-unipotent local monodromies and corresponding Higgs bundles, we obtain one useful generalization as follows:

Theorem 2.3. *Let M be a quasi-projective n-fold with a smooth projective completion $\overline{M}$ such that $D_\infty = \overline{M} - M$ is a reduced normal crossing divisor. Let $\mathbb{V}$ be a polarized $\mathbb{R}$-VHS over M and $\mathcal{F}$ be the bottom filtration of the VHS. Assume all local monodromies of $\mathbb{V}$ are quasi-unipotent. Then, we have:*

1. *There is a unique decomposition*
$$\overline{\mathcal{F}} = \mathcal{A} \oplus \mathcal{U},$$
where $\overline{\mathcal{F}}$ is the Deligne quasi-canonical extension of $\mathcal{F}$ over $\overline{M}$ such that
   a) *$\mathcal{A}$ has no flat quotient even after a finite ramified cover.*
   b) *$\mathcal{U}|_M$ is a unitary bundle on M and there is a covering τ of $\overline{M}$ ramified over D_∞ such that $\tau^*\mathcal{U}$ is unitary (if all local monodromies are unipotent, then $\mathcal{U}$ is unitary).*
2. *For any sufficiently general curve C in $\overline{M}$, we have:*
$$\text{par.deg}(\mathcal{A}|_{C_0}) > 0,$$
where $C_0 = \overline{M} \cap C$. By the definition of parabolic degree, it says that $\text{par.deg}(\mathcal{A}|_M) > 0$. *Moreover, if all local monodromies are unipotent then $\mathcal{A}|_C$ is an ample vector bundle.*

Remark. If all local monodromies are unipotent, $\mathcal{A}$ is semi-positive, i.e., $\deg_T h^*\mathcal{A} \geq 0$ for any morphism $h : T \to \overline{M}$ from a smooth projective curve T.

Proof. Let (E, θ) be the Higgs bundle induced from the polarized VHS $\mathbb{V}$ and h be the Hodge metric on E. The Deligne quasi-canonical extension of (E, θ) is $(\overline{E}, \overline{\theta})$. Omitting the Higgs structure, we have $E^\vee = E$ and the equality of the Deligne quasi-canonical extensions $\overline{\mathcal{H}}^\vee = \overline{\mathcal{H}^\vee}$ for any holomorphic subbundle $\mathcal{H}$ of E.

1. Suppose M is a quasi-projective curve.
   a) Let $\overline{\mathcal{F}} \to \mathcal{Q} \to 0$ be any quotient bundle, the dual exact sequence of holomorphic bundles is $0 \to \mathcal{Q}^\vee \to \overline{\mathcal{F}}^\vee$. $\mathcal{Q}^\vee|_M$ is then a *parabolic* vector bundle. Define $\mathcal{Q}_M$ to $\mathcal{Q}|_M$ and $\mathcal{Q}_M^\vee = \mathcal{Q}^\vee|_M$. The Hodge metric on E induces a singular metric on $\mathcal{Q}^\vee$, we still denote it h. We claim that $\text{par.deg}(\mathcal{Q}_M^\vee) \leq 0$. By Simpson's result, it is equivalent to show that
$$\int_M c_1(\mathcal{Q}_M^\vee, h) \leq 0.$$

Let $\Theta(E,h)$ be the curvature form of the Hodge metric h on E. From Griffiths-Schmid's curvature formula (cf. [18]), we have

$$\Theta(E,h)+\theta\wedge\bar{\theta}_h+\bar{\theta}_h\wedge\theta=0.$$

where $\bar{\theta}_h$ is the complex conjugation of θ with respect to h. On the other hand, we have a $\mathcal{C}^\infty$-decomposition

$$E=\mathcal{Q}_M^\vee\oplus(\mathcal{Q}_M^\vee)^\perp$$

with respect to the metric h. Then,

$$\begin{aligned}\Theta(\mathcal{Q}_M^\vee,h) &= \Theta(E,h)|_{\mathcal{Q}_M^\vee}+\bar{A}_h\wedge A\\ &= -\theta\wedge\bar{\theta}_h|_{\mathcal{Q}_M^\vee}-\bar{\theta}_h\wedge\theta|_{\mathcal{Q}_M^\vee}+\bar{A}_h\wedge A,\end{aligned}$$

where $A\in A^{1,0}(\mathrm{Hom}(\mathcal{Q}_M^\vee,(\mathcal{Q}_M^\vee)^\perp)$ is the second fundamental form of the sub-bundle $\mathcal{Q}_M^\vee\subset E_0$, and $\bar{A}_h$ is its complex conjugate with respect to h. Since $\theta(\mathcal{Q}_M^\vee)=0$ by $\theta(\mathcal{F}^\vee)=0$, we have:

$$\Theta(\mathcal{Q}_M^\vee,h)=-\theta\wedge\bar{\theta}_h|_{\mathcal{Q}_M^\vee}+\bar{A}_h\wedge A.$$

Thus,

$$\int_M c_1(\mathcal{Q}_M^\vee,h)\leq 0.$$

b) That $\mathrm{par.deg}\mathcal{Q}_M^\vee=0$ induces

$$\theta|_{\mathcal{Q}_M^\vee}\equiv\bar{\theta}_h|_{\mathcal{Q}_M^\vee}\equiv 0 \text{ and } \bar{A}_h\equiv A\equiv 0.$$

Because (E,θ) is a ploy-stable *parabolic* Higgs bundle, that

$$\mathrm{par.deg}\mathcal{Q}_M^\vee=0$$

not only implies that $\mathcal{Q}_M^\vee$ is a sub-Higgs bundle of (E,θ) but also show that there is a splitting of the Higgs bundle

$$(E,\theta)=(\mathcal{N},\theta)\oplus(\mathcal{Q}_M^\vee,0).$$

By the Simpson theorem for quasi-projective curve, the Higgs splitting corresponds to a splitting f $\mathbb{C}$-local system

$$\mathbb{V}=\mathbb{V}_{\mathcal{N}}\oplus\mathbb{V}_{\mathcal{Q}_M^\vee},$$

$\mathbb{V}_{\mathcal{Q}_M^\vee}$ corresponds to $\mathcal{Q}_M^\vee$ and it is unitary on M.

c) i. Because (E,θ) is a poly-stable *parabolic* Higgs bundle, there exists a maximal subbundle $\mathcal{B}$ of $\overline{\mathcal{F}}$ such that

$$\mathrm{par.deg}(\mathcal{B}|_M)=0,$$

i.e., any subbundle $\mathcal{G}$ of $\overline{\mathcal{F}}^\vee$ with $\mathrm{par.deg}(\mathcal{G}|_M)=0$ must be contained in $\mathcal{B}$. Denote $\mathcal{U}=\mathcal{B}^\vee$ and $\mathcal{A}$ is the quotient bundle $\overline{\mathcal{F}}/B$. We have the exact sequence of vector bundles

$$0\longrightarrow\mathcal{A}^\vee\longrightarrow\overline{\mathcal{F}}^\vee\longrightarrow\mathcal{U}\longrightarrow 0,$$

and for every quotient $\mathcal{A} \to \mathcal{Q} \to 0$ we have

$$\text{par.deg}(\mathcal{Q}^\vee|_M) < 0$$

even after a generically finite pull back.

ii. We claim that $\text{Hom}(\mathcal{A},\mathcal{U}) = 0$. Otherwise there would exist a nonzero bundle map $0 \neq s \in \text{Hom}(\mathcal{A},\mathcal{U}) = 0$. Let $\mathcal{I}$ be that nonzero vector bundle (we can extend the image sheaf to be a bundle). As $\mathcal{U}$ is a poly-stable *parabolic* vector bundle with *parabolic degree* zero, we have $\text{par.deg}(\mathcal{I}|_M) \leq 0$, i.e.,

$$\int_M c_1(\mathcal{I}, h) \leq 0.$$

It is a contradiction. Thus

$$\overline{\mathcal{F}} = \mathcal{A} \oplus \mathcal{U}$$

by $\text{Ext}^1(\mathcal{U}, \mathcal{A}) = \text{Hom}(\mathcal{A},\mathcal{U}) = 0$, and $\mathcal{A}$ is a *parabolic* bundle with

$$\text{par.deg}(\mathcal{A}|_M) > 0.$$

d) i. In case that all local monodromies of $\mathbb{V}$ around D_∞ are unipotent, we have

$$\text{par.deg}(E) = \deg \overline{E}$$

as the index of the filtered Higgs bundle jumps only at $\alpha = 0$. Because all local monodromies of $\mathbb{U}$ are trivial, $\mathcal{U}$ is a unitary vector bundle over $\overline{M}$ and $\mathcal{A}$ is an ample bundle on $\overline{M}$ by Hartshorne's characterization of ampleness: *A locally free sheaf* $\mathcal{G}$ *over a smooth projective curve* C *is ample if and only if* $\deg_C \mathcal{R} > 0$ *for any nonzero quotient vector bundle* $\mathcal{G} \to \mathcal{R} \to 0$.

ii. In case that all local monodromies for $\mathbb{V}$ around D_∞ are quasi unipotent. Using the method(the Kawamata covering trick) in [9], we can find the cyclic cover τ ramified over D_∞ described in the statement 1.

2. Suppose that M is a higher dimensional quasi-projective manifold. Choose a sufficiently general curve C in $\overline{M}$.

a) We have shown that

$$\overline{\mathcal{F}}|_C = \mathcal{A}' \oplus \mathcal{U}'$$

such that $\mathcal{U}'|_{C_0}$ is unitary on C_0 and any quotient $\mathcal{Q}$ of $\mathcal{A}'$ has $\text{par.deg}(\mathcal{Q}^\vee|_{C_0}) < 0$, and the Higgs splitting

$$(E|_{C_0}, \theta_{C_0}) = (\mathcal{N}', \theta_{C_0}) \oplus ((\mathcal{U}'_{C_0})^\vee, 0)$$

corresponds to a splitting of $\mathbb{C}$-local system

$$\mathbb{V}|_{\mathbb{C}_0} = \mathbb{W}' \oplus \mathbb{B}'$$

over C_0. By the subjectivity of $\pi_1(C_0) \twoheadrightarrow \pi_1(M)$, we have a splitting of harmonic bundle on M:

$$\mathbb{V} = \mathbb{W} \oplus \mathbb{B},$$

such that $\mathbb{B}$ is unitary, and $\mathbb{W}|_{C_0} = \mathbb{W}', \mathbb{B}|_{C_0} = \mathbb{B}'$.

b) The *generalized Donaldson-Simpson-Uhlenbeck-Yau correspondence* for higher dimensional quasi-projective manifolds says that we have

$$(E,\theta) = (\mathcal{N},\theta)\oplus(\mathcal{B}_0,0)$$

over M such that $\mathcal{B}_0^\vee|_{C_0} = \mathcal{U}'|_{C_0}$. Let $\mathcal{B}$ be the quasi-canonical extension of $\mathcal{B}_0$ to $\overline{M}$. Denote $\mathcal{U} = \mathcal{B}^\vee$ and $\mathcal{A} = \overline{\mathcal{F}}/\mathcal{B}$, then $\mathcal{U}|_C = \mathcal{U}'$. Actually the splitting of the Higgs bundle over M is independent of the choice of the curve C. $\mathcal{A}$ and $\mathcal{U}$ are indeed what we ask for.

c) If there is another generic curve C' such that there exists a quotient $\mathcal{Q}$ of $\mathcal{A}|_{C'}$ with $\mathrm{par.deg}_{C'}(\mathcal{Q}^\vee|_{C'_0}) = 0$. It will induce a splitting of the harmonic bundle $\mathbb{V} = \mathbb{G}\oplus\mathbb{K}$ such that $\mathbb{K}$ is unitary over M. Our choice of $\mathcal{U}$ implies that

$$\mathbb{K}|_{C_0}\subset\mathbb{B}_{C_0},\ \mathbb{K}\subset\mathbb{B} \text{ and } \mathrm{par.deg}_C(\mathcal{Q}^\vee|_{C'_0}) = 0,$$

it is a contradiction. Other statements follow directly from the standard Kawamata's covering trick as in [9].

□

Corollary 2.4 (Kollár cf. [12]). *Let C be a smooth projective curve $\mathbb{V}$ be a polarized VHS over a Zariski open set U of C. Let $\mathcal{F}$ be the bottom filtration of the VHS.*

Assume that all local monodromies are unipotent. On the curve C, there is a decomposition of the Deligne canonical extension:

$$\overline{\mathcal{F}} = \mathcal{A}\oplus\mathcal{U}$$

such that $\mathcal{A}$ is an ample vector bundle and $\mathcal{U}$ is a flat vector bundle. Moreover, $\mathcal{A}$ is unique as a subbundle of $\overline{\mathcal{F}}$.

Corollary 2.5 (Kawamata cf. [11]). *Let Y be projective manifold and Y_0 be a dense open set of Y such that $S = Y\setminus Y_0$ is a reduced normal crossing divisor. Let $\mathbb{V}$ be polarized VHS of strict weight k over Y_0 such that all local monodromies are unipotent.*

Assume that

$$T_{Y,p}\longrightarrow \mathrm{Hom}(\mathcal{F}^k_p, \mathcal{F}^{k-1}_p/\mathcal{F}^k_p)$$

is injective at $p\in Y_0$ where $\mathcal{F}^k = F^k\mathbb{V}$ is the bottom filtration of the VHS $\mathbb{V}$. Then, $\det\overline{\mathcal{F}^k}$ *is a big line bundle.*

Proof. Let $\mathcal{N} = \mathcal{F}^\vee$. Then $\theta(\mathcal{N}) = 0$ and $\Theta(\mathcal{N},h) = -\theta\wedge\bar\theta_h|_{\mathcal{N}} + \bar{A}_h\wedge A$. The injectivity of the morphism $T_{Y,p}\to\mathrm{Hom}(\mathcal{F}^k_p,\mathcal{F}^{k-1,1}_p)$ induces that

$$\int_{Y_0} c_1(\mathcal{F},h)^{\dim Y} > 0.$$

Since $\det(\overline{\mathcal{F}})$ is a *nef* and $c_1(\mathcal{F},h)$ is a current due to Cattani- Kaplan-Schmid's theorem, then $\det(\overline{\mathcal{F}})$ is *big* by Sommese-Kawamata-Siu's numerical criterion of bigness:

If L is a hermitian semi-positive line bundle on a compact complex manifold X such that $\int_X \wedge^{\dim X} c_1(L) > 0$, then L is a big line bundle on X. In particular, if the line bundle L is nef with $(L)^{\dim X} > 0$, then L is big.

□

3. Calabi-Yau Manifolds with Semistable Fibre Structures

Our main goal is to study Calabi-Yau varieties with fibre structures. By the motivation from SYZ conjecture, we study the fibration $f : X \to Y$ with connected fibres such that the total space X has trivial canonical line bundle.

Definition 3.1. 1. A projective manifold X is called *Calabi-Yau* if its canonical line bundle ω_X is trivial and $H^0(X, \Omega^p_X) = 0$ for p with $0 < p < \dim X$.
2. A compact Kähler manifold X is called hyperkähler if its dimension is $2n \geq 4$, $H^1(X, \mathcal{O}_X) = 0$ and there is a non-zero holomorphic two form β_X unique up to scalar with $\det(\beta_X) \neq 0$.

(Thus, if X is hyperkähler, then $h^{0,2}(X, \mathcal{O}_X) = 1$ and ω_X is trivial since it has non-zero section $\det \beta_X$).

Some observations. Let $f : X \to C$ be a semistable family from a projective manifold o a smooth projective curve. Since $\omega_C^{-1} = \mathcal{O}_C(\sum t_i - \sum t_j)$ with $\#\{i\} - \#\{j\} = 2 - 2g(C)$, we have:

$$\omega_{X/C} = \omega_X \otimes f^*\omega_C^{-1} = f^*\omega_C^{-1} = \mathcal{O}_X(\sum X_{t_i} - \sum X_{t_j}).$$

Assume X has trivial canonical line bundle. Then, by projection formula $f_*\omega_{X/C} = \omega_C^{-1}$ and so the canonical line bundle of any smooth closed fibre is trivial. In particular, if X is a Calabi-Yau manifold, then a general fibre can be one of Abelian variety, lower dimensional Calabi-Yau variety or hyperkähler variety.

Observation 3.2. Let $f : X \to Y$ be a semistable family of Calabi-Yau varieties over a higher dimension base such that f is smooth over Y_0 and $Y \setminus Y_0$ is a reduced normal crossing divisor. We have:
1. If the induced moduli morphism is generically finite, then $f_*\omega_{X/Y}$ is *big and nef.*
2. If f is smooth and the induced period map has no degenerated point, then $f_*\omega_{X/Y}$ is ample.

Observation 3.3. Let $f : X \to C$ be a semistable non-isotrivial family over a smooth projective curve C. Assume that ω_X is trivial. Then, we have that the line bundle $f_*\omega_{X/C}$ is big and the curve C is a projective line $\mathbb{P}^1$.

The observation 3.3 is a special case of the following proposition 3.4.

Let Z be an algebraic n-fold with trivial canonical bundle. One has an isomorphism

$$H^1(Z, T_Z) \to \mathrm{Hom}(H^0(Z, \Omega^n_Z), H^1(Z, \Omega^{n-1}_Z))$$

from $\Omega^{n-1}_Z \cong T_Z$, i.e., *the infinitesimal Torelli theorem* holds true.

Let $f : X \to Y$ be a semistable proper family smooth over a Zariski open dense set Y_0 such that $S = Y - Y_0$ is a reduced normal crossing divisor. Suppose that X is a projective manifold with trivial canonical line bundle, then a general fibre is a smooth projective manifold with trivial canonical bundle and has certain type 'K'. It is well known that the coarse quasi-projective moduli scheme $\mathfrak{M}_K$ exists for the set of all polarized projective manifolds with trivial canonical line bundle and type 'K' (cf.[24]).

By the infinitesimal Torelli theorem, that the family f satisfies the condition in the corollary 2.5 if and only if the unique moduli morphism $\eta_f : Y_0 \to \mathfrak{M}_K$ for f is a generically finite morphism. Moreover, the condition is equivalent to that f contains no isotrivial subfamily whose base is a subvariety passing through a general point of Y. If Y is a curve, that f satisfies the condition in 2.5 if and only if that f is non-isotrivial.

Proposition 3.4. *Let $f : X \to Y$ be a surjective morphism between two non-singular projective varieties such that every fibre is irreducible and $f : X_0 = X \setminus \Delta \to Y \setminus S$ be the maximal smooth subfamily where $S = Y \setminus Y_0$ is a reduced normal crossing divisor.*

Assume that X is a projective n-fold with trivial canonical line bundle. Let

$$\mathcal{F} = F^{n-1}R^{n-1}f_*(\mathbb{Q}_{X_0})$$

and $\overline{\mathcal{F}}$ be the quasi-canonical extension. If the moduli morphism of f is generically finite, then we have:

1. *The parabolic degree of $\mathcal{F}$ is positive over any sufficient general curve in Y.*
2. *Moreover, if f is weakly semistable (resp. semistable), i.e., $\Delta = f^*S$ is a relative normal crossing divisor in X (resp. reduced divisor), then $f_*\Omega^{n-1}_{X/Y}(\log \Delta)$ is a parabolic line bundle with positive parabolic degree (resp. $f_*\omega_{X/Y}$ is a big and nef line bundle).*

Definition 3.5 (Rationally connected varieties cf. [14], [1])**.** Let X be a smooth projective variety over $\mathbb{C}$ (or any uncountable algebraically closed field of characteristic 0). X is called *rationally connected* if it satisfies the following equivalent conditions:

a) There is an open subset $\emptyset \neq X^0 \subset X$, such that for every $x_1, x_2 \in X^0$, there is a morphism $f : \mathbb{P}^1 \to X$ satisfying $x_1, x_2 \in f(\mathbb{P}^1)$.
b) There is a morphism $f : \mathbb{P}^1 \to X$ such that $H^1(\mathbb{P}^1, f^*T_X(-2)) = 0$. It is equivalent to:

$$f^*T_X = \sum_{i=1}^{\dim X} \mathcal{O}_{\mathbb{P}^1}(a_i) \quad \text{with} \quad a_i \geq 1.$$

   We call this f *very free* (If all $a_i \geq 0$ then $H^1(\mathbb{P}^1, f^*T_X(-1)) = 0$, f is called *free*).
c) There is a smooth variety Y of dimension $\dim X - 1$ and a dominant morphism $F : \mathbb{P}^1 \times Y \to X$ such that $F((0\!:\!1) \times Y)$ is a point. We can also assume that

$$H^1(\mathbb{P}^1, F_y^*T_X(-2)) = 0$$

   for every $y \in Y$ where $F_y := F|_{\mathbb{P}^1 \times \{y\}}$.

The class of rationally connected varieties contains the class of unirational varieties. This class of varieties has many nice properties:

Properties 3.6. (Results for rationally connected varieties).

a) Kollár-Miyaoka-Mori and Campana (cf. [14],[1]) showed:
   i. The class of rationally connected varieties is closed under birational equivalence.
   ii. A smooth projective rationally connected variety X must satisfy

$$H^0(X, (\Omega^1_X)^{\otimes m}) = 0 \text{ with } \forall\ m \geq 1.$$

If $\dim X = 3$, the converse also holds. Thus, rationally connected varieties are simply connected.

iii. Being rationally connected is deformation invariant for smooth projective varieties.

b) Recently, Zhang proved that $\log Q$-Fano varieties are rationally connected (cf. [28]) and it implies that any higher dimensional variety with a *big* and *nef* anticanonical bundle must be rationally connected. Kollár-Miyaoka-Mori obtained this result in case of threefold (cf. [14]).

Observation 3.7. Let $f : X \to Y$ be a semistable proper family between two non-singular projective varieties. Assume that X has a trivial canonical line bundle and the induced moduli morphism η_f is generically finite. Then, Y has a *big* and *nef* anti-canonical line bundle, and is *rationally connected*.

Vanishing of unitary subbundles.

Theorem 3.8. *Let $f : X \to Y$ be semistable family between two non-singular projective varieties with*

$$f : X_0 = f^{-1}(Y_0) \to Y_0$$

smooth, $S = Y \setminus Y_0$ a reduced normal crossing divisor and $\Delta = f^*S$ a relative reduced normal crossing divisor in X. Assume that f satisfies that

a) the polarized VHS $R^k f_*\mathbb{Q}_{X_0}$ is strictly of weight k;

b) $H^k(X, \mathcal{O}_X) = 0$;

c) Y is simply connected.

Then $f_*\Omega^k_{X/Y}(\log\Delta)$ is locally free on Y without flat quotient, and $S \neq \emptyset$. Moreover,

$$\deg_C(f_*\Omega^k_{X/Y}(\log\Delta)) > 0$$

for any sufficiently general curve $C \subset \overline{M}$.

Proof. That $R^k f_*\mathbb{Q}_{X_0}$ is of weight k guarantee that $f_*\Omega^k_{X/Y}(\log\Delta) \neq 0$. Then, we have:

$$f_*\Omega^k_{X/Y}(\log\Delta) = \mathcal{A} \oplus \mathcal{U}$$

such that $\mathcal{A}$ has no flat quotient and $\mathcal{U}$ is flat (so it is trivial here) by 2.3.

It is sufficient to show that $f_*\Omega^k_{X/Y}(\log\Delta)$ has no flat direct summand. Otherwise, there is a nonzero global section $s \in H^0(Y_0, R^k f_*(\mathbb{C}))$ of $(k,0)$-type. Let $y \in Y_0$ be a fixed point. Consider Hodge theory, we have the following commutative diagram:

$$\begin{array}{ccc} H^m(X,\mathbb{C}) & \xrightarrow{\quad i^* \quad} & H^m(X_0,\mathbb{C}) \\ & {\scriptstyle \bar{i}_y^*}\searrow \quad \swarrow{\scriptstyle i_y^*} & \\ & H^m(X_y,\mathbb{C})^{\pi_1(Y_0,y)} & \end{array}$$

where $i_y : X_y \hookrightarrow X_0$, $\bar{i}_y : X_y \hookrightarrow X$ are natural embeddings. For each pair (p,q) with $p+q=m$, the following restriction map induced by $X_y \subset X$ is a Hodge morphism:

$$r_y^{p,q} : H^q(X,\Omega^p_X) \xrightarrow{\hookrightarrow} H^m(X,\mathbb{C}) \xrightarrow{\bar{i}_y^*} H^m(X_y,\mathbb{C})^{\pi_1(Y_0,y)} \xrightarrow{\hookrightarrow} H^m(X_y,\mathbb{C}) \to H^q(X_y,\Omega^p_{X_y}).$$

Since $\overline{i}_y^*$ is a surjective Hodge morphism, we have: *The component of type (p,q) of the group $H^m(X_t,\mathbb{C})^{\pi_1(Y_0,y)}$ is just the image of $H^q(X,\Omega_X^p)$ under $r_y^{p,q}$.*

Let $m=k$. We then have a nonzero lifting $\widetilde{s}\in H^0(X,\Omega_X^k)$ of s, it is a contradiction. With same method, we have $S\neq\emptyset$. □

Corollary 3.9. *Let $f:X\to Y$ be semistable family between two non-singular projective varieties with*

$$f:X_0=f^{-1}(Y_0)\to Y_0$$

smooth, $S=Y\setminus Y_0$ a reduced normal crossing divisor and $\Delta=f^{-1}(S)$ a relative reduced normal crossing divisor in X. Assume that X is a Calabi-Yau n-fold and Y is simply connected. Then,

a) f is a nonisotrivial family with $S\neq\emptyset$;
b) $f_*\omega_{X/Y}$ is an ample line bundle over any sufficiently general curve.

Proof. Consider the polarized VHS $R^{n-1}f_*(\mathbb{Q}_{X_0})$. Suppose that f is isotrivial then the holomorphic period map for the VHS $R^{n-1}f_*(\mathbb{Q}_{X_0})$ is constant over Y_0 by the *infinitesimal Torelli theorem*. Then the line bundle $f_*\omega_{X_0/Y_0}$ is unitary over Y_0, and so $f_*\omega_{X/Y}$ is unitary by the semi-stability of f. Since Y is simply-connected, $f_*\omega_{X/Y}$ then is a trivial line bundle on Y, it is a contradiction to the theorem 3.8. By similar arguments we then obtain $S\neq\emptyset$. □

Remark. Without the assumptions that X is a Calabi-Yau manifold and Y is simply connected, if X has trivial K_X then we have the following results:

1. If f is isotrivial then the line bundle $f_*\omega_{X/Y}$ is unitary.
2. Conversely, if $f_*\omega_{X/Y}$ is unitary and the global Torelli theorem hold for a general fiber (e.g. a general fiber is K3 or Abelian variety) then f is isotrivial.

Corollary 3.10. *Let $f:X\to\mathbb{P}^1$ be semistable family with $f:X_0=f^{-1}(C_0)\to C_0$ smooth, $S=\mathbb{P}^1\setminus C_0$, and $\Delta=f^*S$. Assume that X is a projective manifold with $H^k(X,\mathcal{O}_X)=0$ and the polarized VHS $R^kf_*\mathbb{Q}_{X_0}$ is strictly of weight k. Then, $S\neq\emptyset$ and $f_*\Omega^k_{X/\mathbb{P}^1}(\log\Delta)$ is an ample bundle on $\mathbb{P}^1$.*

Proposition 3.11. *Let $f:X\to C$ be a semistable family over a smooth projective curve with $f:X_0=f^{-1}(C_0)\to C_0$ smooth, $S=C\setminus C_0$, and $\Delta=f^{-1}(S)$. If X is a projective n-fold with trivial ω_X then the following conditions are equivalent:*

(1) *f is a non isotrivial family.*
(2) *$f_*\omega_{X/C}$ is an ample line bundle on C.*
(3) *$C=\mathbb{P}^1$ and $\#S\geqq 3$.*

If one of the following conditions are satisfied then $\#S\geqq 3$.

Proof. Each smooth closed fibre of f has trivial canonical line bundle.

a) The infinitesimal Torelli theorem holds for the VHS $R^{n-1}f_*(\mathbb{Q}_{X_0})$ because there is an isomorphism for any $t\in C_0$:

$$H^1(X_t,T_{X_t})\to\mathrm{Hom}(H^0(X_t,\Omega^{n-1}_{X_t}),H^1(X_t,\Omega^{n-2}_{X_t})).$$

b) f is isotrivial $\Longleftrightarrow f_*\omega_{X_0/C_0}$ is a unitary line bundle on C_0.
c) Since f is semistable, $f_*\omega_{X_0/C_0}$ is unitary on $C_0 \Longleftrightarrow f_*\omega_{X/C} = \omega_C^{-1}$ is unitary on C.
d) ω_C^{-1} is unitary on $C \Longleftrightarrow 0 = \deg \omega_C^{-1} \Longleftrightarrow C$ is elliptic.

Deligne's complete reducible theorem say that the global monodromy is semi-simple. If $C = \mathbb{P}^1$ and $S \leq 2$, then the the global monodromy is same as the locally mondromy around S, a contradiction. Thus $\#S \geqq 3$. □

Remarks. Actually, our proof shows that the following conditions are equivalent:

(1) f is an isotrivial family;
(2) $f_*\omega_{X/C}$ is an unitary line bundle on C;
(3) C is an elliptic curve.

We have an interesting property: It is impossible that $C = \mathbb{P}^1$ with $\#S \leqq 2$ even f is isotrivial.

Question 3.12. Let $f : X \to Y$ be semistable family between two non-singular projective varieties with $f : X_0 = f^{-1}(Y_0) \to Y_0$ smooth, $S = Y \setminus Y_0$ a reduced normal crossing divisor and $\Delta = f^{-1}(S)$ a relative reduced normal crossing divisor in X. Assume that X is a Calabi-Yau n-fold. Are the following two statements equivalent?

(1) The induced moduli map of f is of generally finite.
(2) Y is rationally connected and S is not empty.

In general, "(1) ⇒ (2)". When Y is curve, the question has a positive answer by 3.11.

4. Dimension Counting for Fibered Calabi-Yau Manifolds

Calabi-Yau manifolds fibred by Abelian varieties.

Theorem 4.1. *Let $f : X \to \mathbb{P}^1$ be a semistable family fibred by Abelian varieties such that $f : X_0 = f^{-1}(C_0) \to C_0$ is smooth with finite singular values $S = \mathbb{P}^1 \setminus C_0$ and a normal crossing $\Delta = f^{-1}(S)$. Assume X is a projective manifold with trivial canonical line bundle and $H^1(X, \mathcal{O}_X) = 0$. Then, f is nonisotrivial and* $\dim X \leq 3$.

In particular, if X is a Calabi-Yau manifold, X is one of the following cases:

a) $K3$ with $\#S \geq 6$. Moreover, if $\#S = 6$ then $X \to \mathbb{P}^1$ is modular, i.e., C_0 is the quotient of the upper half plane $\mathcal{H}$ by a subgroup of $\mathrm{SL}_2(\mathbb{Z})$ of finite index.
b) Calabi-Yau threefold with $\#S \geq 4$. Moreover, if $\#S = 4$ then this family is rigid and there exists an étale covering $\pi : Y' \to \mathbb{P}^1$ such that $f' : X' = X \times_{\mathbb{P}^1} Y' \to Y'$ is isogenous over Y' to a product $E \times_{Y'} E$, where $h : E \to Y'$ is a family of semistable elliptic curves and modular.

Proof. Consider the Higgs bundle (E, θ) corresponding to $R^1 f_* \mathbb{Q}_{X_0}$. The semi-stability of f shows that all local monodromies of $R^1 f_* \mathbb{Q}_{X_0}$ are unipotent, thus there is the Deligne canonical extension

$$\overline{E} = f_*\Omega^1_{X/\mathbb{P}^1}(\log \Delta) \bigoplus R^1 f_*(\mathcal{O}_X)$$

where $\Delta = f^*(S)$, and both pieces are locally free.

1. Let $n = \dim f^{-1}(t)$. Then, $n = \mathrm{rk} f_*\Omega^1_{X/\mathbb{P}^1}(\log\Delta)$ as a general fibre is an Abelian variety. It has been shown in 3.10 and 3.9 that f is a non-isotrivial family and the bundle $f_*\Omega^1_{X/\mathbb{P}^1}(\log\Delta)$ is ample on $\mathbb{P}^1$. The Grothendieck splitting theorem says that there is a decomposition:
$$f_*\Omega^1_{X/\mathbb{P}^1}(\log\Delta) = \bigoplus_{i=1}^{n} \mathcal{O}_{\mathbb{P}^1}(d_i),$$
and all integers d_i are positive by the ampleness of the $f_*\Omega^1_{X/\mathbb{P}^1}(\log\Delta)$.
   On the other hand, the commutative diagram of morphisms
$$\begin{array}{ccc} \wedge^n f_*\Omega^1_{X/\mathbb{P}^1}(\log\Delta) & \xrightarrow{\neq 0} & f_*(\wedge^n\Omega^1_{X/}(\log\Delta)) \\ \big\downarrow = & & \big\downarrow = \\ \mathcal{O}_{\mathbb{P}^1}(\sum_{i=1}^n d_i) & \xrightarrow{\neq 0} & f_*\omega_{X/\mathbb{P}^1}. \end{array}$$
induces that
$$n \leq \sum_{i=1}^{n} d_i \leq \deg f_*\omega_{X/\mathbb{P}^1}.$$
By Zariski main theorem, f only having connected fibres is same as $f_*\mathcal{O}_X = \mathcal{O}_{\mathbb{P}^1}$. Hence $\deg f_*\omega_{X/\mathbb{P}^1} = 2$, and so the dimension of a general fibre is less than 3.
2. If X is a Calabi-Yau threefold then
$$f_*\Omega^1_{X/\mathbb{P}^1}(\log\Delta) = \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1).$$
There is so called Arakelov-Yau inequality (cf. [4],[27]).
$$\deg f_*\Omega^1_{X/\mathbb{P}^1}(\log\Delta) \leq \frac{\mathrm{rk} f_*\Omega^1_{X/\mathbb{P}^1}(\log\Delta)}{2}\deg(\Omega^1_{\mathbb{P}^1}(\log S)) = 2g(\mathbb{P}^1) - 2 + \#S.$$
Thus $\#S \geq 4$, and if $\#S = 4$ then there is an étale covering $\pi : Y' \to \mathbb{P}^1$ such that $f' : X' = X \times_{\mathbb{P}^1} Y' \to Y'$ is isogenous over Y' to a product $E \times_{Y'} \cdots \times_{Y'} E$, where $h : E \to Y'$ is a family of semistable elliptic curves reaching the Arakelov bound.
3. If X is a $K3$ surface then
$$f_*\Omega^1_{X/\mathbb{P}^1}(\log\Delta) = f_*\omega_{X/\mathbb{P}^1} = \mathcal{O}_{\mathbb{P}^1}(2).$$
We deduce $\#S \geq 6$ from the Arakelov-Yau inequality for weight one VHS:
$$\deg f_*\Omega^1_{X/\mathbb{P}^1}(\log\Delta) \leq \frac{\mathrm{rk} f_*\Omega^1_{X/\mathbb{P}^1}(\log\Delta)}{2}\deg(\Omega^1_{\mathbb{P}^1}(\log S)) = \frac{\#S}{2} - 1.$$
4. The modularity is due to recent results by Viehweg-Zuo (cf. [26]).

□

Corollary 4.2. *Any hyperkähler manifold can not be fibered by Abelian varieties as a semistable family over* $\mathbb{P}^1$.

Consider a rationally connected manifold Z. Let D_∞ be a reduced normal crossing divisor in Z. It is not difficult that one can choose a very freely rational curve (may not be smooth) C intersecting each component of D_∞ transversely, then

$$\pi_1(C \cap (Z - D_\infty)) \twoheadrightarrow \pi_1(Z - D_\infty) \to 0 \text{ is surjective.}$$

Actually, Kollár's results on fundamental groups show the following fact:

Proposition 4.3 (cf. [15])**.** *Let X be a smooth projective variety and $U \subset Z$ be an open dense subset such that $Z \setminus U$ is a normal crossing divisor. Assume that Z is rationally connected. Then, there exists a very free rational curve $C \subset Z$ such that it intersects each irreducible component of $Z \setminus U$ transversally and the map of topological fundamental groups $\pi_1(C \cap U) \twoheadrightarrow \pi_1(U) \to 0$ is surjective, and*

a) If $\dim Z \geq 3$, C is a smooth rational curve in Z.
b) If $\dim Z = 2$, one has $h : \mathbb{P}^1 \to C \subset Z$ such that h is an immersion.

Warning. Here C is *sufficiently general* does not mean it is complete intersection of hyperplanes.

Theorem 4.4. *Let $f : X \to Y$ be a semistable family of Abelian varieties between two non-singular projective varieties with $f : X_0 = f^{-1}(Y_0) \to Y_0$ smooth, a reduced normal crossing divisor $S = Y \setminus Y_0$ and a relative reduced normal crossing divisor $\Delta = f^*(S)$ in X. Assume that*

a) the period map of the VHS $R^1 f_*(\mathbb{Q}_{X_0})$ is injective at one point in Y_0;
b) the canonical bundle ω_X is trivial and $H^0(X, \Omega^1_X) = 0$.

Then, the dimension of a general fibre is bounded above by a constant dependent on Y.

Proof. The proof follows from the next steps.

1. By 2.3, $f_*\Omega^1_{X/Y}(\log \Delta)$ has no flat quotient. Moreover, $f_*\Omega^1_{X/Y}(\log \Delta)$ is ample on any *sufficiently general* curve in Y. We always have:

$$\wedge^n f_*\Omega^1_{X/Y}(\log \Delta) = \det f_*\Omega^1_{X/Y}(\log \Delta),$$

   where n is the dimension of a general fibre and also is the rank of the locally free sheaf $f_*\Omega^1_{X/Y}(\log \Delta)$. On the other hand, the non-zero map

$$\wedge^n f_*\Omega^1_{X/Y}(\log \Delta) \xrightarrow{\neq 0} f_*(\wedge^n \Omega^1_{X/Y}(\log \Delta)) = f_*\omega_{X/Y},$$

   induces that the line bundle $\omega_Y^{-1} = f_*\omega^1_{X/Y}$ is *big* and *nef*. Thus Y is a rationally connected projective manifold and so $\mathcal{U} = 0$ by 3.8.
2. By 4.3, we have a very free morphism $g : \mathbb{P}^1 \to \overline{M}$ which is sufficiently general, i.e., the image curve $C := g(\mathbb{P}^1)$ satisfies:
   a) C intersects S transversely;
   b) $\pi_1(C_0) \twoheadrightarrow \pi_1(Y_0) \to 0$ is surjective where $C_0 = C \cap Y_0$.

3. If $\dim Y \geq 3$, g is an embedding and we have a very free smooth rational curve $C \subset Y$. Since $f_*\Omega^1_{X/Y}(\log \Delta)$ is ample over C,

$$f_*\Omega^1_{X/Y}(\log \Delta)|_C \cong \bigoplus_{i=1}^{n} \mathcal{O}_{\mathbb{P}^1}(d_i) \text{ with } \forall d_i > 0.$$

Denote $l = -\omega_Y \cdot C$. The commutative diagram of morphisms

$$\begin{array}{ccc} \wedge^n f_*\Omega^1_{X/Y}(\log \Delta)|_C & \xrightarrow{\neq 0} & f_*(\wedge^n \Omega^1_{X/Y}(\log \Delta))|_C \\ \downarrow = & & \downarrow = \\ \mathcal{O}_{\mathbb{P}^1}(\sum_{i=1}^n d_i) & \xrightarrow{\neq 0} & f_*\omega_{X/Y}|_C. \end{array}$$

induces that

$$n \leq \sum_{i=1}^{n} d_i \leq \deg_C f_*\omega_{X/Y} = -\deg_C(\omega_Y) = l.$$

4. If $\dim Y = 1$, Y is then $\mathbb{P}^1$ and $l = 2$. If $\dim Y = 2$, then Y is a smooth Del Pezzo surface and g is an immersion by the theorem 4.3.
   a) Let Γ be the graph of the morphism g, i.e.,

$$\Gamma = \{(x, y) \in \mathbb{P}^1 \times X \mid y = g(x)\} \subset \mathbb{P}^1 \times X.$$

   Γ is a smooth curve, actually it is isomorphic to $\mathbb{P}^1$. We have

$$\begin{array}{ccccc} & & \mathbb{P}^1 \times X & & \\ & \swarrow^{\mathrm{pr}_1} & & \searrow^{\mathrm{pr}_2} & \\ \mathbb{P}^1 & & & & X \end{array}$$

   and the projections $\mathrm{pr}_1, \mathrm{pr}_2$ both are proper morphisms. Hence $\mathrm{pr}_1 : \Gamma \to g(\mathbb{P}^1)$ is a finite morphism. Denote $C = g(\mathbb{P}^1)$. We then have a finite set $B \subset C$ such that
   i. $g^{-1}(B)$ is a finite set in $\mathbb{P}^1$;
   ii. $g : \mathbb{P}^1 \setminus g^{-1}(B) \longrightarrow C \setminus B$ is an étale covering.
   b) Because the real-codimension of B in Y is four, the natural map of topological fundamental groups $\pi_1(Y_0 \setminus B) \xrightarrow{\cong} \pi_1(Y_0)$ is isomorphic. Altogether, we have as a surjective homomorphism $\pi_1(C_0 \setminus B) \twoheadrightarrow \pi_1(Y_0) \to 0$ since $C_0 \setminus B$ is smooth quasi-projective and $\pi_1(C_0 \setminus B) \twoheadrightarrow \pi_1(C_0)$ is surjective.
   c) Denote $T_0 = \mathbb{P}^1 - g^{-1}(B \cup (C - C_0))$ and $\phi = g|_{T_0}$. We have an étale covering $\phi : T_0 \to C_0 \setminus B$, then there is an injective

$$0 \longrightarrow \mathcal{O}_{C_0 \setminus B} \longrightarrow \phi_*\mathcal{O}_{T_0}.$$

   We moreover assume that ϕ is a Galois covering, then the above short sequence has a split and so

$$\phi_*\mathcal{O}_{T_0} = \mathcal{O}_{C_0 \setminus B} \oplus \text{ Galois conjugates.}$$

d) Let $\mathcal{F} = f_*\Omega^1_{X/Y}(\log\Delta)$. The locally free sheaf $\mathcal{L} := g^*(\mathcal{F}|_C)$ on $\mathbb{P}^1$ splits into

$$\mathcal{L} = \bigoplus_{i=1}^{n} \mathcal{O}_{\mathbb{P}^1}(d_i) \text{ with } d_i \geq 0\ \forall i.$$

Suppose that one $d_i = 0$, then $\mathcal{L}$ has a direct summand $\mathcal{O}_{\mathbb{P}^1}$ and $\phi_*(\mathcal{L}|_{T_0})$ has a nonzero flat quotient $\phi_*(\mathcal{L}|_{T_0}) \twoheadrightarrow \mathcal{O}_{C_0\setminus B} \to 0$. On the other hand,

$$0 \longrightarrow \mathcal{F}|_{C_0\setminus B} \longrightarrow \phi_*\phi^*(\mathcal{F}|_{C_0\setminus B}) = \phi_*(\mathcal{L}|_{T_0}),$$

thus $\mathcal{F}|_{C_0\setminus B}$ has a nonzero flat quotient by the projection formula

$$\phi_*\phi^*(\mathcal{F}|_{C_0\setminus B}) = \mathcal{F}|_{C_0\setminus B} \otimes \phi_*(\mathcal{O}_{T_0}).$$

The subjectivity of $\pi_1(C_0 \setminus B) \twoheadrightarrow \pi_1(M) \to 0$ implies that $\mathcal{F}|_M$ has a flat quotient. Moreover, $f_*\Omega^1_{X/Y}(\log\Delta)$ itself has a unitary quotient, it is a contradiction. Hence, all integers d_i are positive and

$$n \leq -\deg_{\mathbb{P}^1} g^*\omega_Y = -g_*[\mathbb{P}^1]\cdot\omega_Y = l.$$

□

Remark. Since g is a very free morphism, $l \geq \dim Y + 1$. However, it seems that we can not find a very free rational curve with $l = \dim Y + 1$ in the theorem.

Calabi-Yau manifolds fibred by hyperkähler varieties. Similarly, we have:

Theorem 4.5. *Let $f : X \to \mathbb{P}^1$ be a semistable family fibred by hyperkähler varieties with $f : X_0 = f^{-1}(C_0) \to C_0$ smooth, $S = \mathbb{P}^1 \setminus C_0$, and $\Delta = f^{-1}(S)$. Assume X is a projective manifold with trivial canonical line bundle and $H^2(X, \mathcal{O}_X) = 0$ (e.g. X is Calabi-Yau). Then, f is nonisotrivial with $\#S \geq 3$ and the dimension of a general fibre is four.*

Proof. Consider the Higgs bundle (E, θ) corresponding to $R^2 f_*\mathbb{Q}_{X_0}$. The semi-stability of f shows that there is the Deligne canonical extension:

$$\overline{E} = f_*\Omega^2_{X/\mathbb{P}^1}(\log\Delta) \bigoplus R^1 f_*\Omega^1_{X/\mathbb{P}^1}(\log\Delta) \bigoplus R^2 f_*(\mathcal{O}_X),$$

such that $f_*\Omega^2_{X/\mathbb{P}^1}(\log\Delta), R^2 f_*(\mathcal{O}_X)$ are line bundles. By 3.10, $f_*\Omega^2_{X/\mathbb{P}^1}(\log\Delta) = \mathcal{O}_{\mathbb{P}^1}(d)$ and d is a positive integer.

Denote F to be be a general fibre and let $\dim F = 2n$. The commutative diagram

$$\begin{array}{ccc} \mathrm{Sym}^n f_*\Omega^2_{X/\mathbb{P}^1}(\log\Delta) & \xrightarrow{\neq 0} & f_*(\wedge^n \Omega^1_{X/\mathbb{P}^1}(\log\Delta)) \\ \big\downarrow = & & \big\downarrow = \\ \mathcal{O}_{\mathbb{P}^1}(nd) & \xrightarrow{\neq 0} & f_*\omega_{X/\mathbb{P}^1}, \end{array}$$

induces $n \leq \deg f_*\omega_{X/\mathbb{P}^1} = 2$. Thus $\dim F = 4$ by the definition, and so

$$f_*\Omega^2_{X/\mathbb{P}^1}(\log\Delta) = \mathcal{O}_{\mathbb{P}^1}(1).$$

$\#S \geq 3$ is a well-known result since f is nonisotrivial. □

Theorem 4.6. *Let $f : X \to Y$ be a semistable family of hyperkähler varieties between two non-singular projective varieties with $f : X_0 = f^{-1}(Y_0) \to Y_0$ smooth, $S = Y \setminus Y_0$ a reduced normal crossing divisor and $\Delta = f^*(S)$ a relative reduced normal crossing divisor in X. Assume that the period map for the VHS $R^2 f_*(\mathbb{Q}_{X_0})$ is injective at one point in Y_0 and X has a trivial canonical line bundle with $H^0(X, \Omega^2_X) = 0$. Then, the dimension of a general fibre is bounded above by a constant depending on Y.*

Remark. If X is a projective irreducible sympletic manifold of dimension $2n$, then by the result of [16] we have $\dim Y = n$. A very recent result of Todorov-Yau shows that if X is hyperkähler then a general fibre is Lagrangian, and so is an Abelian variety (cf. [22]).

We just obtain some necessary conditions for a question asked by N.-C. Leung: *Does there exist a Calabi-Yau manifold fibred by Abelian varieties (resp. hyperkähler varieties)?* Furthermore, we guess that the dimension of a general fibre should be bounded above by a constant depending only on $\dim Y$.

5. Surjective Morphisms from A Calabi-Yau Manifold to A Curve.

Let $f : X \to C$ be a surjective morphism from a projective manifold to a smooth algebraic curve. By the Stein factorization, we have:

$$\begin{array}{ccc} X & \xrightarrow{\;g\;} & B \\ & \searrow^{f} & \downarrow{\scriptstyle\tau} \\ & & C \end{array}$$

where τ is a finite morphism and g has connected fibres. As one can choose a Galois covering C' of C such that $B \times_C C'$ is the disjoint union of copies of C', the Hurwitz formula says that $\tau_*\omega^{\nu}_{B/C}$ is *nef* for all $\nu \geq 0$. On the other hand, it is shown in [24] that $g_*\omega_{X/B}$ is also a locally free sheaf and the Fujita theorem says that $f_*\omega_{X/C}$ and $g_*\omega_{X/B}$ both are *nef*.

Suppose that X has trivial canonical line bundle. The semi-positivity of $g_*\omega_{X/B}$ is equivalent to $g(B) \leq 1$. Thus, we reduce the problem to study the surjective morphism $f : X \to C$ with only connected fibres.

Let n be the dimension of a general fibre and $\Delta = f^*S$. The infinitesimal Torelli holds for the VHS $R^n f_*(\mathbb{Q}_{X_0})$ where $X_0 = f^{-1}(C_0)$. If $f : X \to C$ is weakly semi-stable, the Deligne quasi-canonical extension of $f_*\omega_{X_0/C_0}$ is $f_*\Omega^n_{X/C}(\log \Delta)$ and the sheaf of the relative n-forms $\Omega^n_{X/C}(\log \Delta)$ might be strictly smaller than the relative dualizing sheaf $\omega_{X/C}$. In fact, comparing the first Chern classes of the entries in the tautological sequence

$$0 \longrightarrow f^*\Omega^1_C(\log S) \longrightarrow \Omega^1_X(\log \Delta) \longrightarrow \Omega^1_{X/C}(\log \Delta) \longrightarrow 0,$$

one finds

$$\Omega^n_{X/C}(\log \Delta) = \omega_{X/C}(\Delta_{\mathrm{red}} - \Delta).$$

The effective divisor $D = \Delta - \Delta_{\rm red}$ is zero if and only if the family f is semistable. Denote $\mathcal{L} = f_*\Omega^n_{X/C}(\log \Delta)$. Since $\mathcal{O}_X(\Delta) = f^*(\mathcal{O}_C(S))$, as a sheave morphism we have:

$$\mathcal{L} = f_*(\mathcal{O}_X(\Delta_{red})) \otimes (\omega_C(S))^{-1} \xrightarrow{\subset} f_*\omega_{X/C} = \omega_C^{-1}.$$

If f is not semistable, we still have $f_*\omega_{X_0/C_0} = \mathcal{L}|_{C_0} = \omega_C^{-1}|_{C_0}$. Since $f_*\omega_{X_0/C_0} = \mathcal{A}|_{C_0} \oplus \mathcal{U}|_{C_0}$ such that $\mathcal{U}|_{C_0}$ is unitary and par.deg$(\mathcal{A}|_{C_0}) > 0$, and also f is isotrivial if and only if $f_*\omega_{X_0/C_0} = \mathcal{U}$. Altogether, we have:

Proposition 5.1. *Let $f : X \to C$ be a surjective morphism with connected fibres from a projective manifold X to a smooth algebraic curve C. If f is nonisotrivial and X has a trivial canonical line bundle, then $C = \mathbb{P}^1$.*

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