Classification of Solutions for an Integral Equation

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Abstract Let n be a positive integer and let $0 < \alpha < n$. Consider the integral equation

$$u(x) = \int_{R^n} \frac{1}{|x-y|^{n-\alpha}} u(y)^{(n+\alpha)/(n-\alpha)} dy. \tag{0.1}$$

We prove that every positive regular solution $u(x)$ is radially symmetric and monotone about some point, and therefore assumes the form

$$c(\frac{t}{t^2 + |x - x_o|^2})^{(n-\alpha)/2} \tag{0.2}$$

with some constant $c = c(n, \alpha)$, and for some $t > 0$ and $x_o \in R^n$. This solves an open problem posed by Lieb [L]. The technique we used is the method of moving planes in an integral form, which is quite different from those for differential equations. From the point of view of general methodology, this is another interesting part of the paper.

Moreover, we show that the family of well-known semi-linear partial differential equations

$$(-\Delta)^{\alpha/2} u = u^{(n+\alpha)/(n-\alpha)},$$

is equivalent to our integral equation (0.1), and we thus classify all the solutions of the PDEs.

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1 Introduction

Let R^n be the $n-$dimensional Euclidean space, and let α be a real number satisfying $0 < \alpha < n$. Consider the integral equation

$$u(x) = \int_{R^n} \frac{1}{|x-y|^{n-\alpha}} u(y)^{(n+\alpha)/(n-\alpha)} dy. \tag{1.1}$$

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It arises as an Euler-Lagrange equation for a functional under a constraint in the context of the Hardy-Littlewood-Sobolev inequalities. In [L], Lieb classified the maximizers of the functional, and thus obtained the best constant in the H-L-S inequalities. He then posed the classification of all the critical points of the functional – the solutions of the integral equation (1.1) as an open problem.

This integral equation is also closely related to the following family of semi-linear partial differential equations

$$(-\Delta)^{\alpha/2}u = u^{(n+\alpha)/(n-\alpha)}. \tag{1.2}$$

In the special case $n \geq 3$ and $\alpha = 2$, it becomes

$$-\Delta u = u^{(n+2)/(n-2)}. \tag{1.3}$$

Solutions to this equation were studied by Gidas, Ni, and Nirenberg [GNN]. They proved that all the positive solutions of (1.3) with reasonable behavior at infinity

$$u(x) = O(\frac{1}{|x|^{n-2}}) \tag{1.4}$$

are radially symmetric and therefore assume the form of (0.2). Later, Caffarelli, Gidas, and Spruck [CGS] removed the growth condition (1.4) and obtained the same result. Then Chen and Li [CL], and Li [Li] simplified their proof. Recently, Wei and Xu [WX] generalized this result to the solutions of (1.2) with α being any even numbers between 0 and n.

Apparently, for other real values of α between 0 and n, equation (1.2) is also of practical interest and importance. For instance, it arises as the Euler-Lagrange equation of the functional

$$I(u) = \int_{R^n} |(-\Delta)^{\frac{\alpha}{4}}u|^2 dx / (\int_{R^n} |u|^{\frac{2n}{n-\alpha}} dx)^{\frac{n-\alpha}{n}}.$$

The classification of the solutions would provide the best constant in the inequality of the critical Sobolev imbedding from $H^{\frac{\alpha}{2}}(R^n)$ to $L^{\frac{2n}{n-\alpha}}(R^n)$:

$$(\int_{R^n} |u|^{\frac{2n}{n-\alpha}} dx)^{\frac{n-\alpha}{n}} \leq C \int_{R^n} |(-\Delta)^{\frac{\alpha}{4}}u|^2 dx.$$

We will precisely define $(-\Delta)^{\alpha/2}$ in §6, and we will show that, all the solutions of partial differential equation (1.2) satisfy our integral equation (1.1), and vise versa. Therefore, to classify the solutions of this family of partial differential equations, we only need to work on the integral equations (1.1).

In order either the Hardy-Littlewood-Sobolev inequality or the above mentioned critical Sobolev imbedding to make sense, u must be in $L^{\frac{2n}{n-\alpha}}(R^n)$. Under this assumption, we can show that (see theorem 2.1) a positive solution u is in fact bounded, and therefore possesses higher regularity. Furthermore, by using the method of moving planes, we can show that if a solution is locally $L^{\frac{2n}{n-\alpha}}$, then it is in $L^{\frac{2n}{n-\alpha}}(R^n)$. Hence, in the following, we call a solution u regular if it is locally $L^{\frac{2n}{n-\alpha}}$. It is interesting to notice that if this condition is violated, then a solution may not be bounded. A simple example is $u = \frac{1}{|x|^{(n-\alpha)/2}}$, which is a singular solution. We will study such solutions in our next paper.

We will use the method of moving planes to prove

Theorem 1 *Every positive regular solution $u(x)$ of (1.1) is radially symmetric and decreasing about some point x_o and therefore assumes the form*

$$c(\frac{t}{t^2 + |x - x_0|^2})^{(n-\alpha)/2}, \tag{1.5}$$

with some positive constants c and t.

Consequently, we have also classify the solutions of semi-linear differential equations (1.2):

Theorem 2 *The same conclusion of Theorem 1 holds for the solutions of (1.2).*

Remark. In our earlier version of the paper, we assumed that the solutions u be locally bounded. We would like to thank professor Yanyan Li for pointing out that a more natural condition is locally $L^{\frac{2n}{n-2}}$. Then he [LiY] obtained a regularity result based on this condition, and used the method of moving spheres to prove the same classification result.

The method of the moving planes was invented by the Soviet mathematician Alexanderoff in the early 1950's. Decades later, it was further developed by Serrin [Se], Gidas, Ni, and L.Nirenberg [GNN], Caffarelli, Gidas, and Spruck [CGS], Li [Li], Chen and Li [CL] [CL1], Chang and Yang [CY], and many others. The method has been applied to free boundary problems, semi-linear differential equations, and other problems. Particularly for semi-linear differential equations, there have seen many significant contributions. We refer to [F] for more descriptions on the method.

As is known to people with experiences in the method of moving planes, each problem has its unique difficulty. For partial differential equations, the local properties of the differential operators are used extensively. This lack of knowledge of the local properties prevents us from using many known results, such as maximum principles. However, by exploring various special features possessed by the integral equation in its global form, and through introducing several new ideas which are quite different from those for differential equations, we are still able to establish the symmetry results.

In §2, we use the method of moving planes to obtain the symmetry of the solutions. In §3, we show that all the solutions must assume the form of (1.5). In §4, we define precisely the differential equations (1.2) for any real number α, and prove that the differential equation is equivalent to our integral equation.

For the convenience of presentation we will use c for a general positive constant that depends on n, α, and the solution $u(x)$ itself. Such a c is usually different in different context.

2 Moving Planes-Symmetry of Solutions

For a given real number λ, define

$$\Sigma_\lambda = \{x = (x_1, \ldots, x_n) \mid x_1 \geq \lambda\},$$

and let $x^\lambda = (2\lambda - x_1, x_2, \ldots, x_n)$ and $u_\lambda(x) = u(x^\lambda)$.

Lemma 2.1 *For any solution $u(x)$ of (1.1), we have*

$$u(x) - u_\lambda(x) = \int_{\Sigma_\lambda} (\frac{1}{|x-y|^{n-\alpha}} - \frac{1}{|x^\lambda - y|^{n-\alpha}})(u(y)^{\frac{n+\alpha}{n-\alpha}} - u_\lambda(y)^{\frac{n+\alpha}{n-\alpha}})dy, \tag{2.1}$$

It is also true for the Kelvin type transform $v(x) = \frac{1}{|x|^{n-\alpha}} u(\frac{x}{|x|^2})$ for any $x \neq 0$.

Proof: Since $|x - y^\lambda| = |x^\lambda - y|$, we have

$$\begin{aligned} u(x) &= \int_{\Sigma_\lambda} \frac{1}{|x-y|^{n-\alpha}} u(y)^{\frac{n+\alpha}{n-\alpha}} dy + \int_{\Sigma_\lambda} \frac{1}{|x^\lambda - y|^{n-\alpha}} u_\lambda(y)^{\frac{n+\alpha}{n-\alpha}} dy \\ u(x^\lambda) &= \int_{\Sigma_\lambda} \frac{1}{|x^\lambda-y|^{n-\alpha}} u(y)^{\frac{n+\alpha}{n-\alpha}} dy + \int_{\Sigma_\lambda} \frac{1}{|x - y|^{n-\alpha}} u_\lambda(y)^{\frac{n+\alpha}{n-\alpha}}. \end{aligned}$$

This implies (2.1). The conclusion for $v(x)$ follows similarly since it satisfies the same integral equation (1.1) for $x \neq 0$.

Set $\tau = \frac{n+\alpha}{n-\alpha}$, we use the method of moving planes to prove

Theorem 2.1 *Let $u(x) \in L^{\tau+1}_{local}$ be a positive solution of (1.1). Then it must be radially symmetric and monotone decreasing about some point. Furthermore, u is continuous and*

$$\lim_{|x| \longrightarrow \infty} |x|^{n-\alpha} u(x) = u_\infty, \tag{2.2}$$

for a positive number u_∞.

Since we do not assume any asymptotic behavior of $u(x)$ at infinity, we are not able to carry on the method of moving planes directly on $u(x)$. To overcome this difficulty, we consider $v(x)$, the Kelvin type transform of $u(x)$. It is easy to verify that $v(x)$ satisfies the same equation (1.1), but has a possible singularity at origin, where we need to pay special attention. Since u is locally $L^{\tau+1}$, it is easy to see that $v(x)$ has no singularity at infinity, i.e. for any domain Ω that is a positive distance away from the origin,

$$\int_\Omega v^{\tau+1}(y)\, dy < \infty. \tag{2.3}$$

Let λ be a real number and let the moving plane be $x_1 = \lambda$. We compare $v(x)$ and $v_\lambda(x)$ on $\Sigma_\lambda \setminus \{0\}$. The proof consists of three steps. In step 1, we show that there exists an $N > 0$ such that for $\lambda \leq -N$, we have

$$v(x) \geq v_\lambda(x), \quad \forall x \in \Sigma_\lambda \setminus \{0\}. \tag{2.4}$$

Thus we can start moving the plane continuously from $\lambda \leq -N$ to the right as long as (2.4) holds. If the plane stops at $x_1 = \lambda_o$ for some $\lambda_o < 0$, then $v(x)$ must be symmetric and monotone about the plane $x_1 = \lambda_o$. This implies that $v(x)$ has no singularity at the origin and $u(x)$ has no singularity at infinity. In this case, we can carry on the moving planes on

$u(x)$ directly to obtain the radial symmetry and monotonicity. Otherwise, we can move the plane all the way to $x_1 = 0$, which is shown in step 2. Since the direction of x_1 can be chosen arbitrarily, we deduce that $v(x)$ must be radially symmetric and decreasing about the origin. We will show in step 3 that, in any case, $u(x)$ can not have a singularity at infinity, and hence both u and v are in $L^{\tau+1}(R^n)$. Then by theorem 2.1, v is continuous, and therefore, u satisfies (2.2).

Step 1. Define

$$\Sigma_\lambda^- = \{x | \, x \in \Sigma_\lambda \setminus \{0\}, v(x) < v_\lambda(x)\}. \tag{2.5}$$

Let Σ_λ^C be the compliment of Σ_λ. We show that for sufficiently negative values of λ, Σ_λ^- must be empty. By Lemma 2.1, it is easy to verify that

$$v_\lambda(x) - v(x) \leq C \int_{\Sigma_\lambda^-} \frac{1}{|x-y|^{n-\alpha}} [v_\lambda^{\tau-1}(v_\lambda - v)](y) dy.$$

It follows first from the Hardy-Littlewood-Sobolev inequality and then Holder inequality that, for any $q > \frac{n}{n-\alpha}$,

$$\begin{array}{l} \|v_\lambda - v\|_{L^q(\Sigma_\lambda^-)} \\ \leq C\{\int_{\Sigma_\lambda^-} v_\lambda^{\tau+1}(y)\,dy\}^{\frac{\alpha}{n}} \|v_\lambda - v\|_{L^q(\Sigma_\lambda^-)} \\ \leq C\{\int_{\Sigma_\lambda^C} v^{\tau+1}(y)\,dy\}^{\frac{\alpha}{n}} \|v_\lambda - v\|_{L^q(\Sigma_\lambda^-)} \end{array} \tag{2.6}$$

By condition (2.3), we can choose N sufficiently large, such that for $\lambda \leq -N$, we have

$$C\{\int_{\Sigma_\lambda^C} v^{\tau+1}(y)\,dy\}^{\frac{\alpha}{n}} \leq \frac{1}{2}.$$

Now (2.6) implies that $\|v_\lambda - v\|_{L^q(\Sigma_\lambda^-)} = 0$, and therefore Σ_λ^- must be measure zero, and hence empty.

Step 2. We now move the plane $x_1 = \lambda$ to the right as long as (2.4) holds. Suppose that at a $\lambda_o < 0$, we have $v(x) \geq v_{\lambda_o}(x)$, but $v(x) \not\equiv v_{\lambda_o}(x)$ on $\Sigma_{\lambda_o} \setminus \{0\}$; we show that the plane can be moved further to the right. More precisely, there exists an ϵ depending on n, α, and the solution $v(x)$ itself such that $v(x) \geq v_\lambda(x)$ on $\Sigma_\lambda \setminus \{0\}$ for all λ in $[\lambda_o, \lambda_o + \epsilon)$.

By Lemma 2.2, we have in fact $u(x) > u_{\lambda_o}(x)$ in the interior of Σ_{λ_o}. Let $\overline{\Sigma_{\lambda_o}^-} = \{x \in \Sigma_{\lambda_o} | \, u(x) \leq u_{\lambda_o}(x)\}$. Then obviously, $\overline{\Sigma_{\lambda_o}^-}$ has measure zero, and $\lim_{\lambda \to \lambda_o} \Sigma_\lambda^- \subset \overline{\Sigma_{\lambda_o}^-}$. Let $(\Sigma_\lambda^-)^*$ be the reflection of Σ_λ^- about the plane $x_1 = \lambda$. From the first inequality of (2.6), we deduce,

$$\|v_\lambda - v\|_{L^q(\Sigma_\lambda^-)} \leq C\{\int_{(\Sigma_\lambda^-)^*} v^{\tau+1}(y)\,dy\}^{\frac{\alpha}{n}} \|v_\lambda - v\|_{L^q(\Sigma_\lambda^-)} \tag{2.7}$$

Condition (2.3) ensures that one can choose ϵ sufficiently small, so that for all λ in $[\lambda_o, \lambda_o + \epsilon)$,

$$C\{\int_{(\Sigma_\lambda^-)^*} v^{\tau+1}(y)\,dy\}^{\frac{\alpha}{n}} \leq \frac{1}{2}.$$

Now by (2.7), we have $\|v_\lambda - v\|_{L^q(\Sigma_\lambda^-)} = 0$, and therefore Σ_λ^- must be empty.

Step 3. Finally, we showed that u has the desired asymptotic behavior at infinity, i.e., it satisfies (2.2). Suppose in the contrary, let x^1 and x^2 be any two points in R^n and let x^o be the midpoint of the line segment $\overline{x^1x^2}$. Consider the Kelvin type transform centered at x^o:

$$v(x) = \frac{1}{|x - x^o|^{n-\alpha}} u(\frac{x - x^o}{|x - x^o|^2}).$$

Then $v(x)$ has a singularity at x^o. Carry on the arguments as in Steps 1 and 2, we conclude that $v(x)$ must be radially symmetric about x^o, and in particular, $u(x^1) = u(x^2)$. Since x^1 and x^2 are any two points in R^n, u must be constant. This is impossible. Similarly, The continuity and higher regularity of u follows from standard theory on singular integral operators. This completes the proof of the Theorem.

3 The Uniqueness of Solutions

In the previous section, we proved that all the positive regular solutions of the integral equation (1.1) are radially symmetric and monotone decreasing about some point. Based on this result, we will show, in this section, that all solutions must assume the form of (1.5), and therefore complete the proof of Theorem 1.

The radial symmetry and monotonicity of $u(x)$ and the invariance of solutions under Kelvin type transforms and scaling restrict $u(x)$ to assume the form of (1.5). Lieb made this observation in [L] (also cf. [LL]) on the optimizers in Hardy-Littlewood-Sobolev inequalities and gave a geometric proof. His proof also applies to the solutions of our integral equation (1.1) based on our following Lemma 3.1. Nevertheless, we present an analytic proof here, which is quite different from Lieb's. In the case $n \geq 2$, the proof is simpler and it is relevant to a work of Ou [O].

We need the following lemmas.

Lemma 3.1 *Let w be a solution of (1.1). Let $a \in R^n$ and $s^{n-\alpha} = \frac{w_\infty}{w(a)}$. Then*

$$w(sx + a) = \frac{1}{|x|^{n-\alpha}} w(\frac{sx}{|x|^2} + a). \tag{3.8}$$

Lemma 3.2 *Let $\bar{u}_o(x) = c_o(\frac{1}{1+|x|^2})^{(n-\alpha)/2}$ be the standard solution centered at origin. Assume that w is a solution centered at m, then*

$$w(m)w_\infty = c_o^2. \tag{3.9}$$

Proof of Lemma 3.1. First we consider $a = 0$. Let u be a solution of (1.1) and $s^{n-\alpha} = \frac{u_\infty}{u(0)}$. Let e be any unit vector in R^n. Defined

$$v(x) = \frac{1}{|x|^{n-\alpha}} u_s(\frac{x}{|x|^2} - e).$$

Then $v(0) = v(e)$ and $v(x)$ must be symmetric about $\frac{1}{2}e$. It follows that, for any h,

$$\frac{s^{(n-\alpha)/2}}{|1/2 - h|^{n-\alpha}} u(s \frac{1/2 + h}{1/2 - h} e) = \frac{s^{(n-\alpha)/2}}{|1/2 + h|^{n-\alpha}} u(s \frac{1/2 - h}{1/2 + h} e). \tag{3.10}$$

Let $t = \frac{1/2-h}{1/2+h}$, we arrive at

$$u(ste) = \frac{1}{t^{n-\alpha}} u(\frac{s}{t} e). \tag{3.11}$$

Now (3.8) follows from a translation. This completes the proof of Lemma 3.1.

Proof of Lemma 3.2. Let v be a solution of (1.1). By a translation and rescaling, we may assume that v is centered at origin and $v_\infty = (\bar{u}_o)_\infty$. We show that

$$v(0) = \bar{u}_o(0).$$

Let $u(x) = v(xe)$ and $u_o(x) = \bar{u}_o(xe)$ for any given unit vector e in R^n and any real number x. Suppose $u(0) > u_o(0)$. Then there exits an $a > 0$, such that

$$u(x) > u_o(x), \quad \text{for } -a < x < a; \quad \text{and } u(a) = u_o(a). \tag{3.12}$$

Let $s = (\frac{(u_o)_\infty}{u_o(a)})^{1/(n-\alpha)} = \sqrt{1 + a^2}$. Then Lemma 3.1 and (3.12)imply that

$$u(y) > u_o(y) \quad \forall - \infty < y < -\frac{1 - a^2}{2a}. \tag{3.13}$$

By the symmetry and the assumption $u(a) = u_o(a)$, we have

$$u(y) > u_o(y) \quad \forall \; \frac{1 - a^2}{2a} < y < \infty \quad \text{and} \quad u(\frac{1 - a^2}{2a}) = u_o(\frac{1 - a^2}{2a}). \tag{3.14}$$

If $a > 1$, this already contradicts with (3.12).

Therefore, we must have $a < 1$ and $\frac{1 - a^2}{2a} > a$. Let $b = \frac{1 - a^2}{2a}$ and $s = \sqrt{1 + b^2}$. Then both u_o and u satisfy

$$u(sx + b) = \frac{1}{|x|^{n-\alpha}} u(\frac{s}{x} + b). \tag{3.15}$$

If $b \leq 1$, then(3.13) and (3.15) imply that

$$u(x) > u_o(x) \;\; \forall - \frac{1 - b^2}{2b} < x < b. \tag{3.16}$$

(3.16), (3.14), and the symmetry of u and u_o lead to

$$u(x) > u_o(x) \quad \text{almost everywhere.} \tag{3.17}$$

This is impossible.

If $b > 1$, let $b_o = b$ and $b_n = \frac{b_{n-1}^2 - 1}{2b_{n-1}}$. Then obviously $b_n < \frac{b_{n-1}}{2}$ and $b_n \to 0$, as $n \to \infty$. Then repeat the above argument, we arrive at

$$u(x) > u_o(x) \quad \text{almost everywhere.}$$

Again a contradiction. A similar argument would exclude the possibility that $u(0) < u_o(0)$. Therefore, we must have $u(0) = u_o(0)$, and hence $v(0) = \bar{u}_o(0)$.

This completes the proof of Lemma 3.2.

Completing the Proof of Theorem 1. Let w be a solution of (1.1). By a translation and rescaling, we may assume that w is centered at origin and $w_\infty = w(0)$. We show that

$$w(x) \equiv \bar{u}_o(x). \tag{3.18}$$

First by Lemma 3.2, we have $w(0) = \bar{u}_o(0) = c_o$. Let

$$u(x) = w(xe), \quad \text{and } u_o(x) = \bar{u}_o(xe)$$

for any unit vector $e \in R^n$ and for any real number x. Suppose that there exists $a > 0$, such that $u(a) > u_o(a)$. Let

$$v_o(x) = \frac{1}{|x|^{n-\alpha}} u_o(\frac{1}{x} + a), \; v(x) = \frac{1}{|x|^{n-\alpha}} u(\frac{1}{x} + a).$$

Since $u_\infty = u(0)$ and $(u_o)_\infty = u_o(0)$, by Lemma 3.1, we have

$$u(x) = \frac{1}{|x|^{n-\alpha}} u(\frac{1}{x}) \text{ and } u_o(x) = \frac{1}{|x|^{n-\alpha}} u_o(\frac{1}{x}). \tag{3.19}$$

It follows that

$$v(\frac{-a}{1+a^2}) = (1+a^2)^{n-\alpha} u(a) \text{ and } v_o(\frac{-a}{1+a^2}) = (1+a^2)^{n-\alpha} u_o(a). \tag{3.20}$$

It is easy to verify that $\frac{-a}{1+a^2}$ is the center of $v_o(x)$. Let m be the center of v. If $u(a) > u_o(a)$, then from (3.20),

$$v(m) \geq v(\frac{-a}{1+a^2}) > v_o(\frac{-a}{1+a^2}) \tag{3.21}$$

On the other hand, it follows from $v_\infty = u(a)$, $(v_o)_\infty = u_o(a)$ that

$$v_\infty > (v_o)_\infty. \tag{3.22}$$

By (3.21) and (3.22), we have

$$v(m) v_\infty > v_o(\frac{-a}{1+a^2})(v_o)_\infty = c_o^2.$$

This is a contradiction with Lemma 3.2. Therefore, we must have

$$u(x) \leq u_o(x), \;\; \forall x,$$

and consequently,

$$w(x) \leq \bar{u}_o(x). \tag{3.23}$$

Taking into account that w and $\bar{u}_o$ are solutions of (1.1) and the fact that $w(0) = \bar{u}_o(0)$, we arrive at $w(x) \equiv \bar{u}_o(x)$.

If $u(a) < u_o(a)$, a similar argument will lead to a contradiction. This completes the proof of the Theorem.

4 The Family of Differential Equations

In this section, we study the relations between the family of integral equations and the well-known semi-linear partial differential equations

$$(-\Delta)^{\alpha/2} u = u^{(n+\alpha)/(n-\alpha)}. \tag{4.24}$$

In [WX], Wei and Xu classified the solutions of (4.24) when α is an even number between 0 and n. More precisely, they proved

Proposition 4.1 *(Wei and Xu) Suppose α is an even number between 0 and n, and u is a smooth positive solution of (4.24). Then u is radially symmetric about some point $x_o \in R^n$ and assumes the following form*

$$u(x) = (\frac{2t}{t^2 + |x - x_o|^2})^{(n-\alpha)/2}.$$

They applied the method of moving planes directly on the differential equation. One of their key ingredient is the following lemma.

Lemma 4.1 *(Wei and Xu) Suppose α is an even number between 0 and n and u is a smooth positive solution of (4.24). Then we have*

$$(-\Delta)^i u > 0, \quad i = 1, \cdots, \frac{\alpha}{2} - 1. \tag{4.25}$$

We will show that in this case (when α is even), all the solutions of (4.24) satisfy our integral equation (1.1), and therefore, our theorem includes Wei and Xu's result as a special case. For any real number α between 0 and n, we will define precisely the operator $(-\Delta)^{\alpha/2}$, and show the equivalences between the two equations.

Theorem 4.1 *Every smooth positive solution of PDE (4.24) multiplied by a constant satisfies integral equation (1.1).*

To prove the Theorem, we first establish the following lemma.

Lemma 4.2 *Let u be a solution of (4.24), $\tau = \frac{n+\alpha}{n-\alpha}$, and $v_k = (-\Delta)^k u$ for $k = 1, 2, \cdots, p-1$. Then*

$$\int_{R^n} \frac{1}{|x|^{n-\alpha}} u^\tau(x) dx < \infty, \tag{4.26}$$

$$\int_{R^n} \frac{v_k}{|x|^{n-2k}} dx < \infty \quad \text{for } k = 1, \cdots, p-1; \tag{4.27}$$

and, as a consequence of (4.26), there exists a sequence $r_m \to \infty$, such that

$$\frac{1}{r_m^{n-1}} \int_{\partial B_{r_m}(0)} u(x) d\sigma \to 0 \ . \tag{4.28}$$

Proof of Lemma 4.2

Let $\delta(x)$ be the Dirac Delta function. Let ϕ be the solution of the following boundary value problem

$$\begin{cases} (-\triangle)^p \phi = \delta(x) & x \in B_r(0) \\ \phi = \triangle\phi = \cdots = (-\triangle)^{p-1}\phi = 0 & \text{on } \partial B_r(0). \end{cases} \tag{4.29}$$

By the maximum principle, one can easily verify that

$$\frac{\partial}{\partial \nu}[(-\triangle)^k \phi] \leq 0, \quad k = 0, 1, \cdots, p-1, \quad \text{on } \partial B_r(0). \tag{4.30}$$

Multiply both side of the equation (4.24) by ϕ and integrate on $B_r(0)$. After integrating by parts several times and applying Lemma 4.1 and (4.30), we arrive at

$$\begin{aligned} \int_{B_r(0)} u^\tau(x)\phi(x) dx &= u(0) + \int_{\partial B_r(0)} \sum_{k=0}^{p-1} v_k \frac{\partial}{\partial \nu}[(-\triangle)^{p-1-k}\phi]\} d\sigma \\ &\leq u(0). \end{aligned} \tag{4.31}$$

Now letting $r \to \infty$, one can see that (4.26) is just a direct consequence of the following fact

$$\phi(x) \to \frac{C}{|x|^{n-\alpha}} \tag{4.32}$$

with some constant C.

To verify (4.32), we notice that (4.29) is equivalent to the following system of equations

$$\begin{cases} -\triangle\phi = \psi_1 & \phi \mid_{\partial B_r} = 0 \\ -\triangle\psi_1 = \psi_2 & \psi_1 \mid_{\partial B_r} = 0 \\ \cdots \\ -\triangle\psi_{p-1} = \delta(x) & \psi_{p-1} \mid_{\partial B_r} = 0. \end{cases} \tag{4.33}$$

Applying maximum principle consecutively to ϕ_k for $k = 1, \cdots, p-1$, one derives that:

$$\psi_k(x) \nearrow C \frac{1}{|x|^{n-\alpha+2k}} \ \text{for } k = 0, 1, \cdots, p-1, \ \text{as } r \to \infty.$$

This implies (4.32).

Now, to derive (4.27), we simply apply the above argument to the equation:

$$(-\triangle)^k u = v_k,$$

for each $k = 1, \cdots, p-1$.

Finally, we verify (4.28). Estimate (4.26) implies that there exists a sequence $r_m \to \infty$, such that the average of u^τ on $\partial B_{r_m}(0)$ tends to 0 as $r_m \to \infty$. Then a standard convexity argument shows that the average of u on $\partial B_{r_m}(0)$ approaches to 0 as $r_m \to \infty$.

This completes the proof of the lemma.

Proof of Theorem 4.1.

For each $r > 0$, let $\phi_r(x)$ be the solution of (4.29). Then as in the previous lemma, one verifies that

$$\phi_r(x) = \frac{1}{r^{n-\alpha}} \phi_1(\frac{x}{r}) \tag{4.34}$$

and

$$|\phi_1(x)| \leq \frac{C}{|x|^{n-\alpha}}. \tag{4.35}$$

It follows that,

$$\phi_r(x) \leq \frac{C}{|x|^{n-\alpha}}. \tag{4.36}$$

Also one can verify that, on $\partial B_r(0)$,

$$|\frac{\partial}{\partial \nu}[(-\triangle)^k \phi_r]| \leq \frac{C}{r^{n-\alpha+1+2k}}. \tag{4.37}$$

By virtue of (4.27) and (4.28), through an elementary argument in calculus, one can see that there exist a sequence $r_m \to \infty$, such that each of the boundary integral on $\partial B_{r_m}(x)$ in (4.31) approaches 0 as $r_m \to \infty$. Applying (4.36), (4.26), and the Lesbegue Convergence Theorem to the left hand side of (4.31), and taking limit along the sequence $\{r_m\}$, we conclude that

$$c \int_{R^n} \frac{1}{|y|^{n-\alpha}} u^\tau(y) dy = u(0).$$

By a translation, we see that a constant multiple of $u(x)$ is a solution of (1.1). This completes the proof of the theorem.

So far, we have proved that if α is a even number, then every solution of the PDE (4.24) is a solution of our integral equation. Now for other real values of α, we define the positive solution of (4.24) in the distribution sense, i.e. $u \in H^{\frac{\alpha}{2}}(R^n)$, satisfies

$$\int_{R^n} (-\triangle)^{\frac{\alpha}{4}} u (-\triangle)^{\frac{\alpha}{4}} \phi \, dx = \int_{R^n} u^\tau(x) \phi(x) \, dx, \tag{4.38}$$

for any $\phi \in C_0^\infty$ and $\phi(x) \geq 0$. Here, as usual,

$$\int_{R^n} (-\triangle)^{\frac{\alpha}{4}} u (-\triangle)^{\frac{\alpha}{4}} \phi \, dx$$

is defined by Fourier transform

$$\int_{R^n} |\xi|^\alpha \hat{u}(\xi)\hat{\phi}(\xi)\, d\xi,$$

where $\hat{u}$ and $\hat{\phi}$ are the Fourier transform of u and ϕ respectively.

By taking limits, one can see that (4.38) is also true for any $\phi \in H^{\frac{\alpha}{2}}$.

Theorem 4.2 *Partial differential equation (4.24) as defined above is equivalent to integral equation (1.1).*

Proof. (i) For any $\phi \in C_0^\infty(R^n)$, let

$$\psi(x) = \int_{R^n} \frac{\phi(y)}{|x-y|^{n-\alpha}}\, dy.$$

Then $(-\triangle)^{\alpha/2}\psi = \phi$, consequently $\psi \in H^\alpha \subset H^{\frac{\alpha}{2}}$, and hence (4.38) holds for ψ:

$$\int_{R^n} (-\triangle)^{\frac{\alpha}{4}} u\, (-\triangle)^{\frac{\alpha}{4}} \psi\, dx = \int_{R^n} u^\tau(x)\psi(x)\, dx.$$

Integration by parts of the left hand side and exchange the order of integration of the right hand side yield

$$\int_{R^n} u(x)\phi(x)\, dx = \int_{R^n} \{\int_{R^n} \frac{u^\tau(y)}{|x-y|^{n-\alpha}}\, dy\}\, \phi(x)\, dx.$$

Since ϕ is any nonnegative C_0^∞ function, we conclude that u satisfies the integral equation.

(ii) Now assume that $u \in L^{\frac{2n}{n-\alpha}}(R^n)$ is a solution of the integral equation (1.1). Make a Fourier transform on both sides (cf. [LL], Corollary 5.10), we have

$$\hat{u}(\xi) = c|\xi|^{-\alpha}\hat{u^\tau}(\xi).$$

It follows that

$$\int_{R^n} (-\triangle)^{\frac{\alpha}{4}} u (-\triangle)^{\frac{\alpha}{4}} \phi\, dx = c\int_{R^n} \hat{u^\tau}(\xi)\hat{\phi}(\xi) = c\int_{R^n} u^\tau(x)\phi(x) dx.$$

This completes the proof of the theorem.

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