Qualitative Properties of Solutions for an Integral Equation

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Abstract Let n be a positive integer and let $0 < \alpha < n$. In this paper, we study more general integral equation

$$u(x) = \int_{R^n} \frac{1}{|x - y|^{n-\alpha}} K(y) u(y)^p dy. \qquad (0.1)$$

We establish regularity, radial symmetry, and monotonicity of the solutions. We also consider subcritical cases, super critical cases, and singular solutions in all cases; and obtain qualitative properties for these solutions.

1   Introduction

Let R^n be the $n-$dimensional Euclidean space, and let α be a real number satisfying $0 < \alpha < n$. Consider the integral equation

$$u(x) = \int_{R^n} \frac{1}{|x - y|^{n-\alpha}} u(y)^p dy. \qquad (1.1)$$

When $p = \alpha^* := \dfrac{n+\alpha}{n-\alpha}$, it is the so-called critical case. It arises as an Euler-Lagrange equation for a functional under a constraint in the context of the Hardy-Littlewood-Sobolev inequalities. In his elegant paper [L], Lieb classified the maximizers of the functional, and thus obtained the best constant in the Hardy-Littlewood-Sobolev inequalities. He then posed the classification of all the critical points of the functional – the solutions of the integral equation (1.1) as an open problem.

In our previous paper [CLO], we solved Lieb's open problem by using the method of moving planes. We proved, for $p = \alpha^*$, that

Proposition 1 *Every positive regular solution $u(x)$ of (1.1) is radially symmetric and decreasing about some point x_o and therefore assumes the form*

$$c(\frac{t}{t^2 + |x - x_0|^2})^{(n-\alpha)/2}, \tag{1.2}$$

with some positive constants c and t.

Here we call a solution "regular" if it is locally $L^{\frac{2n}{n-\alpha}}$.

For $p = \alpha^*$, this integral equation is closely related to the following well-known family of semi-linear partial differential equations

$$(-\Delta)^{\alpha/2}u = u^{(n+\alpha)/(n-\alpha)} \ \ x \in R^n. \tag{1.3}$$

In the special case $\alpha = 2$, there have been a series of results concerning the classification of the solutions (cf. [GNN], [CGS], [CL], and [Li]). Recently, Wei and Xu [WX] generalized these results to the cases that α being any even number between 0 and n. Apparently, for any real values of α between 0 and n, equation (1.3) is also of practical interests and importance.

In [CLO], we showed the equivalence between the integral equation and differential equation (1.3), and therefore classified all the solutions of semi-linear differential equations (1.3):

Proposition 2 *The same conclusion of Proposition 1 holds for the solutions of differential equation (1.3).*

This proposition unifies and extends all the previous results on the family of differential equations.

In this paper, we continue to study the integral equation in more general form. We prove regularity, radial symmetry, and monotonicity of the solutions. We also consider subcritical cases $p < \alpha^*$, super critical cases $p > \alpha^*$, and singular solutions in all cases; and obtain qualitative properties of these solutions.

In Section 2 and 3, we consider the following more general integral equation

$$u(x) = \int_{R^n} \frac{K(y)|u(y)|^{p-1}u(y)}{|x-y|^{n-\alpha}} dy \qquad (1.4)$$

In Section 2, we prove a regularity theorem. In section 3, we obtain symmetry and monotonicity of the solutions by using the method of moving planes. These results are not only generalizations of the corresponding results in the critical case (cf. Theorem 1 and Lemma 2.1 in [CLO]), but also apply to subcritical and super critical cases and singular solutions in all cases as we will see in sections 4, 5, and 6.

Theorem 1 *Let u be a solution of (1.4). Assume that*

$$u \in L^{q_o}(R^n) \ \text{ for some } q_o > \min\{\frac{n}{n-\alpha}, \frac{n}{\alpha}\} \qquad (1.5)$$

and

$$\int_{R^n} |K(y)u^{p-1}(y)|^{\frac{n}{\alpha}} dy < \infty. \qquad (1.6)$$

Then u is bounded in R^n.

Remark 1 *i) Here we do not require p to be positive.*
    ii) If

$$p > \frac{n}{n-\alpha}, \ \ |K(x)| \leq M, \ \ \text{ and } u \in L^{\frac{n(p-1)}{\alpha}}(R^n), \qquad (1.7)$$

then conditions (1.5) and (1.6) are satisfied.
    iii) Condition (1.7) is somewhat sharp in the sense that when it is violated, then equation (1.1) possesses singular solutions such as

$$u(x) = \frac{c}{|x|^{\frac{\alpha}{p-1}}}. \qquad (1.8)$$

To obtain symmetry of the solutions, we require $p > 0$.

Theorem 2 *let u be a positive solution of (1.4). Assume that*

$$K(x) \text{ is symmetric in } x_1 \text{ and is monotone decreasing for } x_1 \geq 0 \quad (1.9)$$

$$u \in L^{q_o}(R^n) \text{ for some } q_o > \min\{\frac{n}{n-\alpha}, \frac{n}{\alpha}\} \quad (1.10)$$

and

$$|K(x)u^{p-1}(x)|^{\frac{n}{\alpha}} \text{ is integrable on any domain} \\ \text{that is a positive distance away from the plane } x_1 = 0. \quad (1.11)$$

Then u is symmetric about some plane $x_1 = \lambda_o$.

Remark 2 *i) As a consequence, if K is radially symmetric and monotone, then u is radially symmetric and monotone.*

ii) For $p > \frac{n}{n-\alpha}$ and $K(x) = \frac{1}{|x|^\beta}$ with some $\beta \geq 0$, if u satisfies either one of the following

$$u(x) \leq \frac{C}{(1+|x|)^{\gamma/(p-1)}}, \quad \alpha < \beta + \gamma \quad (1.12)$$

or

$$u \in L^{\frac{n(p-1)}{\alpha}}(R^n). \quad (1.13)$$

Then the conditions of the Theorem are satisfied. As an immediate application, in the subcritical case, after Kelvin type transform, we will arrive an equation with $K(y) = \frac{1}{|y|^\beta}$. This will enable us to carry on the moving planes scheme on solutions without assuming their asymptotic growth at infinity, because their Kelvin type transforms satisfy the desired growth.

Based on the results in Theorem 1 and 2, one can see that if a solution u of (1.1) satisfies

$$\int_\Omega u^{\frac{n(p-1)}{\alpha}}(y) \, dy < \infty \quad (1.14)$$

in any bounded domain Ω, then it is bounded. Consequently, by a standard argument, it is continuous and hence possesses higher regularity. Therefore, we say that a solution of (1.1) is regular, if it satisfies (1.14) for every bounded domain Ω. Naturally, we call a point x^o a singularity of u, if (1.14) is violated in every neighborhood of x^o. For example, the solution given in (1.8) has one and only one singularity at origin.

In Section 4, we consider subcritical cases. We first prove a non-existence theorem.

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Theorem 3 *For $1 < p < \alpha^*$, there does not exist regular positive solutions of (1.1).*

Then we consider solutions with one singularity, and use the method of moving planes to obtain the radial symmetry and monotonicity of the solutions. The result also applies to singular solutions in the critical case. Here we do not count singularity at infinity.

Theorem 4 *For $1 < p \le \alpha^*$, if a solution u of (1.1) has only one singularity at a point x^o, then it must be radially symmetric about the same point.*

In Section 5, we study singular solutions in the critical case and obtain an upper bound for the solutions.

Theorem 5 *Assume that $u(x)$ is a positive solution of (1.1) with only one singularity at x^o, then there is a constant C, such that*

$$u(x) \le \frac{C}{|x - x^o|^{\frac{n-\alpha}{2}}}.$$

In Section 6, we consider super critical cases. Theorem 2 provides some radially symmetric "regular" solutions and (1.8) lists a family of symmetric singular solutions. Then, are there any non-symmetric solutions? We will answer this question affirmatively by constructing examples of non-radially symmetric solutions.

2 Regularity of Solutions

In this section, we prove Theorem 1, which is a generalization of the corresponding result in the critical case (cf. Lemma 2.1 in [CLO]). For simplicity, we write $L^p(R^n)$ as L^p.

Proof of Theorem 1. Define the linear operator

$$T_v w = \int_{R^n} \frac{K(y)|v(y)|^{p-1} w}{|x - y|^{n-\alpha}} dy.$$

For any real number $a > 0$, define

$$u_a(x) = |u(x)|, \quad \text{if } |u(x)| > a; \quad u_a(x) = 0, \quad \text{if } |u(x)| \leq a.$$

Then through an elementary calculation, one can verify that u_a satisfies the equation

$$u_a = T_{u_a} u_a + g(x) \qquad (2.15)$$

with a function $g(x) \in L^\infty \cap L^{q_o}$.

In the case $p > \frac{n}{n-\alpha}$, we first apply Hardy-Littlewood-Sobolev inequality and then Holder inequality. By (1.6), we deduce, for sufficiently large a,

$$
\begin{aligned}
\|T_{u_a} w\|_{L^q} &\leq C(\alpha, n, q) \|K u_a^{p-1} w\|_{L^{\frac{nq}{n+\alpha q}}} \\
&\leq C(\alpha, n, q) \left(\int |K(y) u_a^{p-1}(y)|^{\frac{n}{\alpha}} \, dy \right)^{\frac{\alpha}{n}} \|w\|_{L^q} \leq \tfrac{1}{2} \|w\|_{L^q}.
\end{aligned}
\qquad (2.16)
$$

One can verify that (2.16) is also true for $q > \frac{n}{\alpha}$, if one applies Holder inequality first and then H-L-S inequality.

Applying (2.16) to both the case $q = q_o$ and the case $q = p_o > q_o$, by the Contracting Mapping theorem that the equation

$$w = T_{u_a} w + g(x) \qquad (2.17)$$

has a unique solution in both L^{q_o} and $L^{p_o} \cap L^{q_o}$. From (2.15), u_a is a solution of (2.17) in L^{q_o}. Let w be the solution of (2.17) in $L^{p_o} \cap L^{q_o}$, then w is also a solution in L^{q_o}. By the uniqueness, we must have $u_a = w \in L^{p_o} \cap L^{q_o}$ for any $p_o > \frac{n}{n-\alpha}$. Now an elementary argument will imply that u_a is bounded, and so does u. This completes the proof of the Theorem.

3   Symmetry of Solutions

In this section, we use the method of moving planes to establish general symmetry and monotonicity of the solutions to (1.4).

Proof of Theorem 2. The proof consists two steps. Let λ be a real number and let the moving plane be $x_1 = \lambda$. We denote Σ_λ the region to the right of the moving plane; that is,

$$\Sigma_\lambda = \{ x = (x_1, \ldots, x_n) \mid x_1 \geq \lambda \}.$$

Define
$$x^\lambda = (2\lambda - x_1, x_2, \ldots, x_n), \quad u_\lambda(x) = u(x^\lambda).$$

We compare $u(x)$ and $u_\lambda(x)$ on Σ_λ. In step 1, we show that there exists an $N > 0$ such that for $\lambda \leq -N$, we have

$$u(x) \geq u_\lambda(x), \quad \forall x \in \Sigma_\lambda. \tag{3.18}$$

Thus we can start moving the plane continuously from $\lambda \leq -N$ to the right as long as (3.18) holds. If the plane stops at $x_1 = \lambda_o$ for some $\lambda_o < 0$, then $u(x)$ must be symmetric and monotone about the plane $x_1 = \lambda_o$. Otherwise, we can move the plane all the way to $x_1 = 0$, which is shown in step 2. Since the direction of x_1 can be chosen arbitrarily, we deduce that $u(x)$ must be radially symmetric and decreasing about some plane $x_1 = \lambda$.

Step 1. Define

$$\Sigma_\lambda^- = \{x \,|\, x \in \Sigma_\lambda, u(x) < u_\lambda(x)\}. \tag{3.19}$$

Let Σ_λ^C be the compliment of Σ_λ. We show that for sufficiently negative values of λ, Σ_λ^- must be empty. In fact, it is easy to verify that

$$u_\lambda(x) - u(x) \leq C \int_{\Sigma_\lambda^-} \frac{|K(y^\lambda)|}{|x-y|^{n-\alpha}} [u_\lambda^{p-1}(u_\lambda - u)](y) dy.$$

In the case $p > \frac{n}{n-\alpha}$, it follows first from the Hardy-Littlewood-Sobolev inequality and then Holder inequality that

$$\begin{aligned}
&\|u_\lambda - u\|_{L^{qo}(\Sigma_\lambda^-)} \\
&\leq C\{\textstyle\int_{\Sigma_\lambda^-} [|K(y^\lambda)| \, u_\lambda^{p-1}(y)]^{\frac{n}{\alpha}} \, dy\}^{\frac{\alpha}{n}} \|u_\lambda - u\|_{L^{qo}(\Sigma_\lambda^-)} \\
&\leq C\{\textstyle\int_{\Sigma_\lambda^C} [|K(y)| \, u^{p-1}(y)]^{\frac{n}{\alpha}} \, dy\}^{\frac{\alpha}{n}} \|u_\lambda - u\|_{L^{qo}(\Sigma_\lambda^-)}
\end{aligned} \tag{3.20}$$

One can verify that (3.20) is also true for $q > \frac{n}{\alpha}$, if one applies Holder inequality first and then H-L-S inequality.

By condition (1.11), we can choose N sufficiently large, such that for $\lambda \leq -N$, we have

$$\{\int_{\Sigma_\lambda^C} [|K(y)| \, u^{p-1}(y)]^{\frac{n}{\alpha}} \, dy\}^{\frac{\alpha}{n}} \leq \frac{1}{2}.$$

Now (3.20) implies that $\|u_\lambda - u\|_{L^{q_o}(\Sigma_\lambda^-)} = 0$, and therefore Σ_λ^- must be measure zero, and hence empty.

Step 2. We now move the plane $x_1 = \lambda$ to the right as long as (3.18) holds. Suppose that at a $\lambda_o < 0$, we have $u(x) \geq u_{\lambda_o}(x)$, but $u(x) \not\equiv u_{\lambda_o}(x)$, we show that the plane can be moved further to the right. More precisely, there exists an ϵ depending on n, α, and the solution $u(x)$ itself such that $u(x) \geq u_\lambda(x)$ on Σ_λ for all λ in $[\lambda_o, \lambda_o + \epsilon)$.

By Lemma 2.2 in our previous paper [CLO], we have in fact $u(x) > u_{\lambda_o}(x)$ in the interior of Σ_{λ_o}. Let $\overline{\Sigma_{\lambda_o}^-} = \{x \in \Sigma_{\lambda_o} \mid u(x) \leq u_{\lambda_o}(x)\}$. Then obviously, $\overline{\Sigma_{\lambda_o}^-}$ has measure zero, and $\lim_{\lambda \to \lambda_o} \Sigma_\lambda^- \subset \overline{\Sigma_{\lambda_o}^-}$. Let $(\Sigma_\lambda^-)^*$ be the reflection of Σ_λ^- about the plane $x_1 = \lambda$. From the first inequality of (3.20), we deduce

$$\|u_\lambda - u\|_{L^{q_o}(\Sigma_\lambda^-)} \leq C\{\int_{(\Sigma_\lambda^-)^*} [|K(y)| \, u^{p-1}(y)]^{\frac{n}{\alpha}} \, dy\}^{\frac{\alpha}{n}} \|u_\lambda - u\|_{L^{q_o}(\Sigma_\lambda^-)} \quad (3.21)$$

Condition (1.11) ensures that one can choose ϵ sufficiently small, so that for all λ in $[\lambda_o, \lambda_o + \epsilon)$,

$$\{\int_{(\Sigma_\lambda^-)^*} [|K(y)| \, u^{p-1}(y)]^{\frac{n}{\alpha}} \, dy\}^{\frac{\alpha}{n}} \leq \frac{1}{2}.$$

Now by (3.21), we have $\|u_\lambda - u\|_{L^{q_o}(\Sigma_\lambda^-)} = 0$, and therefore Σ_λ^- must be empty.

This completes the proof of the Theorem.

4 Subcritical Cases

In this section, we prove Theorem 3 and 4.

The main ingredient of the proofs are the Kelvin type transform and the method of moving planes.

Assume that u is a solution of integral equation (1.1). Let

$$v(x) = \frac{1}{|x|^{n-\alpha}} u(\frac{x}{|x|^2}) \quad (4.22)$$

be the Kelvin type transform of $u(x)$. Then it is a straight forward calculation to verify that $v(x)$ satisfies the equation

$$v(x) = \int_{R^n} \frac{1}{|x-y|^{n-\alpha}} |y|^{-(n-\alpha)(\alpha^*-p)} v^p(y) d\,y. \quad (4.23)$$

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The Proof of Theorem 3.

Assume that $u(x)$ is a positive regular solution of the integral equation (1.1). Let x^1 and x^2 be any two points in R^n. Since the integral equation is invariant under translations, we may assume that the midpoint $\dfrac{x^1 + x^2}{2}$ is at the origin. Let $v(x)$ be the Kelvin type transform as defined in (4.22). Then, $v(x)$ satisfies equation (4.23), a special case of (1.4). Apply Theorem 2 to $v(x)$ with $K(y) = |y|^{-(n-\alpha)(\alpha^*-p)}$, we deduce that $v(x)$ is radially symmetric about the origin. Let

$$x^i_* = \frac{x^i}{|x^i|^2} \quad i = 1, 2$$

be the inversions of x^i. Then $v(x^1_*) = v(x^2_*)$; and therefore $u(x^1) = u(x^2)$. Since x^1 and x^2 are any two points in R^n, we conclude that u must be a constant. This is impossible. Therefore, (1.1) does not have any positive regular solution.

The Proof of Theorem 4.

By a translation, we may assume that $u(x)$ has only one singularity at the origin. Let x^1 and x^2 be any two points which are equidistant to the origin. Let Π be the plane that perpendicularly bisects the line segment $\overline{x^1x^2}$. Let $v(x)$ be the Kelvin type transform defined by (4.22); and let x^1_*, x^2_*, and Π_* be the images of x^1, x^2, and Π under the inversion. Applying Theorem 2 to $v(x)$, we conclude that $v(x)$ must be symmetric and monotone decreasing about the plane Π_*. In particular, $v(x^1_*) = v(x^2_*)$, and hence $u(x^1) = u(x^2)$. It follows that $u(x)$ is radially symmetric and monotone decreasing about the origin. This completes the proof of the theorem.

5   Critical Case - Singular Solutions

In this section, we consider singular solutions of the integral equation (1.1) in the critical case when $p = \alpha^*$. As one has seen in the previous section,

$$u(x) = \frac{c}{|x|^{\frac{n-\alpha}{2}}}$$

with a suitable constant c is a singular solution. We will show in fact that any singular solution can not grow faster than this power of x.

Theorem 5.1 *Assume that $u(x)$ is a positive solution of (1.1) with only one singularity at x_o, then there is a constant C, such that*

$$u(x) \leq \frac{C}{|x - x_o|^{\frac{n-\alpha}{2}}}. \tag{5.24}$$

Proof.

Without loss of generality, we may assume that the solution $u(x)$ has only one singularity at the origin. Then as we have shown in section 2, $u(x)$ is radially symmetric and monotone decreasing about the origin. Let e be any point such that $|e| = 1$. Then by the integral equation (1.1), we have, for any $r > 0$,

$$u(re) \geq \int_{B_r(0)} \frac{1}{|re-s\omega|^{n-\alpha}} [u(s)]^{\frac{n+\alpha}{n-\alpha}} s^{n-1} ds d\omega$$
$$\geq [u(r)]^{\frac{n+\alpha}{n-\alpha}} \int_0^r \int_{\partial B_1(0)} \frac{1}{|re-r\omega|^{n-\alpha}} d\omega s^{n-1} ds$$
$$= [u(r)]^{\frac{n+\alpha}{n-\alpha}} r^\alpha \int_0^1 \int_{\partial B_1(0)} \frac{1}{|e-t\omega|^{n-\alpha}} d\omega t^{n-1} dt$$
$$= Cr^\alpha [u(r)]^{\frac{n+\alpha}{n-\alpha}},$$

with some constant C. Here, by the radial symmetry of u, $u(re) = u(r)$ for any e. It follows that

$$u(r) \leq \frac{C}{r^{\frac{n-\alpha}{2}}}.$$

This completes the proof of the theorem.

6 Super Critical Cases

In the super critical case when $p > \alpha^*$, equation (1.1) possesses both symmetric solutions and non-symmetric solutions.

As we mentioned in the previous section,

$$u(x) = \frac{c}{|x|^{\frac{\alpha}{p-1}}}$$

with some appropriate constant c is a singular symmetric solution.

There are examples of regular symmetric solutions as provided by Theorem 2. For instance, Theorem 2 implies that

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Corollary 6.1 *All the solutions that are in $L^{\frac{n(p-1)}{\alpha}}(R^n)$ are radially symmetric and monotone about some point.*

Now we construct a non-radially symmetric solution. Let $x' = (x_1, \cdots, x_{n-1})$, let $u(x)$ be a standard solution in R^{n-1}, i.e.

$$u(x') = c(\frac{1}{1+|x'|^2})^{\frac{n-1-\alpha}{2}}.$$

Then it satisfies

$$u(x') = \int_{R^{n-1}} \frac{1}{|x'-y'|^{n-1-\alpha}} [u(y')]^{\frac{n-1+\alpha}{n-1-\alpha}} dy'. \tag{6.25}$$

Let $x = (x', x_n)$, and define

$$\tilde{u}(x) = u(x').$$

Then one can verify that, for some constant c,

$$\tilde{u}(x) = c \int_{R^n} \frac{1}{|x-y|^{n-\alpha}} [\tilde{u}(y)]^{\frac{n-1+\alpha}{n-1-\alpha}} dy. \tag{6.26}$$

It follows that a constant multiple of $\tilde{u}$ is an n-dimensional solution of the integral equation in super critical case, since $\dfrac{n+\alpha}{n-\alpha} < \dfrac{n-1+\alpha}{n-1-\alpha}$. To see (6.26), one simply need to notice from elementary calculus that

$$\int_{-\infty}^{\infty} \frac{1}{|x-y|^{n-\alpha}} dy_n = \frac{a}{|x'-y'|^{n-1-\alpha}},$$

with some constant a.

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