

Classification of Solutions for a System of Integral Equations

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Abstract

In this paper, we study positive solutions of the following system of integral equations in R^n :

$$\begin{cases} u(x) = \int_{R^n} |x - y|^{\alpha-n} v(y)^q dy \\ v(x) = \int_{R^n} |x - y|^{\alpha-n} u(y)^p dy \end{cases} \quad (0.1)$$

with $\frac{1}{q+1} + \frac{1}{p+1} = \frac{n-\alpha}{n}$. Under the natural integrability conditions $u \in L^{p+1}(R^n)$ and $v \in L^{q+1}(R^n)$, we prove that all the solutions are radially symmetric and monotone decreasing about some point. To prove this result, we introduce an integral form of the method of moving planes which is quite different from the traditional method of moving planes for PDEs. And we expect to see applications of this new method to many other problems.

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1 Introduction

Let $0 < \alpha < n$ and let $s, r > 1$ such that $\frac{1}{r} + \frac{1}{s} = \frac{n+\alpha}{n}$. The well-known Hardy-Littlewood-Sobolev inequality states that:

$$\int_{R^n} \int_{R^n} f(x) |x - y|^{\alpha-n} g(y) dx dy \leq C(n, s, \alpha) \|f\|_r \|g\|_s \quad (1.2)$$

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for any $f \in L^r(R^n)$ and $g \in L^s(R^n)$.

To find the best constant $C = C(n, s, \alpha)$ in the inequality, one can maximize the functional

$$J(f, g) = \int_{R^n} \int_{R^n} f(x) |x - y|^{\alpha-n} g(y) dx dy \quad (1.3)$$

under the constraints

$$\|f\|_r = \|g\|_s = 1. \quad (1.4)$$

Let (f, g) be a maximizer, or more generally, a critical point of (1.3) under the constraints (1.4). Letting $u = \lambda_1 f^{r-1}$, $v = \lambda_2 g^{s-1}$, $p = \frac{1}{r-1}$, $q = \frac{1}{s-1}$, and by a proper choice of constants λ_1 and λ_2 , one can see that (u, v) satisfies the following system of integral equations in R^n :

$$\begin{cases} u(x) = \int_{R^n} |x - y|^{\alpha-n} v^q(y) dy \\ v(x) = \int_{R^n} |x - y|^{\alpha-n} u^p(y) dy \end{cases} \quad (1.5)$$

with $\frac{1}{q+1} + \frac{1}{p+1} = \frac{n-\alpha}{n}$.

The integral system is closely related to the system of partial differential equations

$$\begin{cases} (-\Delta)^{\alpha/2} u = v^q, u > 0, \text{ in } R^n, \\ (-\Delta)^{\alpha/2} v = u^p, v > 0, \text{ in } R^n. \end{cases} \quad (1.6)$$

In the special case where $p = q = \frac{n+\alpha}{n-\alpha}$, and $u(x) = v(x)$, the system becomes:

$$u(x) = \int_{R^n} |x - y|^{\alpha-n} u(y)^{\frac{n+\alpha}{n-\alpha}} dy, \quad u > 0 \text{ in } R^n. \quad (1.7)$$

And the corresponding PDE is the well-known family of semi-linear equations

$$(-\Delta)^{\alpha/2} u = u^{(n+\alpha)/(n-\alpha)}, \quad u > 0, \quad \text{in } R^n \quad (1.8)$$

In particular, when $n \geq 3$, and $\alpha = 2$, (1.8) becomes

$$-\Delta u = u^{(n+2)/(n-2)}, \quad u > 0, \quad \text{in } R^n. \quad (1.9)$$

The classification of the solutions of (1.9) provided an important ingredient in the study of the well-known Yamabe problem and the prescribing

scalar curvature problem. It is also essential in deriving a priori estimates in many related nonlinear elliptic equations.

Solutions to (1.9) were studied by Gidas, Ni, and Nirenberg [GNN]. They proved that all the positive solutions of (1.9) with reasonable behavior at infinity

$$u(x) = O\left(\frac{1}{|x|^{n-2}}\right) \quad (1.10)$$

are radially symmetric and therefore assume the form of

$$c\left(\frac{t}{t^2 + |x - x_o|^2}\right)^{(n-2)/2} \quad (1.11)$$

with some positive constants c and t .

Later, in their fundamental paper [CGS], Caffarelli, Gidas, and Spruck removed the growth condition (1.10) and obtained the same result. Then Chen and Li [CL1], and Li [Li] simplified their proof. Recently, Wei and Xu [WX] generalized this result to the solutions of more general equation (1.8) with α being any even numbers between 0 and n .

Apparently, for other real values of α between 0 and n , equation (1.8) is also of practical interest and importance. For instance, it arises as the Euler-Lagrange equation of the functional

$$I(u) = \int_{R^n} |(-\Delta)^{\frac{\alpha}{4}} u|^2 dx / \left(\int_{R^n} |u|^{\frac{2n}{n-\alpha}} dx \right)^{\frac{n-\alpha}{n}}.$$

The classification of the solutions would provide the best constant in the inequality of the critical Sobolev imbedding from $H^{\frac{\alpha}{2}}(R^n)$ to $L^{\frac{2n}{n-\alpha}}(R^n)$:

$$\left(\int_{R^n} |u|^{\frac{2n}{n-\alpha}} dx \right)^{\frac{n-\alpha}{n}} \leq C \int_{R^n} |(-\Delta)^{\frac{\alpha}{4}} u|^2 dx.$$

In his elegant paper [L], Lieb classified all the maximizers of the functional (1.3) under the constraint (1.4) in the special case where $p = q = \frac{n+\alpha}{n-\alpha}$, and thus obtained the best constant in the H-L-S inequalities in that case. He then posed the classification of all the critical points of the functional – the solutions of the integral equation (1.7) as an open problem.

We solved this open problem in our previous paper [CLO] and proved

Proposition 1 *All solutions of partial differential equation (1.8) satisfy the integral equation (1.7), and vice versa. Every positive solution $u(x) \in L_{loc}^{\frac{2n}{n-\alpha}}(R^n)$*

of (1.7) or (1.8) is radially symmetric and decreasing about some point x_o and therefore assumes the form of (1.11).

In this paper, we consider more general system (1.5) and show that

Theorem 1 *Let the pair (u, v) be a solution of (1.5) and $p, q \geq 1$. Assume that $u \in L^{p+1}(R^n)$ and $v \in L^{q+1}(R^n)$. Then u and v are radially symmetric and decreasing about some point x_o .*

To prove the radial symmetry and monotonicity of the solutions, we use an integral form of the method of moving planes. This was a new idea we introduced in [CLO], now we modify this idea so it can be applied to systems of integral equations. It is entirely different from the traditional methods of moving planes used for partial differential equations. For PDEs, the local properties of the differential operators are exploited extensively. This lack of knowledge of the local properties prevents us from using many known results, such as maximum principles. However, by exploring various special features possessed by the integral equation in its global form, and by estimating certain integral norms, we are still able to establish the symmetry results.

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2 The Proof of Radial Symmetry and Monotonicity

In this section, we use the method of moving planes to prove Theorem 1.

For a given real number λ , define

$$\Sigma_\lambda = \{x = (x_1, \dots, x_n) \mid x_1 \geq \lambda\}.$$

Let $x^\lambda = (2\lambda - x_1, x_2, \dots, x_n)$, $u_\lambda(x) = u(x^\lambda)$ and $v_\lambda(x) = v(x^\lambda)$.

Lemma 2.1 *For any solution $(u(x), v(x))$ of (1.5), we have*

$$u_\lambda(x) - u(x) = \int_{\Sigma_\lambda} (|x - y|^{\alpha-n} - |x^\lambda - y|^{\alpha-n})(v_\lambda^q(y) - v^q(y))dy, \quad (2.12)$$

and

$$v_\lambda(x) - v(x) = \int_{\Sigma_\lambda} (|x - y|^{\alpha-n} - |x^\lambda - y|^{\alpha-n})(u_\lambda^p(y) - u^p(y))dy. \quad (2.13)$$

The proof of this lemma is elementary, and is similar to the one for Lemma 2.1 in our previous paper [CLO].

To prove Theorem 1, we compare $u(x)$ with $u_\lambda(x)$ and $v(x)$ with $v_\lambda(x)$ on Σ_λ . The proof consists of two steps. In step 1, we show that there exists an $N > 0$ such that for $\lambda \leq -N$, we have

$$u(x) \geq u_\lambda(x) \text{ and } v(x) \geq v_\lambda(x) \quad \forall x \in \Sigma_\lambda. \quad (2.14)$$

Thus we can start moving the plane continuously from $\lambda \leq -N$ to the right as long as (2.14) holds. In step 2, we show that if the plane stops at $x_1 = \lambda_o$ for some $\lambda_o < 0$, then $u(x)$ and $v(x)$ must be symmetric and monotone about the plane $x_1 = \lambda_o$; otherwise, we can move the plane all the way to $x_1 = 0$. Since the direction of x_1 can be chosen arbitrarily, we deduce that $u(x)$ and $v(x)$ must be radially symmetric and decreasing about some point.

Step 1. Define

$$\Sigma_\lambda^u = \{x \in \Sigma_\lambda \mid u(x) < u_\lambda(x)\},$$

and

$$\Sigma_\lambda^v = \{x \in \Sigma_\lambda \mid v(x) < v_\lambda(x)\},$$

Let Σ_λ^C be the compliment of Σ_λ . We show that for sufficiently negative values of λ , Σ_λ^u and Σ_λ^v must both be empty. By Lemma 2.1, it is easy to verify that

$$u_\lambda(x) - u(x) \leq C \int_{\Sigma_\lambda^v} |x - y|^{\alpha-n} [v_\lambda^{q-1}(v_\lambda - v)](y)dy.$$

It follows from the Hardy-Littlewood-Sobolev inequality that

$$\|u_\lambda - u\|_{L^{p+1}(\Sigma_\lambda^u)} \leq C \|v_\lambda^{q-1}(v_\lambda - v)\|_{L^{(q+1)/q}(\Sigma_\lambda^v)}. \quad (2.15)$$

Then by the Hölder inequality,

$$\|u_\lambda - u\|_{L^{p+1}(\Sigma_\lambda^u)} \leq C \|v_\lambda\|_{L^{q+1}(\Sigma_\lambda^v)}^{q-1} \|v_\lambda - v\|_{L^{q+1}(\Sigma_\lambda^v)} \quad (2.16)$$

Similarly, one can show that

$$\|v_\lambda - v\|_{L^{q+1}(\Sigma_\lambda^v)} \leq C \|u_\lambda\|_{L^{p+1}(\Sigma_\lambda^u)}^{p-1} \|u_\lambda - u\|_{L^{p+1}(\Sigma_\lambda^u)}. \quad (2.17)$$

Combining (2.16) and (2.17), we arrive

$$\|u_\lambda - u\|_{L^{p+1}(\Sigma_\lambda^u)} \leq C \|v\|_{L^{q+1}(\Sigma_\lambda^C)}^{q-1} \|u\|_{L^{p+1}(\Sigma_\lambda^C)}^{p-1} \|u_\lambda - u\|_{L^{p+1}(\Sigma_\lambda^u)}. \quad (2.18)$$

By the integrability condition $u \in L^{p+1}(R^n)$ and $v \in L^{q+1}(R^n)$, we can choose N sufficiently large, such that for $\lambda \leq -N$, we have

$$C \|v\|_{L^{q+1}(\Sigma_\lambda^C)}^{q-1} \|u\|_{L^{p+1}(\Sigma_\lambda^C)}^{p-1} \leq \frac{1}{2}.$$

Now (2.18) implies that $\|u_\lambda - u\|_{L^{p+1}(\Sigma_\lambda^u)} = 0$, and therefore Σ_λ^u must be measure zero, and hence empty. Similarly, one can show that Σ_λ^v is empty. Therefore (2.14) holds. This completes Step 1.

Step 2. We now move the plane $x_1 = \lambda$ to the right as long as (2.14) holds. Suppose that at a $\lambda_o < 0$, we have, on Σ_{λ_o} ,

$$u(x) \geq u_{\lambda_o}(x) \text{ and } v(x) \geq v_{\lambda_o}(x), \text{ but } u(x) \not\equiv u_{\lambda_o}(x) \text{ or } v(x) \not\equiv v_{\lambda_o}(x);$$

we show that the plane can be moved further to the right. More precisely, there exists an ϵ depending on n, α , and the solution $(u(x), v(x))$ itself such that

$$u(x) \geq u_\lambda(x) \text{ and } v(x) \geq v_\lambda(x) \text{ on } \Sigma_\lambda \text{ for all } \lambda \text{ in } [\lambda_o, \lambda_o + \epsilon). \quad (2.19)$$

In the case

$$v(x) \not\equiv v_{\lambda_o}(x) \text{ on } \Sigma_{\lambda_o},$$

by (2.12), we have in fact $u(x) > u_{\lambda_o}(x)$ in the interior of Σ_{λ_o} . Let

$$\tilde{\Sigma}_{\lambda_o}^u = \{x \in \Sigma_{\lambda_o} \mid u(x) \leq u_{\lambda_o}(x)\}, \quad \text{and} \quad \tilde{\Sigma}_{\lambda_o}^v = \{x \in \Sigma_{\lambda_o} \mid v(x) \leq v_{\lambda_o}(x)\}.$$

Then obviously, $\tilde{\Sigma}_{\lambda_o}^u$ has measure zero, and $\lim_{\lambda \rightarrow \lambda_o} \Sigma_\lambda^u \subset \tilde{\Sigma}_{\lambda_o}^u$. The same is true for that of v . Let D^* be the reflection of the set D about the plane $x_1 = \lambda$. From (2.16) and (2.17), we deduce,

$$\|u_\lambda - u\|_{L^{p+1}(\Sigma_\lambda^u)} \leq C \|v\|_{L^{q+1}((\Sigma_\lambda^v)^*)}^{q-1} \|u\|_{L^{p+1}((\Sigma_\lambda^u)^*)}^{p-1} \|u_\lambda - u\|_{L^{p+1}(\Sigma_\lambda^u)}. \quad (2.20)$$

Again the integrability conditions $u \in L^{p+1}(R^n)$ and $v \in L^{q+1}(R^n)$ ensure that one can choose ϵ sufficiently small, so that for all λ in $[\lambda_o, \lambda_o + \epsilon)$,

$$C\|v\|_{L^{q+1}((\Sigma_\lambda^v)^*)}^{q-1}\|u\|_{L^{p+1}((\Sigma_\lambda^u)^*)}^{p-1} \leq \frac{1}{2}.$$

Now by (2.20), we have $\|u_\lambda - u\|_{L^{p+1}(\Sigma_\lambda^u)} = 0$, and therefore Σ_λ^u must be empty. Similarly, Σ_λ^v must also be empty. This verifies (2.19), and therefore completes the proof of Theorem.

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