L_p Solutions of Vector Refinement Equations with General Dilation Matrix

Song Li and Ruei Fang Hu
Department of Mathematices
Zhejiang University
Hangzhou, Zhejiang, 310027
Xiang Qing Wang
Shaoxing University
Shaoxing, Zhejiang, 312000

Abstract. The purpose of this paper is to investigate the solutions of refinement equations of the form

$$\varphi(x) = \sum_{\alpha \in \mathbb{Z}^s} a(\alpha)\varphi(Mx - \alpha), \qquad x \in \mathbb{R}^s,$$

where the vector of functions $\varphi = (\varphi_1, ..., \varphi_r)^T$ is in $(L_p(\mathbb{R}^s))^r, 0 < p \le \infty, a(\alpha), \alpha \in \mathbb{Z}^s$, is a finitely supported sequence of $r \times r$ matrices called the refinement mask, and M is an $s \times s$ integer matrix such that $\lim_{n\to\infty} M^{-n} = 0$. In this article, we characterize the existence of L_p-solution of refinement equation for $0 < p \le \infty$. Our characterizations are based on the p-norm joint spectral radius.

1. Introduction.

A refinement equation is a functional equation of the form

$$\varphi(x) = \sum_{\alpha \in \mathbb{Z}^s} a(\alpha)\varphi(Mx - \alpha), \qquad x \in \mathbb{R}^s, \tag{1.1}$$

where $\varphi = (\varphi_1, ..., \varphi_r)^T$ is in $(L_p(\mathbb{R}^s))^r, 0 < p \le \infty$, a is a finitely supported refinement mask such that each $a(\alpha)$ is an $r \times r$ (complex) matrix and M is an $s \times s$ integer matrix such that $\lim_{n\to\infty} M^{-n} = 0$. The equation (1.1) is called a homogeneous refinement equation and the matrix M is called a dilation matrix. It is well-known that refinement equations play an important role in wavelet analysis. Most useful wavelets in applications are generated from refinable functions.

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The approximation and smoothness properties of wavelets are determined by the corresponding refinable functions.

For $0 < p \leq \infty$, by $(L_p(\mathbb{R}^s))^r$ we denote the linear space of all vectors $f = (f_1, ..., f_r)^T$ such that $||f||_p < \infty$, where

$$||f||_p := \left(\sum_{j=1}^{r} \int_{\mathbb{R}^s} |f_j|^p dx \right)^{\frac{1}{p}}, \qquad 1 \leq p < \infty,$$

$$||f||_p := \sum_{j=1}^{r} \int_{\mathbb{R}^s} |f_j|^p dx, \qquad 0 < p < 1,$$

and $||f||_\infty$ is the essential supremum of $\max_{1 \leq j \leq r} |f_j|$ on $\mathbb{R}^s$. When $1 \leq p \leq \infty$, $||\cdot||_p$ is a norm and, equipped with this norm, $(L_p(\mathbb{R}^s))^r$ is a Banach space. For $0 < p < 1$, $||\cdot||_p$ is a invariant metric, with this metric, $(L_p(\mathbb{R}^s))^r$ is a complete metric linear space.

The Fourier transform of a vector of functions in $(L_1(\mathbb{R}^s))^r$ is defined by

$$\hat{f}(\xi) := \int_{\mathbb{R}^s} f(x) e^{-ix\cdot\xi} dx, \qquad \xi \in \mathbb{R}^s, \tag{1.2}$$

where $x \cdot \xi$ denotes the inner produce of two vectors x and ξ in $\mathbb{R}^s$.

The shifts of compactly supported functions $f_1, ..., f_r \in L_p(\mathbb{R}^s)(0 < p \leq \infty)$ are said to be L_p-stable if there exist two positive constants C_1 and C_2 such that, for arbitrary $b_1, ..., b_r \in \ell_p(\mathbb{Z}^s)$,

$$C_1 \sum_{j=1}^{r} ||b_j||_p \leq || \sum_{j=1}^{r} \sum_{\alpha \in \mathbb{Z}^s} b_j(\alpha) f_j(\cdot - \alpha)||_p \leq C_2 \sum_{j=1}^{r} ||b_j||_p,$$

where $\ell_p(\mathbb{Z}^s)$ denotes the linear space of all sequence c for which $||c||_p < \infty$, the ℓ_p-norm or quasi-norm of c is defined by

$$||c||_p = \left(\sum_{\alpha \in \mathbb{Z}^s} |c(\alpha)|^p \right)^{1/p}, \quad 0 < p < \infty,$$

and $||c||_\infty$ is the supremum of $|c|$ on $\mathbb{Z}^s$. Clearly, $||\cdot||_p$ is a norm for $1 \leq p \leq \infty$, and is a quasi-norm for $0 < p < 1$.

In [1], Jia and Micchelli established the following characterization for L_p-stability when $1 \leq p \leq \infty$: The shifts of $f_1, ..., f_r$ are L_p-stable if and only if, for any $\xi \in \mathbb{R}^s$, the sequence $(\hat{f}(\xi + 2\beta\pi))_{\beta \in \mathbb{Z}^s} (k = 1, ..., r)$ are linear independent. In [2], Jia obtained a similar characterization for L_p-stability of the shifts of a finite number of compactly supported distributions when $0 < p < 1$.

Taking the Fourier transform of both sides of (1.1), we obtain

$$\hat{\varphi}(\xi) = H((M^T)^{-1}\xi)\hat{\varphi}((M^T)^{-1}\xi), \qquad \xi \in \mathbb{R}^s, \tag{1.3}$$

where M^T denotes the transpose of M, and

$$H(\xi) := \frac{1}{|detM|}\sum_{\alpha\in\mathbb{Z}^s} a(\alpha)e^{-i\alpha\cdot\xi}, \qquad \xi \in \mathbb{R}^s. \tag{1.4}$$

Evidently, $H(\xi)$ is 2π-periodic. If $\hat{\varphi}(0) \neq 0$, then $\hat{\varphi}(0)$ is an eigenvector of the matrix $H(0)$ corresponding to eigenvalue 1.

Let $\varphi = (\varphi_1, ..., \varphi_r)^T$ be a vector of compactly supported functions in $(L_1(\mathbb{R}^s))^r$ satisfying refinement equation (1.1). It was proved in [3] and [4] that if $\hat{\varphi}(0) \neq 0$ and span $\{\hat{\varphi}(2\pi\beta) : \beta \in \mathbb{Z}^s\} = \mathbb{C}^r$. Then $H(0)$ satisfies condition E. We say that a matrix A (or an operator A defined on a finite dimensional linear space) satisfies condition E if $\rho(A) \leq 1$, 1 is the unique eigenvalue on the unit circle and 1 is simple (the spectral radius of A is denoted by $\rho(A)$).

In this paper we assume that $H(0)$ satisfies condition E. Thus, there is a nonsingular matrix V so that $VH(0)V^{-1}$ has the form

$$\begin{pmatrix} 1 & 0 \\ 0 & \Lambda \end{pmatrix}, \tag{1.5}$$

where Λ is an $(r-1)\times(r-1)$ matrix that satisfies $\lim_{n\to\infty}\Lambda^n = 0$. Define $b(\alpha) = Va(\alpha)V^{-1}$, then $\psi = V\varphi$ satisfies the refinement equation

$$\psi = \sum_{\alpha\mathbb{Z}^s} b(\alpha)\psi(M\cdot -\alpha), \qquad x \in \mathbb{R}^s, \tag{1.6}$$

where φ is a solution of (1.1). Therefore, we may assume that the $r\times r$ matrix $H(0)$ has the form (1.5), without losing anything.

For $j = 1, 2, ..., r$, we use e_j to denote the jth column of the $r\times r$ identity matrix. It is easily seen that

$$H(0)e_1 = e_1. \tag{1.7}$$

The question of existence of solutions of refinement equations (1.1) has attracted much attention from mathematicians in approximation theory and wavelet analysis. The existence of the compactly supported distribution solution of (1.1) was considered by Heil and Colella [4], by Cohen, Daubechies and Plonk [5] for the case $s = 1$ and $M = 2$, by Zhou [6] for the case $r = 2, M = 2I$ and by Jiang and Shen [7] for the case $M = 2I$. Micchelli and Prautzsch [8] obtained necessary and sufficient condition for the existence of continuous solutions of refinement equations (1.1) for $r = 1, s = 1$ and $M = 2$. Heil and Colella [9] also characterized the existence of continuous solutions for the case $M = 2$ and $r = 1$. The continuous solutions and L_p-solutions $(1 \leq p < \infty)$ of refinement equations (1.1) were also characterized in [10], [11] and [12] for the case $s = 1$ and $M = 2$. Under the assumption that $H(0)$ has the form (1.5), Jiang [13] provided a characterization of the existence of L_2-solutions of the equation (1.1). It is well-known that if the subdivsion schemes associated with mask a and dilation matrix M converges to some φ in L_p-norm, then the $\lim \varphi$ is a solution of (1.1) in L_p space. Therefore, it is also an efficient way to characterize the existence of the L_p-solution of equation (1.1) by investigating the convergence of the subdivsion schemes associated with equation (1.1) in L_p space. Under the condition that $H(0)$ has the form (1.5), the L_p-convergence of subdivsion schemes was studied by author in [14] for the case $1 \leq p \leq \infty$. However, from example 1 provided by Jia, Lau and Zhou in [11], we know that the L_p-convergence of subdivsion schemes is a sufficient but not a necessary condition for the existence of L_p-solutions. Therefore, it is necessary to give a complete characterization of L_p-solutions of refinement equations (1.1).

The purpose of this article is to give a characterization of the existence of L_p-solutions of refinement equation (1.1) in terms of the p-norm joint spectral radius of a finite collection of some linear operators determined by the sequence a and the set E restricted to a certain invariant subspace when $0 < p \leq \infty$, where the set E is a complete set of representatives of the distinct cosets of the quotient group $\mathbb{Z}^s/M\mathbb{Z}^s$ containing 0. We point out that when $0 < p < 1$, the results given in this paper are new even for $r = 1, s = 1$ and $M = 2$.

2. Characterization of L_p-Solutions.

In this section we give a characterization for the existence of L_p-solutions of the refinement equation (1.1) by using of some ideas of [10], [14] and [15]. Our characterizations are based on p-norm joint spectral radius of a finite collection of some linear operators determined by the sequence a and the set E restricted to a certain invariant subspace, where the set E is a complete set of representatives of the distinct cosets of the quotient group $\mathbb{Z}^s/M\mathbb{Z}^s$ containing 0.

Let $(\ell_0(\mathbb{Z}^s))^r$ denote the linear space of all finitely supported sequences of $r \times 1$ vectors on $\mathbb{Z}^s$.

Let M be a fixed dilation matrix with $m = |detM|$. Then the coset spaces $\mathbb{Z}^s/M\mathbb{Z}^s$ consists of m elements. Let $\gamma_k + M\mathbb{Z}^s, k = 0, 1, ..., m-1$ be the m distinct elements of $\mathbb{Z}^s/M\mathbb{Z}^s$ with $\gamma_0 = 0$. We denote $E = \{\gamma_k, k = 0, 1, ..., m-1\}$. Thus, each element $\alpha \in \mathbb{Z}^s$ can be uniquely represented as $\varepsilon + M\gamma$, where $\varepsilon \in E$ and $\gamma \in \mathbb{Z}^s$. For $\varepsilon \in E$, and $a \in (\ell_0(\mathbb{Z}^s))^{r\times r}$, the linear space of all finitely supported sequences of $r \times r$ matrices, we define the linear operators A_ε on $(\ell_0(\mathbb{Z}^s))^r$ as

$$A_\varepsilon u(\alpha) := \sum_{\beta\in\mathbb{Z}^s} a(\varepsilon + M\alpha - \beta)u(\beta), \qquad \alpha \in \mathbb{Z}^s, u \in (\ell_0(\mathbb{Z}^s))^r. \tag{2.1}$$

The main tool in our study is the joint spectral radius of a finite collection of some linear operators defined by (2.1). The uniform joint spectral radius was introduced by Rota and Strang in [16]. The mean spectral radius was introduced by Wang in [17]. The p-norm joint spectral radius was introduced by Jia in [18] for $1 < p < \infty$ and was used implicitly by Lau and Wang [12] independently.

When $0 < p < 1$, the p-norm joint spectral radius appeared in [19] and [20]. These concepts play an important role in the investigation of wavelets. Zhou [20] provided an efficient formula to compute the p-norm joint spectral radius in terms of the spectral radius of some finite matrix when p is an even integer. Let us review some notions of p-norm joint spectral radius from [21].

Let $\mathcal{A} = \{A_\varepsilon : \varepsilon \in E\}, W = (\ell_0(\mathbb{Z}^s))^r$, where A_ε is given by (2.1). A vector norm $||\cdot||$ on W induces a norm on the linear operators on W as follows:

$$||A|| := \max_{||v||=1} \{||Av||\}.$$

As usual, for $0 < p < \infty$, the ℓ_p norm or quasi-norm of an element $c = (c_1, ..., c_r)$ in $(\ell_0(\mathbb{Z}^s))^r$ is defined by

$$||c||_p = \left(\sum_{j=1}^{r} \sum_{\alpha \in Z^s} |c_j(\alpha)|^p \right)^{1/p},$$

and $||c||_\infty$ is the supremum of $\max_{1\le j\le r} |c_j|$ on $\mathbb{Z}^s$.

A subspace of W is said to be $\mathcal{A}$-invariant if it is invariant under every operator in $\mathcal{A}$. For $w \in W$, we call the intersection of all $\mathcal{A}$-invariant subspace of W containing w the minimal $\mathcal{A}$-invariant subspace generated by w, denoted as $W(w)$. For a positive integer n we denote by $\mathcal{A}^n$ the Cartesian power of $\mathcal{A}$:

$$\mathcal{A}^n = \{(A_1, ..., A_n) : A_1, ..., A_n \in \mathcal{A}\}.$$

When $n = 0$, we interpret $\mathcal{A}^0$ as the set $\{I\}$, where I is the identity mapping on W.

Let

$$||\mathcal{A}^n||_\infty := \max\{||A_1 \cdots A_n|| : (A_1, ..., A_n) \in \mathcal{A}^n\}.$$

Then the uniform joint spectral radius of $\mathcal{A}$ is defined to be

$$\rho_\infty(\mathcal{A}) := \lim_{n\to\infty} ||\mathcal{A}^n||_\infty^{1/n}. \tag{2.2}$$

The p-norm joint spectral radius of $\mathcal{A}$ is defined to be

$$\rho_p(\mathcal{A}) = \lim_{n\to\infty} ||\mathcal{A}^n||_p^{1/n}, \tag{2.3}$$

where

$$||\mathcal{A}^n||_p := \left(\sum_{(A_1,...,A_n)\in\mathcal{A}^n} ||A_1 \cdots A_n||^p \right)^{1/p}, 0 < p < \infty.$$

It is a classical fact that this limit exists and equals the infimum:

$$\lim_{n\to\infty} ||\mathcal{A}^n||_p^{1/n} = \inf_{n\in\mathbb{N}} ||\mathcal{A}^n||_p^{1/n}. \tag{2.4}$$

Clearly, $\rho_p(\mathcal{A})$ is independent of the choice of the vector norm on W.

We denote, for $0 < p < \infty, w \in W$,

$$||\mathcal{A}^n w||_p := \left(\sum\nolimits_{(A_1,\ldots A_n)\in\mathcal{A}^n} ||A_1 \cdots A_n w||^p\right)^{1/p},$$

and, for $p = \infty$,

$$||\mathcal{A}^n w||_\infty := \max\{||A_1 \cdots A_n w|| : (A_1, ..., A_n) \in \mathcal{A}^n\}.$$

It was proved in [21] and [19] that there exist two positive constants C_3 and C_4 such that

$$C_3||\mathcal{A}^n w||_p \leq ||\mathcal{A}^n|_{W(w)}||_p \leq C_4||\mathcal{A}^n w||_p, \quad n \in \mathbb{N}, \quad 0 < p \leq \infty, \tag{2.5}$$

where $W(w)$ denotes the minimal common $\mathcal{A}$- invariant subspace generated by w.

In order to study L_p-solutions of the refinement equation we shall employ the following iteration scheme. Let Q_a be the linear operator on $(L_p(\mathbb{R}^s))^r$ given by

$$Q_a f(x) := \sum_{\alpha\in\mathbb{Z}^s} a(\alpha) f(Mx - \alpha), \quad f \in (L_p(\mathbb{R}^s))^r. \tag{2.6}$$

Let φ_0 be an $r \times 1$ initial vector of functions in $(L_p(\mathbb{R}^s))^r(0 < p \leq \infty)$. For $n = 1, 2, ...,$ let $\varphi_n := Q_a^n \varphi_0$. If $(\varphi_n)_{n=1,2,\ldots}$ converges to some φ in the L_p space for $0 < p \leq \infty$, then the limit φ is a solution of (1.1) in $(L_p(\mathbb{R}^s))^r(0 < p \leq \infty)$. Iterating (2.6) n times gives

$$Q_a^n \varphi_0(x) = \sum_{\alpha\in\mathbb{Z}^s} a_n(\alpha)\varphi_0(M^n x - \alpha), \quad n = 1, 2, ..., \tag{2.7}$$

where for $n = 1, 2, ..., a_1 = a$ and a_n is defined by following iterative relations,

$$a_n(\alpha) = \sum_{\beta\in\mathbb{Z}^s} a_{n-1}(\beta) a(\alpha - M\beta), \quad \alpha \in \mathbb{Z}^s. \tag{2.8}$$

Lemma 2.1. *Let $\mathcal{A} = \{A_\varepsilon : \varepsilon \in E\}$ and $v \in (\ell_0(\mathbb{Z}^s))^r$. Then*

$$||\mathcal{A}^n v||_p = ||a_n * v||_p, \quad 0 < p \leq \infty,$$

where, for $a_n, \in (\ell_0(\mathbb{Z}^s))^{r\times r}$ and $v \in (\ell_0(\mathbb{Z}^s))^r$, the discrete convolution of a_n and v, denoted $a_n * v$, is given by

$$a_n * v(\alpha) = \sum_{\beta\in\mathbb{Z}^s} a_n(\alpha - \beta) w(\beta), \qquad \alpha \in \mathbb{Z}^s, \tag{2.9}$$

and a_n is defined by (2.8).

Proof. Suppose $\alpha = \varepsilon_1 + M\varepsilon_2 + \cdots + M^{n-1}\varepsilon_n + M^n\gamma$, where $\varepsilon_1, ..., \varepsilon_n \in E$ and $\gamma \in \mathbb{Z}^s$. Then by Lemma 2.1 of [22] we have

$$a_n * v(\alpha) = A_{\varepsilon_n} ... A_{\varepsilon_1} v(\gamma).$$

Hence Lemma 2.1 is true for $p = \infty$. When $0 < p < \infty$ we have

$$\begin{aligned} ||a_n * v||^p &= \sum_{\alpha \in \mathbb{Z}^s} |a_n * v(\alpha)|^p \\ &= \sum_{\varepsilon_1, \ldots, \varepsilon_n \in E} \sum_{\gamma \in \mathbb{Z}^s} |A_{\varepsilon_n} ... A_{\varepsilon_1} v(\gamma)|^p. \end{aligned}$$

This verifies Lemma 2.1 for $0 < p < \infty$.

For $\beta \in \mathbb{Z}^s$, we denote by δ_β the sequence on $\mathbb{Z}^s$ given by

$$\delta_\beta(\alpha) = \begin{cases} 1, & \text{for} \quad \alpha = \beta, \\ 0, & \text{for } \alpha \in \mathbb{Z}^s \setminus \{\beta\}. \end{cases}$$

In particular, we write δ for δ_0. For a vector $\lambda \in \mathbb{Z}^s$, the difference operator ∇_λ on $\ell(\mathbb{Z}^s)$ is given by

$$\nabla_\lambda v = v - v(\cdot - \lambda), \qquad \lambda \in \ell(\mathbb{Z}^s).$$

For simplicity, we write ∇_j for $\nabla_{e_j}, j = 1, ..., s$.

We are in a position to give characterizations for the existence of L_p-solutions and continuous solutions of refinement equations.

Theorem 2.2. *Let $\mathcal{A} = \{A_\varepsilon : \varepsilon \in E\}, 1 \leq p \leq \infty$, where A_ε are the linear operators on $(\ell_0(\mathbb{Z}^s))^r$ given by* (2.1). *Let M be a general dilation matrix with $m := |detM|$, $a \in (\ell_0(\mathbb{Z}^s))^{r\times r}$ such that $H(0)$ defined by* (1.4) *has the form* (1.5). *Let V be the minimal common $\mathcal{A}$-invariant subspace generated by $e_1\nabla_j\delta, e_2\nabla_j\delta, ..., e_r\nabla_j\delta$, where $j = 1, 2, ..., s$. If*

$$\rho_p(\mathcal{A}|_V) < m^{1/p}, \tag{2.10}$$

then there exists a compactly supported solution $\varphi \in (L_p(\mathbb{R}^s))^r (\varphi \in (C(\mathbb{R}^s))^r$ in the case $p = \infty)$ of refinement equation (1.1) *with mask a and dilation matrix M subject to $\hat{\varphi}(0) = (1, 0, ..., 0)^T$. Conversely, if $\varphi \in (L_p(\mathbb{R}^s))^r (\varphi \in (C(\mathbb{R}^s))^r$ in the case $p = \infty)$ is a compactly supported solution of* (1.1) *such that the shifts of $\varphi_1, ..., \varphi_r$ are stable, then* (2.10) *holds true.*

Proof. We choose f to be $e_1^T\varphi_0$, where φ_0 is the function given by

$$\varphi_0(x) = \prod_{j=1}^{s} \chi(x_j), \qquad x = (x_1, ..., x_s) \in \mathbb{R}^s,$$

and $\chi(t)$ is the characteristic function of the interval $[0,1)$. We want to prove that $(Q_a^n f)_{n=1,2,...}$ is a Cauchy sequence in $(L_p(\mathbb{R}^s))^r$. From (2.7) we have

$$Q_a^{n+1} f - Q_a^n f = Q_a^n f_0 = \sum_{\alpha\in\mathbb{Z}^s} a_n(\alpha) f_0(M^n x - \alpha),$$

where $f_0 = Q_a f - f$.

By simple computation, we obtain

$$\begin{aligned}||Q_a^n f_0||_p^p &= \int_{\mathbb{R}^s} |(Q_a^n f_0)(x)|^p dx \\ &= \sum_{\beta\in\mathbb{Z}^s} \int_{M^{-n}([0,1)^s+\beta)} |(Q_a^n f_0)(x)|^p dx.\end{aligned}$$

Putting $x = M^{-n}(y+\beta)$ in the above integral, it follows from Lemma 2.1 that

$$\begin{aligned}||Q_a^n f_0||_p^p &= m^{-n} \int_{[0.1)^s} \sum_{\beta\in\mathbb{Z}^s} |\sum_{\alpha\in\mathbb{Z}^s} a_n(\alpha) f_0(y+\beta-\alpha)|^p dy \\ &= m^{-n} \int_{[0,1)^s} ||a_n * v_y||_p^p dy \\ &= m^{-n} \int_{[0,1)^s} ||\mathcal{A}^n v_y||_p^p dy, \qquad (2.11)\end{aligned}$$

where $v_y(\alpha) = f_0(y+\alpha)$ for $\alpha\in\mathbb{Z}^s, y\in[0,1)^s$ and for a vector of functions $\psi = (\psi_1, ..., \psi_r)^T \in (L_p(\mathbb{R}^s))^r$, $|\psi|^p = \sum_{j=1}^r |\psi_j(x)|^p$. By (2.6), we know that for $y\in(0,1]^s, \alpha\in\mathbb{Z}^s$,

$$\begin{aligned}v_y(\alpha) &= Q_a f(y+\alpha) - f(y+\alpha) \\ &= \sum_{\beta\in\mathbb{Z}^s} a(\beta) f(My + M\alpha - \beta) - e_1\delta(\alpha) \\ &= \sum_{\eta\in\mathbb{Z}^s} a(M\beta - \eta) f(My + \eta - \beta) - e_1\delta(\alpha).\end{aligned}$$

Note that for any $y\in(0,1]^s$, there exists a unique $\eta_y\in\mathbb{Z}^s$, such that $My + \eta_y \in (0,1]^s$. It follows from that there exist $\gamma_y\in\mathbb{Z}^s$ and $\gamma_l\in E$ such that

$$\begin{aligned}&v_y(\alpha) = a(M\alpha - \eta_y)e_1 - e_1\delta(\alpha) = a(M\alpha - M\gamma_y + \gamma_l) - e_1\delta(\alpha) \\ &= A_{\gamma_l}(e_1\delta)(\alpha - \gamma_y) - e_1\delta(\alpha).\end{aligned}$$

Let $v_i = A_{\gamma_i}(e_1\delta) - e_1\delta$ for $i = 0, 1, ..., m-1$. Then

$$v_y(\alpha) = v_l(\alpha) + A_{\gamma_l}(e_1\delta_{M\gamma_y})(\alpha) - A_{\gamma_l}(e_1\delta)(\alpha).$$

While $\delta_{M\gamma_y} - \delta$ can be written as a finite linear combination of $\nabla_j \delta_\beta, j = 1, 2, ..., s, \beta \in \mathbb{Z}^s$. By Lemma 2.1, it is easily shown that

$$||\mathcal{A}^{n+1}(e_1 \nabla_j \delta_\beta)||_p = ||\mathcal{A}^{n+1}(e_1 \nabla_j \delta)||_p, \qquad \beta \in \mathbb{Z}^s. \tag{2.12}$$

Following (2.2), (2.3), (2.4) and (2.5), in order to prove that the sequence $(Q_a^n f)_{n=1,2,...}$ converges in the L_p-norm, it suffices to show that for any $y \in (0,1]^s$

$$\lim_{n\to\infty} ||\mathcal{A}^n v_l||_p^{1/n} \le \rho(\mathcal{A}|_V). \tag{2.13}$$

In fact, since $\rho(\mathcal{A}|_V) < m^{1/p}$, we can choose a number σ such that $m^{-1/p}\rho(\mathcal{A}|_V) < \sigma < 1$. Therefore, there exists a constant C independent of n and y such that

$$m^{-1/p}||\mathcal{A}^n v_l||_p \le C\sigma^n.$$

It follows from (2.11) that $(Q_a^n f)_{n=1,2,...}$ converges to some φ in the L_p-norm. In the case $p = \infty$, since each $Q_a^n f$ is continuous, the lim φ is also continuous. Furthermore, by simple computation, we have

$$\hat{Q}_a^n f(0) = H(0)^n \hat{f}(0) = H(0)^n e_1 = (1, 0, ..., 0)^T.$$

Taking the limit as $n \to \infty$ in the above equation, we obtain $\hat{\varphi}(0) = (1, 0, ..., 0)^T$. Let us show (2.13). To prove (2.13), it suffices to show that

$$\lim_{n\to\infty} ||\mathcal{A}^n(v_0 + v_1 + \cdots + v_{m-1})||_p^{1/n} \le \rho(\mathcal{A}|_V) \tag{2.14}$$

and

$$\lim_{n\to\infty} ||\mathcal{A}^n(v_0 - v_i)||_p^{1/n} \le \rho(\mathcal{A}|_V), \quad i = 1, 2, ..., m-1. \tag{2.15}$$

To verify (2.14), we observe that from (1.7)

$$\begin{aligned} v_0 + v_1 + \cdots + v_{m-1} &= \sum_{j=0}^{m-1} \sum_{\beta\in\mathbb{Z}^s} a(M\beta + \gamma_j) e_1 \delta_\beta - m e_1 \delta \\ &= \sum_{j=0}^{m-1} \sum_{\beta\in\mathbb{Z}^s} a(M\beta + \gamma_j) e_1 (\delta_\beta - \delta). \end{aligned}$$

Since $a(\alpha)$ is a finitely supported sequence of $r \times r$ matrices, $\delta_\beta - \delta$ can be written as a finite linear combination of $\nabla_j \delta_\eta, j = 1, 2, ..., s, \eta \in \mathbb{Z}^s$. Therefore, $v_0 + v_1 + \cdots + v_{m-1}$ can be written as a finite linear combination of $e_i(\nabla_j \beta_\eta), i = 1, 2, ..., r, j = 1, 2, ..., s, \eta \in \mathbb{Z}^s$. By using of (2.5) and (2.12), this proves (2.14). To prove (2.15), we observe that

$$\mathcal{A}_{\gamma_0}(e_1\delta)(\alpha) = \sum_{\beta\in\mathbb{Z}^s} a(M\alpha - \beta) e_1 \delta(\beta) = a(M\alpha) e_1 = \mathcal{A}_{\gamma_i}(e_1 \delta_{\gamma_i})(\alpha).$$

It follows that

$$v_0 - v_i = \mathcal{A}_{\gamma_0}(e_1\delta) - \mathcal{A}_{\gamma_i}(e_1\delta) = \mathcal{A}_{\gamma_i}(e_1\delta_{\gamma_i} - e_1\delta).$$

Hence, for $n = 1, 2, ...$, we have

$$||\mathcal{A}^n(v_0 - v_i)||_p \leq ||\mathcal{A}^{n+1}(e_1(\delta_{\gamma_i} - \delta))||_p.$$

Following above discussion, we can prove (2.15). This complete the proof of the sufficiency part of the theorem.

Next, we establish the necessity part of the theorem. Let $\varphi = (\varphi_1, ..., \varphi_r)^T$ be a compactly supported L_p-solution of (1.1). Iterating the refinement equation (1.1) n times, we obtain

$$\varphi = \sum_{\alpha\in\mathbb{Z}^s} a_n(\alpha)\varphi(M^n \cdot -\alpha),$$

where the sequence $a_n (n = 1, 2, ...)$ are defined by (2.8). It follows that

$$\varphi - \varphi(\cdot - M^{-n}e_j) = \sum_{\alpha\in\mathbb{Z}^s} \nabla_j a_n(\alpha)\varphi(M^n \cdot -\alpha).$$

If the shifts of $\varphi_1, ..., \varphi_r$ are stable, then there exists a constant $C_5 > 0$ such that for $n = 1, 2, ...$,

$$m^{-n/p}||\nabla_j a_n||_p \leq C_5||\varphi - \varphi(\cdot - M^{-n}e_j)||_p.$$

Therefore, we obtain

$$m^{-n/p}||\nabla_j a_n e_l||_p \leq C_5||\varphi - \varphi(\cdot - M^{-n}e_j)||_p, \quad j = 1, 2, ..., s, l = 1, 2, ..., r, n \in \mathbb{N}. \tag{2.16}$$

Note that $||\nabla_j a_n e_l||_p = ||a_n * (e_l\nabla_j\delta)||_p$. It follows from Lemma 2.1 that

$$\lim_{n\to\infty} ||\nabla_j a_n e_l||_p^{1/n} = \lim_{n\to\infty} ||\mathcal{A}^n(e_l\nabla_j\delta)||_p^{1/n}.$$

Since $\lim_{n\to\infty} ||\varphi - \varphi(\cdot - M^{-n}e_j)||_p = 0$, then we obtain from (2.5) and (2.16) that $\rho(\mathcal{A}|_V) < m^{1/p}$. This complete the proof of the necessity of the theorem.

Remark 2.3. *We remark that under the stability condition on φ, we give a complete characterization for the existence of the L_p-solution of equation* (1.1) *with* $1 \leq p \leq \infty$. *When* $s = 1$ *and* $M = 2$, *this result was established in* [10]. *In* [11], *Jia, Lau and Zhou gave a complete characterization for the existence of the L_p-solution of equation* (1.1) *without assuming stability when* $s = 1, M = 2$ *and* $1 \leq p \leq \infty$.

By using same method as in the proof of Theorem 2.2, we can obtain following theorem.

Theorem 2.4. *Let $\mathcal{A} = \{A_\varepsilon : \varepsilon \in E\}, 0 < p < 1$, where A_ε are the linear operators on $(\ell_0(\mathbb{Z}^s))^r$ given by* (2.1). *Let M be a general dilation matrix with $m := |detM|$, $a \in (\ell_0(\mathbb{Z}^s))^{r\times r}$ such that $H(0)$ defined by* (1.4) *has the form* (1.5). *Let V be the minimal common $\mathcal{A}$-invariant subspace generated by $e_1\nabla_j\delta, e_2\nabla_j\delta, ..., e_r\nabla_j\delta$, where $j = 1, 2, ..., s$. If*

$$\rho_p(\mathcal{A}|_V) < m^{1/p}, \tag{2.17}$$

then there exists a compactly supported solution $\varphi \in (L_p(\mathbb{R}^s))^r (0 < p < 1)$ of refinement equation (1.1) *with mask a and dilation matrix M. Conversely, if $\varphi \in (L_p(\mathbb{R}^s))^r (0 < p < 1)$ is a compactly supported solution of* (1.1) *such that the shifts of $\varphi_1, ..., \varphi_r$ are stable, then* (2.17) *holds true.*

Remark 2.5. *We remark that Theorem* 2.4 *is new even for $s = 1, M = 2$ and $r = 1$.*

3. Characterization of the existence of continuous solution.

In this section we give a complete characterization for the existence of continuous solution of equation (1.1) without assuming stability when M is an isotropic dilation matrix. A dilation matrix M is isotropic if M is similar to a diagonal matrix $diag(\sigma_1, ..., \sigma_s)$ such that $|\sigma_1| = \cdots = |\sigma_s|$. Some other notions used in this section are same as in section 2. Let $\varphi \in (C(\mathbb{R}^s))^r$ be a nonzero compactly supported solution of (1.1), then

$$\varphi(\alpha) = \sum_{\beta\in\mathbb{Z}^s} a(M\alpha - \beta)\varphi(\beta), \quad \alpha \in \mathbb{Z}^s.$$

By (2.1), we know that

$$A_{\gamma_0}\varphi(\alpha) = \varphi(\alpha).$$

That is, the sequence $u \in (\ell_0(\mathbb{Z}^s))^r$ given by $u(\alpha) = \varphi(\alpha), \alpha \in \mathbb{Z}^s$, is an eigenvector of A_{γ_0} associated with eigenvalue 1.

Given an eigenvector $u \in (\ell_0(\mathbb{Z}^s))^r$ of A_{γ_0} associated with eigenvalue 1, we want to find a continuous solution φ of refinement equations (1.1) for the case in which M is isotropic such that $\varphi|_{\mathbb{Z}^s} = u$. When M is an isotropic dilation matrix, a necessary and sufficient condition for the existence of thus solution is that $\rho_\infty(\mathcal{A}|_U) < 1$, where U is the minimal common $\mathcal{A}$-invariant subspace generated by $\nabla_j u, j = 1, 2, ..., s$.

Theorem 3.1. *Let $\mathcal{A} = \{A_\varepsilon : \varepsilon \in E\}, a \in (\ell_0(\mathbb{Z}^s))^{r\times r}, M$ be defined to be an isotropic dilation matrix and $u \in (\ell_0(\mathbb{Z}^s))^r$ be an eigenvector of A_{γ_0} associated with eigenvalue 1. Then there exists a continuous solution $\varphi = (\varphi_1, ..., \varphi_r)^T$ of refinement equation* (1.1) *such that $\varphi(\alpha) = u(\alpha), \alpha \in \mathbb{Z}^s$, if and only if*

$$\rho_\infty(\mathcal{A}|_U) < 1. \tag{3.1}$$

Proof. *Let φ be a compactly supported continuous solution satisfying* (1.1) *with $\varphi(\alpha) = u(\alpha)$ for every $\alpha \in \mathbb{Z}^s$. Iterating the refinement equation* (1.1) *n times, we have*

$$\varphi = \sum_{\alpha\in\mathbb{Z}^s} a_n(\alpha)\varphi(M^n \cdot -\alpha).$$

Then
$$\varphi(M^{-n}\eta) = \sum_{\alpha\in\mathbb{Z}^s} a_n(\alpha)\varphi(\eta-\alpha) = a_n * u(\eta), \quad \eta\in\mathbb{Z}^s, n\in\mathbb{N}.$$
By the definition of the difference operator ∇_λ, we have
$$\varphi(M^{-n}\eta) - \varphi(M^{-n}(\eta-e_j)) = a_n * \nabla_j u(\eta), \quad \eta\in\mathbb{Z}^s, n\in\mathbb{N}.$$
Since φ is compactly supported, it follows that
$$||a_n * \nabla_j u||_\infty \le ||\varphi - \varphi(\cdot - M^{-n}e_j)||_\infty \to 0, \quad n\to\infty.$$
By (2.4), (2.5) *and Lemma 2.1, we show that*
$$\rho_\infty(\mathcal{A}|_U) < 1.$$
Which implies the proof of the necessity part.

Next,we prove the sufficiency part. We follow the lines of [15]*. Suppose that* (3.1) *holds, then there exist ρ with $\rho_\infty(\mathcal{A}|_U) < \rho < 1$ and constant $C>0$ such that*
$$||\mathcal{A}^n\nabla_j u||_\infty \le C\rho^n, \quad j=1,...,s, \quad n\in\mathbb{N}.$$
By Lemma 2.1, we have
$$||a_n * \nabla_j u||_\infty = ||\nabla_j(a_n * u)||_\infty \le C\rho^n, \quad j=1,...,s, \quad n\in\mathbb{N}. \tag{3.2}$$
We define the solution φ on the sets $\{M^{-n}\mathbb{Z}^s|n\in\mathbb{N}\}$ by
$$\varphi(\eta) := u(\eta), \quad \varphi(M^{-n}\eta) := a_n * u(\eta), \quad \eta\in\mathbb{Z}^s, \quad n\in\mathbb{N}.$$
We observe from $A_{\gamma_0}u = u$ that for $\eta\in\mathbb{Z}^s$ and $n\in\mathbb{N}$,
$$\begin{aligned} a_n * u(M\eta) &= \sum_{\beta\in\mathbb{Z}^s} a_{n-1}(\beta) \sum_{\alpha\in\mathbb{Z}^s} a(\alpha - M\eta)u(M\eta-\alpha) \\ &= \sum_{\beta\in\mathbb{Z}^s} a_{n-1}(\beta) A_{\gamma_0}u(\eta-\beta) \\ &= a_{n-1} * u(\eta). \end{aligned}$$
It follows that φ is well-defined on the sets $\{M^{-n}\mathbb{Z}^s|n\in\mathbb{N}\}$, and $\varphi(\alpha) = u(\alpha), \alpha\in\mathbb{Z}^s$.

When $x\in\mathbb{R}^s$ is not in the sets $\{M^{-n}\mathbb{Z}^s|n\in\mathbb{N}\}$. Note that
$$\mathbb{R}^s = \cup_{\alpha\in\mathbb{Z}^s}(T+\alpha), \tag{3.3}$$
where $T = \{\sum_{j=1}^\infty M^{-j}\eta_{j-1} : \text{each } \eta_j\in E\}$. Then x can be uniquely written as
$$x = \alpha + \sum_{j=1}^\infty M^{-j}\eta_{j-1}, \quad \alpha\in\mathbb{Z}^s, \eta_j\in E.$$

The solution φ *on* $\mathbb{R}^s \setminus \{M^{-n}\mathbb{Z}^s | n \in \mathbb{N}\}$ *is defined by*

$$\varphi(x) = \lim_{n\to\infty} \varphi(\alpha + \sum_{j=1}^{n} M^{-j}\eta_{j-1}) =: \lim_{n\to\infty} \varphi_n(x).$$

To prove the existence of the limit, we only need to show that $(\varphi_n)_{n\in\mathbb{N}}$ *is a Cauchy sequence in* $(L_\infty(\mathbb{R}^s))^r$*. We observe that for* $n \in \mathbb{N}$,

$$\begin{aligned}
&\varphi_{n+1}(x) - \varphi_n(x)\\
&= \varphi(\alpha + \sum_{j=1}^{n+1} M^{-j}\eta_{j-1}) - \varphi(\alpha + \sum_{j=1}^{n} M^{-j}\eta_{j-1})\\
&= a_{n+1} * [u(M^{n+1}\alpha + M^n\eta_0 + M^{n-1}\eta_1 + \cdots + \eta_n)\\
&- u(M^{n+1}\alpha + M^n\eta_0 + M^{n-1}\eta_1 + \cdots + M\eta_{n-1})].
\end{aligned}$$

Since $u(M^{n+1}\alpha + M^n\eta_0 + M^{n-1}\eta_1 + \cdots + \eta_n) - u(M^{n+1}\alpha + M^n\eta_0 + M^{n-1}\eta_1 + \cdots + M\eta_{n-1})$ *can be written as a finite linear combination of* $\nabla_j u(M^{n+1}\alpha + M^n\eta_0 + M^{n-1}\eta_1 + \cdots + M\eta_{n-1} + \beta), j = 1, 2, ..., s, \beta \in \mathbb{Z}^s$*. By* (3.2)*, we see that* $(\varphi_n)_{n\in\mathbb{N}}$ *is a Cauchy sequence in* $(L_\infty(\mathbb{R}^s))^r$ *which implies that the limit exists and* φ *is well-defined for* $x \in \mathbb{R}^s$.

To prove the continuity of φ*, let* $||\cdot||_E$ *be the Euclidean norm on* $\mathbb{R}^s$*. From* [23]*, we know that there exists an equivalent norm* $||\cdot||$ *on* $\mathbb{R}^s$ *(or on* $\mathbb{C}^s$*) such that*

$$||Mx|| = m^{1/s}||x||. \tag{3.4}$$

Let $x, y \in \mathbb{R}^s$*. Suppose that*

$$m^{\frac{-n-1}{s}} \le ||x - y|| < m^{\frac{-n}{s}}$$

for certain $n \in \mathbb{N}$*. There exist* $\alpha, \beta \in \mathbb{Z}^s$ *such that* $M^n x \in T + \alpha$ *and* $M^n y \in T + \beta$*. By* (3.4)*, we have*

$$m^{\frac{-1}{s}} \le ||M^n x - M^n y|| < 1. \tag{3.5}$$

Therefore, there exists an absolute constant C_1 *such that*

$$\begin{aligned}
&||\varphi(M^{-n}\alpha) - \varphi(M^{-n}\beta)||_\infty\\
&\le C_1 \sum_{j=1}^{s} ||a_n * \nabla_j u||_\infty\\
&\le C_1 C s \rho^n.
\end{aligned} \tag{3.6}$$

Since $x \in M^{-n}T + M^{-n}\alpha$ *and* $y \in M^{-n}T + M^{-n}\beta$*. By* (3.2)*, we have for any* $n \in \mathbb{N}$,

$$\begin{aligned}
&||\varphi(x) - \varphi(M^{-n}\alpha)||_\infty\\
&\le \sum_{l=n+1}^{\infty} ||\varphi_{n+1}(x) - \varphi_n(x)||_\infty\\
&\le C \sum_{l=n+1}^{\infty} \rho^l = \frac{C\rho^{n+1}}{1-\rho}
\end{aligned}$$

and

$$\begin{aligned}
&||\varphi(y) - \varphi(M^{-n}\beta)||_\infty \\
&\le \sum_{l=n+1}^{\infty} ||\varphi_{n+1}(y) - \varphi_n(y)||_\infty \\
&\le C \sum_{l=n+1}^{\infty} \rho^l = \frac{C\rho^{n+1}}{1-\rho}.
\end{aligned}$$

Hence

$$\begin{aligned}
&||\varphi(x) - \varphi(y)||_\infty \\
&\le ||\varphi(x) - \varphi(M^{-n}\alpha)||_\infty + ||\varphi(y) - \varphi(M^{-n}\beta)||_\infty \\
&+ ||\varphi(M^{-n}\alpha) - \varphi(M^{-n}\beta)||_\infty \\
&\le \frac{2C\rho^{n+1}}{1-\rho} + CC_1 s\rho^n.
\end{aligned}$$

By (3.5), *it follows that there exists an absolute constant* C_2 *such that*

$$||\varphi(x) - \varphi(y)||_\infty \le C_2 ||x-y||^{-\log_{m^{1/s}} \rho}.$$

This proves the continuity of φ.

Finally, we prove the refinement relation (1.1). *Let* $n \in \mathbb{N}$ *and* $\alpha \in \mathbb{Z}^s$, *it is easy to prove by induction that*

$$a_{n+1}(\alpha) = \sum_{\beta \in \mathbb{Z}^s} a(\beta) a_n(\alpha - M^n\beta), \tag{3.7}$$

where the sequence $(a_n)_{n\in\mathbb{N}}$ *be given by* (2.8). *Hence, for any* $\eta \in \mathbb{Z}^s$

$$\begin{aligned}
\varphi(M^{-n-1}\eta) &= \sum_{\alpha\in\mathbb{Z}^s} \sum_{\beta\in\mathbb{Z}^s} a(\beta) a_n(\alpha - M^n\beta) u(\eta - \alpha) \\
&= \sum_{\beta\in\mathbb{Z}^s} a(\beta) a_n * u(\eta - M^n\beta) \\
&= \sum_{\beta\in\mathbb{Z}^s} a(\beta) \varphi(M^{-n}\eta - \beta).
\end{aligned}$$

This show that φ *satisfies refinement equation* (1.1) *on the set* $\{M^{-n}\mathbb{Z}^s | n \in \mathbb{N}\}$. *By the continuity of* φ, *we know that* φ *satisfies refinement equation* (1.1) *for all* $x \in \mathbb{R}^s$. *We finish the proof of Theorem* 3.1.

Remark 3.2. *Theorem* 3.1 *was established in* [15] *for the case* $s = 1$ *and* $M = 2$.

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