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SOME RESULTS OF THE MARIÑO-VAFA FORMULA

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Abstract. In this paper we derive some new Hodge integral identities by taking the limits of Mariño-Vafa formula. These identities include the formula of $\lambda_1\lambda_g$-integral on $\overline{\mathcal{M}}_{g,1}$, the vanishing result of $\lambda_g \mathrm{ch}_{2l}(\mathbb{E})$-integral on $\overline{\mathcal{M}}_{g,1}$ for $1 \le l \le g-3$. Using the differential equation of Hodge integrals, we give a recursion formula of λ_{g-1}-integrals. Finally, we give two simple proofs of λ_g conjecture and some examples of low genus integral.

1. Introduction

Based on string duality, Mariño and Vafa [10] conjectured a closed formula on certain Hodge integrals in terms of representations of symmetric groups. Recently, C.C. Liu, K. Liu and J. Zhou [6] proved this formula and derived some consequences from it [7]. In this paper we follow their method to derive some new Hodge integral identities. One of the main results of this paper is the following identity: if $1 \le m \le 2g-3$, then

$$\begin{aligned}(1)\qquad & -(2g-2-m)!(-1)^{2g-3-m}\int_{\overline{\mathcal{M}}_{g,1}} \lambda_g \mathrm{ch}_{2g-2-m}(\mathbb{E})\psi_1^m \\ = \;& b_g \sum_{k=0}^{m-1} \frac{(-1)^{2g-1-k}\binom{2g-1}{k}\binom{2g-1-k}{2g-1-m}}{2g-1-k} B_{2g-1-m} \\ + \;& \frac{1}{2}\sum_{g_1+g_2=g, g_1,g_2>0} b_{g_1}b_{g_2} \sum_{k=0}^{\min(2g_2-1,m-1)} \frac{(-1)^{2g_2-1-k}\binom{2g_2-1}{k}\binom{2g-1-k}{2g-1-m}}{2g-1-k} B_{2g-1-m}.\end{aligned}$$

As a consequence, we find a new Hodge integral identity: if $g \ge 2$, then

$$(2)\qquad \int_{\overline{\mathcal{M}}_{g,1}} \lambda_1\lambda_g\psi_1^{2g-3} = \frac{1}{12}\left[g(2g-3)b_g + b_1 b_{g-1}\right],$$

and also a vanishing result: if $g \ge 2$, then for any $1 \le t \le g-1$, we have

$$(3)\qquad \int_{\overline{\mathcal{M}}_{g,1}} \lambda_g \mathrm{ch}_{2t}(\mathbb{E})\psi_1^{2(g-1-t)} = 0.$$

Recently, Liu-Xu [8] derived a generalized formula for Hodge integrals of type (2) by using the λ_g conjecture.

The rest of this paper is organized as follows: In Section 2, we recall the Mariño-Vafa formula and the Mumford's relations. In Section 3, we prove our main theorem and derive a new Hodge integral identity. In Section 4, we give another simple proof

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of λ_g conjecture. In Section 5, we derive a recursion formula of λ_{g-1}-integrals. In the last section, we list some low genus examples.

2. Preliminaries

2.1. Partitions. A partition μ of a positive integer d is a sequence of integers $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{l(\mu)} > 0$ such that

$$\mu_1 + \cdots + \mu_{l(\mu)} = d = |\mu|,$$

for each positive integer i, let

$$m_i(\mu) = |\{j|\mu_j = i, 1 \leq j \leq l(\mu)\}|.$$

The automorphism group $\mathrm{Aut}(\mu)$ of μ consists of possible permutations among the μ_i's, hence its order is given by

$$|\mathrm{Aut}(\mu)| = \prod_i m_i(\mu)!,$$

define the numbers

$$\kappa_\mu = \sum_{i=1}^{l(\mu)} \mu_i(\mu_i - 2i + 1), \qquad z_\mu = \prod_j m_j(\mu)! j^{m_j(\mu)}.$$

The Young diagram of μ has $l(\mu)$ rows of adjacent squares: the i-th row has μ_i squares. The diagram of μ can be defines as the set of points $(i,j) \in \mathbb{Z} \times \mathbb{Z}$ such that $1 \leq j \leq \mu_i$, the conjugate of a partition μ is the partition μ' whose diagram is the transpose of the diagram μ. Finally, we introduce the hook length of μ at the squre $x \in (i,j)$:

$$h(x) = \mu_i + \mu'_j - i - j + 1.$$

Each partition μ of d corresponds to a conjugacy class $C(\mu)$ of the symmetric group S_d and each partition ν corresponds to an irreducible representation R_ν of S_d, let $\chi_\nu(C(\mu)) = \chi_{R_\nu}(C(\mu))$ be the value of the character χ_{R_ν} on the conjugacy class $C(\mu)$.

2.2. Mariño-Vafa formula. Let $\overline{\mathcal{M}}_{g,n}$ denote the Deligne-Mumford moduli stack of stable curves of genus g with n marked points. Let $\pi : \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ be the universal curve, and let ω_π be the relative dualizing sheaf. The Hodge bundle

$$\mathbb{E} = \pi_* \omega_\pi$$

is a rank g vector bundle over $\overline{\mathcal{M}}_{g,n}$ whose fiber over $[(C, x_1, \cdots, x_n)] \in \overline{\mathcal{M}}_{g,n}$ is $H^0(C, \omega_C)$, the complex vector space of holomorphic one forms on C. Let $s_i : \overline{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}_{g,n+1}$ denote the section of π which corresponds to the i-th marked point, and let

$$\mathbb{L}_i = s_i^* \omega_\pi$$

be the line bundle over $\overline{\mathcal{M}}_{g,n}$ whose fiber over $[(C, x_1, \cdots, x_n)] \in \overline{\mathcal{M}}_{g,n}$ is the cotangent line $T^*_{x_i} C$ at the i-th marked point x_i. Consider the Hodge integral

$$\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{j_1} \cdots \psi_n^{j_n} \lambda_1^{k_1} \cdots \lambda_g^{k_g} \tag{4}$$

where $\psi_i = c_1(\mathbb{L}_i)$ is the first Chern class of $\mathbb{L}_i$, and $\lambda_j = c_j(\mathbb{E})$ is the j-th Chern class of $\mathbb{E}$. The dimension of $\overline{\mathcal{M}}_{g,n}$ is $3g-3+n$, hence (4) is equal to zero unless $\sum_{i=1}^n j_i + \sum_{i=1}^g ik_i = 3g-3+n$. Let

$$\Lambda_g^\vee(u) = u^g - \lambda_1 u^{g-1} + \cdots + (-1)^g \lambda_g = \sum_{i=0}^g (-1)^i \lambda_i u^{g-i} \tag{5}$$

be the Chern polynomial of the dual bundle $\mathbb{E}^\vee$ of $\mathbb{E}$. For any partition $\mu : \mu_1 \geq \mu_2 \geq \cdots \geq \mu_{l(u)} > 0$, define

$$\begin{aligned} \mathcal{C}_{g,\mu}(\tau) &= -\frac{\sqrt{-1}^{|\mu|+l(\mu)}}{|\mathrm{Aut}(\mu)|}[\tau(\tau+1)]^{l(\mu)-1} \prod_{i=1}^{l(\mu)} \frac{\prod_{a=1}^{\mu_i-1}(\mu_i\tau+a)}{(\mu_i-1)!} \\ &\quad \cdot \int_{\overline{\mathcal{M}}_{g,n}} \frac{\Lambda_g^\vee(1)\Lambda_g^\vee(-\tau-1)\Lambda_g^\vee(\tau)}{\prod_{i=1}^{l(\mu)}(1-\mu_i\psi_i)}, \end{aligned} \tag{6}$$

$$\mathcal{C}_\mu(\lambda;\tau) = \sum_{g\geq 0} \lambda^{2g-2+l(\mu)} \mathcal{C}_{g,\mu}(\tau), \tag{7}$$

here τ is a formal variable. Note that

$$\int_{\overline{\mathcal{M}}_{g,n}} \frac{\Lambda_g^\vee(1)\Lambda_g^\vee(-\tau-1)\Lambda_g^\vee(\tau)}{\prod_{i=1}^{l(\mu)}(1-\mu_i\psi_i)} = |\mu|^{l(\mu)-3}$$

for $l(\mu) \geq 3$, and we use this expression to extend the definition to the case $l(\mu) < 3$.

Introduce formal variables $p = (p_1, p_2, \cdots, p_n, \cdots)$, and define

$$p_\mu = p_{\mu_1} \cdots p_{\mu_{l(\mu)}}$$

for a partition μ. Define generating functions

$$\mathcal{C}(\lambda;\tau,p) = \sum_{|\mu|\geq 1} \mathcal{C}_\mu(\lambda;\tau) p_\mu, \tag{8}$$

$$\mathcal{C}(\lambda;\tau,p)^\bullet = e^{\mathcal{C}(\lambda;\tau,p)}. \tag{9}$$

In [6], Chiu-chu Melissa Liu, Kefeng Liu and Jian Zhou have proved the following formula which was conjectured by Mariño and Vafa in [10].

Theorem 2.1. (Mariño-Vafa Formula) *For every partition μ, we have*

$$\begin{aligned} \mathcal{C}(\lambda;\tau,p) &= \sum_{n\geq 1} \frac{(-1)^{n-1}}{n} \sum_\mu \left(\sum_{\cup_{i=1}^n \mu^i = \mu} \prod_{i=1}^n \sum_{|\nu^i|=|\mu^i|} \frac{\chi_{\nu^i}(C(\mu^i))}{z_{\mu^i}} e^{\sqrt{-1}(\tau+\frac{1}{2})\kappa_{\nu^i}\lambda/2} V_{\nu^i}(\lambda) \right) p_\mu, \\ \mathcal{C}(\lambda;\tau,p)^\bullet &= \sum_{|\mu|\geq 0} \left(\sum_{|\nu|=|\mu|} \frac{\chi_\nu(C(\mu))}{z_\mu} e^{\sqrt{-1}(\tau+\frac{1}{2})\kappa_\nu\lambda/2} V_\nu(\lambda) \right) p_\mu, \end{aligned}$$

where

$$V_\nu(\lambda) = \prod_{1\leq a<b\leq l(\nu)} \frac{\sin[(\nu_a-\nu_b+b-a)\lambda/2]}{\sin[(b-a)\lambda/2]} \frac{1}{\prod_{i=1}^{l(\nu)}\prod_{v=1}^{\nu_i} 2\sin[(v-i+l(\nu))\lambda/2]}.$$

It is known that

$$V_\nu(\lambda) = \frac{1}{2^{l(\nu)}\prod_{x\in\nu}\sin[h(x)\lambda/2]}.$$

2.3. Mumford's relations. Let $c_t(\mathbb{E}) = \sum_{i=0}^{g} t^i \lambda_i$, then we have

$$c_{-t}(\mathbb{E}) = t^g \Lambda_g^\vee\left(\frac{1}{t}\right).$$

Mumford's relations [11] are given by

$$c_t(\mathbb{E})c_{-t}(\mathbb{E}) = 1, \tag{10}$$

equivalently

$$\Lambda_g^\vee(t)\Lambda_g^\vee(-t) = (-1)^g t^{2g}, \tag{11}$$

then

$$\lambda_k^2 = \sum_{i=1}^{k}(-1)^{i+1} 2\lambda_{k-i}\lambda_{k+i}, \tag{12}$$

where $\lambda_0 = 1$ and $\lambda_k = 0$ for $k > g$. Let $x_1, \cdots, x_g$ be the formal Chern roots of $\mathbb{E}$, the Chern character is defined by

$$\mathrm{ch}(\mathbb{E}) = \sum_{i=1}^{g} e^{x_i} = g + \sum_{n=1}^{+\infty}\sum_{i=1}^{g}\frac{x_i^n}{n!},$$

we write

$$\mathrm{ch}_0(\mathbb{E}) = g, \tag{13}$$

$$\mathrm{ch}_n(\mathbb{E}) = \frac{1}{n!}\sum_{i=1}^{g} x_i^n, \qquad n = 1, 2, \cdots. \tag{14}$$

From the above identities we have the relation between $\mathrm{ch}_n(\mathbb{E})$ and λ_n:

$$n!\mathrm{ch}_n(\mathbb{E}) = \sum_{i+j=n}(-1)^{i-1} i\lambda_i\lambda_j, \qquad n < 2g, \tag{15}$$

$$\mathrm{ch}_n(\mathbb{E}) = 0, \qquad n \geq 2g. \tag{16}$$

It is easy to see that

$$\begin{aligned}
(2g-1)!\mathrm{ch}_{2g-1}(\mathbb{E}) &= (-1)^{g-1}\lambda_{g-1}\lambda_g, \\
(2g-2)!\mathrm{ch}_{2g-2}(\mathbb{E}) &= (-1)^{g-1}\left((2g-2)\lambda_{g-2}\lambda_g - (g-1)\lambda_{g-1}^2\right), \\
(2g-3)!\mathrm{ch}_{2g-3}(\mathbb{E}) &= (-1)^{g-1}\left(3\lambda_{g-3}\lambda_g - \lambda_{g-1}\lambda_{g-2}\right).
\end{aligned}$$

2.4. Bernoulli numbers. The Bernoulli numbers B_m are defined by the following series expansion:

$$\frac{t}{e^t - 1} = \sum_{m=0}^{+\infty} B_m \frac{t^m}{m!}, \tag{17}$$

the first few terms are given by

$$B_0 = 1, \quad B_1 = -\frac{1}{2}, \quad B_2 = \frac{1}{6}, \quad B_3 = 0, \quad B_4 = -\frac{1}{30}, \quad B_5 = 0, \quad B_6 = \frac{1}{42}.$$

Finally we recall two formulas which will be used later:

$$\frac{t/2}{\sin(t/2)} = 1+\sum_{g=1}^{+\infty}\frac{2^{2g-1}-1}{2^{2g-1}}\frac{|B_{2g}|}{(2g)!}t^{2g}, \tag{18}$$

$$\sum_{i=1}^{d-1} i^m = \sum_{k=0}^{m}\frac{\binom{m+1}{k}}{m+1}B_k d^{m+1-k}, \tag{19}$$

where m is a positive integer.

3. Some New Results from Mariño-Vafa Formula

In this section we derive some new results from the Mariño-Vafa formula, we will need two formulas in [7, 2.1 and 5.1].

Theorem 3.1. *We have the following results:*

$$\sum_{g\geq 0}\lambda^{2g}\int_{\overline{\mathcal{M}}_{g,1}}\frac{\frac{d}{d\tau}|_{\tau=0}\left[\Lambda_g^\vee(1)\Lambda_g^\vee(\tau)\Lambda_g^\vee(-\tau-1)\right]}{1-d\psi_1}$$
$$= -\sum_{a=1}^{d-1}\frac{1}{a}\frac{d\lambda/2}{d\sin(d\lambda/2)}+\sum_{i+j=d,i,j\neq 0}\frac{\lambda^2}{8\sin(i\lambda/2)\sin(j\lambda/2)}, \tag{20}$$

$$\frac{d}{d\tau}\Big|_{\tau=0}\left[\Lambda_g^\vee(1)\Lambda_g^\vee(\tau)\Lambda_g^\vee(-\tau-1)\right] = -\lambda_{g-1}-\lambda_g\sum_{k\geq 0}k!(-1)^{k-1}\mathrm{ch}_k(\mathbb{E}). \tag{21}$$

3.1. The coefficient of λ^{2g}. Introduce the notation

$$b_g = \begin{cases} 1, & g=0,\\ \frac{2^{2g-1}-1}{2^{2g-1}}\frac{|B_{2g}|}{(2g)!}, & g>0,\end{cases}$$

then the coefficient of λ^{2g} in $-\sum_{a=1}^{d-1}\frac{1}{a}\frac{d\lambda/2}{d\sin(d\lambda/2)}$ is

$$\left(-\sum_{a=1}^{d-1}\frac{1}{a}\right)\cdot b_g d^{2g-1}. \tag{22}$$

If $g_1, g_2 \geq 0$ and $g_1+g_2=g$, define

$$F_{g_1,g_2}(d) = \sum_{i+j=d,i,j\neq 0} i^{2g_1-1}j^{2g_2-1}. \tag{23}$$

In [6] it is showed that if $g_1, g_2 \geq 1$, then

$$F_{g_1,g_2}(d) = \sum_{k=0}^{2g_2-1}\sum_{l=0}^{2g-2-k}\frac{(-1)^{2g_2-1-k}}{2g-1-k}\binom{2g_2-1}{k}\binom{2g-1-k}{l}B_l d^{2g-1-l}, \tag{24}$$

for the rest case we have

$$
\begin{aligned}
(25)\quad F_{0,g}(d) &= \sum_{i+j=d,i,j\neq 0} i^{-1}j^{2g-1} \\
&= \sum_{i=1}^{d-1} i^{-1}(d-i)^{2g-1} \\
&= \sum_{k=0}^{2g-1}(-1)^{2g-1-k}\binom{2g-1}{k}d^k\sum_{i=1}^{d-1} i^{2g-2-k} \\
&= \sum_{k=0}^{2g-3}(-1)^{2g-1-k}\binom{2g-1}{k}d^k\sum_{l=0}^{2g-k-2}\frac{\binom{2g-1-k}{l}}{2g-1-k}B_l d^{2g-1-k-l} \\
&- (2g-1)d^{2g-2}(d-1)+d^{2g-1}\sum_{i=1}^{d-1}\frac{1}{i} \\
&= \sum_{k=0}^{2g-2}\sum_{l=0}^{2g-k-2}\binom{2g-1}{k}\binom{2g-1-k}{l}\frac{(-1)^{2g-1-k}}{2g-1-k}B_l d^{2g-1-l} \\
&+ (2g-1)d^{2g-2}+d^{2g-1}\sum_{i=1}^{d-1}\frac{1}{i}.
\end{aligned}
$$

Note that $F_{0,g}(d)=F_{g,0}(d)$ and

$$
(26)\qquad \sum_{i+j=d,i,j\neq 0}\frac{\lambda^2}{8\sin(i\lambda/2)\sin(j\lambda/2)} = \frac{1}{2}\sum_{g\geq 0}\lambda^{2g}\left(\sum_{g_1+g_2=g} b_{g_1}b_{g_2}F_{g_1,g_2}(d)\right).
$$

3.2. The Main Theorem. Let

$$
\sum_{g\geq 0}\lambda^{2g}LHS = \sum_{g\geq 0}\lambda^{2g}\int_{\overline{\mathcal{M}}_{g,1}}\frac{\frac{d}{d\tau}|_{\tau=0}\left[\Lambda_g^\vee(1)\Lambda_g^\vee(\tau)\Lambda_g^\vee(-\tau-1)\right]}{1-d\psi_1},
$$

and

$$
\sum_{g\geq 0}\lambda^{2g}RHS = -\sum_{a=1}^{d-1}\frac{1}{a}\frac{d\lambda/2}{d\sin(d\lambda/2)} + \sum_{i+j=d,i,j\neq 0}\frac{\lambda^2}{8\sin(i\lambda/2)\sin(j\lambda/2)},
$$

then we have

$$
\begin{aligned}
(27)\quad LHS &= -\int_{\overline{\mathcal{M}}_{g,1}}(\lambda_{g-1}\psi_1^{2g-1})d^{2g-1} \\
&- \sum_{k=0}^{2g-2}\left[(2g-2-k)!(-1)^{2g-3-k}\int_{\overline{\mathcal{M}}_{g,1}}\lambda_g \mathrm{ch}_{2g-2-k}(\mathbb{E})\psi_1^k\right]d^k, \\
(28)\quad RHS &= -\sum_{a=1}^{d-1}\frac{b_g}{a}d^{2g-1}+b_gF_{0,g}(d)+\frac{1}{2}\sum_{g_1+g_2=g,g_1,g_2>0}b_{g_1}b_{g_2}F_{g_1,g_2}(d).
\end{aligned}
$$

Hence we can derive our main theorem:

Theorem 3.2. *If $1 \leq m \leq 2g-3$ and $g \geq 2$, then*

$$\begin{aligned} & -(2g-2-m)!(-1)^{2g-3-m} \int_{\overline{\mathcal{M}}_{g,1}} \lambda_g \mathrm{ch}_{2g-2-m}(\mathbb{E}) \psi_1^m \\ = & \ b_g \sum_{k=0}^{m-1} \frac{(-1)^{2g-1-k}}{2g-1-k} \binom{2g-1}{k} \binom{2g-1-k}{2g-1-m} B_{2g-1-m} \\ + & \ \frac{1}{2} \sum_{g_1+g_2=g, g_1, g_2>0} b_{g_1} b_{g_2} \sum_{k=0}^{\min(2g_2-1, m-1)} \frac{(-1)^{2g_2-1-k}}{2g-1-k} \binom{2g_2-1}{k} \binom{2g-1-k}{2g-1-m} B_{2g-1-m}. \end{aligned}$$

Remark 3.3. Liu-Liu-Zhou [7] have only considered the cases $m = 2g-1$ and $m = 1$.

3.3. The case of $m = 2g-3$. If $m = 2g-3$, we find that $1!\mathrm{ch}_1(\mathbb{E}) = \lambda_1$, then

$$\begin{aligned} LHS &= -\int_{\overline{\mathcal{M}}_{g,1}} \lambda_g \lambda_1 \psi_1^{2g-3}, \\ RHS &= b_g \sum_{k=0}^{2g-4} \frac{(-1)^{2g-1-k}}{2g-1-k} \binom{2g-1}{k} \binom{2g-1-k}{2} B_2 \\ &+ \frac{1}{2} \sum_{g_1+g_2=g, g_1, g_2>0} b_{g_1} b_{g_2} \sum_{k=0}^{\min(2g_2-1, 2g-4)} \frac{(-1)^{2g_2-1-k}}{2g-1-k} \binom{2g_2-1}{k} \binom{2g-1-k}{2} B_2. \end{aligned}$$

From the above formula we obtain a new result of the Hodge integral.

Theorem 3.4. *If $g \geq 2$, then*

$$\int_{\overline{\mathcal{M}}_{g,1}} \lambda_1 \lambda_g \psi_1^{2g-3} = \frac{1}{12} \left[g(2g-3) b_g + b_1 b_{g-1} \right]. \tag{29}$$

Proof. Note that

$$\begin{aligned} & \int_{\overline{\mathcal{M}}_{g,1}} \lambda_1 \lambda_g \psi_1^{2g-3} = -b_g B_2 \sum_{k=0}^{2g-4} \frac{(-1)^{2g-1-k}}{2g-1-k} \binom{2g-1}{k} \binom{2g-1-k}{2} \\ - & \ \frac{B_2}{2} \sum_{g_1+g_2=g, g_1, g_2>0} b_{g_1} b_{g_2} \sum_{k=0}^{\min(2g_2-1, 2g-4)} \frac{(-1)^{2g_2-1-k}}{2g-1-k} \binom{2g_2-1}{k} \binom{2g-1-k}{2}, \end{aligned}$$

let us write

$$\begin{aligned} A_1 &= \sum_{k=0}^{2g-4} \frac{(-1)^{2g-1-k}}{2g-1-k} \binom{2g-1}{k} \binom{2g-1-k}{2}, \\ f_1(x) &= \sum_{k=0}^{2g-3} (-1)^{2g-1-k} \binom{2g-1}{k} (2g-2-k) x^{2g-3-k}, \\ g_1(x) &= \sum_{k=0}^{2g-2} (-1)^{2g-1-k} \binom{2g-1}{k} x^{2g-2-k}, \end{aligned}$$

then

$$A_1 = \frac{1}{2}\sum_{k=0}^{2g-4}(-1)^{2g-1-k}\binom{2g-1}{k}(2g-2-k),\ xg_1(x) = (1-x)^{2g-1}-1,\ f_1(x) = g_1'(x).$$

Hence

$$f_1(x) = \frac{(2g-1)x(1-x)^{2g-2}-(1-x)^{2g-1}+1}{x^2}, \quad f_1(1) = 1,$$

and we obtain

$$A_1 = \frac{1}{2}\left[f_1(1)-\binom{2g-1}{2g-3}\right] = -\frac{1}{2}\left[\binom{2g-1}{2g-3}-1\right].$$

Similarly, we write

$$A_2 = \sum_{k=0}^{\min(2g_2-1,2g-4)}\frac{(-1)^{2g_2-1-k}}{2g-1-k}\binom{2g_2-1}{k}\binom{2g-1-k}{2},$$

then

$$A_2 = \begin{cases} \frac{1}{2}\sum_{k=0}^{2g_2-1}(-1)^{2g_2-1-k}(2g-2-k)\binom{2g_2-1}{k}, & g_2 \le g-2, \\ \frac{1}{2}\sum_{k=0}^{2g-4}(-1)^{2g-1-k}(2g-2-k)\binom{2g-3}{k}, & g_2 = g-1. \end{cases}$$

Case 1: $g_2 \ge g-2$**.** Let

$$\begin{aligned} f_2(x) &= \sum_{k=0}^{2g_2-1}(-1)^{2g_2-1-k}(2g-2-k)\binom{2g_2-1}{k}x^{2g-3-k}, \\ g_2(x) &= \sum_{k=0}^{2g_2-1}(-1)^{2g_2-1-k}\binom{2g_2-1}{k}x^{2g-2-k}. \end{aligned}$$

Since $g \ge g_2+2$, then $2g-3-(2g_2-1) \ge 2 > 0$ and $g_2'(x) = f_2(x)$. On the other hand

$$g_2(x) = (1-x)^{2g_2-1}x^{2g_1-1},$$

hence

$$g_2'(x) = -(2g_2-1)(1-x)^{2g_2-2}x^{2g_1-1}+(2g_1-1)(1-x)^{2g_2-1}x^{2g_1-2}$$

and

$$f_2(1) = \begin{cases} -1, & g_2 = 1, \\ 0, & 1 < g_2 \le g-2. \end{cases}$$

Case 2: $g_2 = g-1$**.** let

$$\begin{aligned} f_3(x) &= \sum_{k=0}^{2g-4}(-1)^{2g-1-k}(2g-2-k)\binom{2g-3}{k}x^{2g-3-k}, \\ g_3(x) &= \sum_{k=0}^{2g-4}(-1)^{2g-1-k}\binom{2g-3}{k}x^{2g-2-k}. \end{aligned}$$

Since $2g-3-(2g-4) = 1 > 0$,

$$\frac{g_3(x)}{x} = \sum_{k=0}^{2g-4}(-1)^{2g-3-k}\binom{2g-3}{k}x^{2g-3-k} = (1-x)^{2g-3}-1$$

and

$$g_3'(x) = (1-x)^{2g-3} - 1 - (2g-3)(1-x)^{2g-4}x,$$

therefore we have

$$f_3(1) = \begin{cases} -2, & g=2, \\ -1, & g>2. \end{cases}$$

From the values of $f_1(1), f_2(1), f_3(1)$, we obtain

$$\begin{aligned} & \int_{\overline{\mathcal{M}}_{g,1}} \lambda_1 \lambda_g \psi_1^{2g-3} \\ = \quad & -b_g B_2 A_1 - \frac{B_2}{2}\left[\frac{1}{2}\sum_{g_1+g_2=g, 1\le g_2\le g-2} b_{g_1} b_{g_2} f_2(1) + \frac{1}{2} b_1 b_{g-1} f_3(1)\right] \\ = \quad & \frac{B_2}{2}[-b_g A_1 + b_1 b_{g-1}] \\ = \quad & \frac{1}{12}[g(2g-3)b_g + b_1 b_{g-1}] \end{aligned}$$

□

Since $B_n = 0$ for n odd and $n > 1$, we also have the following vanishing result.

Theorem 3.5. *If $g \ge 2$, then for any $1 \le t \le g-1$, we have*

$$\int_{\overline{\mathcal{M}}_{g,1}} \lambda_g \mathrm{ch}_{2t}(\mathbb{E}) \psi_1^{2(g-1-t)} = 0. \tag{30}$$

4. Another Simple Proof of λ_g Conjecture

Let $|\mu| = d, l(\mu) = n$, denote by $[\mathcal{C}_{g,\mu}(\tau)]_k$ the coefficient of τ^k in the polynomial $\mathcal{C}_{g,\mu}(\tau)$, and let

$$\begin{aligned} J^0_{g,\mu}(\tau) &= \sqrt{-1}^{|\mu|-l(\mu)} \mathcal{C}_{g,\mu}(\tau), \\ J^1_{g,\mu}(\tau) &= \sqrt{-1}^{|\mu|-l(\mu)-1} \left(\sum_{\nu \in J(\mu)} I_1(\nu) \mathcal{C}_{g,\nu}(\tau) + \sum_{\nu\in C(\mu)} I_2(\nu) \mathcal{C}_{g-1,\nu}(\tau) \right. \\ & \quad \left. + \sum_{g_1+g_2=g, \nu^1\cup\nu^2 \in C(\mu)} I_3(\nu^1,\nu^2) \mathcal{C}_{g_1,\nu^1}(\tau) \mathcal{C}_{g_2,\nu^2}(\tau) \right). \end{aligned}$$

The set $J(\mu)$ consists of partitions of d of the form

$$\nu = (\mu_1, \cdots, \widehat{\mu}_i, \cdots, \mu_{l(\mu)}, \mu_i + \mu_j)$$

and the set $C(\mu)$ consists of partitions of d of the form

$$\nu = (\mu_1, \cdots, \widehat{\mu}_i, \cdots, \mu_{l(\mu)}, j, k)$$

where $j + k = \mu_i$. The definitions of I_1, I_2 and I_3 can be found in [5]. Liu-Liu-Zhou [6] have proved the following differential equation:

$$\frac{d}{d\tau} J^0_{g,\mu}(\tau) = -J^1_{g,\mu}(\tau). \tag{31}$$

It is straightforward to check that

$$\begin{aligned} [\mathcal{C}_{g,\mu}(\tau)]_{n-1} &= -\frac{\sqrt{-1}^{d+n}}{|\mathrm{Aut}(\mu)|}\int_{\overline{\mathcal{M}}_{g,n}}\frac{\lambda_g}{\prod_{i=1}^{n}(1-\mu_i\psi_i)}, \\ \left[\sum_{\nu\in J(\mu)} I_1(\nu)\mathcal{C}_{g,\nu}(\tau)\right]_{n-2} &= -\frac{\sqrt{-1}^{d+n-1}}{|\mathrm{Aut}(\nu)|}\int_{\overline{\mathcal{M}}_{g,n-1}}\frac{\lambda_g}{\prod_{i=1}^{n-1}(1-\mu_i\psi_i)}, \end{aligned}$$

$$\begin{aligned} \left[\sum_{\nu\in C(\mu)} I_2(\nu)\mathcal{C}_{g-1,\nu}(\tau)\right]_{n-2} &= 0, \\ \left[\sum_{g_1+g_2=g,\nu^1\cup\nu^2\in C(\mu)} I_3(\nu^1,\nu^2)\mathcal{C}_{g_1,\nu^1}(\tau)\mathcal{C}_{g_2,\nu^2}(\tau)\right]_{n-2} &= 0, \end{aligned}$$

hence, from (31) we have the identity

$$\frac{n-1}{|\mathrm{Aut}(\mu)|}\int_{\overline{\mathcal{M}}_{g,n}}\frac{\lambda_g}{\prod_{i=1}^{n}(1-\mu_i\psi_i)} = \sum_{\nu\in J(\mu)}\frac{I_1(\nu)}{|\mathrm{Aut}(\nu)|}\int_{\overline{\mathcal{M}}_{g,n-1}}\frac{\lambda_g}{\prod_{i=1}^{n-1}(1-\nu_i\psi_i)}. \tag{32}$$

Theorem 4.1. *For any partition $\mu : \mu_1 \geq \mu_2 \geq \cdots \geq \mu_n > 0$ of d and $g > 0$, then*

$$\int_{\overline{\mathcal{M}}_{g,n}}\frac{\lambda_g}{\prod_{i=1}^{n}(1-\mu_i\psi_i)} = d^{2g+n-3}b_g. \tag{33}$$

Proof. Recall the definition of $I_1(\nu)$, where $\nu = (\mu_1, \cdots, \widehat{\mu}_i, \cdots, \widehat{\mu}_j, \cdots, \mu_n, \mu_i + \mu_j)$:

$$I_1(\nu) = \frac{\mu_i+\mu_j}{1+\delta^{\mu_i}_{\mu_j}} m_{\mu_i+\mu_j}(\nu),$$

and it is easy to see that

$$\frac{m_{\mu_i+\mu_j}(\nu)}{|\mathrm{Aut}(\nu)|} = \frac{m_{\mu_i}(\mu)(m_{\mu_j}(\mu)-\delta^{\mu_i}_{\mu_j})}{|\mathrm{Aut}(\mu)|}.$$

Let

$$\mu : \underbrace{\mu_{k_1} = \cdots = \mu_{k_1}}_{t_1} > \underbrace{\mu_{k_2} = \cdots = \mu_{k_2}}_{t_2} > \cdots > \underbrace{\mu_{k_s} = \cdots = \mu_{k_s}}_{t_s} > 0,$$

where

$$\sum_{i=1}^{s} t_i = n, \qquad \sum_{i=1}^{s} t_i\mu_{k_i} = d,$$

then

$$\begin{aligned}
&\sum_{\nu\in J(\mu)} \frac{I_1(\nu)}{|\mathrm{Aut}(\nu)|} \\
= &\frac{1}{|\mathrm{Aut}(\mu)|} \sum_{\nu\in J(\mu)} \frac{\mu_i+\mu_j}{1+\delta^{\mu_i}_{\mu_j}}[m_{\mu_i}(\mu)(m_{\mu_j}(\mu)-\delta^{\mu_i}_{\mu_j})] \\
= &\frac{1}{|\mathrm{Aut}(\mu)|}\left[\frac{1}{2}\sum_{i=1}^{s}\sum_{j\neq i}(\mu_{k_i}+\mu_{k_j})m_{\mu_{k_i}}(\mu)m_{\mu_{k_j}}(\mu)+\sum_{i=1}^{s}\mu_{k_i}m_{\mu_{k_i}}(\mu)(m_{\mu_{k_i}}(\mu)-1)\right] \\
= &\frac{1}{|\mathrm{Aut}(\mu)|}\left[\frac{1}{2}\sum_{i=1}^{s}\sum_{j\neq i}(\mu_{k_i}+\mu_{k_j})t_it_j+\sum_{i=1}^{s}\mu_{k_i}t_i(t_i-1)\right] \\
= &\frac{1}{|\mathrm{Aut}(\mu)|}\left[\sum_{i=1}^{s}\sum_{j\neq i}\mu_{k_j}t_jt_i+\sum_{i=1}^{s}\mu_{k_i}t_i^2-d\right] \\
= &\frac{1}{|\mathrm{Aut}(\mu)|}\left[\sum_{i=1}^{s}t_i(d-\mu_{k_i}t_i)+\sum_{i=1}^{s}\mu_{k_i}t_i^2-d\right] \\
= &\frac{(n-1)d}{|\mathrm{Aut}(\mu)|}.
\end{aligned}$$

By the induction of n and the initial value of the Mariño-Vafa formula

$$\int_{\overline{\mathcal{M}}_{g,1}} \frac{\lambda_g}{1-\mu_1\psi_1} = d^{2g-2}b_g,$$

we have

$$\int_{\overline{\mathcal{M}}_{g,n}} \frac{\lambda_g}{\prod_{i=1}^{n}(1-\mu_i\psi_i)} = d\cdot d^{2g+n-1-3}b_g = d^{2g+n-3}b_g.$$

□

Corollary 4.2. *The following λ_g conjecture is true:*

$$\int_{\overline{\mathcal{M}}_{g,n}} \lambda_g \prod_{l=1}^{n}\psi_l^{k_l} = \binom{2g+n-3}{k_1,\cdots,k_n} b_g, \tag{34}$$

where $g>0$.

5. A recursion Formula of the λ_{g-1} Integral

E.Getzler, A.Okounkov and R.Pandharipande have derived explicit formula for the multipoint series of $\mathbb{CP}^1$ in degree 0 from the Toda hierarchy [2], then they obtained certain formulas for the Hodge integrals $\int_{\overline{\mathcal{M}}_{g,n}} \lambda_{g-1}\psi_1^{k_1}\cdots\psi_n^{k_n}$. In this section we give an effective recursion formula of the λ_{g-1} integrals using Mariño-Vafa formula. It is straightforward to check the following lemma.

Lemma 5.1. *We have the following identities*

$$\begin{aligned}
\left[\prod_{i=1}^{n}\frac{\prod_{a=1}^{\mu_i-1}(\mu_i\tau+a)}{(\mu_i-1)!}\right]_0 &= 1,\\
\left[\prod_{i=1}^{n}\frac{\prod_{a=1}^{\mu_i-1}(\mu_i\tau+a)}{(\mu_i-1)!}\right]_1 &= \sum_{i=1}^{n}\sum_{a=1}^{\mu_i-1}\frac{\mu_i}{a},\\
\left[\Lambda_g^{\vee}(1)\Lambda_g^{\vee}(-\tau-1)\Lambda_g^{\vee}(\tau)\right]_0 &= \lambda_g,\\
\left[\Lambda_g^{\vee}(1)\Lambda_g^{\vee}(-\tau-1)\Lambda_g^{\vee}(\tau)\right]_0 &= -\lambda_{g-1}-\sum_{k\geq 0}k!(-1)^{k-1}\mathrm{ch}_k(\mathbb{E})\lambda_g,
\end{aligned}$$

and

$$\begin{aligned}
[\mathcal{C}_{g,\mu}(\tau)]_{n-1} &= -\frac{\sqrt{-1}^{d+n}}{|\mathrm{Aut}(\mu)|}\int_{\overline{\mathcal{M}}_{g,n}}\frac{\lambda_g}{\prod_{i=1}^{n}(1-\mu_i\psi_i)},\\
[\mathcal{C}_{g,\mu}(\tau)]_{n} &= -\frac{\sqrt{-1}^{d+n}}{|\mathrm{Aut}(\mu)|}\left[n-1+\sum_{i=1}^{n}\sum_{a=1}^{\mu_i-1}\frac{\mu_i}{a}\right]\int_{\overline{\mathcal{M}}_{g,n}}\frac{\lambda_g}{\prod_{i=1}^{n}(1-\mu_i\psi_i)}\\
&+ \frac{\sqrt{-1}^{d+n}}{|\mathrm{Aut}(\mu)|}\int_{\overline{\mathcal{M}}_{g,n}}\frac{\lambda_{g-1}+\sum_{k\geq 0}k!(-1)^{k-1}\mathrm{ch}_k(\mathbb{E})\lambda_g}{\prod_{i=1}^{n}(1-\mu_i\psi_i)}.
\end{aligned}$$

Now, we can state our main theorem in this section using equation (31).

Theorem 5.2. *For any partition μ with $l(\mu)=n$, we have the following recursion formula*

$$\begin{aligned}
&\frac{n}{|\mathrm{Aut}(\mu)|}\left[n-1+\sum_{i=1}^{n}\sum_{a=1}^{\mu_i-1}\frac{\mu_i}{a}\right]\int_{\overline{\mathcal{M}}_{g,n}}\frac{\lambda_g}{\prod_{i=1}^{n}(1-\mu_i\psi_i)}\\
-\ &\frac{n}{|\mathrm{Aut}(\mu)|}\int_{\overline{\mathcal{M}}_{g,n}}\frac{\lambda_{g-1}+\sum_{k\geq 0}k!(-1)^{k-1}\mathrm{ch}_k(\mathbb{E})\lambda_g}{\prod_{i=1}^{n}(1-\mu_i\psi_i)}\\
=\ &\sum_{\nu\in J(\mu)}\frac{I_1(\nu)}{|\mathrm{Aut}(\nu)|}\left[n-2+\sum_{i=1}^{n-1}\sum_{a=1}^{\nu_i-1}\frac{\nu_i}{a}\right]\int_{\overline{\mathcal{M}}_{g,n-1}}\frac{\lambda_g}{\prod_{i=1}^{n-1}(1-\nu_i\psi_i)}\\
-\ &\sum_{\nu\in J(\mu)}\frac{I_1(\nu)}{|\mathrm{Aut}(\nu)|}\int_{\overline{\mathcal{M}}_{g,n-1}}\frac{\lambda_{g-1}+\sum_{k\geq 0}k!(-1)^{k-1}\mathrm{ch}_k(\mathbb{E})\lambda_g}{\prod_{i=1}^{n-1}(1-\nu_i\psi_i)}\\
+\ &\sum_{g_1+g_2=g,g_1,g_2\geq 0}\ \sum_{\nu^1\cup\nu^2\in C(\mu)}\frac{I_3(\nu^1,\nu^2)}{|\mathrm{Aut}(\nu^1)||\mathrm{Aut}(\nu^2)|}\\
\cdot\ &\int_{\overline{\mathcal{M}}_{g_1,n_1}}\frac{\lambda_{g_1}}{\prod_{i=1}^{n_1}(1-\nu_i^1\psi_i)}\int_{\overline{\mathcal{M}}_{g_2,n_2}}\frac{\lambda_{g_2}}{\prod_{i=1}^{n_2}(1-\nu_i^2\psi_i)}.
\end{aligned}$$

5.1. The λ_g-Integral. In this subsection, we re-derive the λ_g-integral from theorem 5.2.. Let $\mu_i=Nx_i$ for some $N\in\mathbb{N}$ and $x_i\in\mathbb{R}$, from Kim-Liu[4]'s method and consider the coefficients of $\ln N N^{2g+n-2}$ in theorem 5.2., then

$$\begin{aligned} & n(x_1+\cdots+x_n)\prod_{l=1}^{n} x_l^{k_l}\int_{\overline{\mathcal{M}}_{g,n}}\lambda_g\prod_{l=1}^{n}\psi_l^{k_l} \\ = & \frac{1}{2}\sum_{i=1}^{n}\sum_{j\neq i}(x_i+x_j)^{k_i+k_j}(x_1+\cdots+x_n)\prod_{l\neq i,j} x_l^{k_l}\int_{\overline{\mathcal{M}}_{g,n-1}}\lambda_g\psi^{k_i+k_j-1}\prod_{l\neq i,j}\psi_l^{k_l} \\ + & (x_1+\cdots+x_n)\prod_{l=1}^{n} x_l^{k_l}\int_{\overline{\mathcal{M}}_{g,n}}\lambda_g\prod_{l=1}^{n}\psi_l^{k_l}, \end{aligned}$$

i.e.

$$\begin{aligned} & (n-1)\prod_{l=1}^{n} x_l^{k_l}\int_{\overline{\mathcal{M}}_{g,n}}\lambda_g\prod_{l=1}^{n}\psi_l^{k_l} \\ = & \frac{1}{2}\sum_{i=1}^{n}\sum_{j\neq i}(x_i+x_j)^{k_i+k_j}\prod_{l\neq i,j} x_l^{k_l}\int_{\overline{\mathcal{M}}_{g,n-1}}\lambda_g\psi^{k_i+k_j-1}\prod_{l\neq i,j}\psi_l^{k_l}. \end{aligned} \tag{35}$$

After introducing the formal variables $s_l\in\mathbb{R}^+$ and applying the Laplace transformation

$$\int_0^{+\infty} x^k e^{-x/2s}dx = k!(2s)^{k+1},\quad s>0,$$

we select the coefficient of $\prod_{l=1}^{n}(2s_l)^{k_l+1}$ from the transformation of (35), then we derive

$$(n-1)\int_{\overline{\mathcal{M}}_{g,n}}\lambda_g\prod_{l=1}^{n}\psi_l^{k_l} = \frac{1}{2}\sum_{i=1}^{n}\sum_{j\neq i}\frac{(k_i+k_j)!}{k_i!k_j!}\int_{\overline{\mathcal{M}}_{g,n-1}}\lambda_g\psi^{k_i+k_j-1}\prod_{l\neq i,j}\psi_l^{k_l}. \tag{36}$$

By the induction of n, we obtain the λ_g conjecture

$$\int_{\overline{\mathcal{M}}_{g,n}}\lambda_g\prod_{l=1}^{n}\psi_l^{k_l} = \binom{2g+n-3}{k_1,\cdots,k_n}b_g,$$

in fact, in (36) we have

$$\begin{aligned} RHS &= \frac{1}{2}\sum_{i=1}^{n}\sum_{j\neq i}\frac{(k_i+k_j)!}{k_i!k_j!}\frac{(2g+n-4)!}{\prod_{l\neq i,j}k_l!(k_i+k_j-1)!}b_g \\ &= \frac{1}{2}\sum_{i=1}^{n}\sum_{j\neq i}\frac{k_i+k_j}{2g+n-3}\binom{2g+n-3}{k_1,\cdots,k_n}b_g, \end{aligned}$$

note that $k_1+\cdots+k_n = 2g+n-3$, therefore

$$\begin{aligned} \frac{1}{2}\sum_{i=1}^{n}\sum_{j\neq i}(k_i+k_j) &= \frac{1}{2}\sum_{i=1}^{n}[(n-1)k_i+(2g+n-3-k_i)] \\ &= \frac{1}{2}\left[(n-2)(2g+n-3)+(2g+n-3)n\right] \\ &= \frac{1}{2}[(2n-2)(2g+n-3)] \\ &= (n-1)(2g+n-3). \end{aligned}$$

5.2. The Recursion Formula of λ_{g-1}-integral. We have found the *singular part* $\sum_{i=1}^n \sum_{a=1}^{\mu_i-1} \frac{\mu_i}{a}$ in theorem 5.2., using the following theorem, we can eliminate this part and derive the recursion formula of λ_{g-1}-integral. The notation $[F]_{sing}$ means the singular part of F. First, in theorem 5.2., we have

$$\begin{aligned}
\left[\frac{LHS}{d^{2g+n-4}b_g}\right]_{sing} &= n\sum_{i=1}^{n}\sum_{a=1}^{\mu_i-1}\frac{\mu_i}{a}d, \\
\left[\frac{RHS}{d^{2g+n-4}b_g}\right]_{sing} &= \frac{1}{2}\sum_{i=1}^{n}\sum_{j\neq i}(\mu_i+\mu_j)\left[\sum_{l\neq i,j}\sum_{a=1}^{\mu_l-1}\frac{\mu_l}{a}+(\mu_i+\mu_j)\sum_{a=1}^{\mu_i+\mu_j-1}\frac{1}{a}\right] \\
&+ \sum_{i=1}^{n}\sum_{a=1}^{\mu_i-1}\frac{\mu_i}{a}d-\sum_{i=1}^{n}\sum_{j\neq i}\sum_{a=\mu_j+1}^{\mu_i+\mu_j-1}\frac{\mu_j(\mu_i+\mu_j)}{a}.
\end{aligned}$$

Theorem 5.3. *Under the above notation, we have*

$$\left[\frac{RHS}{d^{2g+n-4}b_g}\right]_{sing} = \left[\frac{LHS}{d^{2g+n-4}b_g}\right]_{sing} + 2(n-1)d.$$

Proof. Since

$$\begin{aligned}
&\sum_{i=1}^{n}\sum_{j\neq i}\sum_{a=\mu_j+1}^{\mu_i+\mu_j-1}\frac{\mu_j(\mu_i+\mu_j)}{a} \\
=& \sum_{i=1}^{n}\sum_{j\neq i}\sum_{a=\mu_j+1}^{\mu_i+\mu_j-1}\frac{(\mu_i+\mu_j)^2}{a}-\sum_{i=1}^{n}\sum_{j\neq i}\sum_{a=\mu_j+1}^{\mu_i+\mu_j-1}\frac{\mu_i(\mu_i+\mu_j)}{a} \\
=& \sum_{i=1}^{n}\sum_{j\neq i}\sum_{a=1}^{\mu_i+\mu_j-1}\frac{(\mu_i+\mu_j)^2}{a}-\sum_{i=1}^{n}\sum_{j\neq i}\sum_{a=1}^{\mu_j}\frac{(\mu_i+\mu_j)^2}{a} \\
-& \sum_{i=1}^{n}\mu_i\sum_{j\neq i}\sum_{a=1}^{\mu_i+\mu_j-1}\frac{\mu_i+\mu_j}{a}+\sum_{i=1}^{n}\mu_i\sum_{j\neq i}\sum_{a=1}^{\mu_j}\frac{(\mu_i+\mu_j)}{a} \\
=& \frac{1}{2}\sum_{i=1}^{n}\sum_{j\neq i}\sum_{a=1}^{\mu_i+\mu_j-1}\frac{(\mu_i+\mu_j)^2}{a}-\sum_{i=1}^{n}\sum_{j\neq i}\sum_{a=1}^{\mu_j-1}\frac{(\mu_i+\mu_j)^2}{a} \\
+& \sum_{i=1}^{n}\mu_i\sum_{j\neq i}\sum_{a=1}^{\mu_j-1}\frac{(\mu_i+\mu_j)}{a}-\sum_{i=1}^{n}\sum_{j\neq i}(\mu_i+\mu_j),
\end{aligned}$$

where we use the identity

$$\begin{aligned}
\sum_{i=1}^{n}\sum_{j\neq i}\sum_{a=1}^{\mu_i+\mu_j-1}\frac{(\mu_i+\mu_j)\mu_i}{a} &= \sum_{i=1}^{n}\sum_{j\neq i}\sum_{a=1}^{\mu_i+\mu_j-1}\frac{(\mu_i+\mu_j)\mu_j}{a} \\
&= \frac{1}{2}\sum_{i=1}^{n}\sum_{j\neq i}\sum_{a=1}^{\mu_i+\mu_j-1}\frac{(\mu_i+\mu_j)^2}{a}.
\end{aligned}$$

Note that $\sum_{i=1}^{n}\sum_{j\neq i}(\mu_i+\mu_j)=2(n-1)d$, hence

$$\begin{aligned}
&\left[\frac{RHS}{d^{2g+n-4}b_g}\right]_{sing}\\
= \;& \frac{1}{2}\sum_{i=1}^{n}\sum_{j\neq i}(\mu_i+\mu_j)\sum_{l\neq i,j}\sum_{a=1}^{\mu_l-1}\frac{\mu_l}{a}+\sum_{i=1}^{n}\sum_{a=1}^{\mu_i-1}\frac{\mu_i}{a}d\\
+ \;& \sum_{i=1}^{n}\sum_{j\neq i}\sum_{a=1}^{\mu_j-1}\frac{(\mu_i+\mu_j)^2}{a}-\sum_{i=1}^{n}\sum_{j\neq i}\mu_i\sum_{a=1}^{\mu_j-1}\frac{(\mu_i+\mu_j)}{a}+\sum_{i=1}^{n}\sum_{j\neq i}(\mu_i+\mu_j)\\
= \;& \left(\sum_{i=1}^{n}\sum_{a=1}^{\mu_i-1}\frac{\mu_i}{a}\right)d+\frac{1}{2}\sum_{i=1}^{n}\sum_{j\neq i}(\mu_i+\mu_j)\sum_{l\neq i,j}\sum_{a=1}^{\mu_l-1}\frac{\mu_l}{a}\\
+ \;& \sum_{i=1}^{n}\sum_{j\neq i}\sum_{a=1}^{\mu_j-1}\frac{\mu_j(\mu_i+\mu_j)}{a}+2(n-1)d,
\end{aligned}$$

it is straightforward to check that

$$\begin{aligned}
\frac{1}{2}\sum_{i=1}^{n}\sum_{j\neq i}(\mu_i+\mu_j)\sum_{l\neq i,j}\sum_{a=1}^{\mu_l-1}\frac{\mu_l}{a} &= (n-2)\sum_{i=1}^{n}\mu_i\sum_{j\neq i}\sum_{a=1}^{\mu_j-1}\frac{\mu_j}{a},\\
\sum_{i=1}^{n}\sum_{j\neq i}\sum_{a=1}^{\mu_j-1}\frac{\mu_j(\mu_i+\mu_j)}{a} &= \sum_{i=1}^{n}\mu_i\sum_{j\neq i}\sum_{a=1}^{\mu_j-1}\frac{\mu_j}{a}+\sum_{i=1}^{n}\sum_{j\neq i}\mu_j\sum_{a=1}^{\mu_j-1}\frac{\mu_j}{a},\\
\sum_{i=1}^{n}\sum_{j\neq i}\mu_j\sum_{a=1}^{\mu_j-1}\frac{\mu_j}{a} &= (n-1)\sum_{i=1}^{n}\mu_i\sum_{a=1}^{\mu_i-1}\frac{\mu_i}{a}.
\end{aligned}$$

Finally, we obtain

$$\begin{aligned}
&\left[\frac{RHS}{d^{2g+n-4}b_g}\right]_{sing}\\
= \;& \left(\sum_{i=1}^{n}\sum_{a=1}^{\mu_i-1}\frac{\mu_i}{a}\right)d+(n-2)\sum_{i=1}^{n}\mu_i\sum_{j\neq i}\sum_{a=1}^{\mu_j-1}\frac{\mu_j}{a}\\
+ \;& \sum_{i=1}^{n}\mu_i\sum_{j\neq i}\sum_{a=1}^{\mu_j-1}\frac{\mu_j}{a}+(n-1)\sum_{i=1}^{n}\mu_i\sum_{a=1}^{\mu_i-1}\frac{\mu_i}{a}+2(n-1)d\\
= \;& \left(\sum_{i=1}^{n}\sum_{a=1}^{\mu_i-1}\frac{\mu_i}{a}\right)d+(n-1)\sum_{i=1}^{n}\mu_i\left(\sum_{j\neq i}\sum_{a=1}^{\mu_i-1}\frac{\mu_j}{a}+\sum_{a=1}^{\mu_i-1}\frac{\mu_i}{a}\right)+2(n-1)d\\
= \;& \left(\sum_{i=1}^{n}\sum_{a=1}^{\mu_i-1}\frac{\mu_i}{a}\right)d+(n-1)\left(\sum_{i=1}^{n}\sum_{a=1}^{\mu_i-1}\frac{\mu_i}{a}\right)d+2(n-1)d\\
= \;& \left[\frac{LHS}{d^{2g+n-4}b_g}\right]_{sing}+2(n-1)d.
\end{aligned}$$

□

Let $\mathbb{R}^k[\mu_1,\cdots,\mu_n]$ be the space of all homogeneous polynomials with real coefficients in $\mu_1,\cdots,\mu_n$ of degree k, then it is the subring of $\mathbb{R}[\mu_1,\cdots,\mu_n]$. From the Theorem 5.3, we obtain the recursion formula of λ_{g-1} Hodge integral.

Theorem 5.4. *For any partition μ with $l(\mu)=n$ and $|\mu|=d$, we have the recursion formula*

$$\begin{aligned}
&\frac{n}{|\mathrm{Aut}(\mu)|}\int_{\overline{\mathcal{M}}_{g,n}}\frac{\lambda_{g-1}}{\prod_{i=1}^n(1-\mu_i\psi_i)}\\
=&\sum_{\nu\in J(\mu)}\frac{I_1(\nu)}{|\mathrm{Aut}(\nu)|}\int_{\overline{\mathcal{M}}_{g,n-1}}\frac{\lambda_{g-1}}{\prod_{i=1}^{n-1}(1-\nu_i\psi_i)}\\
-&\sum_{g_1+g_2=g,\nu^1\cup\nu^2\in C(\mu)}\frac{I_3(\nu^1,\nu^2)}{|\mathrm{Aut}(\nu^1)||\mathrm{Aut}(\nu^2)|}d_1^{2g_1+n_1-3}d_2^{2g_2+n_2-3}b_{g_1}b_{g_2}.
\end{aligned}$$

under the ring $\mathbb{R}^{2g-2+n}[\mu_1,\cdots,\mu_n]$, where $l(\nu^i)=n_i$ and $|\nu^i|=d_i$ for $i=1,2$.

Remark 5.5. When we consider the simplest case $n=1$, the above identity become the formula used in [6].

6. Some Examples of The Main Theorem

In this section we give some examples of theorem 3.2.

6.1. The case of $g=3$. If $g=3$, then $1\le m\le 3$. We consider three cases.

6.1.1. *m=1. LHS* $=-3\int_{\overline{\mathcal{M}}_{3,1}}\lambda_3\mathrm{ch}_3(\mathbb{E})\psi_1$, and $3\mathrm{ch}_3(\mathbb{E})=\sum_{i+j=3}(-1)^{i-1}i\lambda_i\lambda_j=3\lambda_3-\lambda_1\lambda_2$, then we get

$$\begin{aligned}
LHS &= \int_{\overline{\mathcal{M}}_{3,1}}\lambda_3(\lambda_1\lambda_2-3\lambda_3)\psi_1=\int_{\overline{\mathcal{M}}_{3,1}}\lambda_1\lambda_2\lambda_3\psi_1,\\
RHS &= b_3\frac{(-1)^5}{5}\binom{5}{0}\binom{5}{4}B_4+\frac{1}{2}\sum_{g_1+g_2=3,g_1,g_2>0}b_{g_1}b_{g_2}\frac{-1}{5}\binom{2g_2-1}{0}\binom{5}{4}B_4\\
&= -B_4(b_3+b_1b_2).
\end{aligned}$$

Since $b_1=\frac{1}{24},b_2=\frac{7}{5760},b_3=\frac{31}{967680}$, we have

$$\int_{\overline{\mathcal{M}}_{3,1}}\lambda_1\lambda_2\lambda_3\psi_1=\frac{1}{362880}. \tag{37}$$

6.1.2. *m=2.* In this case we have $LHS=2\int_{\overline{\mathcal{M}}_{3,1}}\lambda_3\mathrm{ch}_2(\mathbb{E})\psi_1^2$, and $2!\mathrm{ch}_2(\mathbb{E})=2\lambda_2-\lambda_1^2$, $B_3=0$. Then we have

$$\begin{aligned}
LHS &= \int_{\overline{\mathcal{M}}_{3,1}}(2\lambda_2\lambda_3-\lambda_3\lambda_1^2)\psi_1^2,\\
RHS &= b_3\sum_{k=0}^{1}\frac{(-1)^{4-k}}{5-k}\binom{5}{k}\binom{5-k}{3}B_3\\
&+ \frac{1}{2}\sum_{g_1+g_2=3,g_1,g_2>0}b_{g_1}b_{g_2}\sum_{k=0}^{1}\frac{(-1)^{5-k}}{5-k}\binom{2g_2-1}{k}\binom{5-k}{3}B_3\\
&= 0,
\end{aligned}$$

hence

$$\int_{\overline{\mathcal{M}}_{3,1}} \lambda_3\lambda_1^2\psi_1^2 = 2\int_{\overline{\mathcal{M}}_{3,1}} \lambda_2\lambda_3\psi_1^2.$$

Using the formula $\int_{\overline{\mathcal{M}}_{3,1}} \lambda_2\lambda_3\psi_1^2 = \frac{1}{120960}$, we get

$$\int_{\overline{\mathcal{M}}_{3,1}} \lambda_3\lambda_1^2\psi_1^2 = \frac{1}{60480}. \tag{38}$$

6.1.3. *m=3.* In this case $LHS = -\int_{\overline{\mathcal{M}}_{3,1}} \lambda_3 \mathrm{ch}_1(\mathbb{E})\psi_1^3$ and $\mathrm{ch}_1(\mathbb{E}) = \lambda_1$, hence

$$\begin{aligned}
LHS &= -\int_{\overline{\mathcal{M}}_{3,1}} \lambda_1\lambda_3\psi_1^3, \\
RHS &= b_3\sum_{k=0}^{2}\frac{(-1)^{5-k}}{5-k}\binom{5}{k}\binom{5-k}{2}B_2 \\
&+ \frac{1}{2}\sum_{g_1+g_2=3, g_1,g_2>0} b_{g_1}b_{g_2}\sum_{k=0}^{\min(2g_2-1,2)}\frac{(-1)^{2g_2-1-k}}{5-k}\binom{2g_2-1}{k}\binom{5-k}{2}B_2 \\
&= -\frac{9}{2}b_3B_2 - \frac{1}{2}b_1b_2B_2 \\
&= -\frac{41}{1451520},
\end{aligned}$$

so

$$\int_{\overline{\mathcal{M}}_{3,1}} \lambda_1\lambda_3\psi_1^3 = \frac{41}{145120}. \tag{39}$$

Remark 6.1. The values of (37) and (39) match with the results in [9], the identity (38) is a new result.

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