A Note on the Strong Maximal Operator on R^n *

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Abstract

In this paper,we shall prove that for $f \in L\ln^+ L(R^n)$ with compact support, there is a $g \in L\ln^+ L(R^n)$ such that (a) g and f are equidistributed, (b) $M_S(g) \in L^1(E)$ for any measurable set E of finite measure.

1 Introduction

For a function $f \in L_{loc}(R^n)$, its Hardy-Littlewood maximal function is defined by

$$M(f)(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)|\, dy$$

where Q is a cube with sides parallel to the coordinate axes, its strong maximal function is defined by

$$M_S(f)(x) = \sup_{P \ni x} \frac{1}{|P|} \int_P |f(y)|\, dy$$

where P is a rectangel with sides parallel to the coordinate axes. In addition, let $M^*(f)(x) = M_n \circ \cdots \circ M_1(f)(x)$ where M_j is the Hardy-Littlewood maximal operator on R^1 acting on the $j-th$ coordinate x_j.

It is well-known that for f with compact support,

- $M(f) \in L^1(E)$ for any measurable set E of finite measure$\Leftrightarrow f \in L\ln^+ L(R^n)$. See Stein [5].
- $M^*(f) \in L^1(E)$ for any measurable set E of finite measure$\Leftrightarrow f \in L(\ln^+ L)^n(R^n)$. See Jessen-Marcinkiewicz-Zygmund [4] and Fava-Gatto-Gutiérez [2].
- $f \in L(\ln^+ L)^n(R^n) \Rightarrow M_S(f) \in L^1(E)$ for any measurable set E of finite measure, because $M_S(f) \leq M^*(f)$. It was conjectured that for $f \in L(\ln^+ L)^{n-1}(R^n)$, $M_S(f) \in L^1(E)$ for any measurable set E of finite measure$\Rightarrow$ $f \in L(\ln^+ L)^n(R^n)$. See [2]. In [1] and [3], Bagby and Gomez independently proved that there are many functions $f \in L\ln^+ L(R^2)$ such that $M_S(f) \in L^1(E)$ for any measurable set E of finite measure.

*Supported by 973 project and NSFZJ

In this paper, by a different way which can be easily applied to high dimensions' case, we shall prove that the conjecture is also not true for $n > 2$. An interesting thing is that we do not need $f\in L(\ln^+L)^{n-1}L(R^n)$.

Theorem 1 *For $f \in L\ln^+L(R^n)$ with compact support, there is a $g \in L\ln^+L(R^n)$ such that (a) g and f are equidistributed, (b) $M_S(g) \in L^1(E)$ for any measurable set E of finite measure.*

2 Proof of the Theorem

Before proving the above theorem, we first introduce some notations and give some lemmas. Let

$$\begin{array}{l} A_t = \{(x_1, \cdots, x_n) : \sum_{i=1}^n x_i = t\} \\ D = \{(x_1, \cdots, x_n) : \sum_{i=1}^n x_i \geq n-1, x_i \leq 1 (i = 1, \ldots, n)\} \\ t(x) = \sum_{i=1}^n x_i \\ v(x) = \mu_n(\{y \in D : t(y) < t(x)\}) \end{array}$$

where μ_n denotes the Lebesgue measure on R^n. Without loss of generality, we may assume that

$$\mu_n(\{x \in R^n : |f(x)| > 0\}) \leq \mu_n(D).$$

Take

$$g(x) = \begin{cases} f^*(v(x)) & \text{for} \quad x \in D \\ 0 & \text{for} \quad x \notin D \end{cases}$$

where f^* is the rearrangement function of f, i.e

$$\begin{array}{l} f^*(r) = \lambda_f^{-1}(r) \stackrel{def}{=} \inf\{s : \lambda_f(s) \leq r\} \\ \lambda_f(s) = \mu_n(\{x \in R^n : |f(x)| > s\}) \end{array}$$

for $r > 0$. It is not difficult to show that f and g have the same distribution function, i.e.

$$\mu_n(\{x \in R^n : |f(x)| > s\}) = \mu_n(\{x \in R^n : |g(x)| > s\})$$

for all $s > 0$.

Let $\tilde{g}(s) = \sup\{g(x) : t(x) = s\}$. It is easy to check that $supp(\tilde{g}) \subseteq [n-1, n]$, $g \in L\ln^+L(R^n) \Rightarrow \tilde{g} \in L\ln^+L(R^1)$, and $\tilde{g} \in L\ln^+L(R^1) \Rightarrow g \in L\ln^+L(R^n)$ if $\mu_n(\{x \in R^n : |f(x)| > 0\}) > \mu_n(D)$.

We have

Lemma 2 *$M_S(g)(x) \leq C_n M(\tilde{g})(t(x))$ where M_S is the strong maximal function operator on R^n and M is the Hardy-Littlewood maximal function operator on R^1.*

Proof. For $x \in R^n$, $t \in R^1$, and $P = \prod_1^n [a_i, b_i] \ni x$, let $d_t = \sup_{y\in P} d(y, A_t)$. It is easy to see that if $P \cap A_t \neq \emptyset$, we have

$$d_t \geq \frac{1}{2\sqrt{n}} \left(\sum_1^n b_i - \sum_1^n a_i\right) \text{ and } d_t \cdot \mu_{n-1}(A_t \cap P) \leq \mu_n(P).$$

So, we have

$$\mu_{n-1}(A_t \cap P) \leq 2\sqrt{n} \cdot \mu_n(P) / \left(\sum_1^n b_i - \sum_1^n a_i\right).$$

Now, let $e_0 = (\sqrt{n^{-1}}, \cdots, \sqrt{n^{-1}})$, $L_0 = (R^1 e_0)^{\perp}$, and $R^n \ni x = re_0 \dot{+} z$ where $z \in L_0$. Noting that $P \ni x$ implies that $t(x) \in [\sum_1^n a_i, \sum_1^n b_i]$, we have

$$\begin{aligned}
\frac{1}{\mu_n(P)} \int_P g(y)dy &= \frac{1}{\mu_n(P)} \int_{R^1 e_0 \times L_0} \chi_P(x) g(x) dx \\
&= \frac{1}{\mu_n(P)} \int_{R^1 e_0} \int_{L_0} \chi_D(re_0 \dot{+} z) g(re_0 \dot{+} z) dr dz \\
&\leq \frac{1}{\mu_n(P)} \int_{\sum_1^n a_i/\sqrt{n}}^{\sum_1^n b_i/\sqrt{n}} \mu_{n-1}(\{z : re_0 \dot{+} z \in P\}) \tilde{g}(r\sqrt{n}) dr \\
&\leq \frac{1}{\sqrt{n}\mu_n(P)} \int_{\sum_1^n a_i}^{\sum_1^n b_i} \mu_{n-1}(\{z : \tfrac{r}{\sqrt{n}} e_0 \dot{+} z \in P\}) \tilde{g}(r) dr \\
&= \frac{1}{\sqrt{n}\mu_n(P)} \int_{\sum_1^n a_i}^{\sum_1^n b_i} \mu_{n-1}(A_r \cap P) \tilde{g}(r) dr \\
&\leq \frac{2}{\sum_1^n b_i - \sum_1^n a_i} \int_{\sum_1^n a_i}^{\sum_1^n b_i} \tilde{g}(t) dt \leq 2M(\tilde{g})(t(x)).
\end{aligned}$$

Lemma 3 *For $|x| > 2n$, $M_S(g)(x) \leq C_n |x|^{-1} \|g\|_1$.*

Proof. Without loss of generality, we may assume that $x_1 > \frac{|x|}{n}$ for $|x| > 2n$, and furthermore, we may assume that $a_1 < 1$, $a_1 + \sum_2^n b_i > n-1$ for $P = \prod_1^n [a_i, b_i]$ containning x and satisfying that $P \cap D \neq \emptyset$. Let $z = (1, b_1, b_2, \cdots, b_n)$, we have

$$\begin{aligned}
\frac{1}{\mu_n(P)} \int_P g(y)dy &\leq \frac{1-a_1}{b_1-a_1} \frac{1}{\mu_n([a_1,1] \times \prod_2^n [a_i,b_i])} \int_{[a_1,1] \times \prod_2^n [a_i,b_i]} g(y)dy \\
&\leq \frac{1-a_1}{b_1-a_1} M_S(g)(z) \leq \frac{1-a_1}{b_1-a_1} C_n M(\tilde{g})(t(z)) \\
&\leq \frac{1-a_1}{|x_1|-1} C_n \frac{\sqrt{n}}{t(z)-(n-1)} \|\tilde{g}\|_1 \\
&\leq C_n' \frac{1}{|x|} \frac{1-(n-1-\sum_2^n b_i)}{1+\sum_2^n b_i - (n-1)} \|\tilde{g}\|_1 \leq C_n \frac{1}{|x|} \|g\|_1 .
\end{aligned}$$

From Lemmas 2-3, we can easily get the Theorem.

References

[1] R. Bagby, *A note on the strong maxial function*, **Proc. Amer. Math. Soc.** 88(4)(1983), 648-650.

[2] N. A. Fava, E. A. Gatto and C. Gutierrez, *On the strong maximal function and Zygmund's class $L(log_+L)^n$*, **Studia Math.** 69(1980), 155-158.

[3] M. E. Gomez, *A counterexample for the strong maximal operator*, **Studia Math.** 78(2)(1984), 199-212.

[4] B. Jessen, J. Marcinkiewicz and A. Zugmund, *Note on the differentiability of multiple integrals*, **Fund. Math.** 25(1935), 217-234.

[5] E. M. Stein, *Note on the class $L(logL)$*, **Studia Math.** 32(1969), 305-310.