Generalized Path Coalgebra And Its Application To Dual Gabriel Theorem *

Fang Li†
Gong-xiang Liu‡
Department of Mathematics
Zhejiang University
Hangzhou, Zhejiang 310028, China

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Abstract

As the duality of generalized path algebra (see [4]), we firstly introduce the concept of generalized path coalgebra through assigning a k-coalgebra to each vertex of a given quiver. Some elementary properties of generalized path coalgebras are given. Moreover, we discuss the isomorphism problem. It is shown that two generalized path coalgebras are isomorphic with each other if and only if their quivers are isomorphic with certain condition. For a coalgebra with $\text{Codim}C_0 \leq 1$, the Wedderburn-Malcev Theorem are given, that is, there exists a coalgebra projection of C onto C_0. It is a generalization of the Wedderburn-Malcev Theorem on coalgebras with separable coradicals. As an important application of generalized path coalgebras, the Dual Gabriel Theorem on a pointed coalgebra is generalized to a coalgebra with some conditions, in particular, with separable coradical, so as to embedding such coalgebra into a generalized path coalgebra from the quiver of the cotensor coalgebra of the coalgebra. Lastly, we discuss the uniqueness of the quiver of the cotensor coalgebra under the meaning of embedding. And, it is shown that the quiver is a wide subquiver of another quiver defined by Montgomery[11].

1 Introduction

In this paper, we always suppose that k denotes a field and all linear spaces are over k.

The concept of generalized path algebra was introduced in [4], which is a generalization of path algebra through assigning a k-algebra to each vertex of a given quiver. In [4], some properties of generalized path algebras were given, including the so-called isomorphism

*Project(No.102028) supported by the Natural Science Foundation of Zhejiang Province of China
†fangli@zju.edu.cn
‡gxliu7969@sohu.com

problem for generalized path algebra. As an application, in [9], the classical Gabriel Theorem on elementary algebras was generalized to some finite dimensional algebras by using of generalized path algebras through assigning a k-simple algebra to each vertex of a given quiver.

The aim of this paper is to consider the dual theory for coalgebras.

A coalgebra is called *basic* if its every simple subcoalgebra is the dual of a division algebra. If k is algebraically closed, the basic coalgebra is pointed. In [3], the authors proved that one can associate every coalgebra C with a basic coalgebra B_C and C is Morita-Takeuchi equivalent to B_C. A basic result in path coalgebra says that every path coalgebra is pointed. According to the Dual Gabriel Theorem (see Theorem 4.3 in [3]), for each pointed coalgebra C, one can construct the correspondent quiver $\Gamma(C)$ of C such that C is isomorphic to a "large" subcoalgebra (i.e. contains all group-like and primitive elements) of the path coalgeba $C(\Gamma(C))$. Thus, in the view point of representation theory of coalgebras, if the base field is algebraically closed, it is enough to research representations of pointed coalgebras (which can be thought as subcoalgebras of path coalgrbras).

However, in the view point of structures of coalgebras, researching on coalgebras can not be replaced with that on pointed or basic coalgebras. Therefore, we expect to find a generalization of path coalgebras (which will be called as generalized path coalgebras) so as to obtain a generalized Dual Gabriel Theorem for arbitrary coalgebras to be isomorphic to a subcoalgebra of a generalized path coalgebra.

For this reason, we introduce the concept of generalized path coalgebra in Section 2. Our idea is to assign to each vertex of a given quiver a coalgebra instead of assigning a k. In general, when two generalized path coalgebras are isomorphic, it does not follow that their quivers are isomorphic. In order to avoid such unpleasant case, we will show in Section 3 that it is enough to restrict coalgebras, which are assigned to all vertices of the given quivers, to be simple. In fact, in a large part of this paper, we have to suppose the condition in this case. The most interesting thing is that, in this case, we can generalize the Dual Gabriel Theorem on pointed coalgebras to more general coalgebras. Our main result is that for a coalgebra C with $\mathrm{Codim} C_0 \leq 1$, (a) (Wedderburn-Malcev Theorem on Coalgebra) there exists a coideal I of C such that $C = I \oplus C_0$ as k-spaces; (b) assume that C_1/C_0 is a direct summand of C/C_0 as C_0-bicomodules, then (Generalized Dual Gabriel Theorem) C can be embedded into the generalized path coalgebra $k(\Delta, \mathcal{C})$ satisfying $\varphi(I_1) \subseteq k(\Delta_1, \mathcal{C})$ where Δ is the *quiver of* $\mathrm{CoT}_{C_0}(C_1/C_0)$, $\mathcal{C} = \{S_i | i \in \Lambda\}$ for $C_0 = \oplus_{i\in\Lambda} S_i$ with S_i simple coalgebras for $i \in \Lambda$ and $I_1 = I \cap C_1$.

As preparation, in Section 2, we also give some basic results of generalized path coalgebra and its relation with cotensor coalgebra.

We fix some notations. C will always denote a coalgebra with comultiplication $\triangle$ and counit ε. C^{cop} means the coopposite coalgebra of C. All tensor products are over k. We will write $\triangle(c) = c' \otimes c''$ omitting the notations of its summation and index. $\mathcal{M}^C$ (resp.

${}^C\mathcal{M}$) denotes the category of right (resp. left) C-comodules. For X and Y respectively right and left C-comodules, $X\Box Y$ denotes the cotensor product of X and Y. For another coalgebra B, ${}^B\mathcal{M}^C$ denotes the category of B-C-bicomodules. The "wedge" of subspaces V, W of C is defined as $V\wedge_C W := \triangle^{-1}(C\otimes W + V\otimes C)$. $\wedge_C^2 V$ denotes $V\wedge_C V$. Since $(U\wedge_C V)\wedge_C W = U\wedge_C(V\wedge_C W)$, we can define $\wedge_C^i V$ inductively for $i\geq 1$.

Thanks the authors of [2] for mailing us the preprint when this paper was about to be finished. It shows the interesting fact we obtained independently the result of Theorem 4.7 (b) (i) in the case of coseparable-type coalgebras.

2 On Generalized Path Coalgebras

This section is devoted to define generalized path coalgebra and establish its some basic results.

A *quiver* Δ is given by two sets Δ_0 and Δ_1 together with two maps $s, e: \Delta_1 \to \Delta_0$. The elements of Δ_0 are called *vertices*, while the elements of Δ_1 are called *arrows*. For an arrow $\alpha\in\Delta_1$, the vertex $s(\alpha)$ is the *start vertex* of α and the vertex $e(\alpha)$ is the *end vertex* of α, and we draw $s(\alpha)\xrightarrow{\alpha} e(\alpha)$. A *path* in Δ is $(a|\alpha_1\cdots\alpha_n|b)$, where $\alpha_i\in\Delta_1$, for $i=1,\cdots,n$, and $s(\alpha_1)=a$, $e(\alpha_i)=s(\alpha_{i+1})$ for $i=1,\cdots,n-1$, and $e(\alpha_n)=b$. The *length of a path* is the number of arrows in it. To each arrow α we can assign an edge $\overline{\alpha}$ where the orientation is forgotten. A *walk* between two vertices a and b is given by $(a|\overline{\alpha_1}\cdots\overline{\alpha_n}|b)$, where $a\in\{s(\alpha_1),e(\alpha_1)\}$, $b\in\{s(\alpha_n),e(\alpha_n)\}$, and for each $i=1,\cdots,n-1$, $\{s(\alpha_i),e(\alpha_i)\}\bigcap\{s(\alpha_{i+1}),e(\alpha_{i+1})\}\neq\emptyset$. A quiver is said to be *connected* if for each pair of vertices a and b, there exists a walk between them. More knowledge about quiver and its representations can be found in [1], [5].

Let $\Delta=(\Delta_0,\Delta_1)$ be a quiver and $\mathcal{C}=\{S_i|i\in\Delta_0\}$ be a family of k-coalgebras S_i with comultiplication $\triangle_i$ and counit ε_i, indexed by the vertices of Δ. The elements of $\bigcup_{i\in\Delta_0} S_i$ are called the $\mathcal{C}$ *-path of length zero*, and for each $n\geq 1$, a $\mathcal{C}$*-path of length* n is given by $a_1\beta_1 a_2\beta_2\cdots a_n\beta_n a_{n+1}$, where $(s(\beta_1)|\beta_1\cdots\beta_n|e(\beta_n))$ is a path in Δ of length n, for each $i=1,\cdots,n$, $a_i\in S_{s(\beta_i)}$ and $a_{n+1}\in S_{e(\beta_n)}$. Consider now the quotient R of the k-linear space with basis the set of all $\mathcal{C}$-paths of Δ by the space generated by all the elements of the form

$$a_1\beta_1\cdots\beta_{j-1}(\sum_{l=1}^{m}k_l a_j^l)\beta_j a_{j+1}\cdots a_n\beta_n a_{n+1} - \sum_{l=1}^{m}k_l a_1\beta_1\cdots\beta_{j-1}a_j^l\beta_j a_{j+1}\cdots a_n\beta_n a_{n+1}$$

where $(s(\beta_1)|\beta_1\cdots\beta_n|e(\beta_n))$ is a path in Δ of length n, for each $i=1,\cdots,n$, $a_i\in S_{s(\beta_i)}$, $a_{n+1}\in S_{e(\beta_n)}$, and $k_l\in k$, $a_j^l\in S_{s(\beta_j)}$ for $l=1,\cdots,m$. Define now in R the following comultiplication and counit. Given an element $a_1\beta_1 a_2\beta_2\cdots a_n\beta_n a_{n+1}$, we define

$$\triangle(a_1\beta_1 a_2\beta_2\cdots a_n\beta_n a_{n+1}) = \sum_{i=1}^{n+1} a_1\beta_1\cdots a_{i-1}\beta_{i-1}a_i'\otimes a_i''\beta_i a_{i+1}\cdots a_n\beta_n a_{n+1}$$

In particular, $\triangle(a_i) = \triangle_i(a) = a_i' \otimes a_i''$ for $a_i \in S_i,\ i \in \Delta_0$. The counit ε is defined as

$$\varepsilon(p) = \begin{cases} 0, & \text{if the length of } p \text{ is } n > 0 \\ \varepsilon_i(p), & \text{if } p \in S_i \text{ for some } i \in \Delta_0 \end{cases}$$

It is easy to check that the above comultiplication and counit on R is well-defined and gives to R an structure of k-coalgebra. This coalgebra is called the *$\mathcal{C}$-path coalgebra of* Δ and we denote it by $R = k(\Delta, \mathcal{C})$. In genreal, we call such type coalgebra *generalized path coalgebra* when there is no ambiguity on Δ and $\mathcal{C}$. Clearly, $k(\Delta, \mathcal{C})$ is a graded coalgebra with length grading. That's to say

$$k(\Delta, \mathcal{C}) = k(\Delta_0, \mathcal{C}) \oplus k(\Delta_1, \mathcal{C}) \oplus k(\Delta_2, \mathcal{C}) \oplus \cdots \oplus k(\Delta_i, \mathcal{C}) \oplus \cdots$$

where $k(\Delta_i, \mathcal{C})$ denotes subspace with the basis of all $\mathcal{C}$-paths of length i in $k(\Delta, \mathcal{C})$ and $\triangle(k(\Delta_n, \mathcal{C})) \subseteq \sum_{i=0}^{n} k(\Delta_i, \mathcal{C}) \otimes k(\Delta_{n-i}, \mathcal{C})$.

Remark 2.1 *(i) Observe that if $S_i = k$ for each $i \in \Delta_0$, then the coalgebra $k(\Delta, \mathcal{C})$ defined above is the usual path coalgebra $k\Delta$ of Δ.*

(ii) Any coalgebra C can be realized as a $\mathcal{C}$-path coalgebra $k(\Delta, \mathcal{C})$ by just taking Δ as the quiver consisting of a unique vertex and $\mathcal{C} = \{C\}$. Also, it is not difficult to see that a realization of a k-coalgebra as $\mathcal{C}$-path coalgebra is not necessarily unique. For example, let Δ be the quiver consisting of a unique vertex, Δ' the quiver of two vertices without arrows. Then $S_1 \oplus S_2 \cong k(\Delta, S_1 \oplus S_2) \cong k(\Delta', \mathcal{C} = \{S_1, S_2\})$ for any two coalgebras S_1, S_2. We shall discuss the problem of uniqueness in Section 3 below.

We can give an alternative definition for $\mathcal{C}$-path coalgebra. In order to do it, we give some description about cotensor coalgebra firstly. Given a coalgebra C and C-bicomodule $M \in {}^C\mathcal{M}^C$. The left (right) comodule structure map on M is denoted by δ_L (δ_R). Set

$$\mathrm{CoT}_C(M) = C \oplus M \oplus M^{\Box 2} \oplus \cdots \oplus M^{\Box n} \oplus \cdots$$

Define the counit ε on $\mathrm{CoT}_C(M)$ by

$$\varepsilon|_{M^{\Box i}} = 0 \ \text{ for } \ i \geq 1, \ \text{ and } \ \varepsilon|_C = \varepsilon_C$$

Define $\triangle|_C = \triangle_C,\ \triangle|_M = \delta_L + \delta_R$. In general, for $m_1 \otimes \cdots \otimes m_n \in M^{\Box n}$, define

$$\begin{aligned} \triangle(m_1 \otimes \cdots \otimes m_n) \quad = \quad & \delta_L(m_1) \otimes m_2 \otimes \cdots \otimes m_n + m_1 \otimes \cdots \otimes m_n \\ & + m_1 \otimes \cdots \otimes m_n + \cdots + m_1 \otimes \cdots \otimes m_n \\ & + m_1 \otimes \cdots \otimes \delta_R(m_n) \\ & \in C \otimes M^{\Box n} \oplus M \otimes M^{\Box n-1} \oplus M^{\Box 2} \otimes M^{\Box n-2} \\ & \oplus \cdots \oplus M^{\Box n-1} \otimes M \oplus M^{\Box n} \otimes C \end{aligned}$$

With such structure maps $\triangle$ and ε, $\mathrm{CoT}_C(M)$ is a coalgebra (see [13]) and called the cotensor coalgebra of bicomodule M over C.

Let $R = k(\Delta, \mathcal{C})$ be a $\mathcal{C}$-path coalgebra. For any $x \in k(\Delta, \mathcal{C})$, denote $(x')_i \otimes x''$ to be the summation of all summands in $\triangle(x)$ such that $x' \in k(\Delta_i, \mathcal{C})$. For example, if $x = a + b\beta c$ where $a \in k(\Delta_0, \mathcal{C}), b\beta c \in k(\Delta_1, \mathcal{C})$, then $(x')_0 \otimes x'' = a' \otimes a'' + b' \otimes b''\beta c$. Similarly, we can define $x' \otimes (x'')_i$ to be the summation of all summands in $\triangle(x)$ such that $x'' \in k(\Delta_i, \mathcal{C})$. Clearly, $\triangle(x) = \sum_{i\geq 0}(x')_i \otimes x'' = \sum_{i \geq 0} x' \otimes (x'')_i$.

Proposition 2.2 *(1): For any $\mathcal{C}$-path coalgebra $k(\Delta, \mathcal{C})$, $k(\Delta_n, \mathcal{C})$ is a $k(\Delta_0, \mathcal{C})$-bicomodule for any $n \geq 0$ via, for any $x \in k(\Delta_n, \mathcal{C})$,*

$$\delta_L(x) := (x')_0 \otimes x'' \quad \text{and} \quad \delta_R(x) := x' \otimes (x'')_0$$

(2): We have coalgebra isomorphism $k(\Delta, \mathcal{C}) \cong CoT_{k(\Delta_0,\mathcal{C})}(k(\Delta_1, \mathcal{C}))$.

Proof: (1) Firstly, note that $k(\Delta_0, \mathcal{C})$ is always a subcoalgebra of $k(\Delta, \mathcal{C})$. In fact, it is easy to see that $k(\Delta_0, \mathcal{C}) = \oplus_{i\in\Delta_0} S_i$ if $\mathcal{C} = \{S_i | i \in \Delta_0\}$. For $\sum a_1\beta_1 a_2\beta_2 \cdots a_n\beta_n a_{n+1} \in k(\Delta_n, \mathcal{C})$,

$$\begin{aligned}
(id \otimes \delta_L)\delta_L(\sum a_1\beta_1 a_2\beta_2 \cdots a_n\beta_n a_{n+1}) &= \sum(id \otimes \delta_L)(a_1' \otimes a_1''\beta_1 a_2\beta_2 \cdots a_n\beta_n a_{n+1}) \\
&= \sum a_1' \otimes a_1'' \otimes a_1'''\beta_1 a_2\beta_2 \cdots a_n\beta_n a_{n+1} \\
&= (\triangle \otimes id)\delta_L(\sum a_1\beta_1 a_2\beta_2 \cdots a_n\beta_n a_{n+1})
\end{aligned}$$

$$\begin{aligned}
(\varepsilon \otimes id)\delta_L(\sum a_1\beta_1 a_2\beta_2 \cdots a_n\beta_n a_{n+1}) &= \sum(\varepsilon \otimes id)(a_1' \otimes a_1''\beta_1 a_2\beta_2 \cdots a_n\beta_n a_{n+1}) \\
&= \sum \varepsilon(a_1') \otimes a_1''\beta_1 a_2\beta_2 \cdots a_n\beta_n a_{n+1} \\
&= \sum a_1\beta_1 a_2\beta_2 \cdots a_n\beta_n a_{n+1}
\end{aligned}$$

Thus $k(\Delta_n, \mathcal{C})$ is a left $k(\Delta_0, \mathcal{C})$-comodule with the structure map δ_L. Similarly, we can prove that $k(\Delta_n, \mathcal{C})$ is a right $k(\Delta_0, \mathcal{C})$-comodule with the structure map δ_R. $(id \otimes \delta_R)\delta_L = (\delta_L \otimes id)\delta_R$ can be proved directly. Therefore, $k(\Delta_n, \mathcal{C})$ is a $k(\Delta_0, \mathcal{C})$-bicomodule.

(2) By (1), $k(\Delta_1, \mathcal{C})$ is a $k(\Delta_0, \mathcal{C})$-bicomodule. Thus cotensor coalgebra $\mathrm{CoT}_{k(\Delta_0,\mathcal{C})}(k(\Delta_1, \mathcal{C}))$ can be constructed. Assume $a_1\beta_1 a_2 \otimes a_3\beta_2 a_4 \in k(\Delta_1, \mathcal{C})^{\Box 2}$, i.e. $(id \otimes \delta_L - \delta_R \otimes id)(a_1\beta_1 a_2 \otimes a_3\beta_2 a_4) = 0$. But it is easy to see that $(id \otimes \delta_L - \delta_R \otimes id)(a_1\beta_1 a_2 \otimes a_3\beta_2 a_4) = 0$ if and only if $(\triangle \otimes id)(a_2 \otimes a_3) = (id \otimes \triangle)(a_2 \otimes a_3)$. Similarly, we have $a_1\beta_1 a_2 \otimes \cdots \otimes a_{2n-1}\beta_n a_{2n} \in k(\Delta_1, \mathcal{C})^{\Box n}$ if and only if $(\triangle \otimes id)(a_i \otimes a_{i+1}) = (id \otimes \triangle)(a_i \otimes a_{i+1})$ for $i = 2, 4, \cdots, 2n-2$.

Define $F: \ k(\Delta, \mathcal{C}) \to \mathrm{CoT}_{k(\Delta_0,\mathcal{C})}(k(\Delta_1, \mathcal{C}))$ by

$$F|_{k(\Delta_0,\mathcal{C})\oplus k(\Delta_1,\mathcal{C})} := id, \quad F(a_1\beta_1 a_2\beta_2 a_3 \cdots a_n\beta_n a_{n+1}) := a_1\beta_1\triangle(a_2)\beta_2\triangle(a_3)\cdots\triangle(a_n)\beta_n a_{n+1}$$

where $a_1\beta_1 a_2\beta_2 a_3 \cdots a_n\beta_n a_{n+1} \in k(\Delta_n, \mathcal{C})$ for $n \geq 2$. By coassocitivity of $k(\Delta, \mathcal{C})$, we always have $(\triangle \otimes id)(a_i' \otimes a_i'') = (id \otimes \triangle)(a_i' \otimes a_i'')$ for $i = 2, 3, \cdots, n$. This implies

$a_1\beta_1\triangle(a_2)\beta_2\triangle(a_3)\cdots\triangle(a_n)\beta_n a_{n+1}\in k(\Delta_1,\mathcal{C})^{\Box n}$ by above discussion for $n\geq 2$. Thus F is well-defined. We leave to the reader the verification that F is a coalgebra homomorphism.

In order to prove that F is bijective, we give the inverse map of F. Define G : $\mathrm{CoT}_{k(\Delta_0,\mathcal{C})}(k(\Delta_1,\mathcal{C}))\to k(\Delta,\mathcal{C})$ by

$$G|_{k(\Delta_0,\mathcal{C})\oplus k(\Delta_1,\mathcal{C})} := id$$

and for $a_1\beta_1 a_2\otimes a_3\beta_2 a_4\otimes\cdots\otimes a_{2n-1}\beta_n a_{2n}\in k(\Delta_1,\mathcal{C})^{\Box n}$,

$$G(a_1\beta_1 a_2\otimes a_3\beta_2 a_4\otimes\cdots\otimes a_{2n-1}\beta_n a_{2n}) := a_1\beta_1\varepsilon(a_2)a_3\beta_2\cdots\varepsilon(a_{2n-2})a_{2n-1}\beta_n a_{2n}$$

It is straitforward to prove that $FG=id$ and $GF=id$. $\Box$

The following lemmas means the "universal property" of the cotensor coalgebras, which is very useful for the sequel.

Lemma 2.3 (see Theorem 3.8 in [13]) *Let* $X\stackrel{\psi}{\to}CoT_C(M)$ *be a coalgebra map. Set* $\psi_n := p_n\psi\ :\ X\to M^{\Box n}$ *for* $n\geq 0$, *where* $p_n:\ CoT_C(M)\to M^{\Box n}$ *is the projection. Then*

(1) $\psi_0:\ X\to C$ *is a coalgebra map.*

(2) $\psi_1:\ X\to M$ *is a* C*-bicomodule map, where* X *is the induced* C*-bicomodule via* ψ_0 *naturally.*

(3) For $n\geq 2$, ψ_n *is exactly the* C*-bicomodule map given by*

$$\psi_n:\ X\stackrel{\triangle^{(n-1)}}{\to}X\otimes X\otimes\cdots\otimes X\stackrel{\psi_1^{\otimes n}}{\to}M^{\otimes n}$$

where $\triangle^{(n)} := (\triangle_X^{(n-1)}\otimes id_X)\triangle_X$ *for* $n\geq 2$ *and* $\triangle^{(1)}=\triangle_X$.

(4) $\psi=\psi_0\oplus\psi_1\oplus\cdots\oplus\psi_n\oplus\cdots$. *Thus,* ψ *is uniquely determined by* ψ_0 *and* ψ_1.

Lemma 2.4 (see Theorem 3.9 in [13]) *Let* $\psi_0:\ X\to C$ *be a coalgebra map, and* $\psi_1\ X\to M$ *a* C*-bicomodule map. Let* $\psi_n:\ X\to M^{\otimes n}$ *be the composition*

$$\psi_n:\ X\stackrel{\triangle^{(n-1)}}{\to}X\otimes X\otimes\cdots\otimes X\stackrel{\psi_1^{\otimes n}}{\to}M^{\otimes n},\ \ n\geq 2$$

Then ψ_n *is a* C*-bicomodule map with* $Im(\psi_n)\subseteq M^{\Box n}$.

If for each $x\in X$ *there are only finite* i *such that* $\psi_i(x)\neq 0$, *then* $\psi:\ X\to CoT_C(M)$ *is a coalgebra map, where* $\psi=\sum_{i\geq 0}\psi_i$.

Recall that the coradical filtration $\{C_n\}$ of a coalgebra C is defined as follows:

$$C_0 := \text{the sum of all simple subcoalgebras of } C$$

$$C_n := \triangle^{-1}(C\otimes C_{n-1}+C_0\otimes C)\ \text{ for }\ n\geq 1$$

where C_0 is called the coradical of C. Then we have

$$C_n\subseteq C_{n+1},\ n\geq 0;\ \ C=\cup_{n\geq 0}C_n;\ \ \triangle(C_n)\subseteq\sum_{i=0}^{n}C_i\otimes C_{n-i}$$

see [12], p.60.

Proposition 2.5 *If C is co-semisimple coalgebra, then the coradical $(CoT_C(M))_0$ of $CoT_C(M)$ is exatly C.*

Proof: Clearly, $C \subseteq (\mathrm{CoT}_C(M))_0$. On the other hand, it is straightforward to prove that $\wedge^n_{\mathrm{CoT}_{C(M)}} C = C \oplus M \oplus \cdots \oplus M^{\Box n-1}$ and

$$\triangle(\wedge^n_{\mathrm{CoT}_{C(M)}} C) \subseteq \sum_{i=0}^{n} \wedge^i_{\mathrm{CoT}_{C(M)}} C \otimes \wedge^{n-i}_{\mathrm{CoT}_{C(M)}} C$$

for $n \geq 1$. Thus $\{\wedge^n_{\mathrm{CoT}_{C(M)}} C\}$ is a *coalgebra filtration* (see p.61 of [12]) of $\mathrm{CoT}_C(M)$. By Lemma 5.3.4 in [12], $C \supseteq (\mathrm{CoT}_C(M))_0$. Therefore, $C = (\mathrm{CoT}_C(M))_0$. □

Let $\mathcal{C} = \{\text{simple coalgebras } S_i | i \in \Delta_0\}$ for some quiver $\Delta = (\Delta_0, \Delta_1)$. Then $k(\Delta_0, \mathcal{C}) = \oplus_{i \in \Delta_0} S_i$ is co-semisimple coalgebra. Thus, by Propositions 2.2 and 2.5, we have:

Corollary 2.6 *Let $\mathcal{C}$ be as above. Then the coradical of $k(\Delta, \mathcal{C})$ is $\oplus_{i \in \Delta_0} S_i$.*

3 Isomorphism Problem

As you have seen in Remark 2.1 that, in general, the isomorphism of two generalized path coalgebras does not imply that of their quivers. However, we will find that the isomorphism of quivers can be induced for generalized path coalgebras over simple coalgebras. For this reason, we call $k(\Delta, \mathcal{C})$ a *normal $\mathcal{C}$-path coalgebra* if every S_i is simple in the family of coalgebras $\mathcal{C} = \{S_i | i \in \Delta_0\}$. Moreover in Section 4, it is shown that the Dual Gabriel Theorem can be generalized to some coalgebras through normal $\mathcal{C}$-path coalgebra.

Our main result of this section is as follows:

Theorem 3.1 *Let $k(\Delta, \mathcal{C})$ and $k(\Delta', \mathcal{D})$ be two normal generalized path coalgebras with $\mathcal{C} = \{S_i | i \in \Delta_0\}$ and $\mathcal{D} = \{T_j | j \in \Delta'_0\}$. Then $k(\Delta, \mathcal{C}) \cong k(\Delta', \mathcal{D})$ as coalgebras if and only if there is an isomorphism of quivers $\varphi : \Delta \to \Delta'$ such that $S_i \cong T_{\varphi(i)}$ as coalgebras for $i \in \Delta_0$.*

To complete the proof of this theorem, one needs some preliminary results. Let $D, E \subseteq C$ be two subcoalgebras of C. For a left C-comodule M with the structure map δ_L, denote

$$^D M := \{m \in M | \delta_L(m) \in D \otimes M\}$$

for a right C-comodule M with structure map δ_R, denote

$$M^D := \{m \in M | \delta_R(m) \in M \otimes D\}$$

and for C-bicomodule M, denote

$$^D M^E := \{m \in M | \delta_L(m) \in D \otimes M, \ \ \delta_R(m) \in M \otimes E\}$$

Lemma 3.2 *If $C = \oplus_{i\in I} C_i$ and $M \in \mathcal{M}^C$, then $M = \oplus_{i\in I} M^{C_i}$.*

Proof: Denote the right C-comodule structure map by δ_R and $\delta_R(m) = \sum m_0 \otimes m_1$ for $m \in M$. Clearly, $\delta_R(m) = \sum m_0 \otimes m_1 = \sum\sum_{i\in I} m_{0i} \otimes m_{1i}$ where $m_{0i} \otimes m_{1i}$ equals the summation of all summands in $\delta_R(m)$ satisfying $m_1 \in C_i$. By the identity $(\delta_R \otimes id)\delta_R = (id \otimes \triangle)\delta_R$ we observe that $m_{0i} \in M^{C_i}$ for every m_{0i} in $m_{0i} \otimes m_{1i}$. By using $(id \otimes \varepsilon)\delta_R = id$ we get $m = \sum m_{0i}\varepsilon(m_{1i})$ which implies $m \in \sum_{i\in I} M^{C_i}$. Thus $M = \sum_{i\in I} M^{C_i}$. But, it is clear that $\sum_{i\in I} M^{C_i} = \oplus_{i\in I} M^{C_i}$. Thus $M = \oplus_{i\in I} M^{C_i}$. □

Similarly, we have:

Lemma 3.3 *If $C = \oplus_{i\in I} C_i$ and $M \in {}^C\mathcal{M}$, then $M = \oplus_{i\in I} {}^{C_i}M$.*

Proposition 3.4 *If $C = \oplus_{i\in I} C_i$ and $M \in {}^C\mathcal{M}^C$, then $M = \oplus_{i,j\in I} {}^{C_j}M^{C_i}$.*

Proof: By Lemma 3.2, $M = \oplus_{i\in I} M^{C_i}$. We claim that M^{C_i} is a left C-comodule, and then the assertion follows from Lemma 3.3.

For any $m \in M^{C_i}$, let $\delta_L(m) = \sum m_{-1} \otimes m_0 \in C \otimes M$. It suffices to prove $m_0 \in M^{C_i}$. In fact, by $(id \otimes \delta_R)\delta_L(m) = (\delta_L \otimes id)\delta_R(m)$ we have $m_0 \in M^{C_i}$. □

Let $C = \oplus_{i\in I} C_i$ and ${}^CM^C$, ${}^CN^C \in {}^C\mathcal{M}^C$. Clearly, a C-bicomodule map $f:\ M \to N$ is an isomorphism if and only if $f|_{{}^{C_i}M^{C_j}}:\ {}^{C_i}M^{C_j} \to {}^{C_i}N^{C_j}$ is an isomorphism as C_i-C_j-bicomodules for all $i, j \in I$. Recall the next result (see Lemma 5.3.6 in [12]):

Lemma 3.5 *Let $f:\ C \to D$ be a surjective coalgebra map and let W_1, W_2 be subspaces of C such that $ker(f) \subseteq W_1 \cap W_2$. Then*

$$f(W_1 \wedge W_2) = f(W_1) \wedge f(W_2)$$

Proposition 3.6 *Let $k(\Delta, \mathcal{C})$ and $k(\Delta', \mathcal{D})$ be the normal generalized coalgebras appeared in Theorem 3.1. If $k(\Delta, \mathcal{C}) \overset{\psi}{\cong} k(\Delta', \mathcal{D})$ as coalgebras, then*

(1) There is a bijection $\varphi:\ \Delta_0 \to \Delta'_0$ such that $S_i \cong T_{\varphi(i)}$ for $i \in \Delta_0$

(2) $\psi(k(\Delta_0, \mathcal{C}) \oplus k(\Delta_1, \mathcal{C})) = k(\Delta'_0, \mathcal{D}) \oplus k(\Delta'_1, \mathcal{D})$.

Proof: By Corollary 2.6, the coradicals of $k(\Delta, \mathcal{C})$ and $k(\Delta', \mathcal{D})$ are $\oplus_{i\in\Delta_0} S_i$ and $\oplus_{j\in\Delta'_0} T_j$ respectively. Clearly, $\psi(S_i)$ is a non-zero simple subcoalgebra of $k(\Delta', \mathcal{D})$ since ψ is an injective coalgebra map. Thus there exists a unique $\varphi(i) \in \Delta'_0$ such that $S_i \overset{\psi|_{S_i}}{\cong} T_{\varphi(i)}$. Similarly, for all $j \in \Delta'_0$, $\psi^{-1}(T_j)$ is isomorphic to some S_i of $k(\Delta, \mathcal{C})$. Therefore, $\varphi:\ \Delta_0 \to \Delta'_0$ is a bijection and $S_i \cong T_{\varphi(i)}$. This proves (1).

By the proof of Corollary 2.6 and Lemma 3.5,

$$\psi(k(\Delta_0,\mathcal{C})\oplus k(\Delta_1,\mathcal{C})) = \psi(k(\Delta_0,\mathcal{C})\wedge k(\Delta_0,\mathcal{C})) = \psi(k(\Delta_0,\mathcal{C}))\wedge\psi(k(\Delta_0,\mathcal{C})) = k(\Delta'_0,\mathcal{D})\oplus k(\Delta'_1,\mathcal{D})$$

Thus (2) is proved. □

Proof of Theorem 3.1 :

"If" part: It is clear.

"Only if" part: According to Proposition 3.6, we need only to show that the number of arrows t_{ij} from i to j in Δ equals to that $l_{\varphi(i)\varphi(j)}$ from $\varphi(i)$ to $\varphi(j)$. By proposition 2.2,

$$k(\Delta,\mathcal{C}) \cong \mathrm{CoT}_{k(\Delta_0,\mathcal{C})}(k(\Delta_1,\mathcal{C})) \quad k(\Delta',\mathcal{D}) \cong \mathrm{CoT}_{k(\Delta'_0,\mathcal{D})}(k(\Delta'_1,\mathcal{D}))$$

Denote ${}^{i}(k(\Delta_1,\mathcal{C}))^{j} := {}^{S_i}(k(\Delta_1,\mathcal{C}))^{S_j}$. Clearly, it is a free S_i-S_j-bicomodule, i.e. ${}^{i}(k(\Delta_1,\mathcal{C}))^{j} \cong S_i \otimes (V_{ij}) \otimes S_j$ for some k-space V_{ij} (see Section 4 below), and $t_{ij} = \dim(V_{ij})$ for $i,j \in \Delta_0$. Similarly, $l_{\varphi(i)\varphi(j)} = \dim(W_{\varphi(i)\varphi(j)})$ where $T_{\varphi(i)} \otimes (W_{\varphi(i)\varphi(j)}) \otimes T_{\varphi(j)} \cong {}^{\varphi(i)}(k(\Delta'_1,\mathcal{D}))^{\varphi(j)}$ as $T_{\varphi(i)}$-$T_{\varphi(j)}$-bicomodules for $i,j \in \Delta_0$. By $S_i \overset{\psi}{\cong} T_{\varphi(i)}$, $S_j \overset{\psi}{\cong} T_{\varphi(j)}$, ${}^{i}(k(\Delta_1,\mathcal{C}))^{j}$ becomes a $T_{\varphi(i)}$-$T_{\varphi(j)}$-bicomodules induced by ψ for $i,j \in \Delta_0$. Thus, by above discussion, if we can prove ${}^{T_{\varphi(i)}}(k(\Delta_1,\mathcal{C}))^{T_{\varphi(j)}} \cong {}^{T_{\varphi(i)}}(k(\Delta'_1,\mathcal{D}))^{T_{\varphi(j)}}$ as $T_{\varphi(i)}$-$T_{\varphi(j)}$-bicomodules, then $t_{ij} = l_{\varphi(i)\varphi(j)}$.

Let $C = \oplus_{i\in\Delta_0} S_i$ and $D = \oplus_{j\in\Delta'_0} T_j$. We give the D-bicomodule structure on $k(\Delta_1,\mathcal{C})$ explicitly now. Denote the left (right) D structure map by ρ_L (ρ_R). For any $a\beta b \in k(\Delta_1,\mathcal{C})$,

$$\rho_L(a\beta b) = \psi(a') \otimes a''\beta b, \qquad \rho_R(a\beta b) = a\beta b' \otimes \psi(b'')$$

Denote by $\overline{a\beta b}$ the image of $a\beta b$ in $(k(\Delta_0,\mathcal{C}) \oplus k(\Delta_1,\mathcal{C}))/k(\Delta_0,\mathcal{C})$ under the canonical homomorphism, $\overline{\psi(a\beta b)}$ the image of $\psi(a\beta b)$ in $(k(\Delta'_0,\mathcal{D}) \oplus k(\Delta'_1,\mathcal{D}))/k(\Delta'_0,\mathcal{D})$ under the canonical homomorphism.

Define the D-bicomodule structure on $(k(\Delta_0,\mathcal{C}) \oplus k(\Delta_1,\mathcal{C}))/k(\Delta_0,\mathcal{C})$ by (we also use the notation ρ_L, ρ_R):

$$\rho_L(\overline{a\beta b}) = \psi(a') \otimes \overline{a''\beta b}, \qquad \rho_R(\overline{a\beta b}) = \overline{a\beta b'} \otimes \psi(b'')$$

It is straightforward to prove that they are well-defined and give a D-bicomodule structure on $(k(\Delta_0,\mathcal{C}) \oplus k(\Delta_1,\mathcal{C}))/k(\Delta_0,\mathcal{C})$. With such D-bicomodules on $k(\Delta_1,\mathcal{C})$ and $(k(\Delta_0,\mathcal{C}) \oplus k(\Delta_1,\mathcal{C}))/k(\Delta_0,\mathcal{C})$, it is easy to see that the following canonical isomorphism

$$(k(\Delta_0,\mathcal{C}) \oplus k(\Delta_1,\mathcal{C}))/k(\Delta_0,\mathcal{C}) \overset{\pi_{\mathcal{C}}}{\to} k(\Delta_1,\mathcal{C})$$

is a D-bicomodule map. Similarly, the canonical isomorphism

$$(k(\Delta'_0,\mathcal{D}) \oplus k(\Delta'_1,\mathcal{D}))/k(\Delta'_0,\mathcal{D}) \overset{\pi_{\mathcal{D}}}{\to} k(\Delta'_1,\mathcal{D})$$

is a D-bicomodule map with the D-bicomodule structure on $(k(\Delta'_0,\mathcal{D}) \oplus k(\Delta'_1,\mathcal{D}))/k(\Delta'_0,\mathcal{D})$ by:

$$\delta_L(\overline{c\beta d}) = c' \otimes \overline{c''\beta b}, \qquad \delta_R(\overline{c\beta d}) = \overline{c\beta d'} \otimes d''$$

where $c, d \in D,\ \beta \in \Delta'_1$.

Define $\phi:\ k(\Delta_1, \mathcal{C}) \to k(\Delta'_1, \mathcal{D})$ be the composition of following maps

$$k(\Delta_1,\mathcal{C}) \stackrel{\pi_C^{-1}}{\to} (k(\Delta_0,\mathcal{C})\oplus k(\Delta_1,\mathcal{C}))/k(\Delta_0,\mathcal{C}) \stackrel{\overline{\psi}}{\to} (k(\Delta'_0,\mathcal{D})\oplus k(\Delta'_1,\mathcal{D}))/k(\Delta'_0,\mathcal{D}) \stackrel{\pi_D}{\to} k(\Delta'_1,\mathcal{D})$$

where $\overline{\psi}$ is defined by, for any $\overline{a\beta b} \in (k(\Delta_0, \mathcal{C}) \oplus k(\Delta_1, \mathcal{C}))/k(\Delta_0, \mathcal{C})$, $\overline{\psi}(\overline{a\beta b}) := \overline{\psi(a\beta b)}$. Clearly, $\overline{\psi}$ is an isomorphism as linear spaces since $\psi(k(\Delta_0, \mathcal{C})) = k(\Delta'_0, \mathcal{D})$ and $\psi(k(\Delta_0, \mathcal{C})\oplus k(\Delta_1, \mathcal{C})) = k(\Delta'_0, \mathcal{D}) \oplus k(\Delta'_1, \mathcal{D})$ by Proposition 3.6. We claim that $\overline{\psi}$ is also a D-bicomodule map. In fact, for any $\overline{a\beta b} \in (k(\Delta_0, \mathcal{C}) \oplus k(\Delta_1, \mathcal{C}))/k(\Delta_0, \mathcal{C})$, assume $\psi(a\beta b) = \sum d_i + \sum d_j\beta_j e_j$ for $d_i \in T_i$ $(i \in \Delta'_0)$, $d_j\beta_j e_j \in k(\Delta'_1, \mathcal{D})$. By ψ is a coalgebra map, $(\psi \otimes \psi)\triangle(a\beta b) = \triangle\psi(a\beta b)$, i.e.

$$\psi(a') \otimes \psi(a''\beta b) + \psi(a\beta b') \otimes \psi(b'') = \sum d'_i \otimes d''_i + \sum d'_j \otimes d''_j\beta_j e_j + \sum d_j\beta_j e'_j \otimes e''_j \qquad (*)$$

Denote the canonical projection $k(\Delta'_0, \mathcal{D}) \oplus k(\Delta'_1, \mathcal{D}) \to (k(\Delta'_0, \mathcal{D}) \oplus k(\Delta'_1, \mathcal{D}))/k(\Delta'_0, \mathcal{D})$ by π. Let $id \otimes \pi$ act on the both sides of $(*)$, we get

$$\psi(a') \otimes \overline{\psi(a''\beta b)} = \sum d'_j \otimes \overline{d''_j\beta_j e_j}$$

On the other hand,

$$(id \otimes \overline{\psi})\rho_L(\overline{a\beta b}) = (id \otimes \overline{\psi})(\psi(a') \otimes \overline{a''\beta b}) = \psi(a') \otimes \overline{\psi(a''\beta b)}$$

and

$$\delta_L\overline{\psi}(\overline{a\beta b}) = \delta_L(\overline{\psi(a\beta b)}) = \delta_L(\overline{\sum d_j\beta_j e_j}) = \sum d'_j \otimes \overline{d''_j\beta_j e_j}$$

Then

$$(id \otimes \overline{\psi})\rho_L(\overline{a\beta b}) = \delta_L\overline{\psi}(\overline{a\beta b})$$

This means that $\overline{\psi}$ is a left D-comodule map. Similarly, we have that it is also a right D-comodule map. Thus $\overline{\psi}$ is a D-bicomodule isomorphism. Therefore, $\phi = \pi_D\overline{\psi}\pi_C^{-1}$: $k(\Delta_1, \mathcal{C}) \to k(\Delta'_1, \mathcal{D})$ is a D-bicomodule isomorphism. So

$$^{T_{\varphi(i)}}k(\Delta_1, \mathcal{C})^{T_{\varphi(j)}} \stackrel{\phi}{\cong} {}^{T_{\varphi(i)}}k(\Delta'_1, \mathcal{D})^{T_{\varphi(j)}}$$

for any $i, j \in \Delta_0$. We complete the prove. □

Remark 3.7 *(1) As a special case of Theorem 3.1 for usual path coalgebras $k\Delta$, $k\Delta'$, we have $k\Delta \cong k\Delta'$ if and only if $\Delta \cong \Delta'$ since k is a trivial simple coalgebra.*

(2) The further question is whether the condition that S_i, T_j are simple coalgebras for all $i \in \Delta_0$, $\in \Delta'_0$ always is necessary so as to obtain the result in Theorem 3.1. Recall the fact that the theorem is not true even if any of S_i can be decomposed into direct sum of any two true subcoalgebras, i.e. see the example in Remark 2.1 (ii).

4 Generalized Dual Gabriel's Theorem

In this section, $\mathcal{C} = \{\text{simple coalgebras } S_i | i \in \Delta_0\}$ for some quiver $\Delta = (\Delta_0, \Delta_1)$.

Let V be a k-space, then $V \otimes C$ is a right C-comodule with the structure map $id \otimes \triangle_C$. We denote this comodule by $(V) \otimes C$. Similarly, we can define $C \otimes (V)$, $C \otimes (V) \otimes D$ where D is another coalgebra. A right (left) C-comodle M (N) is called *free* if there exsits a k-space V (W) such that $M \cong (V) \otimes C$ $(N \cong C \otimes (W))$. Similarly, a C-D-bicomodule ${}^C M^D$ is called *free* if ${}^C M^D \cong C \otimes (U) \otimes D$ for some space U.

It is easy to see every right comodule can be embedded in a free comodule (see Corollary 2 in [7]). In fact, for any right C-comodule M, its structure map δ_R is a C-comodule map from M to $(M) \otimes C$. Since $(id \otimes \varepsilon)\delta_R = id$, δ_R is a monomorphism. That's to say, $M \overset{\delta_R}{\hookrightarrow} (M) \otimes C$. Similarly, we have ${}^C M \hookrightarrow C \otimes (M)$, ${}^C M^D \hookrightarrow C \otimes (M) \otimes D$.

Lemma 4.1 *Let $M^C \in \mathcal{M}^C$. Then there exists a k-space V satisfying:*

(1) M can be embedded into $(V) \otimes C$

(2) For any k-space W with dim$(W) <$ dim(V), M can't embedded into $(W) \otimes C$.

The k-space V satisfying (1) (2) is called a *minimal realization of M as a free right C-comodule.*

Proof: Define $\mathcal{F} = \{V | M \hookrightarrow (V) \otimes C,\ \dim(V) \leq \dim(M)\}$. Clearly, $\mathcal{F}$ is a non-vacuous set since $M \hookrightarrow (M) \otimes C$. We define the partial order on $\mathcal{F}$ by $V \leq W$ if and only if $(V) \otimes C \supseteq (W) \otimes C$. If $\{V_i | i \in I\}$ is a chain in $\mathcal{F}$, then $\cap_{i \in I}(V_i \otimes C) = (\cap_{i \in I} V_i) \otimes C$ is an upper bound for the chain $\{V_i | i \in I\}$. By Zorn's Lemma, $\mathcal{F}$ contains a maximal V which is our desire. $\Box$

Similarly, for any ${}^C M^D \in {}^C\mathcal{M}^D$ $({}^C N \in {}^C\mathcal{M})$, we can find a minimal realization of ${}^C M^D$ $({}^C N)$ as a free C-D-bicomodule (left C-comodule).

Assume $M \in {}^C\mathcal{M}^C$ and $C = \oplus_{i \in I} C_i$ as coalgebras. Thus $M = \oplus_{i,j \in I} {}^{C_i}M^{C_j} = \oplus_{i,j \in I} {}^i M^j$ by Proposition 3.4, where ${}^i M^j := {}^{C_i}M^{C_j}$. Therefore, there exsists a minimal realization V_{ij} of ${}^i M^j$ as a free C_i-C_j-bicomodule for $i, j \in I$. We can define a quiver now. Let the set of vertices $\Delta_0 = I$. For $i, j \in I$, let the number of arrows from i to j be the dimension of V_{ij}. Obviously, if ${}^i M^j = 0$, then there are no arrows from i to j. Thus we get a quiver $\Delta = (\Delta_0, \Delta_1)$ called *the quiver of* $\mathrm{CoT}_C(M)$.

Recall a conclusion due to Heyneman-Radford (see [8]):

Lemma 4.2 *If $f: D \to E$ is a coalgebra map, then f is injective if and only if $f|_{D_1}$ is also so.*

Proposition 4.3 *Denote $\mathcal{C} = \{S_i | S_i$ are simple coalgebras, $i \in I\}$. Let $C = \oplus_{i \in I} S_i$, $M \in {}^C\mathcal{M}^C$ and $\Delta = (\Delta_0, \Delta_1)$ be the quiver of $CoT_C(M)$. Then there exists a coalgebra embedding $\psi: CoT_C(M) \hookrightarrow k(\Delta, \mathcal{C})$ such that $\psi(M) \subseteq k(\Delta_1, \mathcal{C})$.*

Proof: By Proposition 2.2, $k(\Delta,\mathcal{C}) \cong \mathrm{CoT}_{k(\Delta_0,\mathcal{C})}(k(\Delta_1,\mathcal{C}))$. So it is enough to prove that there is a coalgebra embedding $\psi: \mathrm{CoT}_C(M) \hookrightarrow \mathrm{CoT}_{k(\Delta_0,\mathcal{C})}(k(\Delta_1,\mathcal{C}))$. Clearly, ${}^ik(\Delta_1,\mathcal{C})^j \cong S_i \otimes (V_{ij}) \otimes S_j$ for $i,j \in \Delta_0$ where V_{ij} is a minimal realization of ${}^iM^j$ as a free C_i-C_j-bicomodule. Thus ${}^iM^j \overset{f_{ij}}{\hookrightarrow} {}^ik(\Delta_1,\mathcal{C})^j$ as S_i-S_j-bicomodules for $i,j \in \Delta_0$.

Denote $p_n: \mathrm{CoT}_C(M) \to M^{\Box n}$ the canonical projection for $n \geq 0$. Since $id: \mathrm{CoT}_C(M) \to \mathrm{CoT}_C(M)$ is a coalgebra map, we have p_0, p_1 are coalgebra map, C-bicomodule map respectively by Lemma 2.3. Define $\psi_0: \mathrm{CoT}_C(M) \to k(\Delta_0,\mathcal{C})$ be the compostion of p_0 and the identity map $id_C: C \to k(\Delta_0,\mathcal{C}) = \oplus_{i\in\Delta_0} S_i = C$. Clearly, ψ_0 is a coalgebra map. For any $x \in \mathrm{CoT}_C(M)$, denote $p_1(x) = \sum_{i,j\in\Delta_0} {}^im_x^j$ for ${}^im_x^j \in {}^iM^j$. Define $\psi_1: \mathrm{CoT}_C(M) \to k(\Delta_1,\mathcal{C})$ by $\psi_1(x) := \sum_{i,j\in\Delta_0} f_{ij}({}^im_x^j)$. ψ_1 is a C-bicomodule map since p_1, f_{ij} for $i,j \in \Delta_0$ are.

Comparing with Lemma 2.4, if we can prove that there are only finite i such that $\psi_i(x) \neq 0$, then $\psi = \sum_{i\geq 0} \psi_i$ is a coalgebra map, where $\psi_i = \psi_1^{\otimes i} \circ \triangle^{(i-1)}$. But it is clear. Thus $\psi = \sum_{i\geq 0} \psi_i$ is a coalgebra map from $\mathrm{CoT}_C(M)$ to $\mathrm{CoT}_{k(\Delta_0,\mathcal{C})}(k(\Delta_1,\mathcal{C}))$. By Proposition 2.5 and its proof, the coradical of $\mathrm{CoT}_C(M)$ is C and $\wedge^2_{\mathrm{CoT}_C(M)} C = C \oplus M$. On the other hand, by the definition of ψ, $\psi|_{C\oplus M}$ is injective. Thus $\psi: \mathrm{CoT}_C(M) \hookrightarrow \mathrm{CoT}_{k(\Delta_0,\mathcal{C})}(k(\Delta_1,\mathcal{C}))$ is injective by Lemma 4.2. Clearly, $\psi(M) \subseteq k(\Delta_1,\mathcal{C})$. □

Recall that a coalgebra C is called to have *separable coradical* (see [12]) if for every simple subcoalgebra D of C, the dual algebra D^* is a separable algebra, or says, the coradical C_0 of C is a coseparable coalgebra (see [7]).

Lemma 4.4 (see Theorem 5.4.2 in [12]) *Let C be a coalgebra with separable coradical. Then there exists a coideal I of C such that $C = I \oplus C_0$ as k-spaces, that is, there exists a coalgebra projection of C onto C_0.*

Let $\{C_n\}$ be the coradical filtration of C. Denote $\pi: C \to C/C_0$ to be the canonocal projection. Since $\triangle(C_0) \subseteq C_0 \otimes C_0$ and $\triangle(C_1) \subseteq C_0 \otimes C_1 + C_1 \otimes C_0$, we have two maps $\delta_L: C_1/C_0 \to C_0 \otimes C_1/C_0$ and $\delta_R: C_1/C_0 \to C_1/C_0 \otimes C_0$, where $\delta_L(\pi(x)) := (id \otimes \pi)\triangle(x)$, $\delta_R(\pi(x)) := (\pi \otimes id)\triangle(x)$ for $x \in C_1$. Clearly, C_1/C_0 is a C_0-bicomodule via δ_L, δ_R defined above.

In [14], the dimension $\mathrm{Dim}A$ of a k-algebra A is defined as $\mathrm{Dim}A = sup\{n : H_k^n(A,M) \neq 0$ for some A-bimodule $M\}$ where $H_k^n(A,M)$ means the n'th Hochschild cohomology module of A with coefficients in M. In particular, $\mathrm{Dim}A = 0$ if and only if A is a separable k-algebra. By Corollary 10.7b of [14], when k is a perfect field (e.g. $\mathrm{char}k = 0$ or k is a finite field), A is separable if and only if A is finite dimensional and semisimple. $\mathrm{Dim}A = 1$ if and only if every factor set of A with values in any A-A-bimodule M is split. According to the famous Wedderburn-Malcev Theorem (see [14]), for a finite dimensional k-algebra A and its radical r, if $\mathrm{Dim}A/r \leq 1$, then A/r can be lifted. Moreover, for a coalgebra C, we define $\mathrm{Codim}C = sup\{\mathrm{Dim}D^* | D$ is any finite dimensional subcoalgebra of $C\}$.

Denote p.dim$({}_AM)$ the projective dimension of $M \in {}_A\mathcal{M}$ as left A-module. Similarly, p.dim(M_A) and p.dim$({}_AM_A)$ denote the projective dimensions of M as right A and A-bimodules respectively. The following lemma is needed:

Lemma 4.5 *(i) If $A = A_1 \oplus \cdots \oplus A_s$ for algebras A, A_i $(i = 1, \cdots, s)$, then* Dim$A$ = *max*$\{\text{Dim}A_i : i = 1, \cdots, s\}$*;*

(ii) If $C = D_1 \oplus \cdots \oplus D_s$ for coalgebras C, D_i $(i = 1, \cdots, s)$, then Codim$C \geq$ *max*$\{\text{Codim}D_i : i = 1, \cdots, s\}$.

Proof: (i) It is well known that the dimension of an algebra B defined above equals the projective dimension of B as B-bimodule (see Section 3 of Chapter X in [10]), i.e. Dim$B$ =p.dim${}_BB_B$. Denote $S = \{1, \ldots, s\}$. Clearly, for $i \in S$, A_i is an A-bimodule by the natural way, and denote this A-bimodule by ${}_A(A_i)_A$. Thus ${}_AA_A = \oplus_{i=1}^s \, {}_A(A_i)_A$ as A-bimodules. So, p.dim$({}_AA_A) = max\{\text{p.dim}({}_A(A_i)_A) | i \in S\}$. Since Dim$A_i$ =p.dim${}_{A_i}(A_i)_{A_i}$, in order to verify (i), it is enough to prove that p.dim$({}_A(A_i)_A)$ =p.dim$({}_{A_i}(A_i)_{A_i})$ for each $i \in S$. In order to prove this, we prove the following claims at first:

Claim 1: For any $M \in {}_A\mathcal{M}_A$, $M = \oplus_{i,j \in S} \, {}_iM_j$, where ${}_iM_j := A_iMA_j$

Claim 2: Let $f : {}_AM_A \to {}_AN_A$ be an A-bimodule map and $f_{ij} := f|_{{}_iM_j}$ for $i, j \in S$. Then $f_{ij}({}_iM_j) \subseteq {}_iN_j$ and $f = \oplus_{i,j \in S} f_{ij}$.

Claim 3: For $i, j \in S$, f_{ij} is an A_i-A_j-bimodule map as well as an A-bimodule map, where we consider the A-bimodule ${}_iM_j$ as an A_i-A_j-bimodule naturally.

Claim 4: Given an A-bimodules complex

$$0 \longrightarrow M \stackrel{f}{\longrightarrow} N \stackrel{g}{\longrightarrow} L \longrightarrow 0$$

then it is exact if and only if the following complexes are exact:

$$0 \longrightarrow {}_iM_j \stackrel{f_{ij}}{\longrightarrow} {}_iN_j \stackrel{g_{ij}}{\longrightarrow} {}_iL_j \longrightarrow 0$$

for $i, j \in S$.

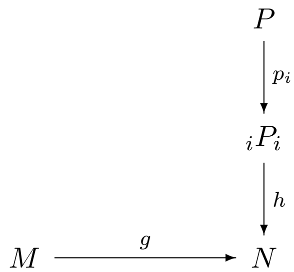
Claim 5: If P is a projective A-bimodule, then ${}_iP_i$ is a projective A_i-bimodule for $i \in S$.

Claim 6: Let ${}_{A_i}P_{A_i}$ be a projective A_i-bimodule, then ${}_{A_i}P_{A_i}$ is a projective A-module where the A-bimodule structure on ${}_{A_i}P_{A_i}$ is induced by the canonical algebra map $\pi_i : A \to A_i$.

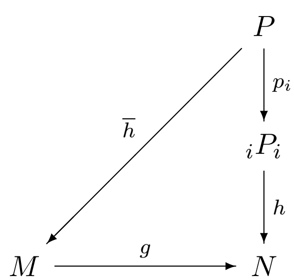
Claims 1,2,3,4 can be proved directly and left to readers. We prove Claims 5,6 now. Given following A_i-bimodules diagram

$$\begin{array}{ccc} & & {}_iP_i \\ & & \downarrow h \\ M & \stackrel{g}{\longrightarrow} & N \end{array}$$

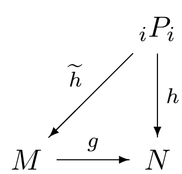
where M, N are A_i-bimodules, $g: M \to N$ is an A_i-bimodule epimorphism and $h: {}_iP_i \to N$ is an A_i-bimodule morphism. Let $p_i: P \to {}_iP_i$ be the canonical A-bimodule projection for $i \in S$. Clearly, through the canonical algebra projection $\pi_i: A \to A_i$, ${}_iP_i$, M, N are A-bimodules and h, g can be considered as A-bimodules maps. Thus we have the following A-bimodules diagram



When P is a projective A-bimodule, there exists an A-bimodule map $\overline{h}: P \to M$ satisfying the following diagram commute.



Let $\iota_i: {}_iP_i \to P$ be the canonical injection as A-bimodules. Define $\widetilde{h}: {}_iP_i \to M$ by $\widetilde{h} := \overline{h}\iota_i$. Thus $\widetilde{h}$ is an A-bimodule map, and, clearly, an A_i-bimodule map. On the other hand, $g\widetilde{h} = g\overline{h}\iota_i = hp_i\iota_i = h$ which implies that we have the following commuting diagram:



Therefore, ${}_iP_i$ is a projective A_i-bimodule. It means the Claim 5. Next, let us prove the Claim 6. Let ${}_{A_i}P_{A_i}$ be a projective A_i-bimodule, then it is a direct summand of a free left $A_i \otimes A_i^{op}$-modules M (where we consider an A_i-bimodule as a left $A_i \otimes A_i^{op}$-module and A_i^{op} denote the opposite algebra of A_i). Thus M is a direct sum of some copies of $A_i \otimes A_i^{op}$. Under the canonical algebra projection $\pi_i: A \to A_i$, ${}_{A_i}P_{A_i}$ and M become A-bimodules and ${}_{A_i}P_{A_i}$ is a direct summand of M as A-bimodules. But, clearly, $A_i \otimes A_i^{op}$ is a direct summand of $A \otimes A^{op}$ as $A \otimes A^{op}$-modules. Therefore, M is a projective A-bimodule and thus ${}_{A_i}P_{A_i}$ is a projective A-bimodule.

Given an projective resolution of A_i as an A-bimodule

$$\cdots \to P_n \to \cdots \to P_1 \to P_0 \to A_i \to 0$$

then by Claim 4 and Claim 5, we have the following projective resolution of A_i as an A_i-bimodule

$$\cdots \to {}_i(P_n)_i \to \cdots \to {}_i(P_1)_i \to {}_i(P_0)_i \to A_i \to 0$$

This implies that p.dim$({}_{A_i}(A_i)_{A_i}) \leq$ p.dim$({}_A(A_i)_A)$. Conversely, given a projective resolution of A_i as an A_i-bimodule

$$\cdots \to P_n \to \cdots \to P_1 \to P_0 \to A_i \to 0$$

then by Claim 6, it is also a projective resolution of A_i as an A-bimodule. So, p.dim$({}_A(A_i)_A)$ $\leq$p.dim$({}_{A_i}(A_i)_{A_i})$. Thus p.dim$({}_A(A_i)_A)$ =p.dim$({}_{A_i}(A_i)_{A_i})$.

(ii) Since every finite dimensional subcoalgbera of any D_i for $i = 1, \ldots, s$ is also a subcoalgbera of C, $CodimC \geq max\{CodimD_i : i = 1, \cdots, s\}$. □

Lemma 4.6 *For a coalgebra C,* Codim$C_0 = 0$ *if and only if C is with separable coradical.*

Proof: "If" part: Any finite dimensional subcoalgebra D of C_0 is a cosemisimple coalgebra. By C is coalgebra with separable coradical and Lemma 4.5, Dim$D^* = 0$. This means Codim$C_0 = 0$.

"Only if" part: This direction can be gotten from Lemma 4.5 and the fact that an algebra A is a separable algebra if and only if Dim$A = 0$. □

Our main result of this section is:

Theorem 4.7 *Let C be a coalgebra satisfying $CodimC_0 \leq 1$. Write $C_0 = \oplus_{i\in\Lambda} S_i$ with S_i simple coalgebras for $i \in \Lambda$. Then*

(a) (Wedderburn-Malcev Theorem on Coalgebra) *There exists a coideal I of C such that $C = I \oplus C_0$ as k-spaces. That is, there exists a coalgebra projection of C onto C_0.*

(b) *Assume that C_1/C_0 is a direct summand of C/C_0 as C_0-bicomodules. Then*

(i) *There is an injective coalgebra map $\psi : C \hookrightarrow CoT_{C_0}(C_1/C_0)$.*

(ii) (Generalized Dual Gabriel Theorem) *Let $\Delta = (\Delta_0, \Delta_1)$ be the quiver of $CoT_{C_0}(C_1/C_0)$, $\mathcal{C} = \{S_i | i \in \Lambda\}$. There is a coalgebra embedding $\varphi : C \hookrightarrow k(\Delta, \mathcal{C})$ with $\varphi(I_1) \subseteq k(\Delta_1, \mathcal{C})$ for $I_1 = I \cap C_1$.*

Proof: (a): We first show that if C is finite-dimensional and D is a subcoalgebra, then a projection π from D to $D_0 = D \cap C_0$ can be extended to a projection from C to C_0. Let $\alpha = \pi' \circ i$ be the composite of the imbedding $i : C_0 \to C$ and the quotient map $\pi' : C \to C/Ker\pi = E$. Then $Ker\alpha = \{0\}$ and $Im\alpha \subseteq E_0$. By Corollary 5.3.5 in [12], $E_0 \subseteq Im\alpha$, thus $E_0 = Im\alpha$. Then $\alpha^* : E^* \to (C_0)^*$ is a surjective algebra morphism.

Now, we can claim that there is a subalgebra B of E^* such that $E^* = B \oplus$ Jac(E^*) and $B \cong (C_0)^*$. According to the well-known Wedderburn-Malcev Theorem, it is enough to prove that $(C_0)^* \cong E^*/$Jac(E^*) and Dim$(C_0)^* \leq 1$, in which the latter is assured by the hypothesis in Theorem. Thus we need only to show that $(C_0)^* \cong E^*/\mathrm{Jac}(E^*)$. Since $\alpha^* : E^* \to (C_0)^*$ is a surjective algebra map and $(C_0)^*$ is semisimple, $Ker\alpha^* \supseteq \mathrm{Jac}(E^*)$ which implies there is a natural epimorphism $E^*/\mathrm{Jac}(E^*) \twoheadrightarrow E^*/Ker\alpha^* \cong (C_0)^*$. Then it is easy to see that $(C_0)^*$ is a direct summand of $E^*/\mathrm{Jac}(E^*)$, i.e. $E^*/\mathrm{Jac}(E^*) = (C_0)^* \oplus A$ for some finite dimensional semisimple algebra A. This follows that there is an algebra epimorphism $\pi : E^* \twoheadrightarrow (C_0)^* \oplus A$. Then $\pi^* : C_0 \oplus A^* = (C_0)^{**} \oplus A^* \hookrightarrow E = E^{**}$ as coalgebras. But, A^* is cosemisimple. So, really, $\pi^* : C_0 \oplus A^* \hookrightarrow E_0$. From the fact that the coradical E_0 of E equals $Im\alpha \cong C_0$, we get $A^* = 0$ and then $(C_0)^* \cong E^*/\mathrm{Jac}(E^*)$.

Therefore, $E^* = B \oplus \mathrm{Jac}(E^*)$ where $B \cong (C_0)^*$ as algebras. Then there exists an algebra projection from $\phi : (C_0)^* \to E^*$ such that $\alpha^* \circ \phi = \mathrm{id}_{(C_0)^*}$. Now set $\widetilde{\pi} = \phi^* \circ \pi'$, it is a projection from C to C_0 and extends π.

Now let $\mathcal{F}$ be the set of all pairs (F, π), where F is a subcoalgebra of C and $\pi : F \to F_0$ a projection. Since (C_0, id) is such a pair, $\mathcal{F} \neq \emptyset$. Moreover, $\mathcal{F}$ is an ordered set via

$$(F', \pi') \leq (F, \pi) \quad \text{if and only if} \quad F' \subset F \text{ and } \pi|_{F'} = \pi'$$

We apply Zorn's Lemma to $\mathcal{F}$ and obtain a maximal element (F, π); we claim that $F = C$.

Suppose $F \neq C$. Then there exists $c \in C$, $c \notin F$; let D be the subcoalgebra generated by c. Then $Im(\pi|_{F\cap D}) = (F \cap D)_0$. Since D is finite dimensional, the argument at the beginning of the proof applied to $F \cap D \subset D$ shows that there exists a projection $\pi_1 : D \to D_0$ which extends $\pi|_{F\cap D}$. Since (D, π_1) and (F, π) have projections which agree on $F \cap D$, they extend to a projection $\pi_2 : F + D \to (F + D)_0$. This contradicts the maximality of F. Thus $F = C$.

(b): (i): By (a), there exists a coalgebra projection $p : C \to C_0$. Denote $I = ker(p)$. Then $C = C_0 \oplus I$. Define ψ_0 be the composition of p and identity map $id : C_0 \to C_0$. Clearly, $\psi_0 : C \to C_0$ is a coalgebra map.

Clearly, C and I are C_0-bimodules via left and right C_0-comodule structure maps $\rho_L := (p \otimes id)\triangle$, $\rho_R := (id \otimes p)\triangle$, and $I \cong C/C_0$ as C_0-bicomodules. Denote $I_1 = I \cap C_1$, then $C_1 = I_1 \oplus C_0$. Since $\triangle(I_1) \subseteq (I \otimes C + C \otimes I) \cap (C_0 \otimes C_1 + C_1 \otimes C_0) \subseteq I_1 \otimes C_0 + C_0 \otimes I_1$, I_1 is a C_0-bicomodule. Clearly, $I_1 \overset{\phi}{\cong} C_1/C_0$. By the assumption in Theorem, there exists a C_0-bicomodule projection from C/C_0 to C_1/C_0. Thus we get a C_0-bicomodule projection $p'' : I \to I_1$ since $I \cong C/C_0$ and $I_1 \cong C_1/C_0$. Denote the canonical C_0-bicomodule projection from C to I by p' and define $\psi_1 = \phi \circ p'' \circ p' : C \to C_1/C_0$. Therefore, ψ_1 is a C_0-bicomodule map.

In order to apply Lemma 2.4, we need to prove there are only finite $\psi_i(x) \neq 0$ for each $x \in C$, where $\psi_i = \psi_1^{\otimes i} \circ \triangle^{i-1}$ for $i \geq 1$. For any $x \in C$, there exists n such that $x \in C_n$ since $\{C_i\}_{i \geq 0}$ is a coalgebra filtration of C. Note that $\triangle^{(j-1)}(C_n) \subseteq \sum_{n1+\cdots+nj=n} C_{n1} \otimes \cdots \otimes C_{nj}$ for $j \geq 1$ and $\psi_1(C_0) = 0$ by the definition of ψ_1, it follows $\psi_m(x) = 0$ for $m > n$. Thus there are only finite $\psi_i(x) \neq 0$ for every x of C.

Applying Lemma 2.4, we have a coalgebra map $\psi = \sum_{i\geq 0} \psi_i : C \to \mathrm{CoT}_{C_0}(C_1/C_0)$. In order to prove ψ is injective, it is enough to prove that $\psi|_{C_1}$ is injective by Lemma 4.2. But it is a easy consequence by the definition of ψ.

(ii): By Proposition 4.3, there exists a coalgebra embedding $\psi' :\ \mathrm{CoT}_{C_0}(C_1/C_0) \hookrightarrow k(\Delta, \mathcal{C})$ such that $\psi'(C_1/C_0) \subseteq k(\Delta_1, \mathcal{C})$. Let $\varphi :\ C \to k(\Delta, \mathcal{C})$ be the composition of

$$C \overset{\psi}{\hookrightarrow} \mathrm{CoT}_{C_0}(C_1/C_0) \overset{\psi'}{\hookrightarrow} k(\Delta, \mathcal{C})$$

It is easy to see φ is our desire map. □

Corollary 4.8 *Let C be a coalgebra satisfying $CodimC_0 = 0$. Write $C_0 = \oplus_{i\in I} S_i$ with S_i simple coalgebras for $i \in \Lambda$. Then*

(i) *There is an injective coalgebra map $\psi :\ C \hookrightarrow CoT_{C_0}(C_1/C_0)$.*

(ii) *Let $\Delta = (\Delta_0, \Delta_1)$ be the quiver of $CoT_{C_0}(C_1/C_0)$, $\mathcal{C} = \{S_i | i \in I\}$. There is a coalgebra embedding $\varphi :\ C \hookrightarrow k(\Delta, \mathcal{C})$ with $\varphi(I_1) \subseteq k(\Delta_1, \mathcal{C})$ for $I_1 = I \cap C_1$.*

Proof: By Theorem 4.7, it is enough to prove that C_1/C_0 is a direct summand of C/C_0 as C_0-bicomodules. But, by assumption and Lemma 4.6, the coradical C_0 of C is a coseparable coalgebra (see [7]). So, $C_0 \otimes C_0^{cop}$ is a cosemisimple coalgebra (by Proposition 12 of [7]). Thus every C_0-bicomodule is cosemisimple (Here we view a C_0-bicomodule as a left $C_0 \otimes C_0^{cop}$- comodule). Therefore, there exists a C_0-bicomodule projection from C/C_0 to C_1/C_0. □

Note that (1): The result of Theorem 4.7 (a) is a generalization of the Wedderburn-Malcev Theorem on Coalgebras with separable coradicals (that is, Lemma 4.4);

(2): The major result in [2], i.e. the so-called Dual Gabriel Theorem on co-separable type coalgebras (i.e. coalgebras with separable coradicals) is just (i) of Corollary 4.8;

(3): Comparing (ii) with (i) in Theorem 4.7, one can see the difference between our understand on Dual Gabriel Theorem in this paper and that in [2] and other papers (e.g. [3]). Our Dual Gabriel Theorem here represents the effect of generalized path coalgebra to dualize the Gabriel Theorem.

Remark 4.9 *The results in Corollary 4.8 always hold respectively under the following simple conditions since any one of them can deduce the separability of the coradical C_0 of C:*

(1) C is a pointed coalgebra;

(2) k is algebraically closed:

(3) $chark = 0$

We call the quiver of $\mathrm{CoT}_{C_0}(C_1/C_0)$ as *the quiver* of C. Denote it by $\Delta_C = ((\Delta_C)_0, (\Delta_C)_1)$. Giving two quivers $\Delta = (\Delta_0, \Delta_1)$, $\Delta' = (\Delta'_0, \Delta'_1)$, We say $\iota : \Delta \hookrightarrow \Delta'$ provided that ι

is a one to one map from Δ_0 to Δ'_0 and the number of arrows from i to j is not larger than that of from $\iota(i)$ to $\iota(j)$ for any two vertices i, j of Δ. The next theorem shows the uniqueness of Δ_C in some sense.

Theorem 4.10 *Assume that C is a coalgebra satisfying $CodimC_0 \leq 1$ and C_1/C_0 is a direct summand of C/C_0 as C_0-bicomodules. Let I be the coideal of C such that $C = I \oplus C_0$ as k-spaces and $I_1 = I \cap C_1$. If there is a quiver $\Delta = (\Delta_0, \Delta_1)$ and a generalized path coalgebra $k(\Delta, \mathcal{D})$ with a set of simple coalgebras $\mathcal{D} = \{T_i | i \in \Delta_0\}$ satisfying $C \overset{\psi}{\hookrightarrow} k(\Delta, \mathcal{D})$ and $\psi(I_1) \subseteq k(\Delta_1, \mathcal{D})$, then $\iota : \Delta_C \hookrightarrow \Delta$ and $S_i \cong T_{\iota(i)}$ for $i \in \Delta_0$.*

Proof: Note that by Theorem 4.7 (a), such I always exists.

Clearly, the image of $\psi|_{S_i}$ is a simple subcoalgebra of $k(\Delta, \mathcal{D})$ for $i \in (\Delta_C)_0$. By Corollary 2.6, $D := \oplus_{j \in \Delta_0} T_j$ is the coradical of $k(\Delta, \mathcal{D})$. Thus $Im(\psi|_{S_i}) = T_{\iota(i)}$ for a unique $\iota(j) \in \Delta_0$. Clearly, $S_i \overset{\psi}{\cong} T_{\iota(j)}$. Therefore, there is one to one map $\iota : (\Delta_C)_0 \to \Delta_0$ by $i \mapsto \iota(i)$ and $S_i \cong T_{\iota(i)}$ as coalgebras.

Next we only need to prove that the number of arrows from i to j is not large than that of from $\iota(i)$ to $\iota(j)$ for any i, j of $(\Delta_C)_0$. Denote $p_i : k(\Delta, \mathcal{D}) = \mathrm{CoT}_{k(\Delta_0, \mathcal{D})}(k(\Delta_1, \mathcal{D})) \to k(\Delta_i, \mathcal{D})$ the canonical projection for $i \geq 0$. Then by Lemma 2.3, $\psi_1 = p_1\psi : C \to k(\Delta_1, \mathcal{D})$ is a D-bicomodule map where C is a D-bicomodule map induced by $p_0\psi$. Thus $\psi_1|_{C_1} : C_1 \to k(\Delta_1, \mathcal{D})$ is also a D-bicomodule map since C_1 is a subcoalgebra. Clearly, $C_1 = C_0 \oplus I_1$.

We claim that $\psi_1|_{I_1}$ is also a D-bicomodule map. In fact, it is enough to show that I_1 is also a D-bicomodule. We only show that it is a left D-comodule since we can prove the right case similarly. By the definition of left D-comodule structure map ρ_L on C, $\rho_L(c) = p_0\psi(c') \otimes c''$ for $c \in C$. We have shown that $\triangle(I_1) \subseteq I_1 \otimes C_0 + C_0 \otimes I_1$ in the proof of Theorem 4.5. Thus $\rho_L(I_1) \subseteq p_0\psi(I_1) \otimes C_0 + p_0\psi(C_0) \otimes I_1 = p_0\psi(C_0) \otimes I_1$ since $\psi(I_1) \subseteq k(\Delta_1, \mathcal{D})$ and thus $p_0\psi(I_1) = 0$. This means $\rho_L(I_1) \subseteq D \otimes I_1$ and I_1 is a left D-comodule.

But $\psi_1|_{I_1} = \psi|_{I_1}$ since $\psi(I_1) \subseteq k(\Delta_1, \mathcal{D})$, which implies $C_1/C_0 = I_1$ can be embedded into $k(\Delta_1, \mathcal{D})$ as D-bicomodules. Therefore, ${}^{S_i}(C_1/C_0)^{S_j} \hookrightarrow {}^{T_{\iota(i)}}k(\Delta_1, \mathcal{D})^{T_{\iota(j)}}$ as $T_{\iota(i)}$-$T_{\iota(j)}$-bicomodules. Let V_{ij} be a minimal realization of ${}^{S_i}(C_1/C_0)^{S_j}$ as a free $T_{\iota(i)}$-$T_{\iota(j)}$-bicomodule. By the definition of $k(\Delta, \mathcal{D})$, there exists a k-space W_{ij} such that ${}^{T_{\iota(i)}}k(\Delta_1, \mathcal{D})^{T_{\iota(j)}} \cong T_{\iota(i)} \otimes (W_{ij}) \otimes T_{\iota(j)}$. Note the number of arrows from i to j in Δ_C is $\dim(V_{ij})$ and that from $\iota(i)$ to $\iota(j)$ is $\dim(W_{ij})$, it follows the desire conclusion since $\dim(V_{ij}) \leq \dim(W_{ij})$. $\square$

In [11], the author defined the quiver $\Gamma(C)$ of C as follows: the set of vertices of $\Gamma(C) = (\Gamma(C)_0, \Gamma(C)_1)$ is the set of simple subcoalgebras of C; and for any two simple subcoalgebras S_1 and S_2, there are exactly $\dim((S_1 \wedge_C S_2)/(S_1 + S_2))$ arrows from S_1 to

S_2. Giving two quivers $\Delta = (\Delta_0, \Delta_1)$, $\Delta' = (\Delta'_0, \Delta'_1)$, we say Δ is a *wide subquiver* of Δ' if Δ is a subquiver of Δ' and $\Delta_0 = \Delta'_0$. The name "wide" comes from groupoid theory. A subgroupoid $G \subseteq G'$ is called a wide subgroupoid if $G_0 = G'_0$ (see p.88 of [6]). The relationship between Δ_C and $\Gamma(C)$ is given as follows:

Theorem 4.11 *Assume that C is a coalgebra satisfying $CodimC_0 \leq 1$. Then Δ_C is a wide subquiver of $\Gamma(C)$.*

Proof: Clearly, $(\Delta_C)_0 = \Gamma(C)_0$. In order to prove Δ_C is a subquiver of $\Gamma(C)$, it is enough to prove that the number arrows from i to j in Δ_C is less than that in $\Gamma(C)$ for any $i, j \in (\Delta_C)_0$.

We claim that $(S_i \wedge_C S_j)/(S_i + S_j) \cong {}^i(C_1/C_0)^j$ as S_i-S_j-bicomodules for $i, j \in (\Delta_C)_0$. This claim has been proved in [2]. But for completeness, we prove it again. Firstly, we show that $(S_i \wedge_C S_j) \cap C_0 = S_i + S_j$. Clearly, $S_i + S_j \subseteq (S_i \wedge_C S_j) \cap C_0$. On the other hand, if there exist a simple coalgebra $S_l \subseteq (S_i \wedge_C S_j) \cap C_0$ with $l \neq i, j$, then $\triangle(S_l) \subseteq S_l \otimes S_l$. Thus S_l does not belong to $S_i \wedge_C S_j$ and then $S_i + S_j \supseteq (S_i \wedge_C S_j) \cap C_0$.

We define a map from $\xi : (S_i \wedge_C S_j)/(S_i + S_j) \to C_1/C_0$ by $x + (S_i + S_j) \mapsto x + C_0$ for $x \in S_i \wedge_C S_j$. Clearly, ξ is well-defined. If $x + C_0 = y + C_0$ for $x, y \in S_i \wedge_C S_j$, then $x - y \in C_0 \cap (S_i \wedge_C S_j) = S_i + S_j$ and thus $x + (S_i + S_j) = y + (S_i + S_j)$. This means that ξ is a injective map. Let $\pi : S_i \wedge_C S_j \to (S_i \wedge_C S_j)/(S_i + S_j)$ be the canonical projection. Define $\delta_L : (S_i \wedge_C S_j)/(S_i + S_j) \to S_i \otimes (S_i \wedge_C S_j)/(S_i + S_j)$ and $\delta_R : (S_i \wedge_C S_j)/(S_i + S_j) \to (S_i \wedge_C S_j)/(S_i + S_j) \otimes S_j$ by $\delta_L(\pi(x)) := (id \otimes \pi)\triangle(x)$, $\delta_R(\pi(x)) := (\pi \otimes id)\triangle(x)$ for $x \in S_i \wedge_C S_j$. It is easy to show that $(S_i \wedge_C S_j)/(S_i + S_j)$ is an S_i-S_j-bicomodule with the structure map δ_L, δ_R and ξ is an S_i-S_j-bicomodule embedding from $(S_i \wedge_C S_j)/(S_i + S_j)$ to ${}^i(C_1/C_0)^j$. Therefore, the claim will follow if we can show ξ is also an epimorphism. In order to do it, we show that $\sum_{i.j \in (\Delta_C)_0} S_i \wedge_C S_j = C_1$ at first. In fact, write $C = C_0 \oplus I$ by Theorem 4.7. For each $i \in \Delta_0$, define $\varepsilon_i \in C^*$ as follows: $\varepsilon_i|_{S_i} = \varepsilon|_{S_i}$, $\varepsilon_i|_{S_j + I} = 0$ for $j \neq i$. For any element x of C_1, we see that $x = \sum_{i,j \in \Delta_0} \varepsilon_i \rightharpoonup x \leftharpoonup \varepsilon_j$ and $\varepsilon_i \rightharpoonup x \leftharpoonup \varepsilon_j \in S_i \wedge_C S_j$. Therefore, $C_1 = \sum_{i.j \in (\Delta_C)_0} S_i \wedge_C S_j$ and thus $C_1/C_0 = \sum_{i.j \in (\Delta_C)_0} (S_i \wedge_C S_j + C_0)/C_0 \subseteq \sum_{i.j \in (\Delta_C)_0} {}^i(C_1/C_0)^j = \oplus_{i.j \in (\Delta_C)_0} {}^i(C_1/C_0)^j = C_1/C_0$, where $(S_i \wedge_C S_j + C_0)/C_0 \subseteq {}^i(C_1/C_0)^j$ follows from that ξ is injective. This implies ξ is an epimorphism.

Denote $t_{ij} = \dim((S_i \wedge_C S_j)/(S_i + S_j))$ for $i, j \in (\Delta_C)_0$. By the isomorphism $(S_i \wedge_C S_j)/(S_i + S_j) \cong {}^i(C_1/C_0)^j$ for $i, j \in (\Delta_C)_0$, we have ${}^i(C_1/C_0)^j \hookrightarrow S_i \otimes ((S_i \wedge_C S_j)/(S_i + S_j)) \otimes S_j$. Note the number of arrows l_{ij} from i to j in Δ_C equals the dimension of a minimal realization of ${}^i(C_1/C_0)^j$, it follows $l_{ij} \leq t_{ij}$. □

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