Characterization Of Left Artinian Algebras Through Pseudo Path Algebras*

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Abstract

In this paper, using of pseudo path algebras, we generalize the Gabriel's Theorem on elementary algebras to left Artinian algebras over a field k when the quotient algebra can be lifted, in particular, when the Dimension of the quotient algebra decided by the n'th Hochschild cohomology is less than 2 (e.g. k is finite or char$k = 0$); and, using of generalized path algebras, the Generalized Gabriel's Theorem is given for finite dimensional algebras with 2-nilpotent radicals. As a tool, the so-called pseudo path algebras are introduced as a new generalization of path algebras, which can cover generalized path algebras (see Fact 2.5).

The main result is that (i) for a left Artinian k-algebra A and $r = r(A)$ the radical of A, when the quotient algebra A/r can be lifted, it holds that $A \cong PSE_k(\Delta, \mathcal{A}, \rho)$ with $J^s \subset \langle\rho\rangle \subset J$ for some s (Theorem 3.2); (ii) for a finite dimensional k-algebra A with $r = r(A)$ 2-nilpotent radical, it holds that $A \cong k(\Delta, \mathcal{A}, \rho)$ with $\widetilde{J}^2 \subset \langle\rho\rangle \subset \widetilde{J}^2 + \widetilde{J} \cap Ker\widetilde{\varphi}$ (Theorem 4.3), where Δ is the quiver of A and ρ is a set of relations.

In all cases we discuss, the uniqueness of such quivers Δ and generalized path algebra/pseudo path algebras satisfying the isomorphism relations is obtained (see Theorem 3.5 and 4.4).

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1 Introduction

We will always suppose that k denotes a field and all modules are unital. A left Artinian algebra implies that it satisfies the descending chain condition on left ideals.

As well-known, for a finite dimensional algebra A over k an algebraically closed field and its nilpotent radical $N = J(A)$, the quotient algebra A/N is semisimple, that is, there are uniquely positive integers $n_1 \leq n_2 \leq \cdots \leq n_r$ such that $A \cong M_{n_1}(k) \oplus \cdots \oplus M_{n_r}(k)$ where $M_{n_i}(k)$ denote the algebras of n_i by n_i matrices with entries in k, which are k-simple algebras. In the special case that A is an *elementary* algebra, due to the definition, every $n_i = 1$, i.e. $M_{n_i} \cong k$, or say, A/N is a direct sum of some k as k-algebras, writing $A/N = \coprod_r (k)$.

It is known that every finite dimensional path algebra is elementary. Conversely, according to the famous Gabriel's Theorem (see [1]), for each elementary algebra Λ, one can construct the correspondent quiver $\Gamma(\Lambda)$ of Λ such that Λ is isomorphic to a quotient algebra of the path algebra

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$k\Gamma(\Lambda)$. On the other hand, the module category of any algebra A must be Morita equivalent to the module category of one certain elementary algebra $\overline{A}$ (see [3]). Therefore, in the view point of representation theory, it may be enough for us to research representations of elementary algebras, i.e. of quotient algebras of path algebras. In particular, it provides the description of the finitely generated modules over some given algebras (see, for instance [1], [5]).

However, in the view point of structures of algebras, researching on finite dimensional algebras can not be replaced with that on elementary algebras. For example, it is important to give a possible classification for finite dimensional algebras.

In this reason, Prof. Shao-xue Liu, one of the authors of [2], raised an interesting problem, that is, how to find a generalization of path algebras so as to obtain a generalized Gabriel Theorem for arbitrary finite dimensional algebras to be isomorphic to a quotient algebra of such a generalized path algebra. The first step on this idea was finished by himself with his cooperator in [2] where the valid concept of generalized path algebras was introduced (see Section 2). But, the further research could not be found before our article.

The aim of this paper is to answer the Liu's problem affirmatively through generalized path algebras under the meaning of [2] and pseudo path algebras. We give some preliminary results in Section 2. In fact, we find generalized path algebras is not enough to characterize more algebras unless the algebras are finite-dimensional with 2-nilpotent radicals. In this reason, the so-called pseudo path algebras are introduced as a new generalization of path algebras, which can cover generalized path algebras (see Fact 2.5). In Section 3, using of pseudo path algebras, we generalize the Gabriel's Theorem on elementary algebras to left Artinian algebras over a field k when the quotient algebra can be lifted, in particular, when the Dimension of the quotient algebra decided by the n'th Hochschild cohomology is less than 2 (e.g. k is finite or char$k = 0$). In Section 4, using of generalized path algebras, the Generalized Gabriel's Theorem is given for finite dimensional algebras with 2-nilpotent radicals. In all cases we discuss, the uniqueness of such quivers Δ and generalized path algebra/pseudo path algebras is obtained (see Theorem 3.5 and 4.4).

It is interesting to note in our Generalized Gabriel's Theorems, the dependence on the ground field is not very necessary under some certain conditions, which implies the possibility to make an approach to modular representations of algebras and groups. Also, note that if $A \cong k(\Delta, \mathcal{A})/\langle\rho\rangle$ or $A \cong PSE_k(\Delta, \mathcal{A})/\langle\rho\rangle$. Then the structure of A is decided by the generated ideal $\langle\rho\rangle$ of the set ρ of some relations. From this, we can try to classify those associative algebras satisfying the theorems, which include many important classes of algebras. All these above are the works we are processing after this paper, which will present the farther significance of this paper.

2 On Generalized Path Algebras And Pseudo Path Algebras

In this section, we firstly introduce the definitions of generalized path algebra from [2] and pseudo path algebra. Then, we discuss their properties and relationship.

A *quiver* Δ is given by two sets Δ_0 and Δ_1 together with two maps $s, e\colon \Delta_1 \to \Delta_0$. The elements of Δ_0 are called *vertices*, while the elements of Δ_1 are called *arrows*. For an arrow $\alpha \in \Delta_1$, the vertex $s(\alpha)$ is the *start vertex* of α and the vertex $e(\alpha)$ is the *end vertex* of α, and we draw $s(\alpha) \xrightarrow{\alpha} e(\alpha)$. A *path* p in Δ is $(a|\alpha_1 \cdots \alpha_n|b)$, where $\alpha_i \in \Delta_1$, for $i = 1, \cdots, n$, and $s(\alpha_1) = a$, $e(\alpha_i) = s(\alpha_{i+1})$ for $i = 1, \cdots, n+1$, and $e(\alpha_n) = b$. $s(\alpha_1)$ and $e(\alpha_n)$ are also called respectively the start vertex and the end vertex of p. Write $s(p) = s(\alpha_1)$ and $e(p) = e(\alpha_n)$. The *length* of a path is the number of arrows in it. To each arrow α, one can assign an edge $\overline{\alpha}$ where the orientation is forgotten. A *walk* between two vertices a and b is given by $(a|\overline{\alpha_1}\cdots\overline{\alpha_n}|b)$, where $a \in \{s(\alpha_1), e(\alpha_1)\}$, $b \in \{s(\alpha_n), e(\alpha_n)\}$, and for each $i = 1, \cdots, n-1$, $\{s(\alpha_i), e(\alpha_i)\} \bigcap \{s(\alpha_{i+1}), e(\alpha_{i+1})\} \neq \emptyset$. A quiver is said to be *connected* if for each pair of vertices a and b, there exists a walk between them.

In this paper, we will always assume a quiver Δ is finite, i.e. the number of vertices $|\Delta_0| \leq \infty$.

Definition 2.1 *For two algebras A and B, the* rank *of a finitely generated A-B-bimodule M is defined as the least number of all generators. In particular, if $M = 0$, we say it is a special finitely generated A-B-bimodule with rank* 0*.*

Clearly, the rank in Definition 2.1 exists uniquely.

Generalized Path Algebra And Tensor Algebra

Let $\Delta = (\Delta_0, \Delta_1)$ be a quiver and $\mathcal{A} = \{A_i : i \in \Delta_0\}$ be a family of k-algebra A_i with identity e_i, indexed by the vertices of Δ. The elements a_i of $\bigcup_{i\in\Delta_0} A_i$ are called the *$\mathcal{A}$-paths of length zero*, whose start vertex $s(a_i)$ and the end vertex $e(a_i)$ both are i. For each $n \geq 1$, an *$\mathcal{A}$-path P of length n* is given by $a_1\beta_1 a_2\beta_2 \cdots a_n\beta_n a_{n+1}$, where $(s(\beta_1)|\beta_1 \cdots \beta_n|e(\beta_n))$ is a path in Δ of length n, for each $i = 1, \cdots, n$, $a_i \in A_{s(\beta_i)}$ and $a_{n+1} \in A_{e(\beta_n)}$. $s(\beta_1)$ and $e(\beta_n)$ are also called respectively the start vertex and the end vertex of P. Write $s(P) = s(\alpha_1)$ and $e(P) = e(\alpha_n)$. Now, consider the quotient R of the k-linear space with basis the set of all $\mathcal{A}$-paths of Δ by the subspace generated by all the elements of the form

$$a_1\beta_1 \cdots \beta_{j-1}(a_j^1 + \cdots + a_j^m)\beta_j a_{j+1} \cdots \beta_n a_{n+1} - \sum_{l=1}^{m} a_1\beta_1 \cdots \beta_{j-1} a_j^l \beta_j a_{j+1} \cdots \beta_n a_{n+1}$$

where $(s(\beta_1)|\beta_1 \cdots \beta_n|e(\beta_n))$ is a path in Δ of length n, for each $i = 1, \cdots, n$, $a_i \in A_{s(\beta_i)}$, $a_{n+1} \in A_{e(\beta_n)}$ and $a_j^l \in A_{s(\beta_j)}$ for $l = 1, \cdots, m$. Define in R the following multiplication. Given two elements $[a_1\beta_1 a_2\beta_2 \cdots a_n\beta_n a_{n+1}]$ and $[b_1\gamma_1 b_2\gamma_2 \cdots b_n\gamma_n b_{n+1}]$ in R, define

$$[a_1\beta_1 a_2\beta_2 \cdots a_n\beta_n a_{n+1}] \cdot [b_1\gamma_1 b_2\gamma_2 \cdots b_n\gamma_n b_{n+1}] = \begin{cases} [a_1\beta_1 a_2\beta_2 \cdots a_n\beta_n(a_{n+1}b_1)\gamma_1 b_2\gamma_2 \cdots b_n\gamma_n b_{n+1}], & \text{if } a_{n+1}, b_1 \in A_i \text{ for same } i \\ 0, & \text{otherwise} \end{cases}$$

It is easy to check that the above multiplication in R is well-defined and gives to R an structure of k-algebra. This algebra R defined above is called the *$\mathcal{A}$-path algebra* of Δ. Denote it by $R = k(\Delta, \mathcal{A})$. Clearly, R is an A-bimodule.

Remark that (i) $R = k(\Delta, \mathcal{A})$ has identity if and only if Δ_0 is finite; (ii) Any path $(s(\beta_1)|\beta_1 \cdot \cdot \beta_n|e(\beta_n))$ in Δ can be considered as an $\mathcal{A}$-path with $a_i = e_i$. Hence the usual path algebra $k\Delta$ can be embedded into the $\mathcal{A}$-path algebra $k(\Delta, \mathcal{A})$. Or say, if $A_i = k$ for each $i \in \Delta_0$, then $k(\Delta, \mathcal{A}) = k\Delta$; (iii) For $R = k(\Delta, \mathcal{A})$, $dim_k R < \infty$ if and only if $dim_k A_i < \infty$ for each $i \in \Delta_0$ and Δ is a finite quiver without oriented cycles.

Associated with the pair $(A, {}_A M_A)$ for a k-algebra A and an A-bimodule M, we write the n-fold A-tensor product $M \otimes_A M \otimes \cdots \otimes_A M$ as M^n, then $T(A, M) = A \oplus M \oplus M^2 \oplus \cdots \oplus M^n \oplus \cdots$ as an abelian group. Writing $M^0 = A$, then $T(A, M)$ becomes a k-algebra with multiplication induced by the natural A-bilinear maps $M^i \times M^j \to M^{i+j}$ for $i \geq 0$ and $j \geq 0$. We call $T(A, M)$ a *tensor algebra*.

Now, we define a special class of tensor algebras so as to characterize generalized path algebras. An *$\mathcal{A}$-path-type tensor algebra* is a tensor algebra $T(A, M)$ satisfying that (i) $A = \bigoplus_{i\in\Delta_0} A_i$ for a family of k-algebras $\mathcal{A} = \{A_i : i \in \Delta_0\}$, (ii) $M = \bigoplus_{i,j\in I} {}_iM_j$ for finitely generated A_i-A_j-bimodules

${}_iM_j$ for all i and j in I and $A_k \cdot_i M_j = 0$ if $k \neq i$ and ${}_iM_j \cdot A_k = 0$ if $k \neq j$. A *free $\mathcal{A}$-path-type tensor algebra* is an $\mathcal{A}$-path-type tensor algebra $T(A, M)$ whose each finitely generated A_i-A_j-bimodule ${}_iM_j$ for i and j in I is a free bimodule with a basis and the rank of this basis equals to the rank of ${}_iM_j$ as finitely generated.

$\mathcal{A}$-path-type tensor algebras and generated path algebras can be constructed each other as follows.

For an $\mathcal{A}$-path algebra $k(\Delta, \mathcal{A})$, let $A = \bigoplus_{i\in\Delta_0} A_i$. For any i and j, let ${}_iM_j^F$ be the free A_i-A_j-bimodule with free generators given by the arrows from i to j. Easily, the number of free generator is the rank of ${}_iM_j^F$ as finitely generated. Define $A_k \cdot_i M_j^F = 0$ if $k \neq i$ and ${}_iM_j^F \cdot A_k = 0$ if $k \neq j$. Let $M^F = \bigoplus_{i\to j} {}_iM_j^F$, which is clearly an A-bimodule. Then we get uniquely the free $\mathcal{A}$-path-type tensor algebras $T(A, M^F)$.

Conversely, assume that $T(A, M)$ is an $\mathcal{A}$-path-type tensor algebra with a family of k-algebras $\mathcal{A} = \{A_i : i \in I\}$ satisfying that $A = \bigoplus_{i\in I} A_i$ and finitely generated A_i-A_j-bimodules ${}_iM_j$ for all i and j in I satisfying $M = \bigoplus_{i,j\in I} {}_iM_j$, $A_k \cdot_i M_j = 0$ if $k \neq i$ and ${}_iM_j \cdot A_k = 0$ if $k \neq j$. Trivially, ${}_iM_j = A_iMA_j$. Let the rank of ${}_iM_j$ be r_{ij}. Now we can associate with $T(A, M)$ a quiver $\Delta = (\Delta_0, \Delta_1)$ and its generalized path algebra $R = k(\Delta, \mathcal{A})$ in the following way. Let the set of vertices $\Delta_0 = I$. For $i, j \in I$, let the number of arrows from i to j in Δ be the rank r_{ij} of the finitely generated A_i-A_j-bimodules ${}_iM_j$. Obviously, if ${}_iM_j = 0$, then there are no arrows from i to j. Thus, we get a quiver $\Delta = (\Delta_0, \Delta_1)$ called *the quiver of* $T(A, M)$, and its $\mathcal{A}$-path algebra $R = k(\Delta, \mathcal{A})$ which is called *the corresponding $\mathcal{A}$-path algebra of* $T(A, M)$.

Remark that since it is possible two non-isomorphic finitely generated bimodules possesses the same rank, there are two $\mathcal{A}$-path-type tensor algebras $T(A, M_1)$ and $T(A, M_2)$, with non-isomorphic M_1 and M_2, inducing a same quiver and its $\mathcal{A}$-path algebra in the way above.

From the above discussion, every $\mathcal{A}$-path-type tensor algebra $T(A, M)$ can be used to construct its corresponding $\mathcal{A}$-path algebra $k(\Delta, \mathcal{A})$; but, from this $\mathcal{A}$-path algebra $k(\Delta, \mathcal{A})$, we can get uniquely the free $\mathcal{A}$-path-type tensor algebra $T(A, M^F)$. Thus, we have:

Lemma 2.1 *Every $\mathcal{A}$-path-type tensor algebra $T(A, M)$ can be used to construct uniquely the free $\mathcal{A}$-path-type tensor algebra $T(A, M^F)$. There is a surjective k-algebra morphism π: $T(A, M^F) \to T(A, M)$ such that $\pi({}_iM_j^F) =_i M_j$ for any $i, j \in I$.*

Proof: It is sufficient to prove the second conclusion. For $T(A, M)$, let the rank of ${}_iM_j$ be r_{ij}. Thus, for the corresponding $\mathcal{A}$-path algebra $k(\Delta, \mathcal{A})$, the number of the arrows from i to j is r_{ij}, and then, in $T(A, M^F)$, the rank of the free generators of ${}_iM_j^F$ given by the arrows is also r_{ij}. Define π: $T(A, M^F) \to T(A, M)$ by giving a bijection between the set of the free generators of ${}_iM_j^F$ and the set of the chosen generators of ${}_iM_j$ with number of the rank. Then π can be expanded to become into a surjective k-algebra morphism with $\pi({}_iM_j^F) =_i M_j$ for any $i, j \in I$.

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Next, we will show in the following Proposition 2.4 that every $\mathcal{A}$-path-type tensor algebra is a homomorphic image of its corresponding $\mathcal{A}$-path algebra.

The following criterion (i.e. Lemma III.1.2 in [1]) for constructing algebra morphisms from tensor algebras to other algebras is useful.

Lemma 2.2 *Let A be a k-algebra and M an A-bimodule. Let Λ be a k-algebra and $f : A \oplus M \to \Lambda$ a map such that the following two conditions are satisfied:*

(i) $f|_A : A \to \Lambda$ is an algebra morphism;

(ii) Viewing $f(M)$ as an A-bimodule via $f|_A : A \to \Lambda$ then $f|_M : M \to f(M) \subset \Lambda$ is an A-bimodule map.
Then there is a unique algebra morphism $\widetilde{f} : T(A, M) \to \Lambda$ such that $\widetilde{f}|_{A\oplus M} = f$ and generally, $\widetilde{f}(\sum_{n=0}^{\infty} m_1^n \otimes \cdots \otimes m_n^n) = \sum_{n=0}^{\infty} f(m_1^n) \cdots f(m_n^n)$ for $m_1^n \otimes \cdots \otimes m_n^n \in M^n$.

Note that it is enough for the proof of (ii) in [1] under the condition that $f(M)$ is an A-bimodule via $f|_A : A \to \Lambda$.

Clearly, all $\mathcal{A}$-paths of length zero, i.e. the elements of $\bigcup_{i\in\Delta_0} A_i$ can generate a subalgebra of $k(\Delta, \mathcal{A})$, denote it by $k(\Delta_0, \mathcal{A})$. Denote by $k(\Delta_1, \mathcal{A})$ the k-linear space consisting of all $\mathcal{A}$-paths of length 1 and J the ideal in a $\mathcal{A}$-path algebra $k(\Delta, \mathcal{A})$ generated by all elements in $k(\Delta_1, \mathcal{A})$. It is easy to see that $k(\Delta_1, \mathcal{A})$ is an A-subbimodule of $k(\Delta, \mathcal{A})$.

Pseudo Path Algebra And Pseudo Tensor Algebra

Let $\Delta = (\Delta_0, \Delta_1)$ be a quiver and $\mathcal{A} = \{A_i : i \in \Delta_0\}$ be a family of k-algebra A_i with identity e_i, indexed by the vertices of Δ. The elements a_i of $\bigcup_{i\in\Delta_0} A_i$ are called the *$\mathcal{A}$-pseudo-paths of length zero*, whose start vertex $s(a_i)$ and the end vertex $e(a_i)$ both are i. For each $n \geq 1$, an *pure $\mathcal{A}$-pseudo-path P of length n* is given by $a_1\beta_1 b_1 \cdot a_2\beta_2 b_2 \cdot ... \cdot a_n\beta_n b_n$, where $(s(\beta_1)|\beta_1 \cdots \beta_n|e(\beta_n))$ is a path in Δ of length n, for each $i = 1, \cdots, n$, $b_{i-1} \in A_{e(\beta_{i-1})}, a_i \in A_{s(\beta_i)}$ with $s(\beta_i) = e(\beta_{i-1})$. $s(\beta_1)$ and $e(\beta_n)$ are also called respectively the *start vertex* and the *end vertex* of P. Write $s(P) = s(\beta_1)$ and $e(P) = e(\beta_n)$. An *general $\mathcal{A}$-pseudo-path Q of length n* is given in the form

$\alpha_1 \cdot c_1 \cdot \alpha_2 \cdot c_2 \cdot ... \cdot c_k \cdot \alpha_k$

or

$c_0 \cdot \alpha_1 \cdot c_1 \cdot \alpha_2 \cdot c_2 \cdot ... \cdot c_k \cdot \alpha_k$

or

$\alpha_1 \cdot c_1 \cdot \alpha_2 \cdot c_2 \cdot ... \cdot c_k \cdot \alpha_k \cdot c_{k+1}$

or

$c_0 \cdot \alpha_1 \cdot c_1 \cdot \alpha_2 \cdot c_2 \cdot ... \cdot c_k \cdot \alpha_k \cdot c_{k+1}$

where α_i is an pure $\mathcal{A}$-pseudo-path of length n_i and $\sum_{i=1}^{k} n_i = n$, and the start vertex of α_{i+1} is just the end vertex of α_i, i.e. $e(\alpha_i) = s(\alpha_{i+1})$ and $c_i \in A_{e(\alpha_i)}$.

Let V be the k-linear space with basis the set of all general $\mathcal{A}$-paths of Δ.

Consider the quotient R of the k-linear space V by the subspace generated by all the elements of the form

$$a_1\beta_1 b_1 \cdot ... \cdot a_j\beta_j(b_j^1 + ... + b_j^m) \cdot \gamma - \sum_{l=1}^{m} a_1\beta_1 b_1 \cdot ... \cdot a_j\beta_j b_j^l \cdot \gamma \qquad (1)$$

$$\alpha \cdot (a_1^1 + ... + a_1^m)\beta_1 b_1 \cdot ... \cdot a_n\beta_n b_n - \sum_{l=1}^{m} \alpha \cdot a_1^l\beta_1 b_1 \cdot ... \cdot a_n\beta_n b_n \qquad (2)$$

$$(ab) \cdot c\beta d - a \cdot (b \cdot c\beta d), \quad a\beta b \cdot (cd) - (a\beta b \cdot c) \cdot d \qquad (3)$$

$$a\beta b \cdot 1 - a\beta b, \quad 1 \cdot a\beta b - a\beta b \qquad (4)$$

where $a, b, c, d, b_j^l, a_1^l \in \bigcup_{i\in\Delta_0} A_i$ and 1 is the identity of A.

Define in R the following multiplication. Given two elements $[a_1\beta_1 b_1 \cdot a_2\beta_2 b_2 \cdot ... \cdot a_n\beta_n b_n]$ and $[c_1\gamma_1 d_1 \cdot c_2\gamma_2 d_2 \cdot ... \cdot c_n\gamma_m d_m]$ in which at least one is of length $n \geq 1$, define

$$[a_1\beta_1 b_1 \cdot a_2\beta_2 b_2 \cdot \ldots \cdot a_n\beta_n b_n] \cdot [c_1\gamma_1 d_1 \cdot c_2\gamma_2 d_2 \cdot \ldots \cdot c_n\gamma_m d_m]$$
$$= \begin{cases} [a_1\beta_1 b_1 \cdot a_2\beta_2 b_2 \cdot \ldots \cdot a_n\beta_n b_n \cdot c_1\gamma_1 d_1 \cdot c_2\gamma_2 d_2 \cdot \ldots \cdot c_n\gamma_m d_m], & \text{if } b_n, c_1 \in A_i \text{ for same } i \\ 0, & \text{otherwise} \end{cases}$$

Given two elements a, b of length zero, i.e. $a, b \in \bigcup_{i\in\Delta_0} A_i$, define

$$a \cdot b = \begin{cases} ab, & \text{if } a, b \text{ are in one same } A_i \\ 0, & \text{otherwise} \end{cases}$$

where ab means the product of a, b in A_i.

It is easy to check that the above multiplication in R is well-defined and gives to R an structure of k-algebra. This algebra R defined above is called the *$\mathcal{A}$-pseudo path algebra* of Δ. Denote it by $R = PSE_k(\Delta, \mathcal{A})$. Clearly, R is an A-bimodule for $A = \bigoplus_{i\in\Delta_0} A_i$.

Remark that (i) $R = PSE_k(\Delta, \mathcal{A})$ has identity if and only if Δ_0 is finite; (ii) Any path $(s(\beta_1)|\beta_1 \cdot \cdot \cdot \beta_n|e(\beta_n))$ in Δ can be considered as an $\mathcal{A}$-path with $a_i = e_i$ the identity of A_i. Hence the usual path algebra $k\Delta$ can be embedded into the $\mathcal{A}$-pseudo path algebra $PSE_k(\Delta, \mathcal{A})$. Or say, if $A_i = k$ for each $i \in \Delta_0$, then $PSE_k(\Delta, \mathcal{A}) = k\Delta$; (iii) For $R = PSE_k(\Delta, \mathcal{A})$, $dim_k R < \infty$ if and only if $dim_k A_i < \infty$ for each $i \in \Delta_0$ and Δ is a finite quiver without oriented cycles.

Associated with the pair $(A, {}_A M_A)$ for a k-algebra A and an A-bimodule M, we write the n-fold k-tensor product $M \otimes_k M \otimes \cdots \otimes_k M$ as M^n; $\sum_{M_1, M_2, \cdots, M_n} M_1 \otimes_k M_2 \otimes_k \cdots \otimes_k M_n$ as $M(n)$ where M_i is either M (at least there exists one) or A but no two A's are neighbouring, then define $\mathcal{PT}(A, M) = A \oplus M(1) \oplus M(2) \oplus \cdots \oplus M(n) \oplus \cdots$ as an abelian group. Denote $M(n, l)$ the sum of these items $M_1 \otimes_k M_2 \otimes_k \cdots \otimes_k M_n$ of $M(n)$ in which there are l M_i's equal to M. Clearly, $(n-1)/2 \le l \le n$ and $M(n) = \sum_{(n-1)/2 \le l \le n} M(n, l)$. Writing $M^0 = A$, then $\mathcal{PT}(A, M)$ becomes a k-algebra with multiplication induced by the natural k-bilinear maps:

$M^i \times M^j \to M^{i+j}$ for $i \ge 1$, $j \ge 1$;

$M^i \times A \to M^i \otimes_k A$ for $i \ge 1$;

$A \times M^j \to A \otimes_k M^j$ for $j \ge 1$

and the natural A-bilinear map:

$A \times A \to A \otimes_A A = A$.

Note that the associative law of $\mathcal{PT}(A, M)$ is from the fact $(A \otimes_A A) \otimes_k M \cong A \otimes_A (A \otimes_k M)$. We call $\mathcal{PT}(A, M)$ a *pseudo tensor algebra*.

Now, we define a special class of pseudo tensor algebras so as to characterize pseudo path algebras. An *$\mathcal{A}$-path-type pseudo tensor algebra* is a pseudo tensor algebra $\mathcal{PT}(A, M)$ satisfying that (i) $A = \bigoplus_{i\in\Delta_0} A_i$ for a family of k-algebras $\mathcal{A} = \{A_i : i \in \Delta_0\}$, (ii) $M = \bigoplus_{i,j\in I} {}_iM_j$ for finitely generated A_i-A_j-bimodules ${}_iM_j$ for all i and j in I and $A_k \cdot_i M_j = 0$ if $k \ne i$ and ${}_iM_j \cdot A_k = 0$ if $k \ne j$. A *free $\mathcal{A}$-path-type pseudo tensor algebra* is an $\mathcal{A}$-path-type pseudo tensor algebra $\mathcal{PT}(A, M)$ whose each finitely generated A_i-A_j-bimodule ${}_iM_j$ for i and j in I is a free bimodule with a basis and the rank of this basis equals to the rank of ${}_iM_j$ as finitely generated.

$\mathcal{A}$-path-type pseudo tensor algebras and pseudo path algebras can be constructed each other as follows.

For an $\mathcal{A}$-pseudo path algebra $PSE_k(\Delta, \mathcal{A})$, let $A = \bigoplus_{i\in\Delta_0} A_i$. For any i and j, let ${}_iM_j^F$ be the free A_i-A_j-bimodule with free generators given by the arrows from i to j. Easily, the number of free generator is the rank of ${}_iM_j^F$ as finitely generated. Define $A_k \cdot_i M_j^F = 0$ if $k \ne i$ and ${}_iM_j^F \cdot A_k = 0$ if $k \ne j$. Let $M^F = \bigoplus_{i\to j} {}_iM_j^F$, which is clearly an A-bimodule. Then we get uniquely the free $\mathcal{A}$-path-type pseudo tensor algebras $\mathcal{PT}(A, M^F)$.

Conversely, assume that $\mathcal{PT}(A, M)$ is an $\mathcal{A}$-path-type pseudo tensor algebra with a family of

k-algebras $\mathcal{A}=\{A_i : i\in I\}$ satisfying that $A=\bigoplus_{i\in I} A_i$ and finitely generated A_i-A_j-bimodules ${}_iM_j$ for all i and j in I satisfying $M=\bigoplus_{i,j\in I}{}_iM_j$, $A_k\cdot_i M_j=0$ if $k\neq i$ and ${}_iM_j\cdot A_k=0$ if $k\neq j$. Trivially, ${}_iM_j=A_iMA_j$. Let the rank of ${}_iM_j$ be r_{ij}. Now we can associate with $\mathcal{PT}(A,M)$ a quiver $\Delta=(\Delta_0,\Delta_1)$ and its pseudo path algebra $R=PSE_k(\Delta,\mathcal{A})$ in the following way. Let the set of vertices $\Delta_0=I$. For $i,j\in I$, let the number of arrows from i to j in Δ be the rank r_{ij} of the finitely generated A_i-A_j-bimodules ${}_iM_j$. Obviously, if ${}_iM_j=0$, then there are no arrows from i to j. Thus, we get a quiver $\Delta=(\Delta_0,\Delta_1)$ which is called *the quiver of* $\mathcal{PT}(A,M)$, and its $\mathcal{A}$-pseudo path algebra $R=PSE_k(\Delta,\mathcal{A})$ which is called *the corresponding $\mathcal{A}$-pseudo path algebra of* $\mathcal{PT}(A,M)$.

Remark that since it is possible two non-isomorphic finitely generated bimodules possesses the same rank, there are two $\mathcal{A}$-path-type pseudo tensor algebras $\mathcal{PT}(A,M_1)$ and $\mathcal{PT}(A,M_2)$, with non-isomorphic M_1 and M_2, inducing a same quiver and its $\mathcal{A}$-pseudo path algebra in the way above.

From the above discussion, every $\mathcal{A}$-path-type pseudo tensor algebra $\mathcal{PT}(A,M)$ can be used to construct its corresponding $\mathcal{A}$-pseudo path algebra $PSE_k(\Delta,\mathcal{A})$; but, from this $\mathcal{A}$-pseudo path algebra $PSE_k(\Delta,\mathcal{A})$, we can get uniquely the free $\mathcal{A}$-path-type pseudo tensor algebra $\mathcal{PT}(A,M^F)$. Thus, we have:

Lemma 2.3 *Every $\mathcal{A}$-path-type pseudo tensor algebra $\mathcal{PT}(A,M)$ can be used to construct uniquely the free $\mathcal{A}$-path-type pseudo tensor algebra $\mathcal{PT}(A,M^F)$. There is a surjective k-algebra morphism π: $\mathcal{PT}(A,M^F)\to\mathcal{PT}(A,M)$ such that $\pi({}_iM_j^F)={}_iM_j$ for any $i,j\in I$.*

Proof: It is sufficient to prove the second conclusion. For $\mathcal{PT}(A,M)$, let the rank of ${}_iM_j$ be r_{ij}. Thus, for the corresponding $\mathcal{A}$-pseudo path algebra $PSE_k(\Delta,\mathcal{A})$, the number of the arrows from i to j is r_{ij}, and then, in $\mathcal{PT}(A,M^F)$, the rank of the free generators of ${}_iM_j^F$ given by the arrows is also r_{ij}. Define π: $\mathcal{PT}(A,M^F)\to\mathcal{PT}(A,M)$ by giving a bijection between the set of the free generators of ${}_iM_j^F$ and the set of the chosen generators of ${}_iM_j$ with number of the rank. Then π can be expanded to become into a surjective k-algebra morphism with $\pi({}_iM_j^F)={}_iM_j$ for any $i,j\in I$.

#

Next, we will show in the following Proposition 2.4 that every $\mathcal{A}$-path-type pseudo tensor algebra is a homomorphic image of its corresponding $\mathcal{A}$-pseudo path algebra.

The following criterion for constructing algebra morphisms from tensor algebras to other algebras is useful, which is alternated from Lemma III.1.2 in [1].

Lemma 2.4 *Let A be a k-algebra and M an A-bimodule. Let Λ be a k-algebra and $f: A\oplus M\to\Lambda$ a k-linear map such that $f|_A: A\to\Lambda$ is an algebra morphism. Then there is a unique algebra homomorphism $\widetilde{f}:\mathcal{PT}(A,M)\to\Lambda$ such that $\widetilde{f}|_{A\oplus M}=f$ and generally, $\widetilde{f}(\sum_{n=0}^{\infty} m_1^n\otimes_k\cdots\otimes_k m_n^n)=\sum_{n=0}^{\infty} f(m_1^n)\cdots f(m_n^n)$ for $m_1^n\otimes_k\cdots\otimes_k m_n^n\in M(n)$.*

Proof: Consider the map $\phi: M\times M\to\Lambda$ defined by $\phi(m_1,m_2)=f(m_1)f(m_2)$ for m_1 and m_2 in M. We have for $\alpha\in k$ that $\phi(m_1\alpha,m_2)=f(m_1\alpha)f(m_2)=f(m_1)f(\alpha m_2)=\phi(m_1,\alpha m_2)$. Hence there is a unique group morphism $f_2: M\otimes_k M\to\Lambda$ such that $f_2(m_1\otimes_k m_2)=f(m_1)f(m_2)$. Of course, f_2 is moreover a k-linear map. It is similar for the map $\phi: M\times A\to\Lambda$ defined by $\phi(m,a)=f(m)f(a)$ for $m\in M$ and $a\in A$ to induce the k-linear map $f_2: M\otimes_k A\to\Lambda$ satisfying $f_2(m\otimes_k a)=f(m)f(a)$.

By induction, one obtains the unique k-linear map $f_n : M(n) \to \Lambda$ satisfying $f_n(v_1 \otimes_k \cdots \otimes_k v_n) = f(v_1) \cdots f(v_n)$. Since $f|_A$ is a k-algebra homomorphism, we define $\widetilde{f} : \mathcal{PT}(A, M) \to \Lambda$ by $\widetilde{f}|_{A \oplus M} = f$ and $\widetilde{f}(\sum_{n=0}^{\infty} m_1^n \otimes_k \cdots \otimes_k m_n^n) = \sum_{n=0}^{\infty} f(m_1^n) \cdots f(m_n^n)$ for $m_1^n \otimes_k \cdots \otimes_k m_n^n \in M(n)$, which can be checked easily to be a k-algebra homomorphism uniquely determined by f.

In fact, for $m_1 \otimes_k \cdots \otimes_k m_n \in M(n)$ and $\bar{m}_1 \otimes_k \cdots \otimes_k \bar{m}_l \in M(l)$, if $m_n, \bar{m}_1 \in A$, then $\widetilde{f}((m_1 \otimes_k \cdots \otimes_k m_n) \cdot (\bar{m}_1 \otimes_k \cdots \otimes_k \bar{m}_l)) = \widetilde{f}(m_1 \otimes_k \cdots \otimes_k m_{n-1} \otimes_k m_n \otimes_A \bar{m}_1 \otimes_k \bar{m}_2 \otimes_k \cdots \otimes_k \bar{m}_l)$ $= \widetilde{f}(m_1 \otimes_k \cdots \otimes_k m_{n-1} \otimes_k m_n \bar{m}_1 \otimes_k \bar{m}_2 \otimes_k \cdots \otimes_k \bar{m}_l) = f(m_1) \cdots f(m_{n-1}) f(m_n \bar{m}_1) f(\bar{m}_2) \cdots f(\bar{m}_l)$ $= f(m_1) \cdots f(m_{n-1}) f(m_n) f(\bar{m}_1) f(\bar{m}_2) \cdots f(\bar{m}_l) = \widetilde{f}(m_1 \otimes_k \cdots \otimes_k m_n) \widetilde{f}(\bar{m}_1 \otimes_k \cdots \otimes_k \bar{m}_l)$;
The other cases are also easy to be proved. #

Comparing the definitions of generalized path algebra, tensor algebra and pseudo path algebra, pseudo tensor algebra, we can find the following facts:

Fact 2.5 (1) *There is a natural surjective homomorphism* $\iota : PSE_k(\Delta, \mathcal{A}) \longrightarrow k(\Delta, \mathcal{A})$ *with*

$$ker\iota = \langle a\beta b \cdot c - a\beta bc,\ c \cdot a\beta b - ca\beta b,\ a\alpha b \cdot c\beta d - a\alpha 1 \cdot bc \cdot 1\beta d \rangle$$

for any $a, b, c, d \in A = \oplus_i A_i$, $\alpha, \beta \in \Delta_1$, *where* 1 *is the identity of* A. *It follows that*

$$PSE_k(\Delta, \mathcal{A})/ker\iota \cong k(\Delta, \mathcal{A})$$

as algebras.

(2) *There is a natural surjective homomorphism* $\tau : \mathcal{PT}(A, M) \longrightarrow T(A, M)$ *with*

$$ker\tau = \langle m \otimes c - mc \otimes 1,\ c \otimes m - 1 \otimes cm,\ mb \otimes cn - m \otimes bc \otimes n \rangle$$

for any $b, c \in A$, $m, n \in M$, *where* 1 *is the identity of* A. *It follows that*

$$\mathcal{PT}(A, M)/ker\tau \cong T(A, M)$$

as algebras.

Clearly, all $\mathcal{A}$-pseudo-paths of length zero (equivalently, $\mathcal{A}$-paths of length zero), i.e. the elements of $\bigcup_{i \in \Delta_0} A_i$ can generate a subalgebra of $PSE_k(\Delta, \mathcal{A})$ (resp. $k(\Delta, \mathcal{A})$), denote it by $PSE_k(\Delta_0, \mathcal{A})$ (resp. $k(\Delta_0, \mathcal{A})$). Then, $PSE_k(\Delta_0, \mathcal{A}) = k(\Delta_0, \mathcal{A})$, or say, $\iota_{PSE_k(\Delta_0, \mathcal{A})} = id$. Denote by $PSE_k(\Delta_1, \mathcal{A})$ (resp. $k(\Delta_1, \mathcal{A})$) the k-linear space consisting of all pure $\mathcal{A}$-pseudo-paths (resp. all $\mathcal{A}$-paths) of length 1 and J (resp. $\widetilde{J}$) the ideal in $PSE_k(\Delta, \mathcal{A})$ (resp. $k(\Delta, \mathcal{A})$) generated by all elements in $PSE_k(\Delta_1, \mathcal{A})$ (resp. $k(\Delta_1, \mathcal{A})$).

It is easy to see that $PSE_k(\Delta_1, \mathcal{A})$ (resp. $k(\Delta_1, \mathcal{A})$) is an A-subbimodule of $PSE_k(\Delta, \mathcal{A})$ (resp. $k(\Delta, \mathcal{A})$), and (i) $\iota(PSE_k(\Delta_1, \mathcal{A})) = k(\Delta_1, \mathcal{A})$, (ii) $\iota J = \widetilde{J}$, $\iota^{-1} \widetilde{J} = J$.

Now, we will show some useful properties which hold similarly for both $\mathcal{A}$-pseudo-path algebra and $\mathcal{A}$-path algebra under the relationships in Fact 2.5.

Lemma 2.6 *Let* $\mathcal{PT}(A, M^F)$ *be the free* $\mathcal{A}$*-path-type pseudo tensor algebra built by a* $\mathcal{A}$*-pseudo path algebra* $PSE_k(\Delta, \mathcal{A})$. *Then there is a* k*-algebra isomorphism* ϕ: $\mathcal{PT}(A, M^F) \to PSE_k(\Delta, \mathcal{A})$ *such that for any* $t \geq 1$,

$$\phi(\bigoplus_{n,l \geq t} M^F(n, l)) = J^t.$$

Proof: By the multiplication in $PSE_k(\Delta, \mathcal{A})$, $[a_i] \cdot [a_j] = 0$ for $i \neq j$ and $a_i \in A_i$, $a_j \in A_j$. Obviously, we have a k-algebra isomorphism f: $A = \bigoplus_{i\in I} A_i \to PSE_k(\Delta_0, \mathcal{A})$ by $f(a_1 + \cdots + a_n) = [a_1] + \cdots + [a_n]$. And, define f: $M^F = \bigoplus_{i,j\in I} {}_iM_j^F \to PSE_k(\Delta_1, \mathcal{A})$ by giving a bijection between a chosen basis for each ${}_iM_j^F$ and the set of arrows from i to j, that is, $f(am_{\alpha_{ij}}b) = a\alpha_{ij}b$ where α_{ij} is an arrow from i to j and $m_{\alpha_{ij}}$ is the correspondent element in the basis of ${}_iM_j^F$, $a, b \in A$. Since $PSE_k(\Delta_0, \mathcal{A})$ is a k-subalgebra of $PSE_k(\Delta, \mathcal{A})$, there is by Lemma 2.2 a k-algebra morphism $\widetilde{f}$: $\mathcal{PT}(A, M^F) \to PSE_k(\Delta, \mathcal{A})$ such that

$$\widetilde{f}|_{A\oplus M^F} = f$$

and

$$\widetilde{f}(\sum_{n=0}^{\infty} m_1^n \otimes \cdots \otimes m_n^n) = \sum_{n=0}^{\infty} f(m_1^n) \cdot \ldots \cdot f(m_n^n)$$

for $m_1^n \otimes \cdots \otimes m_n^n \in M^F(n)$. Thus, $\widetilde{f}((A \otimes_k M^F \otimes_k A)^t) = (A \cdot PSE_k(\Delta_1, \mathcal{A}) \cdot A)^t$ and moreover, $\widetilde{f}(\bigoplus_{n,\, l\geq t} M^F(n, l)) = J^t$, in particular, $\widetilde{f}(\bigoplus_{k\geq 1} M^F(k)) = J$. But, $PSE_k(\Delta, \mathcal{A}) = PSE_k(\Delta_0, \mathcal{A}) \cup J \cup \cdots \cup J^t \cup \cdots$. Hence $\widetilde{f}$ is surjective.

Let $\{x_\lambda\}_\lambda$ denote a k-basis of A. For $M^F(n, l)$, we have a k-basis formed by some elements as

$$x_{\lambda_{i_1}} \otimes x_{\lambda_{j_1}} m_1 x_{\lambda_{k_1}} \otimes x_{\lambda_{i_2}} \otimes x_{\lambda_{j_2}} m_2 x_{\lambda_{k_2}} \otimes \cdots \otimes x_{\lambda_{i_l}} \otimes x_{\lambda_{j_l}} m_l x_{\lambda_{k_l}} \otimes \cdots$$

where there is some $\mathcal{A}$-pseudo-path

$$[x_{\lambda_{i_1}} \cdot x_{\lambda_{j_1}} \beta_1 x_{\lambda_{k_1}} \cdot x_{\lambda_{i_2}} \cdot x_{\lambda_{j_2}} \beta_2 x_{\lambda_{k_2}} \cdot \ldots \cdot x_{\lambda_{i_l}} \cdot x_{\lambda_{j_l}} \beta x_{\lambda_{k_l}} \cdot \ldots]$$

in $PSE_k(\Delta, \mathcal{A})$ such that m_j is amongst the chosen basis elements in ${}_{s(\beta_j)}M^F_{s(\beta_{j+1})}$ of the correspondent arrow β_j for $j = 1, \cdots, t$. Then

$\widetilde{f}(x_{\lambda_{i_1}} \otimes x_{\lambda_{j_1}} m_1 x_{\lambda_{k_1}} \otimes x_{\lambda_{i_2}} \otimes x_{\lambda_{j_2}} m_2 x_{\lambda_{k_2}} \otimes \cdots \otimes x_{\lambda_{i_t}} \otimes x_{\lambda_{j_l}} m_l x_{\lambda_{k_l}} \otimes \cdots)$
$= [x_{\lambda_{i_1}} \cdot x_{\lambda_{j_1}} \beta_1 x_{\lambda_{k_1}} \cdot x_{\lambda_{i_2}} \cdot x_{\lambda_{j_2}} \beta_2 x_{\lambda_{k_2}} \cdot \ldots \cdot x_{\lambda_{i_l}} \cdot x_{\lambda_{j_l}} \beta_l x_{\lambda_{k_l}} \cdot \ldots]$

It implies that distinct basis elements are mapped to distinct $\mathcal{A}$-pseudo-paths. And, for $a_1 + \cdots + a_n \neq 0$ in A, $f(a_1 + \cdots + a_n) = [a_1] + \cdots + [a_n] \neq 0$. Hence $\widetilde{f}$ is injective. Therefore, $\phi = \widetilde{f}$ is a k-algebra isomorphism with the desired properties.

#

From Lemma 2.6, $PSE_k(\Delta, \mathcal{A}) \overset{\phi^{-1}}{\cong} \mathcal{PT}(A, M^F)$, then easily $ker\iota \overset{\phi^{-1}}{\cong} ker\tau$. Thus, a natural induced algebra homomorphism $\overline{\phi^{-1}}$ is obtained from ϕ^{-1} so that $PSE_k(\Delta, \mathcal{A})/ker\iota \overset{\overline{\phi^{-1}}}{\cong} \mathcal{PT}(A, M^F)/ker\tau$. Moreover, by Fact 2.5, we get the following $\widetilde{\phi}$ from $\overline{\phi^{-1}}$ as above so as to gain the result on $\mathcal{A}$-path algebra as similar as on $\mathcal{A}$-pseudo-path algebra:

Lemma 2.7 *Let $T(A, M^F)$ be the free $\mathcal{A}$-path-type tensor algebra built by a $\mathcal{A}$-path algebra $k(\Delta, \mathcal{A})$. Then there is a k-algebra isomorphism $\widetilde{\phi}$: $T(A, M^F) \to k(\Delta, \mathcal{A})$ such that for any $t \geq 1$,*

$$\widetilde{\phi}(\bigoplus_{j\geq t} M^{Fj}) = \widetilde{J}^t.$$

Moreover, we can obtain the commutative diagram:

$$\begin{array}{ccc} \mathcal{PT}(A, M^F) & \xrightarrow{\cong,\ \phi} & PSE_k(\Delta, \mathcal{A}) \\ \downarrow{\scriptstyle \tau} & & \downarrow{\scriptstyle \iota} \\ T(A, M^F) & \xrightarrow{\cong,\ \widetilde{\phi}} & k(\Delta, \mathcal{A}) \end{array} \tag{5}$$

Proposition 2.8 *Let $\mathcal{PT}(A, M)$ be an $\mathcal{A}$-path-type pseudo tensor algebra with the corresponding $\mathcal{A}$-pseudo path algebra $PSE_k(\Delta, \mathcal{A})$. Then there is a surjective k-algebra homomorphism φ: $PSE_k(\Delta, \mathcal{A}) \to \mathcal{PT}(A, M)$ such that for any $t \geq 1$,*

$$\varphi(J^t) = \bigoplus_{n,\ l \geq t} M(n, l).$$

Proof: Let $\mathcal{PT}(A, M^F)$ be the free $\mathcal{A}$-path-type pseudo tensor algebra built by the $\mathcal{A}$-pseudo path algebra $PSE_k(\Delta, \mathcal{A})$. Then by Lemma 2.3, there is a k-algebra isomorphism ϕ: $\mathcal{PT}(A, M^F) \to PSE_k(\Delta, \mathcal{A})$ such that for any $t \geq 1$, $\phi(\bigoplus_{n,\ l \geq t} M^F(n, l)) = J^t$.

On the other hand, by Lemma 2.1, there is a surjective k-algebra morphism π: $\mathcal{PT}(A, M^F) \to \mathcal{PT}(A, M)$ such that $\pi({}_iM^F_j) =_i M_j$ for any $i, j \in I$, so $\pi(M^F) = M$.

Therefore, $\varphi = \pi\phi^{-1}$: $PSE_k(\Delta, \mathcal{A}) \to \mathcal{PT}(A, M)$ is a surjective k-algebra morphism with $\varphi(J^t) = \pi(\bigoplus_{n,\ l \geq t} M^F(n, l)) = \bigoplus_{n,\ l \geq t} M(n, l)$ for any $t \geq 1$.

#

According to $\varphi = \pi\phi^{-1}$ and the description of kerι and kerτ in Fact 2.5, we have φ(kerι) =kerτ. Then, by Proposition 2.8, we induce naturally a surjective k-algebra homomorphism

$\widetilde{\varphi} : PSE_k(\Delta, \mathcal{A})/ker\iota \to \varphi(PSE_k(\Delta, \mathcal{A}))/\varphi(ker\iota) = \mathcal{PT}(A, M)/ker\tau.$

Thus, the similar result holds for $\mathcal{A}$-path-type tensor algebra:

Proposition 2.9 *Let $T(A, M)$ be an $\mathcal{A}$-path-type tensor algebra with the corresponding $\mathcal{A}$-path algebra $k(\Delta, \mathcal{A})$. Then there is a surjective k-algebra homomorphism $\widetilde{\varphi}$: $k(\Delta, \mathcal{A}) \to T(A, M)$ such that for any $t \geq 1$,*

$$\widetilde{\varphi}(\widetilde{J}^t) = \bigoplus_{j \geq t} M^t.$$

Also, we obtain the commutative diagram:

$$\begin{array}{ccc} PSE_k(\Delta, \mathcal{A}) & \xrightarrow{\varphi} & \mathcal{PT}(A, M) \\ \downarrow{\scriptstyle \iota} & & \downarrow{\scriptstyle \tau} \\ k(\Delta, \mathcal{A}) & \xrightarrow{\widetilde{\varphi}} & T(A, M) \end{array} \tag{6}$$

A *relation* σ on an $\mathcal{A}$-pseudo path algebra $PSE_k(\Delta, \mathcal{A})$ (resp. $\mathcal{A}$-path algebra $k(\Delta, \mathcal{A})$) is a k-linear combination of some general $\mathcal{A}$-pseudo paths (resp. some $\mathcal{A}$-paths) P_i with same start vertex and same end vertex, i.e. $\sigma = k_1P_1 + \cdots + k_nP_n$ with $k_i \in k$ and $s(P_1) = \cdots = s(P_n)$ and $e(P_1) = \cdots = e(P_n)$. If $\rho = \{\sigma_t\}_{t \in T}$ is a set of relations on $PSE_k(\Delta, \mathcal{A})$ (resp. $k(\Delta, \mathcal{A})$), the pair $(PSE_k(\Delta, \mathcal{A}), \rho)$ (resp. $(k(\Delta, \mathcal{A}), \rho)$) is called an *$\mathcal{A}$-pseudo path algebra with relations* (resp. *$\mathcal{A}$-path algebra with relations*). Associated with $(PSE_k(\Delta, \mathcal{A}), \rho)$ (resp. $(k(\Delta, \mathcal{A}), \rho)$) is the quotient

k-algebra $PSE_k(\Delta, \mathcal{A}, \rho) \stackrel{\text{def}}{=} PSE_k(\Delta, \mathcal{A})/\langle\rho\rangle$ (resp. $k(\Delta, \mathcal{A}, \rho) \stackrel{\text{def}}{=} k(\Delta, \mathcal{A})/\langle\rho\rangle$), where $\langle\rho\rangle$ denotes the ideal in $PSE_k(\Delta, \mathcal{A})$ (resp. in $k(\Delta, \mathcal{A})$) generated by the set of relations ρ. When the length $l(P_i)$ of each P_i is at least j, it holds $\langle\rho\rangle \subset J^j$ (resp. $\langle\rho\rangle \subset \widetilde{J}^j$).

For an element $x \in PSE_k(\Delta, \mathcal{A})$ (resp. $\in k(\Delta, \mathcal{A})$), we write by $\bar{x}$ the corresponding element in $PSE_k(\Delta, \mathcal{A}, \rho)$ (resp. $k(\Delta, \mathcal{A}, \rho)$).

Fact 2.10 *$\delta \in k(\Delta, \mathcal{A})$ is a relation if and only if all $\sigma \in \iota^{-1}(\delta)$ are relations on $PSE_k(\Delta, \mathcal{A})$.*

This fact can be seen easily from the definition of ι. Note that the lengths of paths in a relation do not been restricted here. So, we have:

Proposition 2.11 *Suppose that Δ is a finite quiver, i.e. $\mid \Delta_0 \mid= n \leq \infty$.*

(i) *Each element x in $PSE_k(\Delta, \mathcal{A})$ (resp. $k(\Delta, \mathcal{A})$) is a sum of some relations;*

(ii) *Every ideal I of $PSE_k(\Delta, \mathcal{A})$ (resp. $k(\Delta, \mathcal{A})$) can be generated by a set of relations.*

Proof: (i) Let 1 be the identity of A, e_i the identity of A_i for $i \in \Delta_0$. Then $1 = \sum_{i\in\Delta} e_i$ is a decomposition into primitive orthogonal idempotents e_i.

$x = 1 \cdot x \cdot 1 = \sum_{i,j\in\Delta_0} e_i \cdot x \cdot e_j$. Due to the multiplication of $\mid \Delta_0 \mid= n \leq \infty$, $e_i \cdot x \cdot e_j$ can be expanded as a k-linear combination of some such $\mathcal{A}$-paths which have the same start vertex i and the same end vertex j, so $e_i \cdot x \cdot e_j$ is a relation on $PSE_k(\Delta, \mathcal{A})$.

(ii) Assume I is generated by $\{x_\lambda\}_{\lambda\in\Lambda}$. By (i), each x_λ is a sum of some relations $\{\sigma_{\lambda,\, i}\}$. Then I is generated by all $\{\sigma_{\lambda,\, i}\}$.

#

From the definition of J, we have

$$PSE_k(\Delta, \mathcal{A}, \rho)/\bar{J} = (PSE_k(\Delta, \mathcal{A})/\langle\rho\rangle)/(J/\langle\rho\rangle) \cong PSE_k(\Delta, \mathcal{A})/J \cong \oplus_{i\in\Delta_0} A_i.$$

Suppose all A_i are k-simple algebras and $J^t \subset \langle\rho\rangle$ for some integer t. Then $PSE_k(\Delta, \mathcal{A}, \rho)/\bar{J} \cong \oplus_{i\in\Delta_0} A_i$ is semisimple and $\bar{J}^t = 0$. It follows that $\bar{J} = \text{rad}PSE_k(\Delta, \mathcal{A}, \rho)$. The similar discussion can be made for $\widetilde{J}$ of $k(\Delta, \mathcal{A})$. Hence we get:

Proposition 2.12 (i) *Let $(PSE_k(\Delta, \mathcal{A}), \rho)$ be an $\mathcal{A}$-pseudo path algebra with relations where A_i are simple for all $i \in \Delta_0$. Assume that $J^t \subset \langle\rho\rangle$ for some t. Then the image $\bar{J}$ of J in $PSE_k(\Delta, \mathcal{A}, \rho)$ is $\text{rad}PSE_k(\Delta, \mathcal{A}, \rho)$, i.e. $\bar{J} = \text{rad}PSE_k(\Delta, \mathcal{A}, \rho)$;*

(ii) *Let $(k(\Delta, \mathcal{A}), \rho)$ be an $\mathcal{A}$-path algebra with relations where A_i are simple for all $i \in \Delta_0$. Assume that $\widetilde{J}^t \subset \langle\rho\rangle$ for some t. Then the image $\bar{\widetilde{J}}$ of $\widetilde{J}$ in $k(\Delta, \mathcal{A}, \rho)$ is $\text{rad}k(\Delta, \mathcal{A}, \rho)$, i.e. $\bar{\widetilde{J}} = \text{rad}k(\Delta, \mathcal{A}, \rho)$.*

Now, suppose A is a left Artinian algebra over k, $r = r(A)$ the radical of A. Then for all $l \geq 0$, the ring r^l/r^{l+1} is an A-bimodule by $a\cdot(r^l/r^{l+1})\cdot b = ar^lb/r^{l+1}$ for $a, b \in A$. From $r\cdot r^l/r^{l+1} = 0$ and $r^l/r^{l+1} \cdot r = 0$, we know that r^l/r^{l+1} is a semisimple left and right A-module. For $\bar{x} = x + r \in A/r$, let $\bar{x} \cdot (r^l/r^{l+1}) \stackrel{\text{def}}{=} x \cdot (r^l/r^{l+1}) = xr^l/r^{l+1}$ and $(r^l/r^{l+1}) \cdot \bar{x} = (r^l/r^{l+1}) \cdot x = r^lx/r^{l+1}$, then r^l/r^{l+1} is also an A/r-bimodule and a semisimple left and right A/r-module.

Proposition 2.13 *Let A be a left Artinian algebra over k, $r = r(A)$ the radical of A. Write $A/r = \bigoplus_{i=1}^s \overline{A}_i$ where $\overline{A}_i$ is a simple subalgebra for each i. Then, for all $l \geq 0$,*

(i) *r^l/r^{l+1} is finitely generated as an A/r-bimodule;*

(ii) *${}_iM_j^{(l)} \stackrel{\text{def}}{=} \overline{A}_i \cdot r^l/r^{l+1} \cdot \overline{A}_j$ is finitely generated as $\overline{A}_i$-$\overline{A}_j$-bimodule for each pair (i, j).*

Proof: (i) Since A is left Artinian, r^l/r^{l+1} is finitely generated as a left A-module by Corollary I.3.2 in [1]. So, we can write $r^l/r^{l+1} = \sum_{p=1}^{w} A\overline{x}_p$ with some $\overline{x}_p \in r^l/r^{l+1}$. But, due to the definitions of actions, $A\overline{x}_p = (A/r)\overline{x}_p$ Then, $r^l/r^{l+1} = \sum_{p=1}^{w}(A/r)\overline{x}_p$. Moreover, $r^l/r^{l+1} = r^l/r^{l+1} \cdot A/r = (\sum_{p=1}^{w}(A/r)\overline{x}_p)(A/r) = \sum_{p=1}^{w}(A/r)\overline{x}_p(A/r)$, which means r^l/r^{l+1} is finitely generated as an A/r-bimodule.

(ii) ${}_iM_j^{(l)} = \overline{A}_i \cdot r^l/r^{l+1} \cdot \overline{A}_j = \overline{A}_i \cdot (\sum_{p=1}^{w}(A/r)\overline{x}_p(A/r)) \cdot \overline{A}_j = \sum_{p=1}^{w}\sum_{u,\,v=1}^{s} \overline{A}_i\overline{A}_u\overline{x}_p\overline{A}_v\overline{A}_j = \sum_{p=1}^{w}\overline{A}_i\overline{x}_p\overline{A}_j$. Hence, ${}_iM_j^{(l)}$ is finitely generated as $\overline{A}_i$-$\overline{A}_j$-bimodule.

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In particular, for $l = 1$, one sees that ${}_iM_j \stackrel{\text{def}}{=} \overline{A}_i \cdot r/r^2 \cdot \overline{A}_j$ is finitely generated as $\overline{A}_i$-$\overline{A}_j$-bimodule for each pair (i, j). Below, the rank of ${}_iM_j$ will be denoted by t_{ij}.

For $k \neq i$, $\overline{A}_k \cdot_i M_j = \overline{A}_k \cdot (\overline{A}_i \cdot r/r^2 \cdot \overline{A}_j) = (\overline{A}_k\overline{A}_i) \cdot (r/r^2 \cdot \overline{A}_j) = 0 \cdot r/r^2 \cdot \overline{A}_j = 0$; similarly, for $k \neq j$, ${}_iM_j \cdot \overline{A}_k = 0$. Thus, we can obtain the $\mathcal{A}$-path-type pseudo tensor algebra $\mathcal{PT}(A/r, r/r^2)$ and the $\mathcal{A}$-path-type tensor algebra $T(A/r, r/r^2)$ and the corresponding $\mathcal{A}$-pseudo path algebra $PSE_k(\Delta, \mathcal{A})$ and $\mathcal{A}$-path algebra $k(\Delta, \mathcal{A})$ respectively, with $\mathcal{A} = \{\overline{A}_i : i \in \Delta_0\}$, where Δ is called as *the quiver of the left Artinian algebra A*.

In the sequel, we will always assume that A is a left Artinian algebra. We will firstly show that under some important conditions, a left Artinian algebra A is isomorphic to some $PSE_k(\Delta, \mathcal{A}, \rho)$.

3 When The Quotient Algebra Can Be Lifted

Firstly, we introduce the concept of the set of primitive orthogonal simple subalgebras of a left Artinian algebra. For a left Artinian algebra A and $A/r = \bigoplus_{i=1}^{s} \overline{A}_i$ with simple subalgebras $\overline{A}_i$ for all i where $r = r(A)$ is the radical of A, assume there are simple k-subalgebras $B_1, \cdots, B_s$ of A satisfying $B_i \cong \overline{A}_i$ as k-algebras for all i under the canonical morphism η: $A \to A/r$ and $B_iB_j = \begin{cases} B_i, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}$. Then we will call $\widehat{B} = \{B_1, \cdots, B_s\}$ *the set of primitive orthogonal simple subalgebras of A*.

Obviously, $\overline{A}_i\overline{A}_j = \begin{cases} \overline{A}_i, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}$. From the definition, $\eta(B_i) = \overline{A}_i$ for any i. Every B_i is a simple k-subalgebra of A, so $B = B_1 + \cdots + B_s$ is a semisimple subalgebra of A.

Our original idea is to introduce the concept of primitive orthogonal simple subalgebras as a generalization of primitive orthogonal idempotents and then transplant the method of primitive orthogonal idempotents in elementary algebras into a left Artinian algebras.

In a left Artinian algebra A, we will show as follows the existence of the set of primitive orthogonal simple k-subalgebras when A/r can be lifted.

An algebra morphism ε: $A/r \to A$ such that $\eta\varepsilon = 1$ will be called a *lifting* of the quotient algebra A/r. In this case, we call A/r *can be lifted*. Evidently, a lifting ε is always a monomorphism and $\mathrm{Im}\varepsilon = B$ is a subalgebra of A while is isomorphic to A/r. Then, B is semisimple. Moreover, $A = B \oplus r$ as a direct sum of k-linear spaces. Hence, A/r can be lifted if and only A is split over its radical r.

Now, we assume that A/r can be lifted such that $A = B \oplus r$ as above. For the canonical morphism η: $A \to A/r$, $\mathrm{Im}\eta|_B = (B + r)/r = A/r$. And, $\mathrm{Ker}\eta|_B = 0$ due to $r \cap B = 0$. Thus, $\eta(B) = A/r$ and $B \stackrel{\eta|_B}{\cong} A/r$ as k-algebras. Since B is semisimple, we can write $B = \bigoplus_{i=1}^{s} B_i$ with simple k-subalgebras B_i for all i. Then $B_iB_j = \begin{cases} B_i, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}$. Moreover, $\eta(B) = \sum_{i=1}^{s} \eta(B_i)$ where $\eta(B_i)$ are simple k-subalgebras of A/r. Denote $\overline{A}_i = \eta(B_i)$. It is seen that $\widehat{B} = \{B_1, \cdots, B_s\}$ is the set of primitive orthogonal simple subalgebras of A.

Lemma 3.1 *Assume A is a left Artinian k-algebra with $r = r(A)$ the radical of A and A/r can be lifted such that $A = B \oplus r$ with $\widehat{B} = \{B_1, \cdots, B_s\}$ the set of primitive orthogonal simple subalgebras of A as constructed above. Write $A/r = \bigoplus_{i=1}^{s} \overline{A}_i$ where $\overline{A}_i$ are simple algebras for all i. The following statements hold:*

(i) *Let $\{r_\lambda : \lambda \in I\}$ be a set of elements in r with the index set I such that the images $\overline{r}_\lambda$ in r/r^2 for all $\lambda \in I$ generate r/r^2 as an A/r-bimodule. Then $B_1 \cup \cdots \cup B_s \cup \{r_\lambda : \lambda \in I\}$ generates A as a k-algebra;*

(ii) *There is a surjective k-algebra homomorphism $\widetilde{f} : \mathcal{PT}(A/r, r/r^2) \to A$ with*

$$\bigoplus_{n \geq rl(A)} \bigoplus_{max\{rl(A),\, (n-1)/2\} \leq l \leq n} M(n,l) \subset Ker\widetilde{f} \subset \bigoplus_{j \geq 2} M(j)$$

where $rl(A)$ denotes the Loewy length of A as a left A-module.

Proof: (i) Since r is nilpotent, there is the least m such that $r^m = 0$ but $r^{m-1} \neq 0$. It is easy to see that m is just the Loewy length $\mathrm{rl}(A)$.

In the follows, we will prove this result by in the induction on m.

When $m = 1$, then $r = 0$ and A is semisimple. Thus $B_i = \overline{A}_i$. Hence A is generated as a k-algebra by $B_1 \cup \cdots \cup B_s$.

When $m = 2$, we have $r^2 = 0$. For the canonical morphism η, $\eta(B_i) = \overline{A}_i$. So, as a k-algebra, A/r can be generated by $(B_1 + r) \cup \cdots \cup (B_s + r)$. Write $A/r = \langle B_1 + r, \cdots, B_s + r\rangle/r$. And, $\langle B_1 + r, \cdots, B_s + r\rangle/r = (\langle B_1, \cdots, B_s\rangle + r)/r$. Thus, $A/r = (\langle B_1, \cdots, B_s\rangle + r)/r$. Hence $A = \langle B_1, \cdots, B_s\rangle + r$. But, $r/r^2 = \sum_{\lambda \in I} A/r \cdot \overline{r}_\lambda = \sum_{\lambda \in I} A/r \cdot (r_\lambda + r^2) = \sum_{\lambda \in I}(Ar_\lambda + r^2) = (\sum_{\lambda \in I} Ar_\lambda) + r^2$. Then from $r^2 = 0$, we get $r = \sum_{\lambda \in I} Ar_\lambda$. It follows $A = \langle B_1, \cdots, B_s\rangle + r = \langle B_1, \cdots, B_s\rangle + \sum_{\lambda \in I}(\langle B_1, \cdots, B_s\rangle + r)r_\lambda = \langle B_1, \cdots, B_s\rangle + \sum_{\lambda \in I}\langle B_1, \cdots, B_s\rangle r_\lambda = \langle B_1 \cup \cdots \cup B_s \cup \{r_\lambda : \lambda \in I\}\rangle$ as a k-algebra.

Assume now that the claim holds for $m = l \geq 2$ and consider the case that $m = l + 1$.

Let P be the k-subalgebra of A generated by $B_1 \cup \cdots \cup B_s \cup \{r_\lambda : \lambda \in I\}$. Firstly, we will show that $P/(P \cap r^l) = A/r^l$.

Since $(A/r^l)/(r/r^l) \cong A/r$ is semisimple, then $r(A/r^l) = r/r^l$. By the induction assumption, $r^{l+1} = 0$ and $r^i \neq 0$ for any $i \leq l$. For any t, $(r/r^l)^t(A/r^l) = r^tA/r^l = r^t/r^l$ since $r^tA = r^t$ due to the existence of identity of A. Thus, $(r/r^l)^t(A/r^l) = 0$ if and only if $t \geq l$. (If there were $t < l$ such that $r^t = r^l$, then $r^{t+1} = r^{l+1} = 0$. It contradicts to $\mathrm{rl}(A) = m = l + 1$). Therefore $\mathrm{rl}(A/r^l) = l$.

Let ζ: $A \to A/r^l$ be the canonical morphism and $\widetilde{B_i} = \zeta(B_i)$ are simple algebras for all i, π the canonical morphism from A/r^l to $(A/r^l)/(r/r^l) = A/r$. Then $\pi\zeta = \eta$. It follows that $\pi(\widetilde{B_i}) = \overline{A}_i$. This means that $\widehat{\widetilde{B}} = \{\widetilde{B_1}, \cdots, \widetilde{B_s}\}$ is the set of primitive radical-orthogonal simple algebras of A/r^l. We have that all elements in $\{\overline{r}_\lambda : \lambda \in I\}$ in r/r^2 generate r/r^2 as an A/r-module. But, $A/r \cong (A/r^l)/(r/r^l)$, $r/r^2 \cong (r/r^l)/(r/r^l)^2$. So, all elements in $\{\overline{r}_\lambda : \lambda \in I\}$ in $(r/r^l)/(r/r^l)^2$ generate $(r/r^l)/(r/r^l)^2$ as an $\widetilde{(A/r^l)/(r/r^l)}$-module. Let $\widetilde{r}_\lambda = \zeta(r_\lambda) \in r/r^l$. Then $\pi(\widetilde{r}_\lambda) = \overline{r}_\lambda$. Thus, by the induction assumption, $\widetilde{B_1} \cup \cdots \cup \widetilde{B_s} \cup \{\widetilde{r}_\lambda : \lambda \in I\}$ generates the k-algebra A/r^l.

On the other hand, $B_1 \cup \cdots \cup B_s \cup \{r_\lambda : \lambda \in I\}$ generates P. Then $\widetilde{B_1} \cup \cdots \cup \widetilde{B_s} \cup \{\widetilde{r_\lambda} : \lambda \in I\}$ generates the k-algebra $P/(P \cap r^l)$. But, $P/(P \cap r^l)$ can be embedded into A/r^l. Therefore, we get that $P/(P \cap r^l) = A/r^l$.

Below it is proved that in fact $P = A$, which means that $B_1 \cup \cdots \cup B_s \cup \{r_\lambda : \lambda \in I\}$ generates A.

Let $x \in A$. Then there exists $y \in P$ such that $x + r^l = y + P \cap r^l$. It follows $x - y \in r^l$. Thus there are $\alpha_i \in r^{l-1}$ and $\beta_i \in r$ such that $x - y = \sum_{i=1}^q \alpha_i \beta_i$. But, $\alpha_i + r^l$ and $\beta_i + r^l$ in A/r^l and $A/r^l = P/(P \cap r^l)$. Then there are a_i and b_i in P such that $\alpha_i + r^l = a_i + P \cap r^l$ and $\beta_i + r^l = b_i + P \cap r^l$. Due to $\alpha_i \in r^{l-1}$ and $\beta_i \in r$, we have $a_i \in r^{l-1}$ and $b_i \in r$. Let $a_i' = \alpha_i - a_i$ and $b_i' = \beta_i - b_i$. Then $a_i', b_i' \in r^l$. Hence $\alpha_i \beta_i = (a_i + a_i')(b_i + b_i') = a_i b_i + a_i' b_i + a_i b_i' + a_i' b_i' = a_i b_i \in P$ for all i where $a_i' b_i \in r^{l+1} = 0$, $a_i b_i' \in r^{2l-1} = 0$, $a_i' b_i' \in r^{2l} = 0$. It follows that $x - y \in P$. Then $x \in P$.

(ii) $r/r^2 = A/r \cdot r/r^2 \cdot A/r = \sum_{i,j=1}^s \overline{A}_i \cdot r/r^2 \cdot \overline{A}_j$ is a direct sum decomposition due to $\overline{A}_i^2 = \overline{A}_i$ and $\overline{A}_i \overline{A}_j = 0$ for $i \neq j$. Corresponding to this, in A, we denote $W = \sum_{i,j=1}^s B_i r B_j$, where $B_i \overset{\eta}{\cong} \overline{A}_i$. W is a direct sum of $B_i r B_j$ due to $B_i^2 = B_i$ and $B_i B_j = 0$ for $i \neq j$. Obviously, W is a subalgebra of r and then of A. And, r/r^2 is a (A/r)-bimodule with the action of A/r as above.

$(A/r) \oplus (r/r^2)$ is a k-algebra in which the multiplication is taken through that of A/r and r/r^2 and the A/r-bimodule action of r/r^2.

For each pair of integers i, j with $1 \leq i, j \leq s$, choose elements $\{y_u^{ij}\}_{u \in \Omega_{ij}}$ in $B_i r B_j$ such that $\{\overline{y}_u^{ij}\}_{u \in \Omega_{ij}}$ is a k-basis for $\overline{A}_i \cdot r/r^2 \cdot \overline{A}_j$ for $\overline{y}_u^{ij} = y_u^{ij} + r^2$ the image in r/r^2. Then $\bigcup_{i,j=1}^s \{\overline{y}_u^{ij}\}_{u \in \Omega_{ij}}$ is a basis for r/r^2. It follows from (i) that $\bigcup_{i,j,u} \{y_u^{i,j}\}_{u \in \Omega_{ij}} \cup B_1 \cup \cdots \cup B_s$ generates A as a k-algebra.

It is easy to see that $\{y_u^{ij}\}_{u \in \Omega_{ij}}$ is k-linear independent in $B_i r B_j$. From the fact that W is a direct sum of $B_i r B_j$, it follows $\bigcup_{i,j=1}^s \{y_u^{ij}\}_{u \in \Omega_{ij}}$ is a k-linear independent set in W.

Define $f : (A/r) \oplus (r/r^2) \to A$ by $f|_{\overline{A}_i} = \eta^{-1}$ and $f(\overline{y}_u^{ij}) = y_u^{ij}$. Then, $f|_{A/r}$: $A/r \to B = f(A/r)$ is a k-algebra isomorphism since $B \overset{\eta|_B}{\cong} A/r$, and $f|_{r/r^2}$: $r/r^2 \to f(r/r^2) (\subset W \subset r)$ is an isomorphism as k-linear spaces. Thus, $f : (A/r) \oplus (r/r^2) \to A$ is a k-linear map. Hence by Lemma 2.4, there is a unique algebra morphism $\widetilde{f}$: $\mathcal{PT}(A/r, r/r^2) \to A$ such that $\widetilde{f}|_{(A/r) \oplus (r/r^2)} = f$. As said above, $\bigcup_{i,j,u} \{y_u^{i,j}\}_{u \in \Omega_{ij}} \cup B_1 \cup \cdots \cup B_s$ generates A as a k-algebra. Therefore, $\widetilde{f}$ is surjective.

From the definition of $\widetilde{f}$, we have $\widetilde{f}((r/r^2)^j) = f(r/r^2)^j \subset r^j \subset r^2$ for $j \geq 2$, where $(r/r^2)^j$ denotes $r/r^2 \otimes_k r/r^2 \otimes_k \cdots \otimes_k r/r^2$ with j copies of r/r^2. And, $f|_{A/r}$ and $f|_{r/r^2}$ are monomorphic, moreover from the definition of f on A/r and r/r^2 respectively, it is easy to see that $\widetilde{f}|_{(A/r) \oplus (r/r^2)}$: $(A/r) \oplus (r/r^2) \to A$ is a monomorphism with image intersecting r^2 trivially. As denoted in Section 2, $M(n) = \sum_{M_1, M_2, \cdots, M_n} M_1 \otimes_k M_2 \otimes_k \cdots \otimes_k M_n$ where M_i is either r/r^2 (at least there exists one) or A/r but no two A/r's are neighbouring, then $\mathcal{PT}(A/r, r/r^2) = A/r \oplus M(1) \oplus M(2) \oplus \cdots \oplus M(n) \oplus \cdots$. It follows that $\mathrm{Ker}\widetilde{f} \subset \bigoplus_{j \geq 2} M(j)$.

On the other hand, $M(n, l)$ equals the sum of these items $M_1 \otimes_k M_2 \otimes_k \cdots \otimes_k M_n$ of $M(n)$ in which there are l M_i's equal to r/r^2 and $M(n) = \sum_{(n-1)/2 \leq l \leq n} M(n, l)$ as in Section 2. $\widetilde{f}((r/r^2)^j) = 0$ for $j \geq rl(A)$ since $r^j = 0$ in this case, it follows $\widetilde{f}(M(n, l)) = 0$ for any n and possible $l \geq rl(A)$. Therefore we get

$$\bigoplus_{n \geq rl(A)} \bigoplus_{max\{rl(A),\, (n-1)/2\} \leq l \leq n} M(n, l) \subset Ker\widetilde{f}.$$

#

Theorem 3.2 (Generalized Gabriel's Theorem Under Lifting)

Assume that A is a left Artinian k-algebra and A/r can be lifted. Then, $A \cong PSE_k(\Delta, \mathcal{A}, \rho)$ with $J^s \subset \langle\rho\rangle \subset J$ for some s, where Δ is the quiver of A and ρ is a set of relations on $PSE_k(\Delta, \mathcal{A})$.

Proof: Let Δ be the associated quiver of A. From Lemma 3.1(ii), we have the surjective k-algebra morphism $\widetilde{f}$: $\mathcal{PT}(A/r, r/r^2) \to A$ with

$$\bigoplus_{n\geq rl(A)} \bigoplus_{max\{rl(A),\,(n-1)/2\}\leq l\leq n} M(n,l) \subset Ker\widetilde{f} \subset \bigoplus_{j\geq 2} M(j).$$

By Proposition 2.8, there is a surjective k-algebra homomorphism φ: $PSE_k(\Delta, \mathcal{A}) \to \mathcal{PT}(A/r, r/r^2)$ such that for any $t \geq 1$, $\varphi(J^t) = \bigoplus_{n,l\geq t} M(n,l)$. Then $\widetilde{f}\varphi$: $k(\Delta, \mathcal{A}) \to A$ is a surjective k-algebra morphism with the kernel $I =$Ker$(\widetilde{f}\varphi) = \varphi^{-1}(Ker\widetilde{f})$.

But, $\varphi(J^{rl(A)}) = \bigoplus_{n,\,l\geq rl(A)} M(n,l)$ and $\varphi(J^2) = \bigoplus_{n,\,l\geq 2} M(n,l)$. So, by Lemma 3.1(ii), $\varphi(J^{rl(A)}) \subset$Ker$\widetilde{f} \subset \varphi(J^2) + M(2,1) + M(3,1)$.

We can show $J^t \subset \varphi^{-1}\varphi(J^t) \subset J^t + \phi(\bigoplus_n \bigoplus_{l\leq t-1} M^F(n,l)) \cap \phi(\mathrm{Ker}\pi)$ for $t \geq 1$. In fact, trivially, $J^t \subset \varphi^{-1}\varphi(J^t)$. On the other hand, $\varphi = \pi\phi^{-1}$ and then $\varphi^{-1} = \phi\pi^{-1}$. By Proposition 2.4, $\varphi(J^t) = \bigoplus_{n,\,l\geq t} M(n,l)$. From the definition of π in Lemma 2.1, it can be seen that $\pi^{-1}(\bigoplus_{n,\,l\geq t} M(n,l)) \subset \bigoplus_{n,\,l\geq t} M^F(n,l) + (\bigoplus_n \bigoplus_{l\leq t-1} M^F(n,l)) \cap \mathrm{Ker}\pi$. Thus, by Lemma 2.3, we have

$\varphi^{-1}\varphi(J^t) = \phi\pi^{-1}(\bigoplus_{n,\,l\geq t} M(n,l)) \subset \phi(\bigoplus_{n,\,l\geq t} M^F(n,l)) + \phi(\bigoplus_n \bigoplus_{l\leq t-1} M^F(n,l)) \cap \phi(\mathrm{Ker}\pi)$
$= J^t + \phi(\bigoplus_n \bigoplus_{l\leq t-1} M^F(n,l)) \cap \phi(\mathrm{Ker}\pi)$.

Hence,

$J^{rl(A)} \subset \varphi^{-1}\varphi(J^{rl(A)}) \subset \varphi^{-1}(\mathrm{Ker}\widetilde{f}) = I \subset \varphi^{-1}\varphi(J^2) + \varphi^{-1}(M(2,1) + M(3,1))$
$\subset J^2 + \phi(M^F(3,1) + M^F(2,1) + M^F(1,1)) \cap \phi(\mathrm{Ker}\pi) + \varphi^{-1}(M(2,1) + M(3,1))$
$= J^2 + A \cdot PSE(\Delta_1, \mathcal{A}) \cdot A$

since $\phi(M^F(1,1)) \cap \phi(\mathrm{Ker}\pi) = 0$ and then $\phi(M^F(3,1) + M^F(2,1) + M^F(1,1)) \cap \phi(\mathrm{Ker}\pi) + \varphi^{-1}(M(2,1) + M(3,1)) = A \cdot PSE(\Delta_1, \mathcal{A}) \cdot A$.

But, it is clear that $J^2 + A \cdot PSE(\Delta_1, \mathcal{A}) \cdot A = J$. Therefore, we get:

$$J^{rl(A)} \subset \varphi^{-1}(Ker\widetilde{f}) = I \subset J$$

Lastly, by Proposition 2.5, there is a set ρ of relations so that I can be generated by ρ, i.e. $I = \langle\rho\rangle$. Hence, $k(\Delta, \mathcal{A}, \rho) = k(\Delta, \mathcal{A})/\langle\rho\rangle \cong A$ with $\langle\rho\rangle =$Ker$(\widetilde{f}\varphi)$ and $J^{rl(A)} \subset \langle\rho\rangle \subset J$.

#

Usually, for a left Artinian algebra A, the set ρ of relations in Theorem 3.2 is infinite. But, when A is finite dimensional, we can show ρ is finite.

In fact, suppose that A is finite dimensional, then $\overline{A}_i$ is finite dimensional for all i. Thus, the k-space consisting of all $\mathcal{A}$-pseudo paths of a certain length is finite dimensional. It follows that $J^{rl(A)}$ is the ideal finitely generated in $PSE_k(\Delta, \mathcal{A})$ by all $\mathcal{A}$-pseudo paths of length $rl(A)$. Similarly, $PSE_k(\Delta, \mathcal{A})/J^{rl(A)}$ is generated finitely as a k-space, under the meaning of isomorphism, by all $\mathcal{A}$-paths of length less than $rl(A)$, so as well as $I/J^{rl(A)}$ as a k-subspace. Then it is easy to know that I is a finitely generated ideal in $PSE_k(\Delta, \mathcal{A})$. Assume $\{\sigma_1, \cdots, \sigma_p\}$ is a set of finite generators for the ideal I. For the identity $\overline{1}$ of A/r, we have the decomposition of orthogonal idempotents $\overline{1} = \overline{e}_1 + \cdots + \overline{e}_s$, where $\overline{e}_i$ is the identity of $\overline{A}_i$. Then $\sigma_l = \overline{1} \cdot \sigma_l \cdot \overline{1} = \sum_{1\leq i,j\leq s} \overline{e}_i \cdot \sigma_l \cdot \overline{e}_j$, where $\overline{e}_i\sigma_l\overline{e}_j$ can be expanded as a k-linear combination of some such $\mathcal{A}$-pseudo paths which have the same start vertex i and the same end vertex j. So, $\sigma^{ilj} = \overline{e}_i \cdot \sigma_l \cdot \overline{e}_j$ is a relation on the $\mathcal{A}$-pseudo path algebra $PSE_k(\Delta, \mathcal{A})$. Moreover, I is generated by all σ^{ilj} due to $\sigma_l = \sum_{i,j} \sigma^{ilj}$. Therefore, we have a finite set $\rho = \{\sigma^{ilj} : 1 \leq i, j \leq s, 1 \leq l \leq p\}$ with $I = \langle\rho\rangle$. so that $k(\Delta, \mathcal{A}, \rho) = k(\Delta, \mathcal{A})/\langle\rho\rangle \cong A$. Therefore we get the following:

Corollary 3.3 *Assume that A is a finite dimensional k-algebra and A/r can be lifted. Then, $A \cong PSE_k(\Delta, \mathcal{A}, \rho)$ with $J^s \subset \langle\rho\rangle \subset J$ for some s, where Δ is the quiver of A and ρ is a finite set of relations on $PSE_k(\Delta, \mathcal{A})$.*

When A is elementary, $A_i = A_j = k$ and ${}_iM_j = r/r^2$ is free as a k-linear space. Thus, π is an isomorphism, then $\text{Ker}\pi = 0$ and $\text{Ker}\varphi = 0$. By the classical Gabriel Theorem, we have $J^{rl(A)} \subset \langle\rho\rangle \subset J^2$, which is a special case of the results of Theorem 3.2 and Corollary 3.3.

According to the famous Wedderburn-Malcev Theorem (see [4]), for a left Artinian k-algebra A and its radical r, if $\text{Dim}A/r \leq 1$, then A/r can be lifted. Here, the dimension $\text{Dim}A$ of a k-algebra A is defined as $\text{Dim}A = sup\{n : H^n_k(A, M) \neq 0$ for some A-bimodule $M\}$ and $H^n_k(A, M)$ means the n'th Hochschild cohomology module of A with coefficients in M. In particular, $\text{Dim}A/r = 0$ if and only if A/r is a separable k-algebra. By Corollary 10.7b of [4], when k is a perfect field (e.g. $\text{char}k = 0$ or k is a finite field), A is separable. So, we have the following:

Proposition 3.4 *Assume that A is a left Artinian k-algebra. Then, $A \cong PSE_k(\Delta, \mathcal{A}, \rho)$ with $J^s \subset \langle\rho\rangle \subset J$ for some s, where Δ is the quiver of A and ρ is a set of relations of $PSE_k(\Delta, \mathcal{A})$, when one of the following conditions holds:*

(i) $\text{Dim}A/r \leq 1$ *for the radical r of A;*

(ii) *A/r is separable;*

(iii) *k is a perfect field (e.g. when* $\text{char}k = 0$ *or k is a finite field).*

Note that in Proposition 3.4, the condition (ii) is a special case of (i), and (iii) is that of (ii).

In Theorem 3.2, $A \cong PSE_k(\Delta, \mathcal{A}, \rho)$ holds where Δ is the quiver of A from the corresponding $\mathcal{A}$-pseudo path algebra of the $\mathcal{A}$-path-type pseudo tensor algebra $\mathcal{PT}(A/r, r/r^2)$ by the definitions in Section 2. Moreover, we will discuss the uniqueness of the correspondent pseudo path algebra and quiver of a left Artinian algebra under isomorphism, that is, if there exists another quiver and its related pseudo path algebra so that the same isomorphism relation is satisfied. In fact, we have the following statement on the uniqueness:

Theorem 3.5 *Assume A is a left Artinian k-algebra. Let $A/r(A) = \bigoplus_{i=1}^p \overline{A}_i$ with simple algebras $\overline{A}_i$ and let $\mathcal{A} = \{A_1, \cdots, A_p\}$. If there is a quiver Δ and a pseudo path algebra $PSE_k(\Delta, \mathcal{B})$ with a set of simple algebras $\mathcal{B} = \{B_1, \cdots, B_q\}$ and ρ a set of relations satisfying that $A \cong PSE_k(\Delta, \mathcal{B}, \rho)$ with $J^t_\Delta \subset \langle\rho\rangle \subset J_\Delta$ for some t and J_Δ the ideal in $PSE_k(\Delta, \mathcal{B})$ generated by all pure paths in $PSE_k(\Delta_1, \mathcal{B})$, then Δ is just the quiver of A and $p = q$ such that $\overline{A}_i \cong B_i$ for $i = 1, \cdots, p$ after reindexing.*

Proof: According to the definition of J_Δ, it holds that $PSE_k(\Delta, \mathcal{B})/J_\Delta = B_1 + \cdots + B_q$. Since $J^t_\Delta \subset \langle\rho\rangle$, it follows that $(J_\Delta/\langle\rho\rangle)^t = J^t_\Delta/\langle\rho\rangle = 0$. And,

$$PSE_k(\Delta, \mathcal{B}, \rho)/(J_\Delta/\langle\rho\rangle) = (PSE_k(\Delta, \mathcal{B})/\langle\rho\rangle)/(J_\Delta/\langle\rho\rangle) = PSE_k(\Delta, \mathcal{B})/J_\Delta = B_1 + \cdots + B_q$$

is semisimple. Hence, $J_\Delta/\langle\rho\rangle$ is the radical of $PSE_k(\Delta, \mathcal{B}, \rho)$. Thus, from $A \cong PSEk(\Delta, \mathcal{B}, \rho)$, it follows that $A/r(A) \cong PSE_k(\Delta, \mathcal{B})/J_\Delta$. But, $A/r(A) = \bigoplus_{i=1}^p \overline{A}_i$ and $PSE_k(\Delta, \mathcal{B})/J_\Delta = B_1 + \cdots + B_q$with $\overline{A}_i$ and B_j are simple algebras. Therefore, $p = q$ and $\overline{A}_i \cong B_i$ for $i = 1, \cdots, p$ after reindexing, according to Wedderburn-Artinian Theorem.

On the other hand, $A/r(A)^2 \cong PSE_k(\Delta,\mathcal{B})/J_\Delta^2$. Then, the quivers of $A/r(A)^2$ and $PSE_k(\Delta,\mathcal{B})/J_\Delta^2$ are same.

But, $PSE_k(\Delta,\mathcal{B})/J_\Delta^2 = (PSE_k(\Delta,\mathcal{B})/\langle\rho\rangle)/(J_\Delta^2/\langle\rho\rangle) = PSE_k(\Delta,\mathcal{B},\rho)/(J_\Delta^2/\langle\rho\rangle)$ and the radical of $PSE_k(\Delta,\mathcal{B},\rho)$ is $J_\Delta/\langle\rho\rangle$. Then the radical of $PSE_k(\Delta,\mathcal{B})/J_\Delta^2$ is $(J_\Delta/\langle\rho\rangle)/(J_\Delta^2/\langle\rho\rangle) \cong J_\Delta/J_\Delta^2$. So, the quivers of $PSE_k(\Delta,\mathcal{B})/J_\Delta^2$ are that of the $\mathcal{A}$-path-type pseudo tensor algebra

$$\mathcal{PT}((PSE_k(\Delta,\mathcal{B})/J_\Delta^2)/(J_\Delta/J_\Delta^2),\ (J_\Delta/J_\Delta^2)/(J_\Delta^2/J_\Delta^2)) \cong \mathcal{PT}(PSE_k(\Delta,\mathcal{B})/J_\Delta, J_\Delta/J_\Delta^2).$$

Now, we consider the quiver Γ of $\mathcal{PT}(PSE_k(\Delta,\mathcal{B})/J_\Delta, J_\Delta/J_\Delta^2)$. From the definition of the quiver associating with an $\mathcal{A}$-path-type pseudo tensor algebra in Section 2, we know that $\Gamma_0 = \{1,\cdots,q\} = \Delta_0$. For any $i,j \in \Gamma_0$, the number of arrows from i to j in Γ is the rank r_{ij} of ${}_iM_j = B_i \cdot J_\Delta/J_\Delta^2 \cdot B_j$ as a finitely generated B_i-B_j-bimodule. However, by the definition of J_Δ, under the meaning of isomorphism, $B_i \cdot J_\Delta/J_\Delta^2 \cdot B_j$ can be constructed as an B_i-B_j-linear expansion of all $\mathcal{A}$-pseudo-paths of length 1 from i to j in $PSE_k(\Delta_1,\mathcal{B})$. It means that r_{ij} is equal to the number of arrows from i to j in Δ. Thus, the number of arrows from i to j in Γ is equal to that of arrows from i to j in Δ. Then $\Gamma_1 = \Delta_1$. Therefore, we get $\Gamma = \Delta$.

From the discussion above, it implies that the quivers of $A/r(A)^2$ is just Δ. Moreover, $A/r(A) = (A/r(A)^2)/(r(A)/r(A)^2)$ and $r(A)/r(A)^2 = (r(A)/r(A)^2)/(r(A)/r(A)^2)^2$, where $r(A)/r(A)^2$ is the radical of $A/r(A)^2$. So, the quivers Δ of $A/r(A)^2$ is also that of

$$\mathcal{PT}((A/r(A)^2)/(r(A)/r(A)^2),\ (r(A)/r(A)^2)/(r(A)/r(A)^2)^2).$$

But,

$$\mathcal{PT}(A/r(A), r(A)/r(A)^2) \cong \mathcal{PT}((A/r(A)^2)/(r(A)/r(A)^2),\ (r(A)/r(A)^2)/(r(A)/r(A)^2)^2).$$

It follows that Δ is the quiver of A.

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According to this theorem, we can see that for a left Artinian algebra A, the existence of a pseudo path algebra such that A is isomorphic to its quotient algebra (i.e. Theorem 3.2) can deduce its uniqueness, that is, it can only be the pseudo path algebra decided by the quiver and the semisimple decomposition of A.

Our main result means when the quotient algebra of a left Artinian algebra is lifted, the algebra can be covered by a pseudo path algebra under an algebra homomorphism. But, a generalized path algebra must be a homomorphic image of a pseudo path algebra and its definition seem to be more concise than that of pseudo path algebra. So, it is natural to ask why we do not look for a generalized path algebra to cover a left Artinian algebra. In fact, this is our first idea. However, unfortunately, in general, as shown by the following counter example, a left Artinian algebra with lifted quotient may not be a homomorphic image of its correspondent $\mathcal{A}$-path-type tensor algebra. Thus, one cannot use the method as above (i.e. through Proposition 2.9) to gain a generalized path algebra to cover the left Artinian algebra. The following counter example was given by W. Crawley-Boevey at University of Leeds. The author thanks him for his effective discussion.

Example 3.1 *There is an example of a finite dimensional algebra A over a field k such that*
(a) *A is split over its radical r, that is, A/r can be lifted;*
(b) *there is* **no** *surjective algebra homomorphism from $T(A/r, r/r^2)$ to A, that is, A cannot be equivalent to some quotient of $T(A/r, r/r^2)$.*

Concretely, we describe A as follows:
(1) Let F/k be a finite field extension, and let $\delta : F \to F$ be a non-zero k-derivation. For example, one can take $k = \mathbf{Z}_2(t)$, $F = \mathbf{Z}_2(\sqrt{t})$ and $\delta(p + q\sqrt{t}) = q$ for $p, q \in \mathbf{Z}_2(t)$ where $\mathbf{Z}_2$ denotes the prime field of characteristic 2. It is easy to check δ as a k-derivation due to $char k = 2$.

(2) Define $E = F \oplus F$ and consider it as an F-F-bimodule with the actions:

$$f(x, y) = (fx, fy) \qquad (x, y)f = (xf + y\delta(f), yf)$$

for $x, y, f \in F$. Let θ and ϕ be F-F-bimodule homomorphisms respectively from F to E and from E to F satisfying

$$\theta(x) = (x, 0) \qquad \phi(x, y) = y$$

for $x, y \in F$. Then we have the non-splitting extension of F-F-bimodules:

$$0 \to F \xrightarrow{\theta} E \xrightarrow{\phi} F \to 0$$

In fact, if there were $\psi : E \to F$ an F-F-bimodule homomorphism with $\psi \cdot \theta = 1_F$, then for all $f \in F$,

$$\delta(f) = \psi\theta(\delta(f)) = \psi(\delta(f), 0) = \psi(\delta, f) - \psi(0, f) = \psi((0, 1)f) - \psi(f(0, 1)) = \psi(0, 1)f - f\psi(0, 1) = 0,$$

it follows $\delta = 0$, which contradicts to the presumption on δ.

(3) Define $A = F \oplus F \oplus E$ with multiplication given by

$$(x, y, e)(x', y', e') = (xx', xy' + yx', xe' + \theta(yy') + ex').$$

Let $S = \{(x, 0, 0) : x \in F\}$. Then S is a subalgebra of A isomorphic to F.
Let $r = \{(0, y, e) : y \in F, e \in E\}$. Then r is an ideal in A with $r^2 = \{(0, 0, e) : e \in Im(\theta)\}$ and $r^3 = 0$. Thus r is the radical of A, and $A = S \oplus r$, so A is split over r.

(4) As an F-F-module, r/r^2 is isomorphic to $F \oplus F$ due to the surjective F-F-module homomorphism $\pi : r \to F \oplus F$ satisfying $\pi(0, y, e) = (y, \phi(e))$ with $\ker\pi = r^2$.

(5) By (3) and (4), the $\mathcal{A}$-path-type tensor algebra $T(A/r, r/r^2) \cong T(F, F \oplus F)$. Let $s = (1, 0)$ and $t = (0, 1)$, then $F \oplus F \cong Fs \oplus Ft$. Thus, $T(F, F \oplus F)$ (equivalently, say $T(A/r, r/r^2)$) can be considered as the free associative algebra $F\langle s, t\rangle$ generated by two variables s, t over F. It follows that the center $Z(T(A/r, r/r^2))$ of $T(A/r, r/r^2)$ is equal to F.

(6) If $(x, y, e) \in Z(A)$ the center of A, then for all $e' \in E$, (x, y, e) commutes with $(0, 0, e')$, thus $(0, 0, xe') = (0, 0, e'x)$, so $xe' = e'x$. Taking $e' = (0, 1)$, we get $x(0, 1) = (0, 1)x$. But by (2), $x(0, 1) = (0, x)$ and $(0, 1)x = (\delta(x), x)$. It follows that $\delta(x) = 0$. Therefore, we have:

$$Z(A) \subset \{(x, y, e) : x, y \in F, e \in E, \delta(x) = 0\}.$$

(7) If L is a subalgebra of $Z(A)$ and is a field, then $\dim_k L \overset{<}{\neq} \dim_k F$.

In fact, the composition $L \hookrightarrow Z(A) \hookrightarrow \{(x,y,e) : x, y \in F, e \in E, \delta(x) = 0\} \to \{x : \delta(x) = 0\}$ is an algebra homomorphism. Assume $l = (x, y, e) \in L$ is in the kernel of this composition, then $x = 0$ and $l = (0, y, e)$, so $l \in r$ the radical of A. By (3), $l^3 = 0$. But L is a field, so $l = 0$. It means that this composition is one-one map. Therefore,

$$dim_k L \leq dim_k\{x : \delta(x) = 0\} \overset{<}{\neq} dim_k F$$

where "$\neq$" is from $\delta \neq 0$.

(8) If there were a surjective algebra homomorphism $\lambda : T(A/r, r/r^2) \to A$, it would induce a homomorphism of the center $Z(T(A/r, r/r^2))$ of $T(A/r, r/r^2)$ into the center $Z(A)$ of A. By (5), $Z(T(A/r, r/r^2)) = F$. Thus, $L = \lambda(F)$ would be a field and a subalgebra of $Z(A)$. By (7), we have $dim_k L \overset{<}{\neq} dim_k F$. On the other hand, if there is x satisfying $0 \neq x \in \mathrm{Ker}\lambda|_F$, i.e. $\lambda(x) = 0$. Since F is a field, we get $\lambda(1) = \lambda(1/x)\lambda(x) = 0$, then $\lambda = 0$ is induced due to λ as an algebra homomorphism. It is impossible since λ is surjective. It means $\mathrm{Ker}\lambda|_F = 0$, i.e. $\lambda|_F$ is injective. So, $F \overset{\lambda|_F}{\cong} L$. It contradicts to $dim_k L \overset{<}{\neq} dim_k F$.

From (1)-(8), we finish the description of Example 3.1. Due to this example, we know a general left Artinian algebra with lifted quotient cannot be covered by its correspondent $\mathcal{A}$-path-type tensor algebra. This is the reason we introduce pseudo path algebra and $\mathcal{A}$-path-type pseudo tensor algebra so as to replace generalized path algebra and $\mathcal{A}$-path-type tensor algebra to cover left Artinian algebras with lifted quotients.

However, there exists still some interesting class of left Artinian algebras which can be covered by its correspondent $\mathcal{A}$-path-type tensor algebra and moreover by a generalized path algebra. This point can be seen in the next section, but we will have to restrict a left Artinian algebra to be finite dimensional.

4 When The Radical Is 2-Nilpotent

In this section, we need the concept of the set of primitive radical-orthogonal simple subalgebras of a finite dimensional algebra. Here we always suppose A is a finite dimensional algebra, write $A/r = \bigoplus_{i=1}^{s} \overline{A}_i$ with simple subalgebras $\overline{A}_i$ for all i where r is the radical of A, assume there exist simple k-subalgebras $B_1, \cdots, B_s$ of A satisfying $B_i \cong \overline{A}_i$ as k-algebras for all i under the canonical morphism η: $A \to A/r$. Then we call $\widehat{B} = \{B_1, \cdots, B_s\}$ *the set of primitive radical-orthogonal simple subalgebras of A.*

In general, we have

Lemma 4.1 *For the set of primitive radical-orthogonal simple subalgebras $\widehat{B} = \{B_1, \cdots, B_s\}$ of a finite dimensional algebra A with radical r, it holds that* $B_iB_j \begin{cases} = B_i, & \text{if } i = j \\ \subset r, & \text{if } i \neq j \end{cases}$.

Proof: From the definition, $\eta(B_i) = \overline{A_i}$ for any i, and from $\eta(B_iB_j) = \overline{A_i}\,\overline{A_j} = \begin{cases} \overline{A_i}, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}$.

#

Every B_i is a simple subalgebra of A, so $B_0 = B_1 + \cdots + B_s$ is also a subalgebra of A.

The existence of the set of primitive radical-orthogonal simple subalgebras can be realized for any finite dimensional algebra A as follows.

For $A/r = \bigoplus_{i=1}^{s} \overline{A_i}$ with simple subalgebras $\overline{A_i}$ for all i and the canonical morphism η: $A \to A/r$, let $A_i = \eta^{-1}(\overline{A_i})$. $\overline{A_i}$ is an ideal of A/r, so A_i is an ideal of A. We can get a series $A_i = A^{(0)} \supset A^{(1)} \supset \cdots \supset A^{(l)} \supset \cdots$ where $A^{(j+1)}$ is an ideal of $A^{(j)}$. Due to $\dim A_i < +\infty$ (this is why we restrict $\dim A < +\infty$), there exists l such that $A^{(l)}$ is a simple algebra. Let $B_i = A^{(l)}$. It follows that $\overline{A_i} = \eta(A_i) = \eta(A^{(0)}) \supset \eta(A^{(1)}) \supset \cdots \supset \eta(A^{(l)}) = \eta(B_i)$ where $\eta(A^{(j+1)})$ is an ideal of $\eta(A^{(j)})$. Due to that $\overline{A_i}$ is simple, so $\eta(B_i) = 0$ or $\eta(B_i) = \overline{A_i}$. Assume that $\eta(B_i) = 0$, then $B_i \subset r$. But, since r is nilpotent, we have $B_i^2 \neq B_i$. Thus $B_i^2 = 0$, which contradicts to the fact that B_i is Artinian simple. Therefore, $\eta(B_i) = \overline{A_i}$. Trivially, $\mathrm{Ker}\eta|_{B_i} = 0$. So, $B_i \cong \overline{A_i}$ for all i. Then $\widehat{B} = \{B_1, \cdots, B_s\}$ is the set of primitive radical-orthogonal simple algebras of A.

In this section, we will always assume the radical r of a finite dimensional algebra A is 2-nilpotent, i.e. $r \neq 0$ but $r^2 = 0$. For $\bar{x} = x + r \in A/r$, let $\bar{x} \cdot r \overset{\mathrm{def}}{=} xr$ and $r \cdot \bar{x} = rx$, then r is a finitely generated A/r-bimodule. For $A/r = \bigoplus_{i=1}^{s} \overline{A_i}$ where $\overline{A_i}$ is a simple subalgebra for each i, r is a finitely generated $\overline{A_i}$-$\overline{A_j}$-bimodule for each pair (i, j), whose rank is written as l_{ij}. Now, $r = A/r \cdot r \cdot A/r = \sum_{i,j=1}^{s} \overline{A_i} \cdot r \cdot \overline{A_j} = \sum_{i,j=1}^{s} {}_iM_j$ where ${}_iM_j \overset{\mathrm{def}}{=} \overline{A_i} \cdot r \cdot \overline{A_j}$. Then, for $k \neq i$, $\overline{A_k} \cdot_i M_j = \overline{A_k} \cdot (\overline{A_i} \cdot r \cdot \overline{A_j}) = B_k B_i r B_j \subset rrB_j = 0$, thus $\overline{A_k} \cdot_i M_j = 0$; similarly, for $k \neq j$, ${}_iM_j \cdot \overline{A_k} = 0$. Hence, we get the $\mathcal{A}$-path-type tensor algebra $T(A/r, r)$ and the corresponding $\mathcal{A}$-path algebra $k(\Delta, \mathcal{A})$ with $\mathcal{A} = \{\overline{A_i} : i \in \Delta_0\}$,. We call Δ the *quiver of* A. As in the case under lifting, we have the similar results:

Lemma 4.2 *Assume A is a finite dimensional k-algebra with 2-nilpotent radical $r = r(A)$ and let $\widehat{B} = \{B_1, \cdots, B_s\}$ be the set of primitive radical-orthogonal simple subalgebras of A as constructed above. Write $A/r = \bigoplus_{i=1}^{s} \overline{A_i}$ where $\overline{A_i}$ are simple algebras for all i. The following statements hold:*

(i) *Let $\{r_1, \cdots, r_t\}$ be a set of generators of the A/r-bimodule r. Then $B_1 \cup \cdots \cup B_s \cup \{r_1, \cdots, r_t\}$ generates A as a k-algebra;*

(ii) *There is a surjective k-algebra homomorphism $\widetilde{f} : T(A/r, r) \to A$ with $Ker\widetilde{f} = \bigoplus_{j\geq 2}(r)^j$, where $(r)^j$ denotes $r \otimes_{A/r} r \otimes_{A/r} \cdots \otimes_{A/r} r$ with j copies of r.*

Proof: It is easy to see that r is a (A/r)-bimodule with the action as $\overline{A_i} \cdot r = B_i r$. Note that $\overline{A_i}\overline{A_j} \cdot r = 0 \cdot r = 0$, and on the other hand, $\overline{A_i}\overline{A_j} \cdot r = (B_i B_j + r) \cdot r = B_i B_j r \subset rr = 0$, therefore the definition of this action is well. The proof of (i) is from that of the part of Lemma 3.1 (i) when $\mathrm{rl}(A) = 2$. Next, we discuss the proof of (ii).

$r = A/r \cdot r \cdot A/r = \sum_{i,j=1}^{s} \overline{A_i} \cdot r \cdot \overline{A_j} = \sum_{i,j=1}^{s} B_i r B_j$ is a direct sum decomposition due to $B_i^2 = B_i$ and $B_i B_j \subset r$ for $i \neq j$.

$(A/r) \oplus r$ is a k-algebra in which the multiplication is taken through the A/r-module action of r and the multiplication of A/r and r.

For each pair of integers i, j with $1 \leq i, j \leq s$, choose elements $\{y_u^{ij}\}$ a k-basis in $B_i r B_j$. Then $\bigcup_{i,j=1}^{s} \{y_u^{ij}\}$ is a basis for r.

Define $f : (A/r) \oplus r \to A$ by $f|_r = id_r$ (i.e. $f(y_u^{ij}) = y_u^{ij}$) and $f|_{\overline{A_i}} = \eta^{-1}$. Then, $f|_{A/r}$: $A/r \to B = f(A/r)$ is a k-algebra isomorphism since $B \overset{\eta|_B}{\cong} A/r$, and $f|_r$: $r \to f(r) = r \subset A$ is an embedded homomorphism of A/r-bimodules. Hence by Lemma 2.2, there is a unique algebra morphism $\widetilde{f}$: $T(A/r, r) \to A$ such that $\widetilde{f}|_{(A/r)\oplus r} = f$.

Firstly, $\bigcup_{i,j,u} \{y_u^{i,j}\} \subset \widetilde{f}(r)$ and $B_1 \cup \cdots \cup B_s \subset \widetilde{f}(A/r)$. From (i), it follows that $\bigcup_{i,j,u} \{y_u^{i,j}\} \cup B_1 \cup \cdots \cup B_s$ generates A as a k-algebra and then $\widetilde{f}$ is surjective. On the other hand, $f|_{A/r}$ and

$f|_r$ are monomorphic, so $\widetilde{f}|_{(A/r)\oplus r}$: $(A/r)\oplus r \to A$ is a monomorphism. Then Ker$\widetilde{f} \subset \bigoplus_{j\geq 2}(r)^j$. Moreover, $\widetilde{f}((r)^j) = 0$ for $j \geq 2$ since $r^j = 0$ in this case. Therefore, $\bigoplus_{j\geq 2}(r)^j \subset$Ker$\widetilde{f}$. Thus, Ker$\widetilde{f} = \bigoplus_{j\geq 2}(r)^j$

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Note that $r^2 = 0$ assures $r = \sum_{i,j=1}^{s} B_i r B_j$ is a direct sum decomposition, so the lifting of the quotient of A is not necessary.

Theorem 4.3 (Generalized Gabriel's Theorem With 2-Nilpotent Radical)

Assume that A is a finite dimensional k-algebra with 2-nilpotent radical $r = r(A)$. Then, $A \cong k(\Delta, \mathcal{A}, \rho)$ with $\widetilde{J}^2 \subset \langle\rho\rangle \subset \widetilde{J}^2 + \widetilde{J}\cap$Ker$\widetilde{\varphi}$ where Δ is the quiver of A and ρ is a set of relations of $k(\Delta, \mathcal{A})$, $\widetilde{\varphi}$ is defined as in Proposition 2.9.

Proof: Let Δ be the associated quiver of A. From Lemma 4.2(ii), we have the surjective k-algebra homomorphism $\widetilde{f}$: $T(A/r, r) \to A$. By Proposition 2.9, there is a surjective k-algebra homomorphism $\widetilde{\varphi}$: $k(\Delta, \mathcal{A}) \to T(A/r, r)$ such that for any $t \geq 1$, $\widetilde{\varphi}(\widetilde{J}^t) = \bigoplus_{j\geq t}(r)^j$. Then $\widetilde{f}\widetilde{\varphi}$: $k(\Delta, \mathcal{A}) \to A$ is a surjective k-algebra morphism where $I =$Ker$(\widetilde{f}\widetilde{\varphi}) = \widetilde{\varphi}^{-1}(\bigoplus_{j\geq 2}(r)^j)$ since Ker$\widetilde{f} = \bigoplus_{j\geq 2}(r)^j = \widetilde{\varphi}(\widetilde{J}^2)$.

Just like the correspondent part of the proof of Theorem 3.2, as in a special case, we have $\widetilde{J}^t \subset \widetilde{\varphi}^{-1}\widetilde{\varphi}(\widetilde{J}^t) \subset \widetilde{J}^t + \widetilde{\phi}(\bigoplus_{j\leq t-1}(r)^{Fj}) \cap \widetilde{\phi}(\text{Ker}\pi)$ for $t \geq 1$. Hence,

$\widetilde{J}^2 \subset \widetilde{\varphi}^{-1}\widetilde{\varphi}(\widetilde{J}^2) = \widetilde{\varphi}^{-1}(\text{Ker}\widetilde{f}) = I \subset \widetilde{J}^2 + \widetilde{\phi}(\bigoplus_{j\leq 1}(r)^{Fj}) \cap \widetilde{\phi}(\text{Ker}\pi) = \widetilde{J}^2 + \widetilde{J} \cap \widetilde{\phi}(\text{Ker}\pi)$.

But, $\widetilde{\phi}(\text{Ker}\pi) = \widetilde{\phi}(\pi^{-1}(0)) = \widetilde{\varphi}^{-1}(0) =Ker\widetilde{\varphi}$. Then we get $\widetilde{J}^2 \subset I \subset \widetilde{J}^2 + \widetilde{J}\cap$Ker$\widetilde{\varphi}$.

$\widetilde{J}^2$ is the ideal finitely generated in $k(\Delta, \mathcal{A})$ by all $\mathcal{A}$-paths of length 2. $k(\Delta, \mathcal{A})/\widetilde{J}^2$ is generated finitely as a k-space by all $\mathcal{A}$-paths of length less than 2, so as well as $I/\widetilde{J}^2$ as a k-subspace. Then I is a finitely generated ideal in $k(\Delta, \mathcal{A})$, assume $\{\sigma_1, \cdots, \sigma_p\}$ is its set of finite generators. Moreover, $\sigma_l = \sum_{1\leq i,j\leq s} \bar{e}_i\sigma_l\bar{e}_j$ where $\bar{e}_i\sigma_l\bar{e}_j$ is a relation on the $\mathcal{A}$-path algebra $k(\Delta, \mathcal{A})$. Therefore, for $\rho = \{\bar{e}_i\sigma_l\bar{e}_j : 1 \leq i, j \leq s, 1 \leq l \leq p\}$, we get $I = \langle\rho\rangle$. Hence $k(\Delta, \mathcal{A}, \rho) = k(\Delta, \mathcal{A})/\langle\rho\rangle \cong A$ with $\langle\rho\rangle =$Ker$(\widetilde{f}\widetilde{\varphi})$ and $\widetilde{J}^2 \subset \langle\rho\rangle \subset \widetilde{J}^2 + \widetilde{J}\cap$Ker$\widetilde{\varphi}$.

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In the proof of this theorem, since $f|_r = id_r$, it is naturally an A/r-homomorphism. So, the condition of Lemma 2.2 is satisfied by $T(A/r, r)$. This is not true for $T(A/r, r)$ when $r^2 \neq 0$ generally.

Similarly as in Section 3, there is also the uniqueness of the correspondent $\mathcal{A}$-path algebra and quiver of a finite dimensional algebra under isomorphism, that is, if there exists another quiver and its related $\mathcal{A}$-path algebra so that the same isomorphism relation is satisfied. It is the following statement:

Theorem 4.4 *Assume A is a finite dimensional k-algebra. Let $A/r(A) = \bigoplus_{i=1}^{p} \overline{A}_i$ with simple algebras $\overline{A}_i$ and let $\mathcal{A} = \{A_1, \cdots, A_p\}$. If there is a quiver Δ and a generalized path algebra $k(\Delta, \mathcal{B})$ with a set of simple algebras $\mathcal{B} = \{B_1, \cdots, B_q\}$ and ρ a set of relations satisfying that $A \cong k(\Delta, \mathcal{B}, \rho)$ with $J_\Delta^t \subset \langle\rho\rangle \subset J_\Delta$ for some t and J_Δ the ideal in $k(\Delta, \mathcal{B})$ generated by all elements in $k(\Delta_1, \mathcal{B})$, then Δ is the quiver of A and $p = q$ such that $\overline{A}_i \cong B_i$ for $i = 1, \cdots, p$ after reindexing.*

This theorem can be proved fully similarly with Theorem 3.5. It is enough to replace $\mathcal{A}$-path-type tensor algebra and $\mathcal{A}$-path with $\mathcal{A}$-path-type pseudo tensor algebra and $\mathcal{A}$-pseudo path respectively.

By Fact 2.5, $\mathcal{A}$-path-type tensor algebra and $\mathcal{A}$-path algebra can be covered respectively by $\mathcal{A}$-path-type pseudo tensor algebra and $\mathcal{A}$-pseudo path algebra. So, we can also describe the Generalized Gabriel's Theorem with 2-nilpotent radical through $\mathcal{A}$-pseudo path algebra. As a corollaries of Theorem 4.3 respectively, one has the following:

Proposition 4.5 *Assume that A is a finite dimensional k-algebra with 2-nilpotent radical $r = r(A)$. Then, $A \cong PSE_k(\Delta, \mathcal{A}, \rho)$ with $J^2 \subset \langle\rho\rangle \subset J$ where Δ is the quiver of A and ρ is a set of relations on $PSE_k(\Delta, \mathcal{A})$.*

Proof: We have the composition of surjective homomorphisms: $PSE_k(\Delta, \mathcal{A}) \xrightarrow{\iota} k(\Delta, \mathcal{A}) \xrightarrow{\widetilde{f}\widetilde{\varphi}} A$. Then $A \cong PSE_k(\Delta, \mathcal{A})/Ker(\widetilde{f}\widetilde{\varphi}\iota)$, where $Ker(\widetilde{f}\widetilde{\varphi}\iota) = \iota^{-1}(Ker(\widetilde{f}\widetilde{\varphi}))$.

From Theorem 4.3, $\widetilde{J}^2 \subset Ker(\widetilde{f}\widetilde{\varphi}) \subset \widetilde{J}^2 + \widetilde{J} \cap Ker\widetilde{\varphi}$. Thus,

$$\iota^{-1}(\widetilde{J}^2) \subset \iota^{-1}(Ker(\widetilde{f}\widetilde{\varphi})) \subset \iota^{-1}(\widetilde{J}^2) + \iota^{-1}(\widetilde{J} \cap Ker\widetilde{\varphi}).$$

But, since $\iota^{-1}(\widetilde{J}) = J$, it follows $\iota^{-1}(\widetilde{J}^2) = J^2$ and $\iota^{-1}(\widetilde{J} \cap Ker\widetilde{\varphi}) \subset J$. Thus, we get

$$J^2 \subset Ker(\widetilde{f}\widetilde{\varphi}\iota) \subset J.$$

By Proposition 2.11 (ii), there is a set ρ of relations on $PSE_k(\Delta, \mathcal{A})$ such that $Ker(\widetilde{f}\widetilde{\varphi}\iota) = \langle\rho\rangle$. Then, $A \cong PSE_k(\Delta, \mathcal{A})/Ker(\widetilde{f}\widetilde{\varphi}\iota) = PSE_k(\Delta, \mathcal{A})/\langle\rho\rangle = PSE_k(\Delta, \mathcal{A}, \rho)$ and $J^2 \subset \langle\rho\rangle \subset J$.
#

So far, in Section 3 and this section, we have established the isomorphisms between an algebra and its $\mathcal{A}$-pseudo path algebra with relations (see Theorem 3.2 and Proposition 4.5) in the cases either this algebra is left Artinian with splitting over its radical or it is finite-dimensional with 2-nilpotent. However, it seems to be difficult to discuss the same question for an arbitrary algebra. Our illusion is whether it is possible to characterize an arbitrary finite-dimensional through the combination of the two methods for a left Artinian algebra with splitting over its radical or a finite-dimensional algebra with 2-nilpotent respectively.

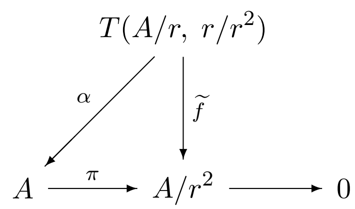
In fact, for a finite-dimensional algebra A, we can start from $B = A/r^2$ where $r = r(A)$ the radical of A. Consider $r(A/r^2) = r/r^2$, denoted as $\widehat{r}$. Then $\widehat{r}^2 = r^2/r^2 = 0$. By Lemma 4.2(ii), there is a surjective homomorphism of algebras $\widetilde{f}: T((A/r^2)/(r/r^2),\ r/r^2) \to A/r^2$.

But, we have $(A/r^2)/(r/r^2) \cong A/r$. So,

$$\widetilde{f}: T(A/r,\ r/r^2) \to A/r^2$$

is a surjective homomorphism of algebras.

On the other hand, according to the method in Section 3, in order to gain the correspondent Gabriel Theorem for this A, the key is to give a homomorphism of algebras α as $\widetilde{f}$ in Lemma 3.1. Therefore, this question may be thought to find a surjective homomorphism of algebras α such that the following diagram commutes:



where π denotes the natural homomorphism. If such α exists, the generalized Gabriel Theorem should hold for this finite-dimensional algebra A.

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References

[1] M.Auslander, I.Reiten, S.O.Smalϕ, **Representation Theory of Artin Algebras**, Cambridge University Press, 1995.

[2] F.U.Coelho, S.X.Liu, **Generalized path algebras**, Interactions between ring theory and representations of algebras (Murcia), 53–66, Lecture Notes in Pure and Appl. Math., 210, Dekker, New York, 2000.

[3] Y.A.Drozd, V.V.Kirichenko, **Finite Dimensional Algebras**, Springer-Verlag, Berlin, 1994.

[4] R.S.Pierce, **Associative Algebras**, Springer-Verlag, New York, 1982.

[5] C.M.Ringel, **Tame algebras and integral quadratic forms**, Lecture Notes in Math., Springer-Verlag, 1099 (1984).