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# $S^1$-fixed-points in hyper-Quot-schemes and an exact mirror formula for flag manifolds from the extended mirror principle diagram

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## Abstract

In the series of work [L-L-Y1, III: Sec. 5.4] on mirror principle, two of the current authors (K.L. and S.-T.Y.) with Bong H. Lian developed a method to compute the integral $\int_X \tau^* e^{H\cdot t} \cap \mathbf{1}_d$ for a flag manifold $X = Fl_{r_1,\ldots,r_I}(\mathbb{C}^n)$ via an extended mirror principle diagram. This integral determines the fundamental hypergeometric series $HG[\mathbf{1}]^X(t)$ and is also related to the computation of the Gromov-Witten invariants (string world-sheet instanton numbers) on $X$. This method turns the required localization computation on the augmented moduli stack $\overline{\mathcal{M}}_{0,0}(\mathbb{C}\mathrm{P}^1 \times X)$ of stable maps to a localization computation on a hyper-Quot-scheme $HQuot(\mathcal{E}^n)$ of inclusion sequences of subsheaves of a trivialized trivial bundle $\mathcal{E}^n$ of rank $n$ on $\mathbb{C}\mathrm{P}^1$. In this article, the detail of this localization computation on $HQuot(\mathcal{E}^n)$ is carried out. The necessary ingredients in the computation, notably, the $S^1$-fixed-point components and the distinguished ones $E_{(A;0)}$ in $HQuot(\mathcal{E}^n)$, the $S^1$-equivariant Euler class of $E_{(A;0)}$ in $HQuot(\mathcal{E}^n)$, and a push-forward formula of cohomology classes involved in the problem from the total space of a restrictive flag manifold bundle to its base manifold are given. With these, an exact expression of $\int_X \tau^* e^{H\cdot t} \cap \mathbf{1}_d$ is obtained. When $X$ is a Grassmannian manifold, the same route reproduces the known exact expression for $HG[\mathbf{1}]^X(t)$. For a general flag manifold $X$, our expression determines $HG[\mathbf{1}]^X(t)$ implicitly. Remarks on what it suggests for general Hori-Vafa formula are given. Due to the technical necessity, a discussion on the general construction of restrictive flag manifold bundle, its natural embedding in a flag manifold bundle, and the Thom class of this embedding is also given. This work generalizes the result in [L-L-L-Y]. This work gives explicit formulas for mirror principle computations of Calabi-Yau manifolds in flag manifolds.


**Key words:** mirror principle, flag manifold, hyper-Quot-scheme, $S^1$-fixed-point, hypergeometric series, restrictive flag manifold, tangent stack, Euler sequence, mirror symmetry, Hori-Vafa formula, Young tableau combinatorics.

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# 0. Introduction and outline.

**Introduction.**

In the series of work [L-L-Y1, III: Sec. 5.4] on Mirror Principle, two of the current authors (K.L. and S.-T.Y.) with Bong H. Lian developed a method to compute the integral $\int_X \tau^* e^{H\cdot t} \cap \mathbf{1}_d$ for a flag manifold $X = Fl_{r_1,\ldots,r_I}(\mathbb{C}^n)$ via an extended mirror principle diagram, cf. Sec. 1 and Sec. 3.1.

This integral determines the fundamental hypergeometric series $HG[\mathbf{1}]^X(t)$ and is also related to the computation of the Gromov-Witten invariants (string world-sheet instanton numbers) on $X$. This method turns the required localization computation on the augmented moduli stack $\overline{\mathcal{M}}_{0,0}(\mathbb{CP}^1 \times X)$ of stable maps to a localization computation on a hyper-Quot-scheme $HQuot(\mathcal{E}^n)$ of inclusion sequences of subsheaves of a trivialized trivial bundle $\mathcal{E}^n$ of rank $n$ on $\mathbb{CP}^1$. The major purpose of the current work is to carry out the full detail of this localization computation on $HQuot(\mathcal{E}^n)$.

As a first step, the $S^1$-fixed-point components in $HQuot(\mathcal{E}^n)$ are described and the distinguished ones $E_{(A;0)}$ are identified, cf. Sec. 2.1, Sec. 2.2, and Sec. 3.1. In particular, $E_{(A;0)}$ admits a tower of fibrations with fiber restrictive flag manifolds. Also, by construction, there are canonical morphisms from all these $E_{(A;0)}$ to the flag manifold $X$.

For technical necessity, we study a general construction of a restrictive flag manifold bundle $W$ over a base manifold $Y$ and its associated flag manifold bundle $W'$ over the same base, cf. Sec. 3.2. $W$ naturally embeds in $W'$ and we work out the Thom class of $W$ in $W'$ with respect to this embedding.

The second step involves the computation of the $S^1$-equivariant Euler class of the normal bundle $\nu(E_{(A;0)}/HQuot(\mathcal{E}^n))$ of $E_{(A;0)}$ in $HQuot(\mathcal{E}^n)$. This involves both the understanding of the restriction of the tangent bundle $T_*HQuot(\mathcal{E}^n)$ of the hyper-Quot-scheme to $E_{(A;0)}$ and the tangent bundle of $E_{(A;0)}$. Some deformation-theoretical aspects of these spaces and their natural decompositions in the $K$-group of $E_{(A;0)}$ are studied. The computation of the $S^1$-equivariant Euler class of $\nu(E_{(A;0)}/HQuot(\mathcal{E}^n))$ then follows. Cf. Sec. 3.3 and Sec. 3.5.

The third step involves a push-forward formula of the cohomology classes involved in the problem from the total space of a restrictive flag manifold bundle to its base manifold. Consecutive applications of this push-forward formula via the tower fibration of $E_{(A;0)}$ give rise to an exact expression of the integral $\int_X \tau^* e^{H\cdot t} \cap \mathbf{1}_d$. Cf. Sec. 3.4 and Sec. 3.6.

For a general flag manifold, our expression can be interpreted as arising from the fundamental hypergeometric series for a product of Grassmannian manifolds that contains the flag manifold combined with the effect of the Thom class of the induced inclusion of $HQuot(\mathcal{E}^n)$ in a product of Quot-schemes. When the flag manifold is a Grassmannian manifold, the same route reproduces the known expression of $HG[\mathbf{1}]^X(t)$ in [B-CF-K] and hence the Hori-Vafa formula, conjectured in [H-V] and studied also in [B-CF-K] following [L-L-L-Y], for the case Grassmannian manifolds. Cf. Sec. 4.

The current work generalizes the results in [L-L-L-Y].

**Outline.**



# 1 Essential background and notations for physicists.

Essential background or its main references used in this article and notations for objects involved are collected in this section for the convenience of readers. The list extends that in [L-L-L-Y].

• **Hyper-Quot-scheme.** Fix the ample line bundle $\mathcal{O}_{\mathbb{CP}^1}(1)$ over $\mathbb{CP}^1$. Let $\mathcal{E}^n$ be the trivialized trivial bundle $\mathcal{O}_{\mathbb{CP}^1} \otimes \mathbb{C}^n$ of rank $n$ over $\mathbb{CP}^1$, $P = (P_1, \ldots, P_I)$ be a finite sequence of integral polynomials $P_i(t) = (n-r_i)t + d_i + (n-r_i)$ with $r_1 < \ldots < r_I$. Then *the hyper-Quot-scheme* $HQuot_P(\mathcal{E}^n)$ is the fine moduli space that parameterizes the set of successive quotients

$$\mathcal{E}^n \twoheadrightarrow \mathcal{E}^n/\mathcal{V}_1 \twoheadrightarrow \cdots \twoheadrightarrow \mathcal{E}^n/\mathcal{V}_I$$

with Hilbert polynomial $P(\mathcal{E}^n/\mathcal{V}_i, t) = P_i(t)$. It is the scheme that represents the *hyper-Quot*-functor - which generalizes Grothendieck's Quot-functor - for $\mathcal{E}^n$, cf [Gr3].

• **Hyper-Quot-scheme compactification of** $Hom(\mathbb{CP}^1, Fl_{r_1,\ldots,r_I}(\mathbb{C}^n))$**.** (Cf. [CF1], [Kim], [La], and [Str].) Let $C = \mathbb{CP}^1$ with the very ample line bundle $\mathcal{O}_{\mathbb{CP}^1}(1)$, $\mathcal{E}^n = \mathcal{O}_C \otimes \mathbb{C}^n$ be a trivialized trivial bundle of rank $n$ over $C$, $Fl_{r_1,\ldots,r_I}(\mathbb{C}^n)$, $r_1 < \cdots < r_I$, be the flag manifold that parameterizes inclusion sequences $V_\bullet : V_1 \hookrightarrow \cdots \hookrightarrow V_I$ of planes $V_i$ in $\mathbb{C}^n$ of dimension $r_i$, and $Hom(\mathbb{CP}^1, Fl_{r_1,\ldots,r_I}(\mathbb{C}))$ be the space of morphisms from $\mathbb{CP}^1$ to $Fl_{r_1,\ldots,r_I}(\mathbb{C}^n)$. Then an element $(f : \mathbb{CP}^1 \rightarrow Fl_{r_1,\ldots,r_I}(\mathbb{C}^n))$ in $Hom(\mathbb{CP}^1, Fl_{r_1,\ldots,r_I}(\mathbb{C}))$ determines a unique inclusion sequence (i.e. filtration of $\mathcal{E}^n$) $\mathcal{V}_\bullet : \mathcal{V}_1 \hookrightarrow \cdots \hookrightarrow \mathcal{V}_I$ of subbundles $\mathcal{V}_i$ of rank $r_i$ in $\mathcal{E}^n$, which corresponds in turn to the element $\mathcal{E}^n \twoheadrightarrow \mathcal{E}^n/\mathcal{V}_1 \twoheadrightarrow \cdots \twoheadrightarrow \mathcal{E}^n/\mathcal{V}_I$ (i.e. cofiltration of $\mathcal{E}^n$) in $HQuot(\mathcal{E}^n)$.

This gives a natural embedding of $Hom(\mathbb{C}\mathrm{P}^1, Fl_{r_1,\ldots,r_I}(\mathbb{C}^n))$ in $HQuot(\mathcal{E}^n)$. The component of $Hom(\mathbb{C}\mathrm{P}^1, Fl_{r_1,\ldots,r_I}(\mathbb{C}^n))$ that contains degree $d = (d_1, \ldots d_I)$ image curves in $Fl_{r_1,\ldots,r_I}(\mathbb{C}^n)$ is embedded in $HQuot_P(\mathcal{E}^n)$ with the Hilbert polynomial $P = (P_1(t), \ldots, P_I(t))$, where $P_i(t) = (n-r_i)t+d_i+(n-r_i)$. This gives a compactification of $Hom(\mathbb{C}\mathrm{P}^1, Fl_{r_1,\ldots,r_I}(\mathbb{C}^n))$ via hyper-Quot-schemes, other than the moduli space $\overline{M}_{0,0}(Fl_{r_1,\ldots,r_I}(\mathbb{C}^n), d)$ of stable maps from $\mathbb{C}\mathrm{P}^1$ to $Fl_{r_1,\ldots r_I}(\mathbb{C}^n)$. Recall also that $HQuot_P(\mathcal{E}^n)$ is a smooth, irreducible, projective variety of dimension $\sum_{i=1}^I (n-r_i)(r_i - r_{i-1}) + \sum_{i=1}^I d_i(n_{i+1} - n_{i-1})$. The $S^1$-action on $\mathbb{C}\mathrm{P}^1$ induces an $S^1$-action on $Hom(\mathbb{C}\mathrm{P}^1, Fl_{r_1,\ldots,r_I}(\mathbb{C}^n))$ and $HQuot_P(\mathcal{E}^n)$ respectively. The two actions coincide under the natural embedding of $Hom(\mathbb{C}\mathrm{P}^1, Fl_{r_1,\ldots,r_I}(\mathbb{C}^n))$ in $HQuot(\mathcal{E}^n)$.

• **Mirror principle diagram for flag manifolds.** For the details of Mirror Principle, readers are referred to [L-L-Y1 : I, II, III, IV]. Some survey is given in [L-L-Y2]. To avoid digressing too far away, here we shall take [L-L-Y1, III : Sec. 5.4] as our starting point and restrict to the case that the target manifold of stable maps is $X = Gr_r(\mathbb{C}^n)$. Recall the embedding of

$$\tau : X := Fl_{r_1,\ldots,r_I}(\mathbb{C}^n) \hookrightarrow Y := \mathbb{C}\mathrm{P}^{\binom{n}{r_1}-1} \times \cdots \times \mathbb{C}\mathrm{P}^{\binom{n}{r_I}-1}$$

induced from the Plücker embeddings $\tau_i : Gr_{r_i}(\mathbb{C}^n) \rightarrow \mathbb{C}\mathrm{P}^{\binom{n}{r_i}-1}$. $\tau$ induces an isomorphism between the divisor class groups $\tau^* : A^1(Y) \xrightarrow{\sim} A^1(X)$.

Recall next the Mirror Principle diagram for $X = Fl_{r_1,\ldots,r_I}(\mathbb{C}^n)$. The geometric objects involved are contained in the following diagram :

$$\begin{array}{ccccccccccc}
V & & U_d & & V_d & & \mathcal{U}_d & & & & \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & & & \\
X & \stackrel{ev}{\longleftarrow} & \overline{M}_{0,1}(X,d) & \stackrel{\rho}{\longrightarrow} & \overline{M}_{0,0}(X,d) & \stackrel{\pi}{\longleftarrow} & M_d & \stackrel{\varphi}{\longrightarrow} & W_d & \stackrel{\psi}{\longleftarrow} & \mathcal{Q}_d := HQuot_P\mathcal{E}^n \\
 & & & & & & \cup & & \cup & & \cup \\
 & & & & & & F_0 & \stackrel{ev^Y}{\longrightarrow} & Y_0\ (\supset X_0 = X) & \stackrel{g}{\longleftarrow} & E_0 \ = \ \cup_s E_{0s}\,, \\
 & & & & & & \downarrow ev^X & & \wr\! \| & & \\
 & & & & & & X & \stackrel{\tau}{\longrightarrow} & Y & &
\end{array}$$

where

(1) *Moduli spaces*: $\overline{M}_{0,0}(X,d)$ is the moduli space of genus-0 stable maps of degree $d = (d_1, \ldots, d_I)$ into $X$, $\overline{M}_{0,1}(X,d)$ is the moduli space of genus-0, 1-pointed stable maps of degree $d$ into $X$, $M_d = \overline{M}_{0,0}(\mathbb{C}\mathrm{P}^1 \times X, (1,d))$, $W_d$ is the linearized moduli space at degree $d$, which can be chosen to be the product of projective spaces: $\prod_{i=1}^I \mathbb{P}(H^0(\mathbb{C}\mathrm{P}^1, \mathcal{O}_{\mathbb{C}\mathrm{P}^1}(d_i)) \otimes \Lambda^{r_i}\mathbb{C}^n)$ for $X = Fl_{r_1,\ldots,r_I}(\mathbb{C}^n)$, and $\mathcal{Q}_d = HQuot_P(\mathcal{E}^n)$ with $P = (P_1(t), \cdots, P_I(t))$, where $P_i(t) = (n-r_i)t + d_i + (n-r_i)$.

(2) *Group actions*: there are $\mathbb{C}^\times$-actions on $M_d$, $W_d$, and $\mathcal{Q}_d$ respectively that are compatible with the morphisms among these moduli spaces; these $\mathbb{C}^\times$-actions induce $S^1$-actions on these moduli spaces by taking the subgroup $U(1) \subset \mathbb{C}^\times$.

(3) *Morphisms*: $ev$ is the evaluation map, $\rho$ is the forgetful map, $\pi$ is the contracting morphism, $\varphi$ is the collapsing morphism, and $\psi$ is an $S^1$-equivariant resolution of singularities of $\varphi(M_d)$. $\varphi$ and$\psi$ are discussed in detail in [L-L-L-Y: Sec. 3.1] when $X$ is a Grassmannian manifold. Their generalization to flag manifolds will be discussed in Sec. 3.1.

(4) *Bundles*: $V$ is a vector bundle over $X$, $V_d = \rho_! ev^* V$, $U_d = \rho^* V_d$, and $\mathcal{U}_d = \pi^* V_d$.

(5) *Special $S^1$-fixed-point locus*: $F_0 \simeq \overline{M}_{0,1}(X,d)$ is the special $S^1$-fixed-point component in $M_d$ that corresponds to gluing stable maps $(C', f', x')$ to $\mathbb{C}\mathrm{P}^1$ at $x' \in C'$ and $\infty \in \mathbb{C}\mathrm{P}^1$, $Y_0$ is the special $S^1$-fixed-point component in $W_d$ such that $\varphi^{-1}(Y_0) = F_0$, and $E_0$ is the $S^1$-fixed-point locus in $\psi^{-1}(Y_0)$ and is called the distinguished $S^1$-fixed-point locus or components in $\mathcal{Q}_d$. There is a natural smooth morphism $p$ from each component $E_{0s}$ of $E_0$ onto the flag manifold $X$.

(6) *Relation of $\psi$ and $\phi$*. It will be shown in Sec. 3.1 that $\varphi(M_d) = \psi(\mathcal{Q}_d)$ and that $\psi$ is a resolution of singularities of $\varphi(M_d)$. This implies that [L-L-Y1, III: Lemma 5.5] holds.

Associated to each $(V, b)$, where $b$ is a multiplicative characteristic class, is the Euler series $A(t) \in A^*(X)(\alpha)[t]$:

$$
\begin{aligned}
A(t) \;&:=\; A^{V,b} \;:=\; e^{-H\cdot t/\alpha} \textstyle\sum_d A_d\, e^{d\cdot t}\,,\\
A_d \;&=\; i_0^*\, b(\mathcal{U}_d) \;:=\; ev_*^X \left(\frac{\rho^* b(V_d)\cap[M_{0,1}(d,X)]}{e_{\mathbb{C}^\times}(F_0/M_d)}\right) \;=\; \frac{(i_{X_0}^*\varphi_* b(\mathcal{U}_d))\cap[X_0]}{e_{\mathbb{C}^\times}(X_0/W_d)}\,, \text{ denoted } \frac{\Theta_d}{e_{\mathbb{C}^\times}(X_0/W_d)}\,,\\
&=\; g_*\left(\textstyle\sum_s \frac{(i_{E_{0s}}^*\, g^*\, i_{X_0}^* \varphi_* b(\mathcal{U}_d))\cap[E_{0s}]}{e_{\mathbb{C}^\times}(E_{0s}/\mathcal{Q}_d)}\right)\,, \text{ denoted } g_*\left(\textstyle\sum_s \frac{\Xi_{d,s}}{e_{\mathbb{C}^\times}(E_{0s}/\mathcal{Q}_d)}\right)\,,
\end{aligned}
$$

where $\alpha = c_1(\mathcal{O}_{\mathbb{C}\mathrm{P}^\infty})(1)$ is the generator for $H^*_{\mathbb{C}^\times}(pt)$. On the other hand, one has the intersection numbers and their generating function

$$
\begin{aligned}
K_d \;&=\; K_d^{V,b} \;=\; \textstyle\int_{M_{0,0}(d,X)} b(V_d)\,,\\
\Phi \;&=\; \Phi^{V,b} \;=\; \textstyle\sum_d K_d\, e^{d\cdot t}\,.
\end{aligned}
$$

In the good cases, $K_d$ and $\Phi$ can be obtained from $A_d$ and $A(t)$ by appropriate integrals of the form $\int_X e^{-H\cdot t/\alpha} A_d$, where $H = (H_1, \cdots, H_I)$ is the restriction to $X$ of the hyperplane classes, also denoted by $H$, on $Y$ from its product projective space components, e.g. [L-L-Y1, III: Theorem 3.12]. This integral can be turned into an integral on $E_0$:

$$
\int_X \tau^* e^{H\cdot t}\cap A_d \;=\; \int_{Y_0} e^{H\cdot t}\cap g_*\left(\sum_s \frac{\Xi_{d,s}}{e_{\mathbb{C}^\times}(E_{0s}/\mathcal{Q}_d)}\right) \;=\; \sum_s \int_{E_{0s}} \frac{g^* e^{H\cdot t}\cap \widehat{\Xi}_{d,s}}{e_{\mathbb{C}^\times}(E_{0s}/\mathcal{Q}_d)}\,,
$$

where $\widehat{\Xi}_{d,s}$ is the Poincaré dual of $\Xi_{d,s}$ with respect to $[E_{0,s}]$. As will be discussed in Sec. 3.1, $E_{0s}$ is a flag manifold fibred over $X$ and, hence, $g^* e^{H\cdot t}$ can be read off from the natural fibration of flag manifolds $E_{0s} \to X$.

Following [L-L-Y1, III: Sec. 5.4], in the case that $b = 1$ the above integral is reduced to the integral

$$\int_X \tau^* e^{H\cdot t} \cap \mathbf{1}_d \;=\; \sum_s \int_{E_{0s}} \frac{g^*\psi^* e^{\kappa\cdot\zeta}}{e_{\mathbb{C}^\times}(E_{0s}/\mathcal{Q}_d)}\,,$$

where $\kappa = (\kappa_1, \cdots, \kappa_I)$ is the tuple of hyperplane classes in $W_d$ from its product projective space components. Via the natural smooth morphism $p : E_{0s} \to X$, one can integrate out the fiber of $p$ and lead to an integral over $X$.

In this article, we work out all the equivariant Euler classes $e_{\mathbb{C}^\times}(E_{0s}/\mathcal{Q}_d)$ and hence an exact expression of this integral. This determines $A(t)$ with $b = 1$ by [L-L-Y1, II: Lemma 2.5]. In the case of Grassmannian manifolds, the discussion gives also the known expression of $A(t)$ in [B-CF-K], via which the Hori-Vafa formula for Grassmannian manifolds was checked. For general $b$ induced from a concavex bundle on $X$, our method gives an explicit formula for the hypergeometric series in the mirror formulas.

• **Conventions and notation.**

(1) For historical reason, due to the relation of the Euler series with hyper-geometric series when $X$ is a toric manifold, $A(t)$ will also be denoted by $HG[b]^{X,V}(t)$ and be called a hypergeometric series.

(2) All the dimensions are *complex* dimensions unless otherwise noted.

(3) The $S^1$-actions involved in this article are induced from $\mathbb{C}^\times$-actions and both have the same fixed-point locus. In many places, it is more convenient to phrase things in term of $\mathbb{C}^\times$-action and we will not distinguish the two actions when this ambiguity causes no harm.

(4) A locally free sheaf and its associated vector bundle are denoted the same.

(5) An $I \times J$ matrix whose $(i, j)$-entry is $a_{ij}$ is denoted by $(a_{ij})_{i,j}$ when the position of an entry is emphasized and by $[a_{ij}]_{I\times J}$ when the size of the matrix is emphasized.

(6) From Section 2 on, the smooth curve $C$ will be $\mathbb{C}\mathrm{P}^1$ unless other noted.

(7) All the products of $\mathbb{C}$-schemes are products over $Spec\,\mathbb{C}$.

(8) For notation simplicity, the structure sheaf of a scheme is denoted also by $\mathcal{O}$ when the scheme is clear from the contents.

# 2 The $S^1$-fixed-point components on hyper-Quot-schemes.

The $S^1$-fixed-point components on the hyper-Quot-scheme $HQuot_P(\mathcal{E}^n)$ and their topology are studied in this section.

## 2.1 Inclusion pairs of $S^1$-invariant subsheaves of $\mathcal{E}^n$.

A characterization of $S^1$-invariant subsheaves of $\mathcal{E}^n$ is given in [L-L-L-Y: Sec. 2.1] and is summarized into the following fact:

**Fact 2.1.1 [$S^1$-invariant subsheaf].** (Cf. [L-L-L-Y: Sec. 2.1].)

(1) *An $S^1$-invariant subsheaf $\mathcal{V}$ of $\mathcal{E}^n$ with Hilbert polynomial of $\mathcal{E}^n/\mathcal{V}$ being $P(t) = (n-r)\,t + d + (n-r)$ is characterized by the following data $(\,V_\bullet^{(0)}, \alpha_\bullet\,;\, V_\bullet^{(\infty)}, \beta_\bullet\,)$ :*

  (*i*) *A pair of flags $(V_\bullet^{(0)}, V_\bullet^{(\infty)})$ of $\mathbb{C}^n$ :*

$$V_\bullet^{(0)}\;:\; V_1^{(0)} \hookrightarrow \cdots \hookrightarrow V_k^{(0)} \hookrightarrow \mathbb{C}^n \quad \textit{and} \quad V_\bullet^{(\infty)}\;:\; V_1^{(\infty)} \hookrightarrow \cdots \hookrightarrow V_l^{(\infty)} \hookrightarrow \mathbb{C}^n$$

  *with $V_k^{(0)} = V_l^{(\infty)}$, both of dimension $r$.*

  (*ii*) *A pair of integer sequences* (cf. [L-L-L-Y: Definition 2.1.6])

$$\alpha_\bullet : 0 \le \alpha_1 \le \cdots \le \alpha_r \qquad \textit{and} \qquad \beta_\bullet : 0 \le \beta_1 \le \cdots \le \beta_r$$

  *that satisfy* $(\alpha_1 + \cdots + \alpha_r) + (\beta_1 + \cdots + \beta_r) = d$.

(2) *For any $S^1$-invariant coordinate system on $\mathbb{C}\mathrm{P}^1$:*

$$\mathbb{C}\mathrm{P}^1 = U_0 \cup U_\infty\,, \quad \textit{where} \quad U_0 = \mathit{Spec}\,\mathbb{C}[z] \quad \textit{and} \quad U_\infty = \mathit{Spec}\,\mathbb{C}[w]$$

*with the gluing*

$$\mathit{Spec}\,\mathbb{C}[z] \hookleftarrow \mathit{Spec}\,\mathbb{C}[z, z^{-1}] \stackrel{z \leftrightarrow w^{-1}}{\simeq} \mathit{Spec}\,\mathbb{C}[w, w^{-1}] \hookrightarrow \mathit{Spec}\,\mathbb{C}[w]\,.$$

*with the $S^1$-action: $z \mapsto e^{i\theta} z$ and $w \mapsto e^{-i\theta} z$, a data $(V_\bullet^{(0)}, \alpha_\bullet\,;\, V_\bullet^{(\infty)}, \beta_\bullet)$ in Item* (1) *determines local subsheaves $(\mathcal{V}^{(0)}, \mathcal{V}^{(\infty)})$ of $(\mathcal{E}^n|_{U_0}, \mathcal{E}^n|_{U_\infty})$, which automatically glue together over $U_0 \cap U_\infty$ via the canonical isomorphism*

$$\mathcal{V}^{(0)}|_{U_0 \cap U_\infty} \;\simeq\; \mathcal{E}^n|_{U_0 \cap U_\infty} \;\simeq\; \mathcal{E}^n|_{U_0 \cap U_\infty}$$

*as $\mathcal{O}_{U_0 \cap U_\infty}$-modules, and hence an $S^1$-invariant subsheaf $\mathcal{V}$ of $\mathcal{E}^n$ with the required Hilbert polynomial for $\mathcal{E}^n/\mathcal{V}$.*

(3) *Given the data $(V_\bullet^{(0)}, \alpha_\bullet\,;\, V_\bullet^{(\infty)}, \beta_\bullet)$ in Item* (1)*, write $(\alpha_\bullet\,;\, \beta_\bullet)$ as $(a_\bullet, m_\bullet\,;\, b_\bullet, n_\bullet)$:*

$$\begin{array}{ccccccccccccccc} 0 & \le & a_1\,(=\alpha_1) & < & \cdots & < & a_k\,(=\alpha_r)\,; & \quad & 0 & \le & b_1\,(=\beta_1) & < & \cdots & < & b_l\,(=\beta_r) \\ & & m_1 & & \cdots & & m_k & & & & n_1 & & \cdots & & n_l \end{array}$$

*with the multiplicity of $a_i$, $b_j$ indicated. Recall the notation $M^\sim$ for the coherent sheaf on $\mathit{Spec}\,A$ associated to an $A$-module $M$, cf.* [Ha]*. Then, in Item* (2)*,*

$$\mathcal{V}^{(0)} \;=\; \left( z^{a_1}\, W_1^{(0)} \,+\, \cdots \,+\, z^{a_k}\, W_k^{(0)} \right)^\sim\,,$$

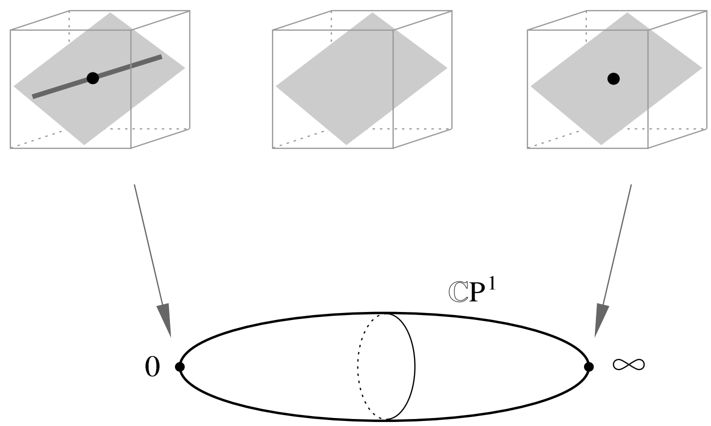


FIGURE 2-1-1. An $S^1$-invariant subsheaf $\mathcal{V}$ of a trivialized trivial bundle $\mathcal{E}^n$ is characterized by a pair of flags with identical last element, together with integral labels on elements in the flags.

*where* $W_i^{(0)}$ *is any subspace in* $V_i^{(0)} - V_{i-1}^{(0)}$ *of rank* $m_i$*, and*

$$\mathcal{V}^{(\infty)} \;=\; \left( w^{b_1}\, W_1^{(\infty)} \,+\, \cdots \,+\, w^{b_l}\, W_l^{(\infty)} \right)^{\sim} ,$$

*where* $W_j^{(\infty)}$ *is any subspace in* $V_j^{(\infty)} - V_{j-1}^{(\infty)}$ *of rank* $n_j$*.*

(Cf. FIGURE 2-1-1; see also FIGURE 2-1-2.)

The goal of this subsection is to generalize the above result to the case of inclusion sequences of $S^1$-invariant subsheaves of $\mathcal{E}^n$.

**Definition 2.1.2 [(integrally) labelled flag].** An (*integrally*) *labelled flag* of $\mathbb{C}^n$

$$V_\bullet(s_\bullet) \;:\; V_1(s_1) \;\hookrightarrow\; \cdots \;\hookrightarrow\; V_k(s_k)$$

is an ordinary flag $V_1 \hookrightarrow \cdots \hookrightarrow V_k \hookrightarrow \mathbb{C}^n$ together with a label $s_i \in \mathbb{Z}$ attached to each $V_i$ such that $s_1 < \cdots < s_k$.

With the same notation, Fact 2.1.1 Item (1) can be rephrased as follows.

**Fact 2.1.1′ [$S^1$-invariant subsheaf].** *An* $S^1$*-invariant subsheaf* $\mathcal{V}$ *of* $\mathcal{E}^n = \mathcal{O}_{\mathbb{CP}^1} \otimes \mathbb{C}^n$ *is characterized by a pair of labelled flags* $(V_\bullet^{(0)}(s_\bullet)\,,\, V_\bullet^{(\infty)}(t_\bullet))$*, where the label* $s_i$ *of* $V_i^{(0)}$ *is* $\alpha_{\dim V_i^{(0)}}$ *and the label* $t_j$ *of* $V_j^{(\infty)}$ *is* $\beta_{\dim V_j^{(\infty)}}$*.*

**Definition 2.1.3 [admissible inclusion].** Given a labelled flag

$$V_\bullet \;:\; V_1(s_1) \;\hookrightarrow\; \cdots \;\hookrightarrow\; V_k(s_k) \;\hookrightarrow\; V_{k+1}(\infty) := \mathbb{C}^n(\infty)$$

and a labelled vector subspace $\Pi(s) \subset \mathbb{C}^n$, we say that $\Pi(s)$ is *admissibly contained* in $V_i(s_i)$ for some $i$ if $\Pi \subset V_i$ and $s \geq s_i$. Since the sequence of the labels of the flag is non-decreasing, there is a maximal $i \leq k$ such that $\Pi(s)$ is admissibly contained in $V_i(s_i)$ but not in $V_{i+1}(s_{i+1})$. In this case, we say that $\Pi(s)$ is *admissibly and critically* contained in $V_i(s_i)$.

The following lemma is the inductive step in understanding an inclusion sequence of $S^1$-invariant subsheaves of $\mathcal{E}^n$.

**Lemma 2.1.4** [**$S^1$-invariant pair**]. *Let $\mathcal{V}_1$ and $\mathcal{V}_2$ be $S^1$-invariant subsheaves of $\mathcal{E}^n$ characterized by the data $(V_{1,\bullet}^{(0)}(s_{1,\bullet})\,;\, V_{1,\bullet}^{(\infty)}(t_{1,\bullet}))$ and $(V_{2,\bullet}^{(0)}(s_{2,\bullet})\,;\, V_{2,\bullet}^{(\infty)}(t_{2,\bullet}))$ respectively. Then $\mathcal{V}_1$ is a subsheaf of $\mathcal{V}_2$ if and only if each labelled subspace in the sequence $V_{1,\bullet}^{(0)}(s_{1,\bullet})$ (resp. $V_{1,\bullet}^{(\infty)}(t_{1,\bullet})$) can be admissibly contained in a labelled subspace in the sequence $V_{2,\bullet}^{(0)}(s_{2,\bullet})$ (resp. $V_{2,\bullet}^{(\infty)}(t_{2,\bullet})$).*

**Definition 2.1.5** [**order/precedence**]. When the condition in Lemma 2.1.4 is met, we shall say that $(V_{1,\bullet}^{(0)}(s_{1,\bullet})\,;\, V_{1,\bullet}^{(\infty)}(t_{1,\bullet}))$ *precedes* $(V_{2,\bullet}^{(0)}(s_{2,\bullet})\,;\, V_{2,\bullet}^{(\infty)}(t_{2,\bullet}))$. In notation, $(V_{1,\bullet}^{(0)}(s_{1,\bullet})\,;\, V_{1,\bullet}^{(\infty)}(t_{1,\bullet})) \preccurlyeq (V_{2,\bullet}^{(0)}(s_{2,\bullet})\,;\, V_{2,\bullet}^{(\infty)}(t_{2,\bullet}))$.

*Proof of Lemma 2.1.4.* Given a pair of flags in $\mathbb{C}^n$

$$(V_\bullet, V'_{\bullet'}) \;:=\; (\, V_1 \hookrightarrow \cdots \hookrightarrow V_k \hookrightarrow \mathbb{C}^n \,,\, V'_1 \hookrightarrow \cdots \hookrightarrow V'_{k'} \hookrightarrow \mathbb{C}^n \,)\,,$$

by considering either the double filtration or the double graded object of $\mathbb{C}^n$ associated to the pair of flags, one can show that there exists a direct-sum decomposition $\mathbb{C}^n = \oplus_m E_m$ of $\mathbb{C}^n$ such that any $V_i$, $V'_{i'}$ is a sum of some direct summands in this decomposition:

$$V_i \;=\; \oplus_j\, E_{i_j} \qquad \text{and} \qquad V'_{i'} \;=\; \oplus_{j'}\, E_{i'_{j'}}\,.$$

Such a decomposition of $\mathbb{C}^n$ is said to be *compatible* with the pair of flags $(V_\bullet, V_{\bullet'})$.

Apply this to our problem first with

$$(V_\bullet(s_\bullet), V'_\bullet(s'_\bullet)) \;=\; \Big(\, V_{1,\bullet}^{(0)}(s_{1,\bullet})\;,\; V_{2,\bullet}^{(0)}(s_{2,\bullet}) \,\Big)$$

and choose $W_i \subset V_i - V_{i-1}$ and $W'_{i'} \subset V'_{i'} - V'_{i'-1}$, as defined in Fact 2.1.1 (3), to be also direct sums with the summands some $E_m$'s:

$$W_i \;=\; \oplus_j\, E_{i_j} \quad \text{and} \quad W'_{i'} \;=\; \oplus_{j'}\, E_{i'_{j'}}\,.$$

Recall Fact 2.1.1 (3), Fact 2.1.1$'$ and the notations therein. Then

$$\mathcal{V}^{(0)} \;:=\; \mathcal{V}_1|_{U_0} \;=\; (\, \oplus_i\, z^{a_i} W_i\,)^{\sim} \;=\; \Big(\, \oplus_i\; \oplus_j\;\, z^{a_i} E_{i_j} \,\Big)^{\sim}$$

while

$$\mathcal{V}'^{\,(0)} \;:=\; \mathcal{V}_2|_{U_0} \;=\; \Big(\, \oplus_{i'}\, z^{a'_{i'}} W'_{i'} \Big)^{\sim} \;=\; \Big(\, \oplus_{i'}\; \oplus_{j'}\;\, z^{a'_{i'}} E_{i'_{j'}} \Big)^{\sim}\,.$$

Each $E_{i_j}$ in $W_i$ appears exactly once as some $E_{i'_{j'}}$ in some $W_{i'}$. Consequently, $\mathcal{V}^{(0)}$ is a subsheaf of $\mathcal{V}'^{(0)}$ if and only if

$$a_i \;\geq\; a'_{i'} \quad \text{whenever} \quad E_{i_j} = E_{i'_{j'}}\,.$$

But this means precisely that $V_{1,\bullet}^{(0)}(s_{1,\bullet}) \preccurlyeq V_{2,\bullet}^{(0)}(s_{2,\bullet})$. (Cf. FIGURE 2-1-2.)

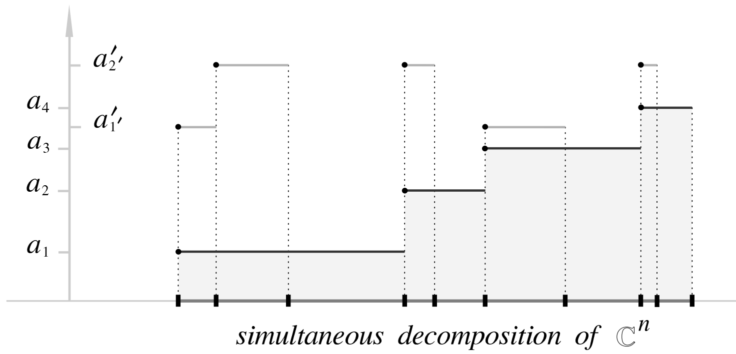


FIGURE 2-1-2. The relation of the characterization data of successive $S^1$-invariant subsheaves $\mathcal{V}_1 \hookrightarrow \mathcal{V}_2$ of $\mathcal{E}^n$.

Apply the same argument next to

$$(V_\bullet(s_\bullet), V'_\bullet(s'_\bullet)) \;=\; \Big(V_{1,\bullet}^{(\infty)}(t_{1,\bullet})\,,\; V_{2,\bullet}^{(\infty)}(t_{2,\bullet})\Big)$$

to conclude that $V_\bullet^{(\infty)}(t_{1,\bullet}) \preccurlyeq V_\bullet^{(\infty)}(t_{2,\bullet})$ as well. This completes the proof.

□

To better describe the structure of $S^1$-fixed-point components in $HQuot_P(\mathcal{E}^n)$, we introduce a couple of definitions in the passing.

**Definition/Lemma 2.1.6 [admissible pair of $(\alpha_\bullet;\beta_\bullet)$].** *Let $(\alpha_{2,\bullet};\beta_{2,\bullet})$ be from the characterization data of the $S^1$-invariant subsheaf $\mathcal{V}_2$ of $\mathcal{E}^n$, $P_1 = P_1(t) = (n-r_1)t + d_1 + (n-r_1)$ be an integral polynomial and $(\alpha_{1,\bullet};\beta_{1,\bullet})$ be admissible to $P_1$. Then there exists an $S^1$-invariant subsheaf $\mathcal{V}_1$ of $\mathcal{V}_2$ whose labels in its characterization data comes from $(\alpha_{1,\bullet};\beta_{1,\bullet})$ if and only if*

$$r_1 \;\leq\; r_2\,, \quad \alpha_{1,i} \;\geq\; \alpha_{2,i} \quad \text{and} \quad \beta_{1,i} \;\geq\; \beta_{2,i} \quad \text{for } i = 1,\,\ldots,\,r_1\,.$$

*This condition depends only on the data $(\alpha_{2,\bullet};\beta_{2,\bullet})$, not on any other detail of $\mathcal{V}_2$. We say that $(\alpha_{1,\bullet};\beta_{1,\bullet})$ is admissible to $(\alpha_{2,\bullet};\beta_{2,\bullet})$. In notation, $(\alpha_{1,\bullet}\,;\,\beta_{1,\bullet}) \rightarrow (\alpha_{2,\bullet}\,;\,\beta_{2,\bullet})$.*

**Definition/Lemma 2.1.7 [characteristic chain of subspaces].** *Following Definition/Lemma 2.1.6, given $(\alpha_{1,\bullet}\,;\,\beta_{1,\bullet}) \rightarrow (\alpha_{2,\bullet}\,;\,\beta_{2,\bullet})$, let $\mathcal{V}_1$ be an $S^1$-invariant subsheaf of*

*$\mathcal{V}_2$ with characterization data $(V_{1,\bullet}^{(0)}(s_{1,\bullet})\,;\,V_{1,\bullet}^{(\infty)}(t_{1,\bullet}))$ equivalent to $(V_{1,\bullet}^{(0)},\alpha_{1,\bullet}\,;\,V_{1,\bullet}^{(\infty)},\beta_{1,\bullet})$. Recall that each element in $V_{1,\bullet}^{(0)}$ (resp. $V_{1,\bullet}^{(\infty)}$) is admissibly and critically contained in a unique element in $V_{2,\bullet}^{(0)}(s_{2,\bullet})$ (resp. $V_{2,\bullet}^{(\infty)}(t_{2,\bullet})$). These latter elements form a sequence of successive subspaces*

$$\Pi_\bullet^{(0)} \qquad (\,resp.\;\; \Pi_\bullet^{(\infty)}\,)\,.$$

*Then the pair $(\Pi_\bullet^{(0)}\,;\,\Pi_\bullet^{(\infty)})$ depends only on $(\alpha_{1,\bullet}\,;\,\beta_{1,\bullet})$, not on the choice of $\mathcal{V}_1$.*

**Definition 2.1.8 [restrictive flag manifold].** Given an inclusion sequence of subspaces $\Pi_\bullet : \Pi_1 \subset \cdots \subset \Pi_s \subset \mathbb{C}^n$ (with $\Pi_i = \Pi_{i+1}$ allowed) and a strictly increasing sequence of integers $0 < k_1 < \cdots < k_s$ with $k_i \le dim\,\Pi_i =: l_i$, the subspace $Fl_{k_1,\cdots,k_s}(\mathbb{C}^n,\Pi_\bullet)$ of the flag manifold $Fl_{k_1,\cdots,k_s}(\mathbb{C}^n)$ defined by

$$Fl_{k_1,\cdots,k_s}(\mathbb{C}^n,\Pi_\bullet) \;:=\; \{\,\text{flags } V_\bullet : V_1 \hookrightarrow \cdots \hookrightarrow V_s \hookrightarrow \mathbb{C}^n \;\mid\, dim\,V_i = k_i \;\text{ and }\; V_i \subset \Pi_i\,\}$$

is called a *restrictive flag manifold associated to* $\Pi_\bullet$.

**Lemma 2.1.9 [$Fl_{k_1,\cdots,k_s}(\mathbb{C}^n,\Pi_\bullet)$ smooth].** *$Fl_{k_1,\cdots,k_s}(\mathbb{C}^n,\Pi_\bullet)$ is a projective, connected, smooth manifold of dimension $k_1(l_1-k_1)+(k_2-k_1)(l_2-k_2)+\cdots+(k_s-k_{s-1})(l_s-k_s)$.*

*Proof.* Projectivity of $Fl_{k_1,\cdots,k_s}(\mathbb{C}^n,\Pi_\bullet)$ follows from the prejectivity of $Fl_{k_1,\cdots,k_s}(\mathbb{C}^n)$. By construction, $Fl_{k_1,\cdots,k_s}(\mathbb{C}^n,\Pi_\bullet)$ admits a tower of fibrations by Grassmannian manifolds and hence is connected. The fibrations in the tower is not topologically locally trivial, in particular the fibers of a fixed fibration in the tower can vary; so it is not immediate from this fibration tower that $Fl_{k_1,\cdots,k_s}(\mathbb{C}^n,\Pi_\bullet)$ is smooth.

To prove the last statement, let $F_\bullet\mathbb{C}^n$ be the filtration of $\mathbb{C}^n$ by $\Pi_\bullet$, then one can show that at each point $[V_\bullet]$ of $Fl_{k_1,\cdots,k_s}(\mathbb{C}^n,\Pi_\bullet)$, the space $T_{[V_\bullet]}Fl_{k_1,\cdots,k_s}(\mathbb{C}^n,\Pi_\bullet)$ of the first order deformations of $[V_\bullet]$ as a restrictive flag fits into an exact sequence of complex vector spaces of the form

$$0 \;\longrightarrow\; Hom(V_\bullet,V_\bullet) \;\longrightarrow\; Hom(V_\bullet,F_\bullet\mathbb{C}^n) \;\longrightarrow\; T_{[V_\bullet]}Fl_{k_1,\ldots,k_s}(\mathbb{C}^n,\Pi_\bullet) \;\longrightarrow\; 0\,.$$

(Cf. See Sec. 3.5 for more related details in the construction of an Euler sequence for $T_*Fl_{k_1,\cdots,k_s}(\mathbb{C}^n,\Pi_\bullet)$.) It follows that

$$dim\,T_{[V_\bullet]}Fl_{k_1,\ldots,k_s}(\mathbb{C}^n,\Pi_\bullet) \;=\; k_1(l_1-k_1)\,+\,(k_2-k_1)(l_2-k_2)\,+\,\cdots\,+\,(k_s-k_{s-1})(l_s-k_s)\,.$$

Since this is independent of $[V]$ and all elements in $T_{[V_\bullet]}Fl_{k_1,\ldots,k_s}(\mathbb{C}^n,\Pi_\bullet)$ are realizable from a family of restrictive flags over a small disc, $Fl_{k_1,\cdots,k_s}(\mathbb{C}^n,\Pi_\bullet)$, as a scheme, must be reduced everywhere and hence is smooth of the above dimension. This concludes the lemma.

□

With these preparations, we can now describe first the topology of the connected components of the $S^1$-fixed-point locus in the special Quot-scheme $Quot_{P_1}(\mathcal{V}_2 \hookrightarrow \mathcal{E}^n)$ and then the topology of the connected components of the $S^1$-fixed-point components in the general hyper-Quot-scheme $HQuot_P(\mathcal{E}^n)$.

**Lemma 2.1.10 [$S^1$-invariant subsheaves of a fixed $S^1$-invariant sheaf].** *Fix an $S^1$-invariant subsheaf $\mathcal{V}_2$ of $\mathcal{E}^n$ with characterization data $(\,V_{2,\bullet}^{(0)}(s_{2,\bullet})\,;\,V_{2,\bullet}^{(\infty)}(t_{2,\bullet})\,)$ equivalent to $(\,V_{2,\bullet}^{(0)},\alpha_{2,\bullet}\,;\,V_{2,\bullet}^{(\infty)},\beta_{2,\bullet}\,)$ : and the Hilbert polynomial of $\mathcal{E}^n/\mathcal{V}_2$ being $P_2=P_2(t)=(n-r_2)\,t+d_2+(n-r_2)$.*

(1) *The space $F_{P_1}^{S^1}(\mathcal{V}_2\hookrightarrow\mathcal{E}^n)$ of $S^1$-invariant subsheaves $\mathcal{V}_1$ of $\mathcal{V}_2$ with the Hilbert polynomial of $\mathcal{E}^n/\mathcal{V}_1$ being $P_1=P_1(t)=(n-r_1)\,t+d_1+(n-r_1)$ is non-empty if and only if there exists $(\alpha_{1,\bullet}\,;\,\beta_{1,\bullet})$ admissible both to $P_1$ and $(\alpha_{2,\bullet}\,;\,\beta_{2,\bullet})$.*

(2) *The set of connected components of $F_{P_1}^{S^1}(\mathcal{V}_2\hookrightarrow\mathcal{E}^n)$ is in one-to-one correspondence with the set of pairs $(\alpha_{1,\bullet}\,;\,\beta_{1,\bullet})$ that is admissible both to $P_1$ and $(\alpha_{2,\bullet}\,;\,\beta_{2,\bullet})$. Let $(A;B)$ be a pair of incomplete matrices defined by*

$$A\;=\;\left[\begin{array}{cccccc}\alpha_{1,1} & \cdots & \alpha_{1,r_1} & & & \\ \alpha_{2,1} & \cdots & \alpha_{2,r_1} & \cdots & \alpha_{2,r_2}\end{array}\right]_{2\times r_2}\quad\text{and}\quad B\;=\;\left[\begin{array}{cccccc}\beta_{1,1} & \cdots & \beta_{1,r_1} & & & \\ \beta_{2,1} & \cdots & \beta_{2,r_1} & \cdots & \beta_{2,r_2}\end{array}\right]_{2\times r_2}.$$

*(Note that $\alpha_{1,j}$ and $\beta_{1,j}$ for $j>r_1$ is left undefined/blank; this is why we call $A$ and $B$ incomplete matrices.) The corresponding component of $F_{P_1}^{S^1}(\mathcal{V}_2\hookrightarrow\mathcal{E}^n)$ will be denoted by $F_{(A;B)}$.*

(3) *Each $(A;B)$ in Item (2) determines a pair $(\Pi_{2,\bullet}^{(0)}\,;\,\Pi_{2,\bullet}^{(\infty)})$ of chains of subspaces by Definitio/Lemma 2.1.7 and hence a pair of restrictive flag manifolds*

$$Fl_{(\alpha_{1,\bullet})}(\mathbb{C}^n,\Pi_\bullet^{(0)})\;:=\;Fl_{m_{1,1},m_{1,1}+m_{1,2},\,\ldots,\,r_1}(\mathbb{C}^n,\Pi_{2,\bullet}^{(0)})$$

*and*

$$Fl_{(\beta_{1,\bullet})}(\mathbb{C}^n,\Pi_\bullet^{(\infty)})\;:=\;Fl_{n_{1,1},n_{1,1}+n_{1,2},\,\ldots,\,r_1}(\mathbb{C}^n,\Pi_{2,\bullet}^{(\infty)})\,.$$

*Notice that both $Fl_{(\alpha_{1,\bullet})}(\mathbb{C}^n,\Pi_{2,\bullet}^{(0)})$ and $Fl_{(\beta_{1,\bullet})}(\mathbb{C}^n,\Pi_{2,\bullet}^{(\infty)})$ fiber over $Gr_{r_1}(\mathbb{C}^n)$. The topology of $F_{(A;B)}$ is then given by*

$$F_{(A;B)}\;=\;Fl_{(\alpha_{1,\bullet})}(\mathbb{C}^n,\Pi_{2,\bullet}^{(0)})\times_{Gr(n,r_1)}Fl_{(\beta_{1,\bullet})}(\mathbb{C}^n,\Pi_{2,\bullet}^{(\infty)})\,.$$

*It depends only on the connected $S^1$-fixed-point component $V'$ belongs to in the Quot-scheme $Quot_{P'}(\mathcal{E}^n)$. In other words, it depends only on the admissible incomplete matrix $(A;B)$. (This justifies our notation.)*

*Proof.* These follow immediately from our discussion in this subsection.

□

**Corollary 2.1.11 [$S^1$-fixed-point locus in $HQuot_{P_1,P_2}(\mathcal{E}^n)$].** *The set of connected components of $S^1$-fixed-point locus in $HQuot_{P_1,P_2}(\mathcal{E}^n)$ is in a natural one-to-one correspondence with the set of pairs*

$$\left\{\left(\begin{array}{c}(\alpha_{1,\bullet}\,;\,\beta_{1,\bullet})\\(\alpha_{2,\bullet}\,;\,\beta_{2,\bullet})\end{array}\right)\;\middle|\;\begin{array}{l}(\alpha_{1,\bullet}\,;\,\beta_{1,\bullet})\text{ admissible to }P_1,\ (\alpha_{2,\bullet}\,;\,\beta_{2,\bullet})\text{ admissible to }P_2\\ \text{and }(\alpha_{1,\bullet}\,;\,\beta_{1,\bullet})\;\rightarrow\;(\alpha_{2,\bullet}\,;\,\beta_{2,\bullet})\end{array}\right\},$$

i.e. *the set of pairs of incomplete matrices* $(A;B)$ *given in Lemma 2.1.10, Item* (2). *Denote the* $S^1$*-fixed-point component corresponding to* $(A;B)$ *by* $E_{(A;B)}$. *Then* $E_{(A;B)}$ *is smooth.*

*Proof.* Recall from [L-L-L-Y: Sec. 2.1] that the pair $(\alpha_{1,\bullet};\beta_{1,\bullet})$ (resp. $(\alpha_{2,\bullet};\beta_{2,\bullet})$) admissible to $P_1$ (resp. $P_2$) determines a unique $S^1$-fixed-point component $E_{(\alpha_{1,\bullet};\beta_{1,\bullet})}$ (resp. $E_{(\alpha_{2,\bullet};\beta_{2,\bullet})}$) in $Quot_{P_1}(\mathcal{E}^n)$ (resp. $Quot_{P_2}(\mathcal{E}^n)$), whose topology is a fibered product of flag manifolds and hence is connected and smooth.

Consider the natural inclusion $\iota : HQuot_{P_1,P_2}(\mathcal{E}^n) \hookrightarrow Quot_{P_1}(\mathcal{E}^n) \times Quot_{P_2}(\mathcal{E}^n)$. Let $E_{(A;B)}$ be the $S^1$-fixed-point sublocus in $HQuot_{P_1,P_2}(\mathcal{E}^n)$ that consists of points associated to pairs of $S^1$-invariant subsheaves of $\mathcal{E}^n$ whose discrete part of the characterization data is given by $(A;B)$. Then

$$E_{(A;B)} \;=\; (\, E_{(\alpha_{1,\bullet};\beta_{1,\bullet})} \times E_{(\alpha_{2,\bullet};\beta_{2,\bullet})}\,) \cap \iota\left( HQuot^{S^1}_{P_1,P_2}(\mathcal{E}^n) \right) .$$

With respect to the ambient product structure via $\iota$, $E_{(A;B)}$ fibers over $E_{(\alpha_{2,\bullet};\beta_{2,\bullet})}$ with fiber $F_{(A;B)}$ in Lemma 2.1.10, Item (2). Since both $E_{(\alpha_{2,\bullet};\beta_{2,\bullet})}$ and $F_{(A;B)}$ are connected and smooth, so is $E_{(A;B)}$. This completes the proof.

□

## 2.2 The $S^1$-fixed-point locus in $HQuot_P(\mathcal{E}^n)$.

Successive applications of Lemma 2.1.4, Lemma 2.1.10 and Corollary 2.1.11 give rise to the following description of the topology of the $S^1$-fixed-point components in $HQuot_P(\mathcal{E}^n)$.

**Proposition 2.2.1** [**$S^1$-fixed-point component in** $HQuot_P(\mathcal{E}^n)$]**.** *Recall the sequence of Hilbert polynomials*

$$P \;:\; P_1\,,\,\cdots\,,\,P_I\,.$$

(1) *For each admissible sequence of pairs of finite sequences:*

$$(\alpha_{1,\bullet}\,;\,\beta_{1,\bullet}) \;\rightarrow\; \cdots \;\rightarrow\; (\alpha_{I,\bullet}\,;\,\beta_{I,\bullet}) \qquad \textit{with}\;\; (\alpha_{i,\bullet}\,;\,\beta_{i,\bullet}) \;\; \textit{admissible to} \;\; P_i\,,$$

*define the* $I \times r_I$ *incomplete matrices*

$$A \;=\; \left[\begin{array}{ccccc} \alpha_{1,1} & \cdots & \alpha_{1,r_1} & & \\ \cdots & \cdots & \cdots & \cdots & \\ \alpha_{I,1} & \cdots & \cdots & \cdots & \alpha_{I,r_I} \end{array}\right]_{I\times r_I} \qquad \textit{and} \quad B \;=\; \left[\begin{array}{ccccc} \beta_{1,1} & \cdots & \beta_{1,r_1} & & \\ \cdots & \cdots & \cdots & \cdots & \\ \beta_{I,1} & \cdots & \cdots & \cdots & \beta_{I,r_I} \end{array}\right]_{I\times r_I} .$$

*Then there is a natural one-to-one correspondence between the set of* $(A;B)$ *defined above and the set of* $S^1$*-fixed-point components of* $HQuot_P(\mathcal{E}^n)$.

(2) *Let*

$$(A_1;B_1) = (A;B)\,,\; (A_2;B_2)\,,\,\cdots\,,\,(A_I;B_I)$$

*be a sequence of pairs of incomplete matrices defined by*

$$A_i \;=\; \left[\begin{array}{ccccc} \alpha_{i,1} & \cdots & \alpha_{i,r_i} & & \\ \cdots & \cdots & \cdots & \cdots & \\ \alpha_{I,1} & \cdots & \cdots & \cdots & \alpha_{I,r_I} \end{array}\right]_{I\times r_I} \qquad \textit{and} \quad B_i \;=\; \left[\begin{array}{ccccc} \beta_{i,1} & \cdots & \beta_{i,r_i} & & \\ \cdots & \cdots & \cdots & \cdots & \\ \beta_{I,1} & \cdots & \cdots & \cdots & \beta_{I,r_I} \end{array}\right]_{I\times r_I} .$$

*Then $E_{(A,B)}$ admits a tower of fibrations that is compatible with the tower of fibrations of the flag manifold $X = Fl_{r_1,\ldots,r_I}(\mathbb{C}^n)$:*

$$\begin{array}{ccccccccc} E_{(A,B)} = E_{(A_1;B_1)} & \xrightarrow{f_1} & \cdots & \xrightarrow{f_{i-1}} & E_{(A_i;B_i)} & \xrightarrow{f_i} & \cdots & \xrightarrow{f_{I-1}} & E_{(A_I;B_I)} \\ \downarrow g_1 & & \cdots & & \downarrow g_i & & \cdots & & \downarrow g_I \\ X = Fl_{r_1,\ldots,r_I}(\mathbb{C}^n) & \xrightarrow{\underline{f}_1} & \cdots & \xrightarrow{\underline{f}_{i-1}} & Fl_{r_i,\ldots,r_I}(\mathbb{C}^n) & \xrightarrow{\underline{f}_i} & \cdots & \xrightarrow{\underline{f}_{I-1}} & Gr_{r_I}(\mathbb{C}^n)\,. \end{array}$$

*The fiber $F_{i,i+1}$ of $f_i : E_{(A_i;B_i)} \to E_{(A_{i+1};B_{i+1})}$ is the fibered product obtained in Lemma 2.1.10 (3) with $(A;B)$ in Lemma 2.1.10 (3) consisting of the $i$-th and the $(i+1)$-th rows of $(A;B)$ in Item (1) above. We shall denote $E_{A_i;B_i}$ by $E^{(i)}_{(A;B)}$.*

(3) *In particular, $E_{(A,B)}$ is smooth for all $(A;B)$ in Item (1) above.*

*Remark 2.2.2* [*tower of fibrations*]. The tower of fibrations in Proposition 2.2.1, Item (2), is exactly the one induced from the tower of trivial fibrations

$$\prod_{i=1}^{I} Quot_{P_i}(\mathcal{E}^n) \longrightarrow \prod_{i=2}^{I} Quot_{P_i}(\mathcal{E}^n) \longrightarrow \cdots \longrightarrow Quot_{P_I}(\mathcal{E}^n)$$

and the inclusion

$$HQuot_P(\mathcal{E}^n) \hookrightarrow \prod_{i=1}^{I} Quot_{P_i}(\mathcal{E}^n)\,.$$

# 3 An exact computation of $\int_X \tau^* e^{H\cdot t} \cap \mathbf{1}_d$ from the mirror principle diagram.

With the $S^1$-fixed-points locus in $HQuot_P(\mathcal{E}^n)$ understood in Sec. 2, we proceed now to compute the fundamental hypergeometric series $HG[\mathbf{1}]^X(t)$ reviewed in Sec. 1. There are many technical details involved in the process and we study them in Sec. 3.1 - Sec. 3.5, following the logic order toward an exact expression in Sec. 3.6.

## 3.1 The extended Mirror Principle diagram and the distinguished $S^1$-fixed-point components in the hyper-Quot-scheme $\mathcal{Q}_d$.

To make the comparison immediate, here we follow the notations in [L-L-Y1: III, Sec. 5.4]. Recall the following approach ibidem to compute $A(t)$ when there is a commutative diagram:

$$\begin{array}{ccccc} F_0 & \xrightarrow{e^Y} & Y_0 & \xleftarrow{g} & E_0 \\ \downarrow i & & \downarrow j & & \downarrow k \\ M_d & \xrightarrow{\varphi} & W_d & \xleftarrow{\psi} & \mathcal{Q}_d \end{array} ,$$

where $\mathcal{Q}_d$ is an $S^1$-manifold, $\psi : \mathcal{Q}_d \to W_d$ is an $S^1$-equivariant resolution of singularities of $\varphi(M_d)$, $E_0$ is the set of fixed-points in $\psi^{-1}(Y_0)$ and is called the distinguished $S^1$-fixed-point locus, and $\varphi_*[M_d] = \psi_*[\mathcal{Q}_d]$ in $A_*^{S^1}(W_d)$ .

In the case that $X$ is the Grassmannian manifold $Gr_r(\mathbb{C}^n)$, $\mathcal{Q}_d$ is the Quot-scheme $Quot_{P(t)=(n-r)t+(d+n-r)}(\mathcal{E}^n)$, and the linearized moduli space $W_d$ for $X$ is the projective space $\mathbb{P}(H^0(C, \mathcal{O}_C(d)) \otimes \Lambda^r\mathbb{C}^n)$ of $\binom{n}{r}$-tuple of degree-$d$ homogeneous polynomials on $C$. This is a linearized moduli space for $\mathbb{P}(\Lambda^r\mathbb{C}^n)$ that is turned into a linearized moduli space for $X$ via the Plücker embedding $Gr_r(\mathbb{C}^n) \to \mathbb{P}(\Lambda^r\mathbb{C}^n)$. An element $[\mathcal{E}^n \to \mathcal{E}^n/\mathcal{V}]$ in $\mathcal{Q}_d$ can be represented by an $n \times r$-matrix $A_{\mathcal{V}}$ of homogeneous polynomials in $z_0$, $z_1$ of degree $d$. The map $\psi : Quot_P(\mathcal{E}^n) \to W_d = \mathbb{P}(H^0(\mathbb{C}\mathrm{P}^1, \mathcal{O}(d)) \otimes \Lambda^r\mathbb{C}^n)$ is given by taking the $\binom{n}{r}$-tuple of $r \times r$-minors of $A_{\mathcal{V}}$. From this we deduce that the distinguished $S^1$-fixed-point components are exactly those labelled by admissible $(\alpha_\bullet\,;\,0_\bullet)$. (Cf. See [L-L-L-Y: Sec. 3.1] for more details and some related references.)

In the current case, $X$ is the flag manifold $Fl_{r_1,\ldots,r_I}(\mathbb{C}^n)$ and the following two natural embeddings

$$\iota_1 \,:\, Fl_{r_1,\ldots,r_I}(\mathbb{C}^n) \;\longrightarrow\; Gr_{r_1}(\mathbb{C}^n) \,\times\, \cdots \,\times\, Gr_{r_I}(\mathbb{C}^n)$$

and

$$\iota_2 \,:\, HQuot_P(\mathcal{E}^n) \;\longrightarrow\; Quot_{P_1}(\mathcal{E}^n) \,\times\, \cdots \,\times\, Quot_{P_I}(\mathcal{E}^n)$$

give rise to the following choices of spaces and morphisms for the diagram at the beginning of the subsection: (To save burden of notations, the projection map of a product to its $i$-th component will be denoted by $\mathrm{pr}_i$, regardless of which space is in question.)

(1) The embedding

$$\tau \,=\, (\tau_1 \circ \mathrm{pr}_1 \circ \iota_1,\, \ldots,\, \tau_I \circ \mathrm{pr}_I \circ \iota_1) \,:\, X \,=\, Fl_{r_1,\ldots,r_I}(\mathbb{C}^n) \;\longrightarrow\; Y \,=\, \prod_{i=1}^{I} \mathbb{C}\mathrm{P}^{\binom{n}{r_i}-1}$$

where $\tau_i : Gr_{r_i}(\mathbb{C}^n) \to \mathbb{C}\mathrm{P}^{\binom{n}{r_i}-1}$ is the Plücker embedding.

(2) $\mathcal{Q}_d$ is the hyper-Quot-scheme $HQuot_P(\mathcal{E}^n)$, where $P = (P_1, \ldots, P_I)$ with $P_i = P_i(t) = (n - r_i)t + d_i + (n - r_i)$.

(3) The linearized moduli space $W_d$ for $X$ is the product of projective spaces

$$W_d \,=\, \prod_{i=1}^{I} W_{d_i} \,=\, \prod_{i=1}^{I} \mathbb{P}(H^0(C,\, \mathcal{O}_C(d_i)) \otimes \Lambda^{r_i}\mathbb{C}^n)$$

of $\binom{n}{r_i}$-tuple of degree-$d_i$ homogeneous polynomials on $C$. This is a linearized moduli space for $\prod_{i=1}^{I} \mathbb{P}(\Lambda^{r_i}\mathbb{C}^n)$ that is turned into a linearized moduli space for $X$ via $\tau : X \to Y$.

(4) The collapsing morphism $\varphi = (\varphi_1 \circ \iota_3, \ldots, \varphi_I \circ \iota_3)$, where

$$\iota_3 : M_d(X) \hookrightarrow M_{d_1}(Gr_{r_1}(\mathbb{C})) \times \cdots \times M_{d_I}(Gr_{r_I}(\mathbb{C}^n))$$

is the embedding induced by $\iota_1$ and

$$\varphi_i : M_d(Gr_{r_i}(\mathbb{C}^n)) \longrightarrow \mathbb{P}(H^0(C, \mathcal{O}_C(d_i)) \otimes \Lambda^{r_i}\mathbb{C}^n)$$

is the collapsing morphism when $X = Gr_{r_i}(\mathbb{C}^n)$.

(5) The morphism $\psi = (\psi_1 \circ \mathrm{pr}_1 \circ \iota_2, \ldots, \psi_I \circ \mathrm{pr}_I \circ \iota_2)$, where

$$\psi_i : Quot_{P_i}(\mathcal{E}^n) \longrightarrow \mathbb{P}(H^0(C, \mathcal{O}_C(d_i)) \otimes \Lambda^{r_i}\mathbb{C}^n)$$

is the map $\psi$ when $X$ is $Gr_{r_i}(\mathbb{C}^n)$ and the degree of curves in question is $d_i$, as reviewed in the beginning of this subsection.

**Lemma 3.1.1 [identical image].** *$\varphi(M_d) = \psi(\mathcal{Q}_d)$ in $W_d$ and $\psi : \mathcal{Q}_d \to \varphi(M_d)$ is a resolution of singularities of $\varphi(M_d)$.*

*Proof.* On the stable map side, $\mathbb{C}\mathrm{P}^1 \times Fl_{r_1,\ldots,r_I}(\mathbb{C}^n)$ is nonsingular, projective, and convex; thus, $M_d$ contains the space $W_d^0 := Mor_{(1,d)}(\mathbb{C}\mathrm{P}^1, \mathbb{C}\mathrm{P}^1 \times Fl_{r_1,\ldots,r_I}(\mathbb{C}^n))$ of morphisms from $\mathbb{C}\mathrm{P}^1$ to $\mathbb{C}\mathrm{P}^1 \times Fl_{r_1,\ldots,r_I}(\mathbb{C}^n)$ of degree $(1,d)$ as an open dense subset (e.g. [F-P]). On the coherent sheaf side, $\mathcal{Q}_d$ contains an open dense subset $\mathcal{Q}_d^0$ that consists of sequences of vector bundle inclusions $\mathcal{V}_1 \hookrightarrow \cdots \hookrightarrow \mathcal{V}_I \hookrightarrow \mathcal{E}^n$ (e.g. [CF1]). Consequently, we only need to show that $\varphi(M_d^0) = \psi(\mathcal{Q}_d^0)$ in $W_d$. Furthermore, since $M_d^0$ is a smooth Deligne-Mumford stack and $W_d^0$ is a smooth scheme, we only need to check the above identity at the level of atlas variety and, hence, only on the related sets of closed points.

Let $Fl = Fl_{r_1,\ldots,r_I}(\mathbb{C}^n)$ and $S_1 \hookrightarrow \cdots \hookrightarrow S_I \hookrightarrow \mathcal{O}_{Fl} \otimes \mathbb{C}^n$ be the tautological subbundles on $Fl$. The map $\varphi : M_d^0 \to W_d$ can be described as follows: The dual vector bundle morphisms: $\mathcal{O}_{Fl} \otimes \mathbb{C}^n \to S_i^\vee$ induces a morphism $\mathcal{O}_{Fl} \times \Lambda^{r_i}(\mathbb{C}^n) \to Det\, S_i^\vee$ produce the Plücker embedding of $Gr_{r_i}(\mathbb{C}^n)$ in $\mathbb{C}\mathrm{P}^{\binom{n}{r_i}-1}$. The universal $\Delta$-collection of $\mathbb{C}\mathrm{P}^{\binom{n}{r_i}-1}$ as a toric variety ([Cox]) pulls back to a $\Delta$-collection on $Gr_{r_i}(\mathbb{C}^n)$ and then on $Fl_{r_1,\ldots,r_I}(\mathbb{C}^n)$. This $\Delta$-collection is given exactly by the morphism $\mathcal{O}_{Fl} \otimes \Lambda^{r_i}\mathbb{C}^n$ above. (Here, the comparison data $\{c_m\}_m$ in [Cox] of different line bunles in a $\Delta$-collection is implicit by consider only $Det\, S_i^\vee$ instead of $\binom{n}{r_i}$ isomorphic copies of it.) Given a morphism $f : \mathbb{C}\mathrm{P}^1 \to \mathbb{C}\mathrm{P}^1 \times Fl_{r_1,\ldots,r_I}(\mathbb{C}^n)$ of degree $(1,d)$, one has then a pull-back $\Delta$-collection on $\mathbb{C}\mathrm{P}^1$ with $\Delta$ the fan associated to the toric manifold $\mathbb{C}\mathrm{P}^1 \times \mathbb{C}\mathrm{P}^{\binom{n}{r_1}-1} \times \cdots \times \mathbb{C}\mathrm{P}^{\binom{n}{r_I}-1}$. The morphism $\varphi$ is then the collapsing morphism, as constructed in [L-L-Y1, II] for toric variety and elaborated in [L(CH)-L-Y], that relates maps to a toric variety to the linearized moduli space $W_d$ via $\Delta$-collections of [Cox].

On the other hand, the map $\psi : \mathcal{Q}_d^0 \to W_d$ can be described as follows. Fix a presentation $\mathbb{C}\mathrm{P}^1 = \mathrm{Proj}\,\mathbb{C}[z_0, z_1]$ and identify $\mathcal{O}_{Fl} \otimes \mathbb{C}^n$ with $\mathbb{C}[z_0, z_1]^{\oplus n}$. Then an element

$[\mathcal{V}_\bullet] \in \mathcal{Q}_d$ is identified with an $I$-tuple of $\mathbb{C}[z_0, z_1]$-valued matrices $(A_{\mathcal{V}_1}, \cdots A_{\mathcal{V}_I})$ as reviewed in the beginning of this subsection. Taking the minors of these matrices with appropriate order gives the map $\psi : \mathcal{Q}_d \to W_d$.

Now, the matrices $A_{\mathcal{V}_i}$ that appear in the definition of $\psi$ corresponds to the inclusion $\mathcal{V}_i \hookrightarrow \mathcal{O} \otimes \mathbb{C}^n$ on $\mathbb{CP}^1$. For $[\mathcal{V}_\bullet] \in \mathcal{Q}_d^0$, taking minors corresponds to a morphism $Det\mathcal{V}_i \to \mathcal{O} \otimes \Lambda^{r_i}\mathbb{C}^n$ on $\mathbb{CP}^1$. The dual of these, $\mathcal{O} \otimes \Lambda^{r_i}\mathbb{C}^n \to Det\mathcal{V}_i^\vee$, is associated to taking the minors of the transpose $A_{\mathcal{V}_i}^T$ of $A_{\mathcal{V}_i}$ in the corresponding order. They constitute a $\Delta$-collection on $\mathbb{CP}^1$ that corresponds to a map $\mathbb{CP}^1 \to \mathbb{CP}^{\binom{n}{r_1}-1} \times \cdots \mathbb{CP}^{\binom{n}{r_I}-1}$ that factors through the morphism $f_{\mathcal{V}_\bullet} : \mathbb{CP}^1 \to Fl$ associated to $[\mathcal{V}_\bullet]$. It follows from the key steps and ingredients reviewed above in the construction of $\varphi$ and $\psi$ that $\varphi(f_{\mathcal{V}_\bullet}) = \psi([\mathcal{V}_\bullet])$ in $W_d$ since the minors of $A_{\mathcal{V}_\bullet}$ and $A_{\mathcal{V}_\bullet}^T$ give the same tuple when taken in a consistent respective order. Conversely, an $f \in M_d^0$ induces a $[\mathcal{V}_\bullet^f] \in \mathcal{Q}_d^0$ by pulling back the tautological subbundles $S_i$ on $Fl$ and one can check that $\varphi(f) = \psi([\mathcal{V}_\bullet^f])$ in $W_d$ as well.

Since the correspondence $M_d^0 \to \mathcal{Q}_d^0$ with $f \mapsto [\mathcal{V}_\bullet^f]$ is surjective, this shows that $\varphi(M_d^0) = \psi(\mathcal{Q}_d^0)$ at the set of closed points of the domain stack/variety of $\varphi$ and $\psi$ respectively and, hence, at the whole domain stack/variety.

Since $\mathcal{Q}_d$ is smooth, to show that $\psi : \mathcal{Q}_d \to \varphi(M_d)$ is a resolution of singularities of $\varphi(M_d)$, one only have to show that the morphism $\psi : \mathcal{Q}_d \to \varphi(M_d)$ is birational. But this follows from the details of $\varphi$ and $\psi$ reviewed above that both $\mathcal{Q}_d$ and $\varphi(M_d)$ contains a copy of $Mor(\mathbb{CP}^1, Fl_{r_1,\ldots r_I}(\mathbb{C}^n))$ as an open dense subset and that the restriction of $\psi$ to this subset is the identity map.

This concludes the proof.

□

Consequently, Lemma 5.5 of [L-L-Y1, III] holds and one can compute the fundamental hypergeometric series $HG[\mathbf{1}]^X(t)$ for $X = Fl_{r_1,\ldots,r_I}(\mathbb{C}^n)$ via the localization method on the hyper-Quot-scheme $\mathcal{Q}_d$ rather than on $W_d$ directly.

The first task now is to identify the distinguished $S^1$-fixed-point components $E_0$ in $\mathcal{Q}_d$ that appear in the extended Mirror Principle diagram.. Since the linearized moduli space $W_d$ is a product of $W_{d_i}$, its $S^1$-fixed-point components $Y_u$ must also come from products of $S^1$-fixed-point components in $W_d$:

$$Y_u \;=\; Y_{u_1} \times \cdots \times Y_{u_I}\,,$$

where $u = (u_1, \ldots, u_I)$, $0 \le u_i \le d_i$, and $Y_{u_i}$ is the $S^1$-fixed-point component in $W_{d_i}$ labelled by $u_i$, as given in [L-L-L-Y: Sec. 3.1]. The $Y_0$ of the Mirror Principle diagram, given in the beginning of this subsection corresponds to $u = (0, \ldots, 0)$.

**Lemma 3.1.2 [image of $E_{(A;B)}$].** $\psi(E_{(A;B)}) \subset Y_u$ *with* $u_i = \beta_{i,1} + \cdots + \beta_{i,r_i}$.

*Proof.* Under the embedding

$$\iota_2 \;:\; HQuot_P(\mathcal{E}^n) \;\longrightarrow\; Quot_{P_1}(\mathcal{E}^n) \times \cdots \times Quot_{P_I}(\mathcal{E}^n)\,,$$

$E_{(A;B)}$ (is the unique $S^1$-fixed-point component in $HQuot_P(\mathcal{E}^n)$ that) goes into the $S^1$-fixed-point component $E_{(\alpha_{1,\bullet}\,;\,\beta_{1,\bullet})}\times\cdots\times E_{(\alpha_{I,\bullet}\,;\,\beta_{I,\bullet})}$ in $Quot_{P_1}(\mathcal{E}^n)\times\,\cdots\,\times\, Quot_{P_I}(\mathcal{E}^n)$. From [L-L-L-Y: Lemma 3.1.1], $\psi_i(Quot_{P_i}(\mathcal{E}^n))\subset Y_{\beta_{i,1}+\,\cdots\,+\beta_{i,r_i}}$. This implies the lemma.

□

**Corollary 3.1.3 [distinguished components].** *The distinguished $S^1$-fixed-point locus $E_0$ in the Mirror Principle diagram is given by*

$$E_0 \;=\; \coprod_A E_{(A;0)}\,,$$

*where $A$ runs over all the incomplete matrices $(\alpha_{i,j})_{i,j}$ with the $i$-th row $(\alpha_{i,\bullet})$ being a partition $0\le\alpha_{i,1}\le\;\cdots\;\le\alpha_{i,r_i}$ of $d_i$ into non-negative integers of length $r_i$ and $\alpha_{i,j}\ge\alpha_{i+1,j}$ for $j=1,\,\ldots,\,r_i$. The set of such $A$ is always non-empty.* (*Cf.* Figure 3-1-1.)

| | | |
|---|---|---|
| 0 | 2 | |
| 0 | 0 | 6 |

| | | |
|---|---|---|
| 0 | 2 | |
| 0 | 1 | 5 |

| | | |
|---|---|---|
| 0 | 2 | |
| 0 | 2 | 4 |

| | | |
|---|---|---|
| 1 | 1 | |
| 0 | 0 | 6 |

| | | |
|---|---|---|
| 1 | 1 | |
| 0 | 1 | 5 |

| | | |
|---|---|---|
| 1 | 1 | |
| 1 | 1 | 4 |

Figure 3-1-1. The set of distinguished $S^1$-fixed-point component $E_{(A;0)}$ in $HQuot_P(\mathcal{E}^n)$, with $P=(P_1,\,\ldots,\,P_I)$, $P_i=P_i(t)=(n-r_i)t+d_i+(n-r_i)$ is the same as the set of Young tableaux whose entries satisfy some monotonous conditions both horizontally and vertically. Illustrated here is such a set for $Fl_{2,3}(\mathbb{C}^n)$ with $g=0$ stable maps of multiple degree $(d_1,d_2)=(2,6)$ in consideration.

We will come back in Sec. 3.4 to work out the hyperplane-induced classes on $E_{(A;0)}$ from the map $\psi$ after understanding $E_{(A;0)}$ better.

## 3.2 Bundles with fiber restrictive flag manifolds and the class $\Omega(\mathcal{P}_\bullet)$.

The discussion in the previous subsection together with Proposition 2.2.1 implies that a distinguished $S^1$-fixed-point component $E_{(A;0)}$ in $HQuot_P(\mathcal{E}^n)$ admits a tower of fibrations by restrictive flag manifolds. For this reason and quantities that will be needed in later subsections, we digress in this subsection to take a look at bundles with fiber restrictive flag manifolds and their associated bundle with fiber ordinary flag manifolds.

Let $\mathcal{E}$ be a vector bundle of rank $n$ on a smooth variety $Y$, $k_\bullet : 1\le k_1<\,\cdots\,<k_s<n$ with $k_i$ integers, and $\mathcal{P}_\bullet : \mathcal{P}_1\hookrightarrow\;\cdots\;\hookrightarrow\mathcal{P}_s\hookrightarrow\mathcal{E}$ be a (not necessarily strict) inclusion sequence of vector subbundles of $\mathcal{E}$. Let $l_i$ be the rank of $\mathcal{P}_i$. Then associated to the triple $(\mathcal{E},k_\bullet,\mathcal{P}_\bullet)$ is a bundle $g:W\to Y$ over $Y$, whose fiber $W_y$ over $y\in Y$ is the restrictive flag manifold $Fl_{k_1,\ldots,k_s}(\mathcal{E}_y,\mathcal{P}_{\bullet,y})$, where $(\,\cdot\,)_y$ denotes the fiber of $(\,\cdot\,)$ at $y$. By construction,

$W \to Y$ is naturally a subbundle of a flag manifold bundle $g' : W' \to Y$, whose fiber $W'_y$ at $y$ is the flag manifold $Fl_{k_1,\ldots,k_s}(\mathcal{E}_y)$. Denote the tautological inclusion $W \hookrightarrow W'$ over $Y$ by $\iota$. By construction, $g = g' \circ \iota$. Let $\mathcal{S}_j$ (resp. $\mathcal{S}'_j$), $j = 1, \ldots, s$, be the tautological bundles on $W$ (resp. $W'$), whose fiber at $w \in g^{-1}(y)$ (resp. $w' \in {g'}^{-1}(y)$) is the $j$-th element of the flag $w \in Fl_{k_1,\ldots,k_s}(\mathcal{E}_y, \mathcal{P}_{\bullet,y})$ (resp. $w' \in Fl_{k_1,\ldots,k_s}(\mathcal{E}_y)$). Note that $\iota^*\mathcal{S}'_\bullet = \mathcal{S}_\bullet$.

Define $\Omega(\mathcal{P}_\bullet)$ to be (the Poincaré dual of) $[W]$ in $A_*(W')$. We will discuss below how to express $\Omega(\mathcal{P}_\bullet)$ in terms of the Chern roots of $\mathcal{S}'_j$ and $\mathcal{P}_j$.

**A pedagogic discussion of a basic excess example.**

Consider the sequence of bundle morphisms obtained from composition

$$\varphi_j \;:\; \mathcal{S}'_j \;\hookrightarrow\; g'^*\mathcal{E} \;\longrightarrow\; g'^*\mathcal{E}/g'^*\mathcal{P}_j$$

of $\mathcal{O}_{W'}$-modules. Then $Ker\,\varphi_j$ is an $\mathcal{O}_{W'}$-module, which is not locally free in general. The reduced scheme associated to the intersection $Z := \cap_{j=1}^s Z_j$ of the minimal stratum $Z_j$ of the flattening stratification of $Ker\,\varphi_j$ is exactly $W$ and $(Ker\,\varphi_j)|_W = \iota^*\mathcal{S}'_j$.

**Lemma 3.2.1** [$Z = W$ **as schemes**]. *$Z$ is reduced and hence $Z = W$ as schemes.*

*Proof.* Since $Z_j$ is the minimal stratum of the flattening stratification of $Ker\,\varphi_j$, it is a closed subscheme of $W'$ defined by the Fitting ideal sheaf $\mathcal{I}_j$ whose local sections are generated by the set of all entries of any local presentation of the $\mathcal{O}_{W'}$-module moprhism $\varphi_j$. Since the problem is local, we may assume that $Y = Spec\,R$, where $R$ is a local ring with the residue field extending $\mathbb{C}$, and that $\mathcal{E}$ and $\mathcal{P}_\bullet$ are free $R$-modules with $\mathcal{E} = \oplus R$ and $\mathcal{P}_j$ being the direct sum of the first $l_j$ direct summands in the decomposition of $\mathcal{E}$. Under such specification, the quotients $\mathcal{E}/\mathcal{P}_j$ are realized as the direct sum of the the last $(n - l_j)$ direct summand in the decomposition of $\mathcal{E}$.

Given a point $w'$ in $W'$, after a $GL_n(R)$-transformation $M = (r_{\mu\nu})_{n\times n}$ one can represent $w'$ by the $n \times k_s$ matrix $[w']$ obtained from enlarging the $k_s \times k_s$ diagonal matrix $Diag[\,I_{k_1}, I_{k_2-k_1}, \cdots, I_{k_s-k_{s-1}}\,]$, where $I_\bullet$ is the $\bullet\times\bullet$ identity matrix, to an $n \times k_s$ matrix by adding zero entries.

In terms of this presentation, $w'$ is contained in an affine chart

$$U \;:=\; Spec\,R\,[x_{\mu\nu} \;:\; 1 \le \nu \le k_s,\, k_j < \mu \le n \;\text{ if } k_{j-1} < \nu \le k_j,\, j = 1, \ldots, s\,]\,,$$

where $x_{\mu\nu}$ are indeterminants that appear as the entries of the block lower triangular matrix determined by $[w']$, as indicated in FIGURE 3-2-1. Let $\Xi$ be the $n \times k_s$ matrix $([w']_{\mu\nu}) + (x_{\mu\nu})$. Then, over $U$, $\mathcal{S}'_j$ is generated by the column vectors of $\Xi$, $g'^*\mathcal{E}/g'^*\mathcal{P}_j$ is generated by the column vectors of $M^{(j)}$, the lower $(n - l_j) \times n$ part of the matrix $M$, and $\varphi_j$ are represented by the matrix, (cf. the projection to the quotient $g'^*\mathcal{E}/g'^*\mathcal{P}_j$)

$$(M\,\Xi)^{(j)}\,,$$

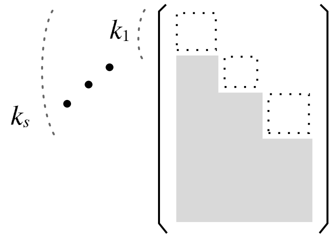


FIGURE 3-2-1. The lower-triangular matrices determined by $[w']$.

the lower $(n-l_j)\times k_s$ part of the $n\times k_s$ matrix $M\Xi$. Thus on $U$, the ideal sheaf of $Z$ is generated by $k_1(n-l_1)+k_2(n-l_2)+\cdots+k_s(n-l_s)$-many degree-1 elements in the polynomial ring

$$R\,[\,x_{\mu\nu}\;:\;1\le\nu\le k_s,\,k_j<\mu\le n\ \text{ if } k_{j-1}<\nu\le k_j,\,j=1,\,\ldots,\,s\,]$$

over $R$. Since the intersection of linear subvarieties is always smooth, this implies that $Z\cap U$ is smooth over $Y$. Since $w'$ is arbitrary, this shows that $Z$ is smooth over $Y$; in particular, it is reduced. Consequently, $Z=W$ as schemes.

□

In this way, $W$ is realized as the degeneracy locus of the bundle morphism

$$\phi\;:=\;(\varphi_1,\,\cdots,\,\varphi_s)\;:\;\mathcal{S}'\;:=\;\oplus_{j=1}^{s}\mathcal{S}'_j\;\longrightarrow\;\mathcal{P}'\;:=\;\oplus_{j=1}^{s}g'^*\mathcal{E}/g'^*\mathcal{P}_j$$

on $W'$, over which the rank of $\phi$ is 0.

Let us perform some dimension count: if $\phi$ were generic while sending $\mathcal{S}'_j$ into $g'^*\mathcal{E}/g'^*\mathcal{P}_j$, its minimal degeneracy locus has codimension in $W'$ equal to

$$k_1(n-l_1)+k_2(n-l_2)+\;\cdots\;+k_s(n-l_s);$$

on the other hand the codimension of $W$ in $W'$ is the same as the codimension of $Fl_{k_1,\ldots,k_s}(\mathbb{C}^n,\Pi_\bullet)$ in $Fl_{k_1,\ldots,k_s}(\mathbb{C}^n)$, which is

$$k_1(n-l_1)+(k_2-k_1)(n-l_2)+\;\cdots\;+(k_s-k_{s-1})(n-l_s)\,.$$

Thus we see that $W$ is an excess degeneracy locus of $\phi$ and we cannot apply the Thom-Porteous formula directly to represent $[W]$ in $W'$ in terms of Chern classes of $\mathcal{S}'$ and $\mathcal{P}'$. We now discuss how to remedy this.

**Removal of excess via nesting restrictions.**

Consider the following sequence of morphisms

$$\phi^{(j)}:=(\varphi_1,\,\cdots,\,\varphi_j)\;:\;\mathcal{S}'^{,(j)}\;:=\;\oplus_{j'=1}^{j}\mathcal{S}'_{j'}\;\longrightarrow\;\mathcal{P}'^{,(j)}\;:=\;\oplus_{j'=1}^{j}g'^*\mathcal{E}/g'^*\mathcal{P}_{j'}$$

and the associated sequence of minimal degeneracy locus $W_{(j)}$. Then

$$W' =: W_{(0)} \supset W_{(1)} \supset W_{(2)} \supset \cdots \supset W_{(s)} = W$$

and, similar to the previous discussion, all the $W_{(j)}$ are smooth; indeed they are all restrictive flag manifold bundle over $Y$. Observe that the codimension of $W_{(j)}$ in $W_{(j-1)}$ is $(k_j - k_{j-1})(n - l_j)$.

Consider the morphism

$$\overline{\varphi}_j \;:\; \mathcal{S}'_j/\mathcal{S}'_{j-1} \longrightarrow g'^*\mathcal{E}/g'^*\mathcal{P}_j \quad \text{on} \quad W_{(j-1)}$$

induced from the restriction of $\varphi_j$ to related sheaves over $W_{(j-1)}$. Though not well-defined on any bigger domain, $\overline{\varphi}_j$ is well-defined on $W_{(j-1)}$ since $\varphi_j$ maps $\mathcal{S}'_{j-1}$ into $g'^*\mathcal{P}_{j-1} \subset g'^*\mathcal{P}_j$ when restricted to $W_{(j-1)}$. The minimal degeneracy locus of $\overline{\varphi}_j$ on $W_{(j-1)}$ is exactly $W_{(j)}$. Since the codimension of $W_{(j)}$ in $W_{(j-1)}$ is the same as $\mathit{rank}\,(\mathcal{S}'_j/\mathcal{S}'_{j-1}) \cdot \mathit{rank}\,(g'^*\mathcal{E}/g'^*\mathcal{P}_j)$, $W_{(j)}$ is now a proper degeneracy locus of $\overline{\varphi}_j$, cf. FIGURE 3-2-2. Thus, the Thom-Porteous

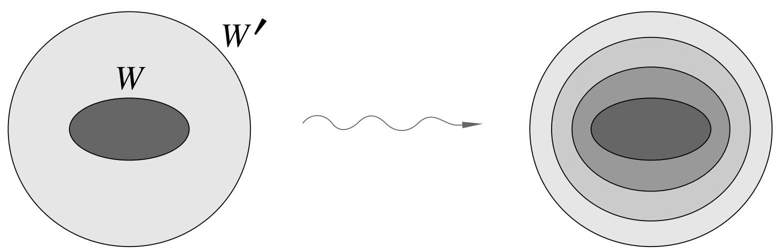


FIGURE 3-2-2. A restrictive flag manifold bundle $W$ is not directly realizable as a critical degeneracy locus of a bundle morphism on the associated flag manifold bundle $W'$. The nesting construction removes the excess degeneracy.

formula applies and one can express (the Poincaré dual $\Omega_{(j)}$ of) $[W_{(j)}]$ in $A_*(W_{(j-1)})$ in terms of the Chern classes of $\mathcal{S}'_j/\mathcal{S}'_{j-1}$ and $g'^*\mathcal{E}/g'^*\mathcal{P}_j$ and, hence, in terms of Chern roots of $\mathcal{S}'_j/\mathcal{S}'_{j-1}$ and $\mathcal{E}/\mathcal{P}_j$ via determinantal identities. (Cf. [Fu1].)

Let

$$\begin{aligned} c(\mathcal{S}'_j/\mathcal{S}'_{j-1})(t) &= \textstyle\prod_{j''=1}^{k_j-k_{j-1}}(1+y_{j;j''}t) \qquad \text{and} \\ c(\mathcal{E}/\mathcal{P}_j)(t) &= \textstyle\prod_{j'=1}^{n-l_j}(1+q_{j;j'}t) \end{aligned}$$

be the Chern polynomials of bundles involved in terms of their Chern roots. It follows from [Fu1: Chapter 14 and Lemma A.9.1] that

$$\Omega_{(j)} = \Delta_{n-l_j}^{(k_j-k_{j-1})}\left(\frac{c\left(g'^*\mathcal{E}/g'^*\mathcal{P}_j\right)}{c\left(\mathcal{S}'_j/\mathcal{S}'_{j-1}\right)}\right) = \Delta_{n-l_j}^{(k_j-k_{j-1})}\left(\frac{\prod_{j'=1}^{n-l_j}(1+q_{j;j'})}{\prod_{j''=1}^{k_j-k_{j-1}}(1+y_{j;j''})}\right)$$

$$= \prod_{j'=1}^{n-l_j} \prod_{j''=1}^{k_j-k_{j-1}} (q_{j;j'} - y_{j;j''})\,.$$

Recall now the natural morphism

$$\cdot\,\Omega_{(j)} \;:\; A^*(W_{(j)}) \;\longrightarrow\; A^{*+(k_j-k_{j-1})(n-l_j)}(W_{(j-1)})$$

dual to the intersection product

$$\cdot\,[W_{(j)}] \;:\; A_*(W_{(j-1)}) \;\longrightarrow\; A_{*-(k_j-k_{j-1})(n-l_j)}(W_{(j)})\,,$$

for $j = 1, \ldots, s$. Thus, the Poincaré dual of $[W_{(j)}]$ in $A^*(W_{(j-2)})$ is given by $\Omega_{(j)} \cdot \Omega_{(j-1)}$. Iterating this procedure, one concludes the following lemma:

**Lemma 3.2.2 [the class $\Omega(\mathcal{P}_\bullet)$].** *(The Poincaré dual of) $[W]$ in $W'$ is given by*

$$\Omega(\mathcal{P}_\bullet) \;=\; \Omega_{(s)} \cdot \Omega_{(s-1)} \cdot \cdots \cdot \Omega_{(1)} \;=\; \prod_{j=1}^{s} \prod_{j'=1}^{n-l_j} \prod_{j''=1}^{k_j-k_{j-1}} (q_{j;j'} - y_{j;j''})$$

*in terms of the Chern roots of $\mathcal{S}'_j/\mathcal{S}'_{j-1}$ and $\mathcal{E}/\mathcal{P}_j$, $j = 1, \ldots, s$..*

*Remark 3.2.3 [rational presentation of $\Omega$].* Let $c(\mathcal{E})(t) = \prod_{j'=1}^{n}(1 + e_{j'}t)$ and $c(\mathcal{P}_j)(t) = \prod_{j'=1}^{l_j}(1 + p_{j;j'}t)$ be the Chern polynomials of bundles involved in terms of their Chern roots. When the ratio makes sense, a rational presentation of $\Omega_{(j)}$ (resp. $\Omega(\mathcal{P}_\bullet)$) is given by

$$\Omega_{(j)} \;=\; \prod_{j''=1}^{k_j-k_{j-1}} \frac{\prod_{j'=1}^{n}(e_{j'} - y_{j;j''})}{\prod_{j'=1}^{l_j}(p_{j;j'} - y_{j;j''})} \quad \text{(resp. } \;\Omega(\mathcal{P}_\bullet) \;=\; \prod_{j=1}^{s} \prod_{j''=1}^{k_j-k_{j-1}} \frac{\prod_{j'=1}^{n}(e_{j'} - y_{j;j''})}{\prod_{j'=1}^{l_j}(p_{j;j'} - y_{j;j''})}\,).$$

## 3.3 Tautological sheaves on $E_{(A;0)}$ and $E_{(A;0)} \times \mathbb{C}\mathrm{P}^1$.

**Tautological sheaves and filtrations on $E_{(A;0)}$.**

Recall from Proposition 2.2.1 the tower of fibrations of $E_{(A;0)}$

$$E_{(A;0)} = E^{(1)}_{(A;0)} \xrightarrow{f_1} \cdots \xrightarrow{f_{i-1}} E^{(i)}_{(A;0)} \xrightarrow{f_i} \cdots \xrightarrow{f_{I-1}} E^{(I)}_{(A;0)} = Fl_{m_{I,\bullet}}(\mathbb{C}^n)$$

with the fiber $F^{(i)}$ of $f_i$ being the restrictive flag manifold $Fl_{m_{i,\bullet}}(\mathbb{C}^n, \Pi_{i+1,\bullet})$. The two systems of tautological vector bundles on $Fl_{m_{i,\bullet}}(\mathbb{C}^n, \Pi_{i+1,\bullet})$, one comes from the restriction of the sequence of tautological subbundles on $Fl_{m_{i.\bullet}}(\mathbb{C}^n)$ and the other from the data $\Pi_{i+1,\bullet}$, gives rise to two inclusion sequences of locally free sheaves $\mathcal{S}_{i,\bullet}$ and $f_i^*\mathcal{P}_{i+1,\bullet}$ on

$E^{(i)}_{(A;0)}$ with $\mathcal{S}_{i,j} \hookrightarrow f_i^*\mathcal{P}_{i+1,j}$, $j = 1, \ldots, K_i$. From the discussion in Sec. 2.1, each $\mathcal{P}_{i+1,j}$ is an $\mathcal{S}_{i+1,j'}$ for some $\mathcal{S}_{i+1,j'}$ on $E^{(i+1)}_{(A;0)}$. The pull-back of $\mathcal{S}_{i,\bullet}$ and $f_i^*\mathcal{P}_{i+1,\bullet}$ on $E^{(i)}_{(A;0)}$ to the whole $E_{(A;0)}$ will be denoted by $\widehat{\mathcal{S}}_{i,\bullet}$ and $\widehat{\mathcal{P}}_{i+1,\bullet}$ respectively. Recall also the tautological embedding of $E_{(A;0)}$ into a product flag manifolds (cf. Sec. 3.1, the discussion before Lemma 3.1.2). It is good to keep in mind that both $\widehat{\mathcal{S}}_{i,j}$ and $\widehat{\mathcal{P}}_{i+1,j}$ are the restriction to $E_{(A;0)}$ of tautological bundles on this product.

In terms of Sec. 3.2, given $E^{(i+1)}_{(A;0)}$ with the tautological subbundles $\mathcal{S}_{i+1,\bullet}$, the smooth bundle map $f_i : E^{(i)}_{(A;0)} \to E^{(i+1)}_{(A;0)}$ with fiber restrictive flag manifolds can be constructed as in that subsection from a sequence of integers $r_{i,1} < \cdots < r_{i,K_i} = r_i$ determined by $(\alpha_{i,1}, \cdots, \alpha_{i,r_i})$ and a sequence of subbundles $\mathcal{P}_{i+1,j} = \mathcal{S}_{i+1,j'}$ on $E^{(i+1)}_{(A;0)}$ with $j'$ determined by the submatrix from $A$:

$$\left[\begin{array}{lllll} \alpha_{i,1} & \cdots & \alpha_{i,r_i} & & \\ \alpha_{i+1,1} & \cdots & \alpha_{i+1,r_i} & \cdots & \alpha_{i+1,r_{i+1}} \end{array}\right]_{2\times r_{r+1}} .$$

For later use, denote this $j'$ as the value $I_A(i,j)$ of a function, denoted by $I_A$, on index pairs $(i,j)$.

**Definition 3.3.1 [index function asociated to $A$].** $I_A$ will be called the (*first*) *index function* associated to the Young tableau $A$.

*Remark 3.3.2 [more index functions].* Later in Sec. 3.5, there will also be the *second* and the *third index function* that appear in the discussion. They are all determined by the Young tableau $A$ and will be denoted by $I'_A$ and $I''_A$.

Introduce also the Chern roots of the components of the associated graded vector bundles on $E_{(A;0)}$:

$$c\left(\widehat{\mathcal{S}}_{i,j}/\widehat{\mathcal{S}}_{i,j-1}\right) = \prod_{k=1}^{m_{i,j}} (1 + y_{i,j;k}) \quad \text{for } i = 1, \ldots, I\,,$$

where $\widehat{\mathcal{S}}_{i,0} := 0$, $\widehat{\mathcal{S}}_{i,K_i+1} := \mathcal{O}_{E_{(A;0)}} \otimes \mathbb{C}^n$, and $m_{i,K_i+1} ;= n - r_i$.

**Tautological sheaves and filtrations on $E_{(A;0)} \times \mathbb{C}\mathrm{P}^1$.**

Let $\pi_1 : E_{(A;0)} \times \mathbb{C}\mathrm{P}^1 \to E_{(A;0)}$ and $\pi_2 : E_{(A;0)} \times \mathbb{C}\mathrm{P}^1 \to \mathbb{C}\mathrm{P}^1$ be the two projection maps and $\mathcal{E} := \pi_2^*\,\mathcal{E}^n$. Then one has the following filtrations by locally free sheaves: (Caution that this is *not* a double filtration.)

$$
\begin{array}{ccccccccccc}
 & & \mathcal{E}_{1,\bullet} & & \cdots & & \mathcal{E}_{i,\bullet} & & \cdots & & \mathcal{E}_{I,\bullet} & & \\
 & & .. & & & & .. & & & & .. & & \\
\mathcal{E}_\bullet\; : & & \mathcal{E}_1 & \hookrightarrow & \cdots & \hookrightarrow & \mathcal{E}_i & \hookrightarrow & \cdots & \hookrightarrow & \mathcal{E}_I & \hookrightarrow & \mathcal{E}\,, \\
 & & \| & & \vdots & & \| & & \vdots & & \| & & \\
 & & \mathcal{E}_{1,K_1} & & \vdots & & \mathcal{E}_{i,K_i} & & \vdots & & \mathcal{E}_{I,K_I} & & \\
 & & \cup & & \vdots & & \cup & & \vdots & & \cup & & \\
 & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \\
 & & \cup & & \vdots & & \cup & & \vdots & & \cup & & \\
 & & \mathcal{E}_{1,1} & & \vdots & & \mathcal{E}_{i,1} & & \vdots & & \mathcal{E}_{I,1} & &
\end{array}
$$

where the horizontal filtration $F_\bullet\mathcal{E} := \mathcal{E}_\bullet$ on the first line comes from the universal filtration of $\mathcal{E}$ on $HQuot_P(\mathcal{E}^n) \times \mathbb{C}\mathrm{P}^1$ while the vertical filtration $F_\bullet\mathcal{E}_i := \mathcal{E}_{i,\bullet}$ of $\mathcal{E}_i$ is determined by the labelled flag $V_{i,\bullet}(s_{i,\bullet})$ in $\mathbb{C}^n$ that characterizes $\mathcal{E}_i$. (In particular, the length of the vertical filtration $F_\bullet\mathcal{E}_i$ of $\mathcal{E}_i$ depends on $i$.)

From the above diagram of various tautological sheaves on $E_{(A;0)} \times \mathbb{C}\mathrm{P}^1$, one has the following two types of associated graded objects on $E_{(A;0)} \times \mathbb{C}\mathrm{P}^1$:

(1) From the horizontal filtration: $\oplus_{i=1}^{I+1}\mathcal{E}_i/\mathcal{E}_{i-1}$.

  (1.1) The quotient $\mathcal{E}_i/\mathcal{E}_{i-1}$ in general is not a direct sum of a locally free and a torsion $\mathcal{O}_{E_{(A;0)}\times\mathbb{C}\mathrm{P}^1}$-module.

  (1.2) For each $i$, there is a natural stratification of $E_{(A;0)}$ by locally closed subsets in $E_{(A;0)}$. The strata are labelled by the isomorphism type of $\mathcal{E}_i/\mathcal{E}_{i-1}$ on each fiber $\mathbb{C}\mathrm{P}^1$. When restricted to each stratum, $\mathcal{E}_i/\mathcal{E}_{i-1}$ is of the form of a direct sum of a locally free shaef and a torsion sheaf. There is a unique open stratum in this stratification.

  (1.3) By taking intersections of strata of the stratifications associated to different $i$, one obtains a stratification of of $E_{(A;0)}$ that gives a minimal common refinement of all the stratifications in Item (1.2). From Item (1.2) this common refinement contains a unique open stratum. Set $\mathcal{E}_0 = 0$ and $\mathcal{E}_{I+1} = \mathcal{E}$, then the graded sheaf $\oplus_{i=1}^{I+1}\mathcal{E}_i/\mathcal{E}_{i-1}$ is of the form of a direct sum of a locally free sheaf and a torsion sheaf over each stratum of the common stratification. (Cf. See [CF1], [CF2], and [M-M] for more related studies.)

(2) For each vertical filtration,

$$\mathcal{E}_{i,j}/\mathcal{E}_{i,j-1} \;=\; \left(\pi_1^*\left(\widehat{\mathcal{S}}_{i,j}/\widehat{\mathcal{S}}_{i,j-1}\right)\right)(-a_{i,j}\,z)\,,$$

where $z$ here represents the divisor $[E_{(A;0)} \times \{0\}]$ on $E_{(A;0)} \times \mathbb{C}\mathrm{P}^1$ (and will be omitted in the following discussion). The $S^1$-action on $\mathbb{C}\mathrm{P}^1$ induces a natural $S^1$-action on the trivialized trivial bundle $\mathcal{E}$, which induces in turn an $S^1$-action on $\mathcal{E}_{i,\bullet}$ and hence on the graded bundle $\oplus_{j=1}^{K_i} \mathcal{E}_{i,j}/\mathcal{E}_{i,j-1}$, where we set $\mathcal{E}_{i,0} = 0$. These graded objects from the vertical filtration in the diagram will play crucial roles in our later discussions.

Recall also the restricting bundles $\Pi_\bullet$ in the discussion that are related to restrictvie flag manifolds. Since $\widehat{\mathcal{P}}_{i+1,j} = \widehat{\mathcal{S}}_{i+1,I_A(i,j)}$, they are covered in the above discussion. In particular, $\widehat{\mathcal{P}}_{i+1,K_i} = \widehat{\mathcal{S}}_{i+1,K_{i+1}}$, ( i.e. $I_A(i,K_i) = K_{i+1}$ ) always holds.

### 3.4 The hyperplane-induced classes on $E_{(A;0)}$.

Recall from Sec. 3.1 the commutative diagram

$$\begin{array}{ccccc} E_{(A;0)} & \stackrel{k}{\hookrightarrow} & \mathcal{Q}_d & \stackrel{\psi}{\longrightarrow} & W_d \\ \downarrow & & \downarrow & & \| \\ \prod_{i=1}^I E_{(\alpha_{i,\bullet};0)} & \longrightarrow & \prod_{i=1}^I Quot_{P_i}(\mathcal{E}^n) & \longrightarrow & \prod_{i=1}^I W_{d_i}(Gr_{r_i}(\mathbb{C}^n))\,. \end{array}$$

The following lemma follows from the result of [L-L-L-Y: Sec. 3.1] for the hyperplane-induced class in the case of Grassmannian manifolds and the discussion and notations of the tautological bundles on $E_{(A;0)}$ in Sec. 3.3.

**Lemma 3.4.1 [hyperplane-induced classes].** *Let $\kappa_1, \ldots, \kappa_I$ be the hyperplane classes on $W_d$ from the product structure $W_d = \prod_{i=1}^I W_{d_i}(Gr_{r_i}(\mathbb{C}^n))$. Then*

$$k^*\psi^*\kappa_i \;=\; -c_1(\widehat{\mathcal{S}}_{i,K_i}).$$

*In terms of Chern roots of $\widehat{\mathcal{S}}_{i,j}$,*

$$k^*\psi^*\kappa_i \;=\; -(\, y_{i,1;1} + \cdots + y_{i,1;m_{i,1}} \;+\; \cdots \;+\; y_{i,K_i;1} + \cdots + y_{i,K_i;m_{i,K_i}} \,)\,.$$

### 3.5 An exact computation of $e_{\mathbb{C}^\times}(E_{(A;0)}/HQuot_P(\mathcal{E}^n))$.

In this subsection we work out an exact expresion of $e_{\mathbb{C}^\times}(E_{(A;0)}/HQuot_P(\mathcal{E}^n))$ in terms of Chern roots $y_{i,j;k}$ and $S^1$-weights of $\widehat{\mathcal{S}}_{i,j}/\widehat{\mathcal{S}}_{i,j-1}$.

**An Euler sequence for $(T_*HQuot_P(\mathcal{E}^n))|_{E_{(A;0)}}$.**

Recall the projection maps $\pi_1 : E_{(A;0)} \times \mathbb{C}\mathrm{P}^1 \to E_{(A;0)}$ and $\pi_2 : E_{(A;0)} \times \mathbb{C}\mathrm{P}^1 \to \mathbb{C}\mathrm{P}^1$, the subsheaves

$$0 \hookrightarrow \mathcal{E}_1 \overset{j_1}{\hookrightarrow} \cdots \overset{j_{I-1}}{\hookrightarrow} \mathcal{E}_I \overset{j_I}{\hookrightarrow} \mathcal{E} = \pi_2^*(\mathcal{E}^n)$$

and the quotient sheaves

$$\mathcal{E} \overset{p_1}{\twoheadrightarrow} \mathcal{E}/\mathcal{E}_1 \overset{p_2}{\twoheadrightarrow} \cdots \overset{p_I}{\twoheadrightarrow} \mathcal{E}/\mathcal{E}_I \to 0\,.$$

on $E_{(A;0)} \times \mathbb{C}\mathrm{P}^1$. Let $\mathcal{K}$ be the kernel of the following morphism of $\mathcal{O}_{E_{(A;0)} \times \mathbb{C}\mathrm{P}^1}$-modules

$$\oplus_{i=1}^{I} \mathcal{H}om(\,\mathcal{E}_i\,,\,\mathcal{E}/\mathcal{E}_i\,) \longrightarrow \oplus_{i=1}^{I-1} \mathcal{H}om(\,\mathcal{E}_i\,,\,\mathcal{E}/\mathcal{E}_{i+1}\,)$$

defined by

$$(\varphi_i)_{i=1}^{I} \longmapsto (p_{i+1} \circ \varphi_i - \varphi_{i+1} \circ j_i)_{i=1}^{I-1}$$

on each open subset $U$ of $E_{(A;0)} \times \mathbb{C}\mathrm{P}^1$, cf. [CF1: Appendix]. Then

**Lemma 3.5.1** **[**$(T_* HQuot_P(\mathcal{E}^n))|_{E_{(A;0)}}$ **as push-forward].**

$$\pi_{1*}\mathcal{K} = (T_* HQuot_P(\mathcal{E}^n))|_{E_{(A;0)}}\,.$$

*Proof.* In [CF1: Appendix], a fiberwise statement is given. Here we strengthen his result to a global statement.

Let us outline first the approach of the proof. Let $\mathcal{HQ}$ be the stack associated to the hyper-Quot scheme $HQuot_P(\mathcal{E}^n)$. Then the tangent stack $T_*\mathcal{HQ}$ is the stackification of the prestack that associates to each affine $\mathbb{C}$-scheme $U$ the groupoid $\mathcal{HQ}(U_\varepsilon)$, where $U_\varepsilon := U \times_{\mathbb{C}} \mathbb{C}[\varepsilon]$, with $\varepsilon^2 = 0$. $T_*\mathcal{HQ}$ is represented by the scheme associated to the

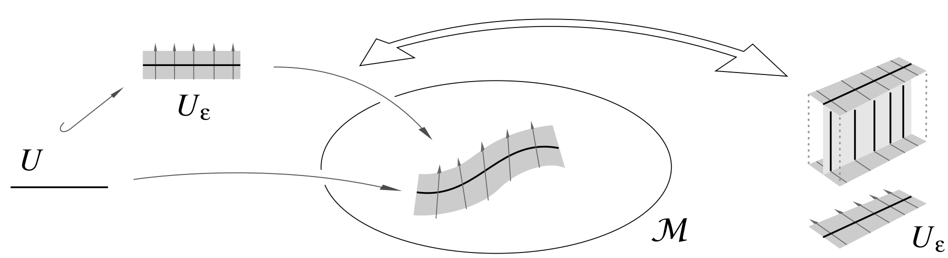


FIGURE 3-5-1. The functor that gives rise to the tangent space $T_*\mathcal{M}$ of a moduli stack $\mathcal{M}$. A morphism from $U$ (resp. $U_\varepsilon$) to the moduli stack $\mathcal{M}$ is the same as a flat family over $U$ (resp. $U_\varepsilon$) of the objects $\mathcal{M}$ parameterizes (cf. the right third of the figure).

tangent sheaf (a vector bundle in our case) $T_* HQuot_P(\mathcal{E}^n)$. Associated to the scheme

$T_*HQuot_P(\mathcal{E}^n)|_{E_{(A;0)}}$ is then the stack $T_*\mathcal{HQ}|_{E_{(A;0)}}$ over the subcategory of $E_{(A;0)}$-schemes from the stackification of the prestack that associates to each $U \to E_{(A;0)}$, where $U$ is affine, the groupoid $T_*\mathcal{HQ}(U) = \mathcal{HQ}(U_\varepsilon)$. In our current case, $HQuot_P(\mathcal{E}^n)$ and hence $E_{(A;0)}$ are projective. Thus $E_{(A;0)}$ can be covered by an atlas that consists of finitely many affine schemes such that any of their intersections are also affine. One shows first that the statement of the lemma holds for any affine open subset $U$ of $E_{(A;0)}$. Since all the isomorphisms of the groupoids $\mathcal{HQ}(U)$ and $\mathcal{HQ}(U_\varepsilon)$ are identity maps, both $\mathcal{HQ}(U)$ and $\mathcal{HQ}(U_\varepsilon)$ are sets canonically. Thus, to say that the lemma holds for $U$ means that

$$(T_*\mathcal{HQ}|_{E_{(A;0)}})(U \hookrightarrow E_{(A;0)}) \;\simeq\; H^0(\pi_1^{-1}(U), \mathcal{K}|_{\pi_1^{-1}(U)})$$

as sets, where $(T_*\mathcal{HQ}|_{E_{(A;0)}})(U \hookrightarrow E_{(A;0)})$ is the groupoid (set) of union of all groupoids $\mathcal{HQ}(U_\varepsilon \to E_{(A;0)})$ with $U_\varepsilon \to E_{(A;0)}$ extending the given inclusion $U \hookrightarrow E_{(A;0)}$. This set isomorphism will be constructed in a canonical/functorial way. Once this is achieved, then since the collection of groupoids $T_*\mathcal{HQ}|_{E_{(A;0)}}(U \hookrightarrow E_{(A;0)})$ glue to give the stack $T_*\mathcal{HQ}|_{E_{(A;0)}}$ via the Isom-functor construction and the Grothendieck descent, the collection $\pi_{1*}(\mathcal{K}|_{\pi_1^{-1}(U)})$ must glue to give the restriction of tangent bundle $(T_*HQuot_P(\mathcal{E}^n))|_{E_{(A;0)}}$, which represents the stack $T_*\mathcal{HQ}|_{E_{(A;0)}}$.

Note that one may prove the Lemma first for the whole $HQuot_P(\mathcal{E}^n)$ and then discuss the restriction to $E_{(A;0)}$. But then one has to deal with the issue of commutativity of push-forward and restriction to a closed subscheme, which in general does not hold but has to be checked case by case. The above setting incorporates this issue into the discussion directly.

*Case (a)* : $X = Gr_r(\mathbb{C}^n)$. In this case $HQuot_P(\mathcal{E}^n)$ is the Quot-scheme $Quot_P(\mathcal{E}^n)$. Let $U$ be an open affine subscheme of $E_{(A;0)}$. Since $E_{(A;0)}$ is smooth, we will assume that $U$ is smooth and quasi-projective. Let $U \stackrel{i}{\hookrightarrow} U_\varepsilon \stackrel{\pi}{\to} U$ be the natural morphisms whose composition is the identity map on $U$. (The corresponding morphisms $U \times \mathbb{C}\mathrm{P}^1 \to U_\varepsilon \times \mathbb{C}\mathrm{P}^1 \to U \times \mathbb{C}\mathrm{P}^1$ will be denoted the same.) Let $\mathcal{V}$ be the tautological subbundle of $\mathcal{E}$ on $U \times \mathbb{C}\mathrm{P}^1$, $\mathcal{E}_\varepsilon = \pi^*\mathcal{E}$ on $U_\varepsilon \times \mathbb{C}\mathrm{P}^1$, and $\mathcal{V}'$ be a subsheaf of $\mathcal{E}_\varepsilon$ of Hilbert polynomial $P$ with its restriction to $U \times \mathbb{C}\mathrm{P}^1$ being $\mathcal{V}$. Then $\pi_*\mathcal{V}'$ is a locally free subsheaf of $\pi_*\mathcal{E}_\varepsilon = \mathcal{E} \oplus \mathcal{E} \otimes \varepsilon$ (canonically) on $U \times \mathbb{C}\mathrm{P}^1$ with the Hilbert polynomial of the associated quotient sheaf being $2P$ and one has a canonical exact sequence

$$0 \longrightarrow \mathcal{V} \otimes \varepsilon \longrightarrow \pi_*\mathcal{V}' \longrightarrow \mathcal{V} \longrightarrow 0\,.$$

(This sequence splits non-canonically; thus $\pi_*\mathcal{V}' \simeq \mathcal{V} \oplus \mathcal{V} \otimes \varepsilon$ non-canonically.)

The above sequence together with projection of the locally free subsheaves $\pi_*\mathcal{V}'$ of $\pi_*\mathcal{E}_\varepsilon$ into the direct summands, $\mathcal{E}$ and $\mathcal{E} \otimes \varepsilon$, of $\pi_*\mathcal{E}_\varepsilon$ induces the following diagram of canonical morphisms and isomorphisms

$$\begin{array}{ccc} \pi_*\mathcal{V}' & \longrightarrow & \mathcal{E} \otimes \varepsilon \simeq \mathcal{E} \\ \downarrow & & \downarrow \\ \mathcal{V} & & (\mathcal{E} \otimes \varepsilon)/(\mathcal{V} \otimes \varepsilon) \simeq \mathcal{E}/\mathcal{V} \end{array}$$

with both of the vertical arrows epimorphisms. Any local section $s$ of $\mathcal{V}$ can be lifted to a local section $s'$ in $\pi_*\mathcal{V}'$. The latter then maps to a local section in $Hom(\mathcal{V}, \mathcal{E}/\mathcal{V})$ by following the above diagram. The image of $s$ in $Hom(\mathcal{V}, \mathcal{E}/\mathcal{V})$ depends only on $s$, not on the choice of its lifting $s'$. Thus, one obtains a canonical homomorphism

$$\varphi_{\mathcal{V}'} \;:\; \mathcal{V} \;\longrightarrow\; \mathcal{E}/\mathcal{V}\,.$$

The correspondence $\mathcal{V}' \mapsto \varphi_{\mathcal{V}'}$ gives a map from $(T_*\mathcal{HQ}|_{E_{(A;0)}})(U \hookrightarrow E_{(A;0)})$ to $Hom(\mathcal{V}, \mathcal{E}/\mathcal{V})$ (which is $H^0(\pi_1^{-1}(U), \mathcal{K}|_{\pi_1^{-1}(U)})$ in the current case).

Conversely, given a $\varphi : \mathcal{V} \longrightarrow \mathcal{E}/\mathcal{V}$, let $\mathcal{V}''_\varphi$ be the locally free subsheaf of $\pi_*\mathcal{E}_\varepsilon = \mathcal{E} \oplus \mathcal{E} \otimes \varepsilon$, whose elements in fibers of $\mathcal{V}''_\varphi$ are given by

$$\{\, (v_0, v') \,|\, v_0 \in \mathcal{V}\,,\ v' \text{ is mapped to } \varphi(v_0) \text{ under } \mathcal{V}\otimes\varepsilon \to (\mathcal{E}\otimes\varepsilon)/(\mathcal{V}\otimes\varepsilon) \simeq \mathcal{E}/\mathcal{V}\,\}\,.$$

Then $\mathcal{V}''_\varphi$ fits into the exact sequence of $\mathcal{O}_{\pi_1^{-1}(U)}$-modules

$$0 \;\longrightarrow\; \mathcal{V}\otimes\varepsilon \;\longrightarrow\; \mathcal{V}''_\varphi \;\longrightarrow\; \mathcal{V} \;\longrightarrow\; 0$$

and is invariant under the action of $\varepsilon$ on $\pi_*\mathcal{E}_\varepsilon$ induced from the multiplication of $\varepsilon$ on $\mathcal{E}_\varepsilon$. Thus $\mathcal{V}''_\varphi = \pi_*\mathcal{V}'_\varphi$ for a unique locally free (and hence flat) $\mathcal{O}_{U_\varepsilon\times\mathbb{C}\mathrm{P}^1}$-submodule $\mathcal{V}'_\varphi$ of $\mathcal{E}_\varepsilon$ on $U_\varepsilon \times \mathbb{C}\mathrm{P}^1$. This gives a map from $Hom(\mathcal{V}, \mathcal{E}/\mathcal{V})$ to $(T_*\mathcal{HQ}|_{E_{(A;0)}})(U \hookrightarrow E_{(A;0)})$. One can check that the correspondences, $\mathcal{V}' \mapsto \varphi_{\mathcal{V}'}$ and $\varphi \mapsto \mathcal{V}'_\varphi$, are inverse to each other. These constructions are canonical and functorial; thus

$$(T_*HQuot_P(\mathcal{E}^n))|_{E_{(A;0)}} \;=\; \pi_{1*}\mathcal{H}om(\mathcal{S}, \mathcal{E}/\mathcal{S})\,,$$

where $\mathcal{S}$ is the tautological subsheaf of $\mathcal{E}$ on $E_{(A;0)} \times \mathbb{C}\mathrm{P}^1$.

*Case (b)* : $X = Fl_{r_1,\ldots,r_I}(\mathbb{C}^n)$. Repeat the same discussion for nested sequence of subsheaves over $U_\varepsilon \times \mathbb{C}\mathrm{P}^1$ gives an embedding

$$(T_*\mathcal{HQ}|_{E_{(A;0)}})(U \hookrightarrow E_{(A;0)}) \;\hookrightarrow\; \oplus_{i=1}^I\, Hom_{U\times\mathbb{C}\mathrm{P}^1}(\,\mathcal{E}_i, \mathcal{E}/\mathcal{E}_i\,)\,.$$

We shall now check that its image coincide with the set $H^0(U \times \mathbb{C}\mathrm{P}^1, \mathcal{K}|_{U\times\mathbb{C}\mathrm{P}^1})$ as the subset

$$\{\, (\varphi_i)_{i=1}^I \,|\, (p_{i+1}\circ\varphi_i - \varphi_{i+1}\circ j_i)_{i=1}^{I-1} \;=\; 0\,\}$$

of $\oplus_{i=1}^I Hom(\,\mathcal{E}_i\,,\, \mathcal{E}/\mathcal{E}_i\,)$. By induction, we only need to consider the case $I = 2$.

Let $\mathcal{E}'_1 \hookrightarrow \mathcal{E}_\varepsilon$ (resp. $\mathcal{E}'_2 \hookrightarrow \mathcal{E}_\varepsilon$) be a flat subsheaf extension of $\mathcal{E}_1 \hookrightarrow \mathcal{E}$ (resp. $\mathcal{E}_2 \hookrightarrow \mathcal{E}$) to $U_\varepsilon \times \mathbb{C}\mathrm{P}^1$. Suppose that $\mathcal{E}'_1$ is contained in $\mathcal{E}'_2$. Then $\varphi_{\mathcal{E}'_2}\circ j_1$ is the restriction of $\varphi_{\mathcal{E}'_2}$ (defined on $\mathcal{E}_2$) to $\mathcal{E}_1$. In choosing the lifting sections $s'$ in $\pi_*\mathcal{E}'_2$ for local sections $s$ in $\mathcal{E}_1$ to define $\varphi_{\mathcal{E}'_2}(s)$, one may choose $s'$ a local section in $\pi_*\mathcal{E}_1 \hookrightarrow \pi_*\mathcal{E}_2$ since $\varphi_{\mathcal{E}'_2}(s)$ is independent of the choice of liftings. Consequently, for $s$ a local section of $\mathcal{E}_1$ $\varphi_{\mathcal{E}'_2}(s) = \varphi_{\mathcal{E}'_1}(s)$ modulo $\mathcal{V}_2$, which is exactly $p_2\circ\varphi_{\mathcal{E}'_1}(s)$. This shows that $\varphi_{\mathcal{E}'_2}\circ j_1 = p_2\circ\varphi_{\mathcal{E}'_1}$ and hence that $(T_*\mathcal{HQ}|_{E_{(A;0)}})(U \hookrightarrow E_{(A;0)})$ embeds in $H^0(U \times \mathbb{C}\mathrm{P}^1, \mathcal{K}|_{U\times\mathbb{C}\mathrm{P}^1})$.

Conversely, suppose that $\varphi_{\mathcal{E}_2'} \circ j_1 = p_2 \circ \varphi_{\mathcal{E}_1'}$. From Part (a) of the proof, consider the canonical quotients $\delta_1 : \pi_* \mathcal{E}_1' \to \mathcal{E}_1$ and $\delta_2 : \pi_* \mathcal{E}_2' \to \mathcal{E}_2$. Treating all these locally free sheaves as vector bundles, then for a given $v \in \mathcal{E}_1 \subset \mathcal{E}_2$, the assumption that $\varphi_{\mathcal{E}_2'} \circ j_1 = p_2 \circ \varphi_{\mathcal{E}_1'}$. implies that the projection of $\delta_1^{-1}(v)$ in $\mathcal{E} \otimes \varepsilon$ is contained in the projection of $\delta_2^{-1}(v)$ in $\mathcal{E} \otimes \varepsilon$. Since the projection of $\delta_1^{-1}(v)$ (resp. $\delta_2^{-1}(v)$) to $\mathcal{E} \otimes \varepsilon$ is injective, this shows that $\delta_1^{-1}(v) \hookrightarrow \delta_2^{-1}(v)$ for all $v \in \mathcal{E}_1$ and hence that $\mathcal{E}_1'$ is contained in $\mathcal{E}_2'$. Together with the previous discussion, this proves that

$$(T_* \mathcal{HQ}|_{E_{(A;0)}})(U \hookrightarrow E_{(A;0)}) \;=\; H^0(U \times \mathbb{C}\mathrm{P}^1, \mathcal{K}|_{U \times \mathbb{C}\mathrm{P}^1})\,.$$

Consequently, the collection $\pi_{1*}(\mathcal{K}|_{\pi_1^{-1}(U)})$ glue to give $(T_* HQuot_P(\mathcal{E}^n))|_{E_{(A;0)}}$ and we conclude the proof.

□

Recall from [CF1: Appendix] that there is a sheaf morphism

$$\mathcal{H}om_{E_{(A;0)} \times \mathbb{C}\mathrm{P}^1}(\mathcal{E}, \mathcal{E}) \;\longrightarrow\; \mathcal{H}om_{E_{(A;0)} \times \mathbb{C}\mathrm{P}^1}(\mathcal{E}_i, \mathcal{E}/\mathcal{E}_i)$$

given by (denote $\mathcal{E}_i \hookrightarrow \mathcal{E}$ and $\mathcal{E} \to \mathcal{E}/\mathcal{E}_i$ also by $j_i$ and $p_i$ respectively)

$$\psi \;\longmapsto\; (p_i \circ \psi \circ j_i)_{i=1}^I$$

that factors through $\mathcal{K}$ and is generically surjective (i.e. surjective at the stalk - or equivalently the fiber - at the generic point in the Zariski topology) onto $\mathcal{K}$. An investigation of the non-surjectivity onto $\mathcal{K}$ of this morphism at the stalk at points on $E_{(A;0)} \times \{0\}$ motivates the following construction.

Consider the sheaf morphism

$$\Psi \;:\; \oplus_{i=1}^I \mathcal{H}om_{E_{(A;0)} \times \mathbb{C}\mathrm{P}^1}(\mathcal{E}_i, \mathcal{E}) \;\longrightarrow\; \oplus_{i=1}^I \mathcal{H}om_{E_{(A;0)} \times \mathbb{C}\mathrm{P}^1}(\mathcal{E}_i, \mathcal{E}/\mathcal{E}_i)$$

given by the following map of local sections on any open subset of $E_{(A;0)} \times \mathbb{C}\mathrm{P}^1$:

$$(\psi_i)_{i=1}^I \;\longmapsto\; (p_i \circ \psi_i)_{i=1}^I\,.$$

Let $\mathcal{G}$ be the subsheaf of $\oplus_{i=1}^I \mathcal{H}om_{E_{(A;0)} \times \mathbb{C}\mathrm{P}^1}(\mathcal{E}_i, \mathcal{E})$ defined by the local sections $(\psi_i)_{i=1}^I$ such that the image sheaf $(\psi_i - \psi_{i+1} \circ j_i)(\mathcal{E}_i)$ of $\mathcal{E}_i$ in $\mathcal{E}$ lies in $\mathcal{E}_{i+1}$ (on open subsets of $E_{(A;0)} \times \mathbb{C}\mathrm{P}^1$).

**Lemma 3.5.2 [locally free resolution of $\mathcal{K}$].**

(1) *The morphism $\Psi$ maps $\mathcal{G}$ surjectively onto $\mathcal{K}$.*

(2) *$\mathcal{G}$ is locally free of rank $(r_1 r_2 + \cdots + r_{I-1} r_I + r_I n)$. Along each $\mathbb{C}\mathrm{P}^1$-fiber, $\mathcal{G}$ is a direct sum of non-negative line bundles.*

(3) *The kernel of $\mathcal{G} \xrightarrow{\Psi} \mathcal{K}$ is given by $\oplus_{i=1}^I \mathcal{H}om(\mathcal{E}_i, \mathcal{E}_i)$, which is locally free of rank $r_1^2 + \cdots + r_I^2$.*

*Thus, in particular,*

$$0 \longrightarrow \oplus_{i=1}^{I}\mathcal{H}om\,(\mathcal{E}_i,\mathcal{E}_i) \longrightarrow \mathcal{G} \longrightarrow \mathcal{K} \longrightarrow 0$$

*is a locally free resolution of* $\mathcal{K}$*.*

*Proof.* Item (1) and Item (3) follow immediately by construction. For Item (2), we only need to check that $\mathcal{G}$ is locally free of the rank claimed along each $\mathbb{C}\mathrm{P}^1$-fiber over a closed point of $E_{(A;0)}$. Since both $E_{(A;0)}\times\mathbb{C}\mathrm{P}^1$ and $E_{(A;0)}$ are smooth and the rank is independent of the $\mathbb{C}\mathrm{P}^1$-fibers, this then implies that $\mathcal{G}$ is locally free.

Consider the sheaves $\mathcal{E}_i$ restricted to a $\mathbb{C}\mathrm{P}^1$-fiber. Fix a realization $\mathbb{C}\mathrm{P}^1 = \mathrm{Proj}\,\mathbb{C}[z_0,z_1]$ that is compatible with the $S^1$-action with $0\in\mathbb{C}\mathrm{P}^1$ corresponding to $[0:1]$. Recall from the proof of Lemma 2.1.4 the simultaneous $S^1$-weight subspace decomposition of an adjacent pair $\mathcal{E}_i\hookrightarrow\mathcal{E}_{i+1}$ on $\mathbb{C}\mathrm{P}^1$. Incorporating these into presentation, one can identify $\mathcal{E}_i$ and $\mathcal{E}_{i+1}$ as graded $\mathbb{C}[z_0,z_1]$-modules:

$$\mathcal{E}_i \;=\; \left(\oplus_{j=1}^{r_i}\, z_0^{a_j}\cdot\mathbb{C}[z_0,z_1]\right)^{\sim}\,,\quad \mathcal{E}_{i+1} \;=\; \left(\oplus_{j'=1}^{r_{i+1}}\, z_0^{a'_{j'}}\cdot\mathbb{C}[z_0,z_1]\right)^{\sim}\,,\quad \mathcal{E}^n \;=\; \left(\mathbb{C}[z_0,z_1]^{\oplus n}\right)^{\sim}$$

such that $a_j\ge a'_j$, $j=1,\ldots,r_i$, and that the inclusion $\mathcal{E}_i\hookrightarrow\mathcal{E}_{i+1}$ is induced from the natural inclusions of graded modules $z_0^{a_j}\cdot\mathbb{C}[z_0,z_1]\hookrightarrow z_0^{a'_j}\cdot\mathbb{C}[z_0,z_1]$, $j=1,\ldots,r_i$, from the identity map $\mathbb{C}[z_0,z_1]\hookrightarrow\mathbb{C}[z_0,z_1]$. In terms of these, the local sections $\psi_i$ and $\psi_{i+1}$ of the Hom-sheaves $\mathcal{H}om\,(\mathcal{E}_i,\mathcal{E})$ and $\mathcal{H}om\,(\mathcal{E}_{i+1},\mathcal{E})$ are represented respectively as (degree-0 part of the localization of) $\mathbb{C}[z_0,z_1]$-valued matrices of the following block form:

$$\psi_i \;=\; B_i \qquad\text{and}\qquad \psi_{i+1} \;=\; [B_{i+1},*]_{n\times r_{i+1}}\,,$$

where both $B_i$ and $B_{i+1}$ are $n\times r_i$ matrices. $B_{i+1}$ corresponds to the composition $\psi_{i+1}\circ j_i$. Thus,

$$(\psi_i-\psi_{i+1}\circ j_i)(\mathcal{E}_i)\subset\mathcal{E}_{i+1} \qquad\Longleftrightarrow\qquad B_i-B_{i+1} \;=\; \left[\begin{array}{c} C_{i,i+1}\\ 0\end{array}\right]_{n\times r_i}\,,$$

where $C_{i,i+1}=(c_{kl}(z_0,z_1))_{k,l}$ is an $r_{i+1}\times r_i$-matrix that satisfies

$$deg_{\,z_0}\,c_{kl}(z_0,z_1)\;\ge\; a'_l-a_l\,,$$

Since, for a fixed $\psi_{i+1}$, the space of $C_{i,i+1}$ that satisfy the above condition is a free graded $\mathbb{C}[z_0,z_1]$-module of rank $r_ir_{i+1}$. Let $i$ run from 1 to $I$, this show that the restriction of $\mathcal{G}$ to each $\mathbb{C}\mathrm{P}^1$-fiber is locally free of rank $r_1r_2+\cdots+r_{I-1}r_I+r_In$ and hecne the first half of Item (2).

Since $a'_l-a_l\le 0$ for each $l=1,\ldots,r_i$, this proves the second half of Item (2).

This completes the proof.

□

The above lemma, Item (2), implies that $R^1\pi_{1*}\mathcal{G}=0$ by Grauert theorem, cf. [Ha: III. Corollary 12.9]. Consequently,

**Corollary 3.5.3 [Euler sequence of $(T_*HQuot_P(\mathcal{E}^n))|_{E_{(A;0)}}$].** *The restriction $(T_*HQuot_P(\mathcal{E}^n))|_{E_{(A;0)}} = \pi_{1*}\mathcal{K}$ fits into the following exact sequence*

$$0 \longrightarrow \pi_{1*}\left(\oplus_{i=1}^I \mathcal{H}om(\mathcal{E}_i,\mathcal{E}_i)\right) \longrightarrow \pi_{1*}\mathcal{G} \longrightarrow \pi_{1*}\mathcal{K} \longrightarrow R^1\pi_{1*}\left(\oplus_{i=1}^I \mathcal{H}om(\mathcal{E}_i,\mathcal{E}_i)\right) \longrightarrow 0\,.$$

**An Euler sequence for the vertical tangent bundle $T_*^{(vert,i)}E_{(A;0)}$.**

**Lemma 3.5.4 [Euler sequence for $T_*Fl_{k_1,\ldots,k_s}(\mathbb{C}^n,\Pi_\bullet)$].** *Let $M := Fl_{k_1,\ldots,k_s}(\mathbb{C}^n,\Pi_\bullet)$ for notation, $\mathcal{E}=\mathcal{O}_M\otimes\mathbb{C}^n$, $F_\bullet S : S_1 \hookrightarrow \cdots \hookrightarrow S_s =: S$ be the tautological filtration of subsheaves on $Fl_{k_1,\ldots,k_s}(\mathbb{C}^n,\Pi_\bullet)$, and $F_\bullet\mathcal{E}$ be the filtration of $\mathcal{E}$ by $\mathcal{O}_M\otimes\Pi_\bullet$. Then $T_*Fl_{k_1,\ldots,k_s}(\mathbb{C}^n,\Pi_\bullet)$ fits into the following exact sequences of locally free $\mathcal{O}_M$-modules*

(1) (*compact form*) :

$$0 \longrightarrow \mathcal{H}om_{\mathcal{O}_M}(F_\bullet S, F_\bullet S) \longrightarrow \mathcal{H}om_{\mathcal{O}_M}(F_\bullet S, F_\bullet \mathcal{E}) \longrightarrow T_*Fl_{k_1,\ldots,k_s}(\mathbb{C}^n,\Pi_\bullet) \longrightarrow 0\,;$$

(2) (*splitted form*) :

$$0 \longrightarrow \oplus_{j=1}^s \mathcal{H}om_{\mathcal{O}_M}(S_j,S_j) \longrightarrow \mathcal{H} \longrightarrow T_*Fl_{k_1,\ldots,k_s}(\mathbb{C}^n,\Pi_\bullet) \longrightarrow 0\,,$$

*where $\mathcal{H}$ is a subsheaf of $\oplus_{j=1}^s\mathcal{H}om_{\mathcal{O}_M}(S_j,\mathcal{O}_M\otimes\Pi_j)$ consisting of local sections $(\psi_j)_{j=1}^s$ of sheaf morphisms such that the image sheaf $(\psi_j-\psi_{j+1}\circ\iota_j)(S_j)$ in $\mathcal{O}_M\otimes\Pi_j$ lies in $S_{j+1}$. (Here $\iota_j : S_j\rightarrow S_{j+1}$ is the inclusion morphism.)*

*Proof.* Recall the deformation of flags in $\mathbb{C}^n$ and the construction of the Euler sequence for the tangent bundle of the usual flag manifolds and that for $(T_*HQuot_P(\mathcal{E}^n))|_{E_{(A;0)}}$. Statement (1) follows by the case of ordinary flag manifolds but take into account the restriction that the subspace $S_j$ are now restricted to move only in $\Pi_j$. (Note that the morphisms in the splitted form of the Euler sequence already apear in the construction of Euler sequence for $(T_*HQuot_P(\mathcal{E}^n))|_{E_{(A;0)}}$. )

Locally-freeness of the sheaf of modules involved in both Statement (1) and Statement (2) follows from the fact that all the sheaves involved are locally free and hence can be identified as vector bundles and that all the filtration involved are filtrations by subbundles.

This concludes the proof.

□

Recall the tower of fibrations of $E_{(A,0)}$

$$E_{(A;0)} = E_{(A;0)}^{(1)} \longrightarrow \cdots \longrightarrow E_{(A;0)}^{(i)} \longrightarrow \cdots \longrightarrow E_{(A;0)}^{(I)} = Fl_{m_{I,\bullet}}(\mathbb{C}^n)$$

with the fiber of $E^{(i)}_{(A;0)} \to E^{(i+1)}_{(A;0)}$ being the restrictive flag manifold $Fl_{m_{i,\bullet}}(\mathbb{C}^n, \Pi_{i+1,\bullet})$. Denote the vertical tangent bundle of the fibration $E^{(i)}_{(A;0)} \to E^{(i+1)}_{(A;0)}$ by $T_*^{(vert)} E^{(i)}_{(A;0)}$ (i.e. the subbundle of $T_* E^{(i)}_{(A;0)}$ consisting of tangent vectors along the fibers of the fibration $f_i$) and its pull-back to $E_{(A;0)}$ by $T_*^{(vert,i)} E_{(A;0)}$. Recall also the various tautological subsheaves $\widehat{\mathcal{S}}_{i,\bullet}$ and $\widehat{\mathcal{P}}_{i+1,\bullet}$ on $E_{(A;0)}$, $i = 1, \ldots, I$. Let $\widehat{\mathcal{S}}_i = \widehat{\mathcal{S}}_{i,K_i}$, $F_\bullet \widehat{\mathcal{S}}_i$ be the filtration $\widehat{\mathcal{S}}_{i,\bullet}$ of $\widehat{\mathcal{S}}_i$, and $F_\bullet^{(i)} \mathcal{E}$ be the filtration $\widehat{\mathcal{P}}_{i+1,\bullet}$ of $\mathcal{E} = \mathcal{O}_{E_{(A;0)}} \otimes \mathbb{C}^n$. Then Lemma 3.5.4 together with the locally-freeness of the sheaf of modules involved imply immediately the following:

**Corollary 3.5.5 [Euler sequence for $T_*^{(vert,i)} E_{(A;0)}$].** *The $i$-th vertical tangent bundle $T_*^{(vert,i)} E_{(A;0)}$ fits into the following exact sequence of $\mathcal{O}_{E_{(A;0)}}$-modules;*

(1) (*compact form*) :

$$0 \longrightarrow \mathcal{H}om\,(F_\bullet \widehat{\mathcal{S}}_i, F_\bullet \widehat{\mathcal{S}}_i) \longrightarrow \mathcal{H}om\,(F_\bullet \widehat{\mathcal{S}}_i, F_\bullet^{(i)} \mathcal{E}) \longrightarrow T_*^{(vert,i)} E_{(A;0)} \longrightarrow 0\,;$$

(2) (*splitted form*) :

$$0 \longrightarrow \oplus_{\mathrm{j}=1}^{K_i} \mathcal{H}om\,(\widehat{\mathcal{S}}_{i,j}, \widehat{\mathcal{S}}_{i,j}) \longrightarrow \mathcal{H}^{(i)} \longrightarrow T_*^{(vert,i)} E_{(A;0)} \longrightarrow 0\,,$$

*where $\mathcal{H}^{(i)}$ is a subsheaf of $\oplus_{j=1}^{K_i} \mathcal{H}om\,(\widehat{\mathcal{S}}_{i,j}, \widehat{\mathcal{P}}_{i+1,j})$ consisting of local sections $(\psi_j)_{j=1}^{K_i}$ of sheaf morphisms such that the image sheaf $(\psi_j - \psi_{j+1} \circ \iota_j)(\widehat{\mathcal{S}}_{i,j})$ in $\widehat{\mathcal{P}}_{i+1,j}$ lies in $\widehat{\mathcal{S}}_{i,j+1}$. (Here $\iota_j : \widehat{\mathcal{S}}_{i,j} \to \widehat{\mathcal{S}}_{i,j+1}$ is the inclusion morphism.)*

**A decomposition of $\nu(E_{(A;0)}/HQuot_P(\mathcal{E}^n))$ in the $K$-group of $E_{(A;0)}$.**

We give a decomposition of $T_* E_{(A;0)}$ and $(T_* HQuot_P(\mathcal{E}^n))|_{E_{(A;0)}}$ in the $K$-group of $E_{(A;0)}$. The decomposition of $\nu(E_{(A;0)}/HQuot_P(\mathcal{E}))$ follows from

$$[\nu(E_{(A;0)}/HQuot_P(\mathcal{E}))] \;=\; [(T_* HQuot_P(\mathcal{E}^n))|_{E_{(A;0)}}] \,-\, [T_* E_{(A;0)}]\,.$$

(*a*) *The $[T_* E_{(A;0)}]$-part*: Recall that associated to a smooth morphism of smooth variety $f : X \to Y$ is the exact sequence of $\mathcal{O}_X$-modules

$$0 \longrightarrow T_*^{vert}(X/Y) \longrightarrow T_* X \longrightarrow f^* T_* Y \longrightarrow 0\,.$$

This gives rise to the decomposition

$$[T_* X] \;=\; [T_*^{vert}(X/Y)] + [f^* T_* Y]$$

in the $K$-group of $X$. Since $f_i : E^{(i)}_{(A;0)} \to E^{(i+1)}_{(A;0)}$ is projective, $f_i^*$ is an exact functor on the category of coherent sheaves. Thus one can employ the above identity iteratingly to the tower of fibrations of $E_{(A;0)}$ by restrictive flag manifolds. Corollary 3.5.5 and the fact that all the graded sheaves of modules involved are locally free together imply the following decomposition of $T_*E_{(A;0)}$ in the $K$-group of $E_{(A;0)}$ :

$$\begin{aligned}
[T_*E_{(A;0)}] \;&=\; \sum_{i=1}^{I}[T_*^{(vert,i)}E_{(A;0)}]\,, \qquad \text{where } T_*^{(vert,I)}E_{(A;0)} := \text{ the pull-back of } T_*Fl_{m_{I,\bullet}}(\mathbb{C}^n)\,,\\
&=\; \sum_{i=1}^{I}[\mathcal{H}om\,(F_\bullet\widehat{\mathcal{S}}_i, F_\bullet^{(i)}\mathcal{E})] - \sum_{i=1}^{I}[\mathcal{H}om\,(F_\bullet\widehat{\mathcal{S}}_i, F_\bullet\widehat{\mathcal{S}}_i)]\\
&=\; \sum_{i=1}^{I}\sum_{1\le j'\le j\le K_i}\left[\mathcal{H}om\left(\widehat{\mathcal{S}}_{i,j}/\widehat{\mathcal{S}}_{i,j-1}\,,\,\widehat{\mathcal{P}}_{i+1,j'}/\widehat{\mathcal{P}}_{i+1,j'-1}\right)\right]\\
&\qquad\qquad -\sum_{i=1}^{I}\sum_{1\le j'\le j\le K_i}\left[\mathcal{H}om\left(\widehat{\mathcal{S}}_{i,j}/\widehat{\mathcal{S}}_{i,j-1}\,,\,\widehat{\mathcal{S}}_{i,j'}/\widehat{\mathcal{S}}_{i,j'-1}\right)\right]\\
&=\; \sum_{i=1}^{I}\sum_{1\le j'\le j\le K_i}\left[\mathcal{H}om\left(\widehat{\mathcal{S}}_{i,j}/\widehat{\mathcal{S}}_{i,j-1}\,,\,\widehat{\mathcal{S}}_{i+1,I_A(i,j')}/\widehat{\mathcal{S}}_{i+1,I_A(i,j'-1)}\right)\right]\\
&\qquad\qquad -\sum_{i=1}^{I}\sum_{1\le j'\le j\le K_i}\left[\mathcal{H}om\left(\widehat{\mathcal{S}}_{i,j}/\widehat{\mathcal{S}}_{i,j-1}\,,\,\widehat{\mathcal{S}}_{i,j'}/\widehat{\mathcal{S}}_{i,j'-1}\right)\right]\\
&=\; \sum_{i=1}^{I}\sum_{1\le j'\le j\le K_i}\;\sum_{I_A(i,j'-1)+1\le k\le I_A(i,j')}\left[\mathcal{H}om\left(\widehat{\mathcal{S}}_{i,j}/\widehat{\mathcal{S}}_{i,j-1}\,,\,\widehat{\mathcal{S}}_{i+1,k}/\widehat{\mathcal{S}}_{i+1,k-1}\right)\right]\\
&\qquad\qquad -\sum_{i=1}^{I}\sum_{1\le j'\le j\le K_i}\left[\mathcal{H}om\left(\widehat{\mathcal{S}}_{i,j}/\widehat{\mathcal{S}}_{i,j-1}\,,\,\widehat{\mathcal{S}}_{i,j'}/\widehat{\mathcal{S}}_{i,j'-1}\right)\right].
\end{aligned}$$

(*b*) *The* $[(T_*HQuot_P(\mathcal{E}^n))|_{E_{(A;B)}}]$*-part*: Recall the exact sequences

$$0 \longrightarrow \oplus_{i=1}^{I}\mathcal{H}om\,(\mathcal{E}_i,\mathcal{E}_i) \longrightarrow \mathcal{G} \longrightarrow \mathcal{K} \longrightarrow 0\,,$$

$$0 \longrightarrow \pi_{1*}\left(\oplus_{i=1}^{I}\mathcal{H}om\,(\mathcal{E}_i,\mathcal{E}_i)\right) \longrightarrow \pi_{1*}\mathcal{G} \longrightarrow \pi_{1*}\mathcal{K} \longrightarrow R^1\pi_{1*}\left(\oplus_{i=1}^{I}\mathcal{H}om\,(\mathcal{E}_i,\mathcal{E}_i)\right) \longrightarrow 0\,,$$

and the filtered sheaves $F_\bullet\,\mathcal{E}_i := \mathcal{E}_{i,\bullet}$ with

$$\mathcal{E}_{i,j}/\mathcal{E}_{i,j-1} \;=\; \left(\pi_1^*\left(\widehat{\mathcal{S}}_{i,j}/\widehat{\mathcal{S}}_{i,j-1}\right)\right)(-a_{i,j}\,z)\,,$$

which is locally free of rank $m_{i,j}$. Recall also that if $f: V \to W$ is a proper morphism and $\mathcal{F}$ is a coherent sheaf on $V$ then the map $f_!(\mathcal{F}) := \sum_i(-1)^i\,[R^if_*\mathcal{F}]$ extends to a morphism of $K$-groups $f_!: K(V)\to K(W)$.

In the $K$-group of $HQuot_P(\mathcal{E}^n)\times\mathbb{C}\mathrm{P}^1$

$$\begin{aligned}
[\mathcal{H}om\,(\mathcal{E}_i,\mathcal{E}_i)] \;&=\; \sum_{j,j'=1}^{K_i}\left[\mathcal{H}om\left(\mathcal{E}_{i,j}/\mathcal{E}_{i,j-1}\,,\,\mathcal{E}_{i,j'}/\mathcal{E}_{i,j'-1}\,\right)\right]\\
&=\; \sum_{j,j'=1}^{K_i}\left[\mathcal{H}om\left((\,\pi_1^*(\widehat{\mathcal{S}}_{i,j}/\widehat{\mathcal{S}}_{i,j-1}))(-a_{i,j}\,z)\,,\,(\,\pi_1^*(\widehat{\mathcal{S}}_{i,j'}/\widehat{\mathcal{S}}_{i,j'-1}))(-a_{i,j'}\,z)\right)\right].
\end{aligned}$$

By the definition of $\mathcal{G}$, $\mathcal{G}$ admits a filtration $F_\bullet\mathcal{G}$ with the associated graded coherent sheaf $\oplus_{i=1}^{I}\mathcal{H}om\,(\mathcal{E}_i,\mathcal{E}_{i+1})$, where $\mathcal{E}_{I+1}$ is set to be $\mathcal{E}$. Together with the tautological filtration of $\mathcal{E}_i$, this gives rise to the identity

$$
\begin{aligned}
[\mathcal{G}] \;&=\; \sum_{i=1}^{I}[\mathcal{H}om\,(\mathcal{E}_i,\mathcal{E}_{i+1})] \;=\; \sum_{i=1}^{I}\;\sum_{\substack{1\le j\le K_i\\ 1\le j'\le K_{i+1}}}\left[\mathcal{H}om\,\left(\mathcal{E}_{i,j}/\mathcal{E}_{i,j-1}\,,\,\mathcal{E}_{i+1,j'}/\mathcal{E}_{i+1,j'-1}\,\right)\right]\\
&=\; \sum_{i=1}^{I}\;\sum_{\substack{1\le j\le K_i\\ 1\le j'\le K_{i+1}}}\left[\mathcal{H}om\,\left(\,(\,\pi_1^*(\widehat{\mathcal{S}}_{i,j}/\widehat{\mathcal{S}}_{i,j-1})\,)(-a_{i,j}\,z)\,,\,(\,\pi_1^*(\widehat{\mathcal{S}}_{i+1,j'}/\widehat{\mathcal{S}}_{i+1,j'-1})\,)(-a_{i+1,j'}\,z)\right)\right].
\end{aligned}
$$

(By convention, $\mathcal{E}_{I+1}=\mathcal{E}$ with trivial filtration and $[\mathcal{H}om\,(\mathcal{E}_I),\mathcal{E}_{I+1}]=\sum_{j=1}^{K_I}[\mathcal{E}_{I,j}/\mathcal{E}_{I,j-1}\,,\,\mathcal{E}]=\sum_{j=1}^{K_I}[(\,\pi_1^*(\widehat{\mathcal{S}}_{I,j}/\widehat{\mathcal{S}}_{I,j-1})\,)(-a_{I,j}\,z)\,,\;\,,\mathcal{E}]$ in the summation.)

Recall Lemma 3.5.2 and [CF1: Appendix]. It follows that

$$
H^1(HQuot_{\,P}(\mathcal{E})\times\mathbb{C}\mathrm{P}^1,\mathcal{G}) \;=\; H^1(HQuot_{\,P}(\mathcal{E})\times\mathbb{C}\mathrm{P}^1,\mathcal{K}) \;=\; 0\,.
$$

Consequently,

$$
\begin{aligned}
[(T_*HQuot_P(\mathcal{E}^n))|_{E_{(A;0)}}] \;&=\; \pi_{1!}\,[\mathcal{K}] \;=\; \pi_{1!}[\mathcal{G}]-\pi_{1!}\sum_{i=1}^{I}[\mathcal{H}om\,(\mathcal{E}_i,\mathcal{E}_i)]\\
&=\; [\pi_{1*}\mathcal{G}] \,-\, \sum_{i=1}^{I}[\pi_{1*}\mathcal{H}om\,(\mathcal{E}_i,\mathcal{E}_i)] \,+\, \sum_{i=1}^{I}[R^1\pi_{1*}\mathcal{H}om\,(\mathcal{E}_i,\mathcal{E}_i)]
\end{aligned}
$$

in the $K$-group of $E_{(A;0)}$.

Putting all these together, expressing $\mathcal{H}om$ of locally free sheaves by tensors with duals and applying the projection formula (e.g. [Ha: III, Exercise 8.3]):

$$
R^i\pi_{1*}\,(\,\pi_1^*(\,\cdot\,)\otimes\mathcal{O}(m)\,) \;=\; (\,\cdot\,)\,\otimes\,R^i\pi_{1*}\,(\mathcal{O}(m))\,,
$$

one leads to the decomposition

$$
\begin{aligned}
&[(T_*HQuot_P(\mathcal{E}^n))|_{E_{(A;0)}}]\\
&=\sum_{i=1}^{I}\;\sum_{\substack{1\le j\le K_i\\ 1\le j'\le K_{i+1}}}\left[\left(\widehat{\mathcal{S}}_{i,j}/\widehat{\mathcal{S}}_{i,j-1}\right)^\vee\otimes\left(\widehat{\mathcal{S}}_{i+1,j'}/\widehat{\mathcal{S}}_{i+1,j'-1}\right)\otimes\pi_{1*}\mathcal{O}(a_{i,j}-a_{i+1,j'})\right]\\
&\quad-\sum_{i=1}^{I}\sum_{j,j'=1}^{K_i}\left[\left(\widehat{\mathcal{S}}_{i,j}/\widehat{\mathcal{S}}_{i,j-1}\right)^\vee\otimes\left(\widehat{\mathcal{S}}_{i,j'}/\widehat{\mathcal{S}}_{i,j'-1}\right)\otimes\pi_{1*}\mathcal{O}(a_{i,j}-a_{i,j'})\right]\\
&\quad+\sum_{i=1}^{I}\sum_{j,j'=1}^{K_i}\left[\left(\widehat{\mathcal{S}}_{i,j}/\widehat{\mathcal{S}}_{i,j-1}\right)^\vee\otimes\left(\widehat{\mathcal{S}}_{i,j'}/\widehat{\mathcal{S}}_{i,j'-1}\right)\otimes R^1\pi_{1*}\mathcal{O}(a_{i,j}-a_{i,j'})\right].
\end{aligned}
$$

Since $\pi_1 : E_{(A;0)} \times \mathbb{C}\mathrm{P}^1 \to E_{(A;0)}$ is the projection map,

$$\pi_{1*}\mathcal{O}(a) \;=\; \mathcal{O}_{E_{(A;0)}} \otimes H^0(\mathbb{C}\mathrm{P}^1, \mathcal{O}(a)) \;=\; \begin{cases} \mathcal{O}_{E_{(A;0)}} \otimes \mathbb{C}^{a+1} & \text{for } a \geq 0 \\ 0 & \text{else} \end{cases}$$

and

$$R^1\pi_{1*}\mathcal{O}(a) \;=\; \mathcal{O}_{E_{(A;0)}} \otimes H^1(\mathbb{C}\mathrm{P}^1, \mathcal{O}(a)) = \begin{cases} \mathcal{O}_{E_{(A;0)}} \otimes \mathbb{C}^{-a-1} & \text{for } a \leq -2 \\ 0 & \text{else} \end{cases} .$$

Observe also that, for a fixed $i = 1, \ldots, I$, the set of indices in $\mathbb{N} \times \mathbb{N}$

$$\{ (j, j') \,|\, 1 \leq j \leq K_i,\, 1 \leq j' \leq K_{i+1},\, a_{i,j} - a_{i+1,j'} \geq 0 \}$$

coincides with the set of indices

$$\{ (j, j') \,|\, 1 \leq j \leq K_i,\, I_A(i, j'' - 1) + 1 \leq j' \leq I_A(i, j'') \quad \text{with } j'' \text{ running over } [1, j] \} ,$$

(cf. Figure 2-1-2). Incorporating these, one has the final expression

$$\begin{aligned}
&[(T_* HQuot_P(\mathcal{E}^n))|_{E_{(A;0)}}] \\
&\quad = \sum_{i=1}^{I} \sum_{1 \leq j'' \leq j \leq K_i} \sum_{I_A(i,j''-1) \leq j' \leq I_A(i,j'')} \left[ \left( \widehat{\mathcal{S}}_{i,j} / \widehat{\mathcal{S}}_{i,j-1} \right)^{\vee} \otimes \left( \widehat{\mathcal{S}}_{i+1,j'} / \widehat{\mathcal{S}}_{i+1,j'-1} \right) \otimes \pi_{1*}\mathcal{O}(a_{i,j} - a_{i+1,j'}) \right] \\
&\qquad - \sum_{i=1}^{I} \sum_{1 \leq j' \leq j \leq K_i} \left[ \left( \widehat{\mathcal{S}}_{i,j} / \widehat{\mathcal{S}}_{i,j-1} \right)^{\vee} \otimes \left( \widehat{\mathcal{S}}_{i,j'} / \widehat{\mathcal{S}}_{i,j'-1} \right) \otimes \pi_{1*}\mathcal{O}(a_{i,j} - a_{i,j'}) \right] \\
&\qquad + \sum_{i=1}^{I} \sum_{1 \leq j < j' \leq K_i} \left[ \left( \widehat{\mathcal{S}}_{i,j} / \widehat{\mathcal{S}}_{i,j-1} \right)^{\vee} \otimes \left( \widehat{\mathcal{S}}_{i,j'} / \widehat{\mathcal{S}}_{i,j'-1} \right) \otimes R^1\pi_{1*}\mathcal{O}(a_{i,j} - a_{i,j'}) \right] .
\end{aligned}$$

(c) *The decomposition of* $\nu(E_{(A;0)}/HQuot_P(\mathcal{E}^n))$: Combining Part (a) and Part (b), one obtains

$$\begin{aligned}
&[\nu(E_{(A;0)}/HQuot_P(\mathcal{E}^n))] \\
&\quad = \sum_{i=1}^{I} \sum_{1 \leq j'' \leq j \leq K_i} \sum_{I_A(i,j''-1) \leq j' \leq I_A(i,j'')} \left[ \left( \widehat{\mathcal{S}}_{i,j} / \widehat{\mathcal{S}}_{i,j-1} \right)^{\vee} \otimes \left( \widehat{\mathcal{S}}_{i+1,j'} / \widehat{\mathcal{S}}_{i+1,j'-1} \right) \otimes \pi_{1*}\mathcal{O}(a_{i,j} - a_{i+1,j'}) \right] \\
&\qquad - \sum_{i=1}^{I} \sum_{1 \leq j' \leq j \leq K_i} \left[ \left( \widehat{\mathcal{S}}_{i,j} / \widehat{\mathcal{S}}_{i,j-1} \right)^{\vee} \otimes \left( \widehat{\mathcal{S}}_{i,j'} / \widehat{\mathcal{S}}_{i,j'-1} \right) \otimes \pi_{1*}\mathcal{O}(a_{i,j} - a_{i,j'}) \right] \\
&\qquad + \sum_{i=1}^{I} \sum_{1 \leq j < j' \leq K_i} \left[ \left( \widehat{\mathcal{S}}_{i,j} / \widehat{\mathcal{S}}_{i,j-1} \right)^{\vee} \otimes \left( \widehat{\mathcal{S}}_{i,j'} / \widehat{\mathcal{S}}_{i,j'-1} \right) \otimes R^1\pi_{1*}\mathcal{O}(a_{i,j} - a_{i,j'}) \right]
\end{aligned}$$

$$-\sum_{i=1}^{I}\sum_{1\le j'\le j\le K_i}\sum_{I_A(i,j'-1)+1\le k\le I_A(i,j')}\left[\left(\widehat{\mathcal{S}}_{i,j}/\widehat{\mathcal{S}}_{i,j-1}\right)^\vee\otimes\left(\widehat{\mathcal{S}}_{i+1,k}/\widehat{\mathcal{S}}_{i+1,k-1}\right)\right]$$
$$+\sum_{i=1}^{I}\sum_{1\le j'\le j\le K_i}\left[\left(\widehat{\mathcal{S}}_{i,j}/\widehat{\mathcal{S}}_{i,j-1}\right)^\vee\otimes\left(\widehat{\mathcal{S}}_{i,j'}/\widehat{\mathcal{S}}_{i,j'-1}\right)\right].$$

**An exact expression of the $S^1$-equivariant Euler class $e_{S^1}(E_{(A;0)}/HQuot_P(\mathcal{E}^n))$.**

Since $\widehat{\mathcal{S}}_{i,j}/\widehat{\mathcal{S}}_{i,j-1}$ are bundles on $E_{(A;0)}$ rather than on $E_{(A;0)}\times\mathbb{C}\mathrm{P}^1$, the $S^1$-action on $\mathbb{C}\mathrm{P}^1$ induces only the trivial action on them. Thus in terms of Chern roots and $S^1$-weights

$$c_{S^1}\left(\widehat{\mathcal{S}}_{i,j}/\widehat{\mathcal{S}}_{i,j-1}\right)\;=\;c\left(\widehat{\mathcal{S}}_{i,j}/\widehat{\mathcal{S}}_{i,j-1}\right)\;=\;\prod_{k=1}^{m_{i,j}}(1+y_{i,j;k}).$$

The $S^1$-action on $\mathcal{E}_{i,j}/\mathcal{E}_{i,j-1}$ induces an $S^1$-action on their dual, tensor products, and also direct image sheaves of any of these: $\pi_{1*}(\,\cdot\,)$ and $R^1\pi_{1*}(\,\cdot\,)$ on $E_{(A;0)}$. This induced $S^1$-action on $\pi_{1*}(\,\cdot\,)$ and $R^1\pi_{1*}(\,\cdot\,)$ coincides with the $S^1$-action induced from that on the related $H^0(\mathbb{C}\mathrm{P}^1,\mathcal{O}(m))$ and $H^1(\mathbb{C}\mathrm{P}^1,\mathcal{O}(m))$ respectively. The $S^1$-weight system for the latter can be computed directly by the Čech representation of sheaf cohomologies, e.g. [Ha: III.5]:

$$H^0(\mathbb{C}\mathrm{P}^1,\mathcal{O}(m)),\;m\ge 0\;:\;\{0,\,1,\,\cdots,\,m\},$$

represented by $1,\,z,\,\ldots,\,z^m$ on $U_0$,

$$H^1(\mathbb{C}\mathrm{P}^1,\mathcal{O}(m)),\;m\le -2\;:\;\{m+1,\,m+2,\,\ldots,\,-2,\,-1\},$$

represented by $z^{m+1},\,z^{m+2},\,\ldots,\,z^{-2},\,z^{-1}$ on $U_0\cap U_\infty$, and the $S^1$-weight system of the sheaf cohomology groups is the empty set for any other choice of $m$. Denote the irreducible representation of $S^1=U(1)$ with weight $w$ by $\gamma^{(w)}(\simeq\mathbb{C})$ and define $I'_A(i,j)$ to be the maximal $l$, $1\le l\le K_{i+1}$ such that $\widehat{\mathcal{S}}_{i,j}\subset\widehat{\mathcal{S}}_{i+1,l}$ with $a_{i,j}\le a_{i+1,l}-1$ and $I''_A(i,j)$ to be the minimal $l$ such that $a_{i,j}\le a_{i,l}-2$. Then, after cancellation of identical terms, one can express $[\nu(E_{(A;0)}/HQuot_P(\mathcal{E}^n))]$ as

$$\begin{aligned}
&[\nu(E_{(A;0)}/HQuot_P(\mathcal{E}^n))]\\
&=\sum_{i=1}^{I}\sum_{1\le j''\le j\le K_i}\sum_{I_A(i,j''-1)\le j'\le I'_A(i,j'')}\left[\left(\widehat{\mathcal{S}}_{i,j}/\widehat{\mathcal{S}}_{i,j-1}\right)^\vee\otimes\left(\widehat{\mathcal{S}}_{i+1,j'}/\widehat{\mathcal{S}}_{i+1,j'-1}\right)\otimes\left(\gamma^{(1)}\oplus\cdots\oplus\gamma^{(a_{i,j}-a_{i+1,j'})}\right)\right]\\
&\quad-\sum_{i=1}^{I}\sum_{1\le j'<j\le K_i}\left[\left(\widehat{\mathcal{S}}_{i,j}/\widehat{\mathcal{S}}_{i,j-1}\right)^\vee\otimes\left(\widehat{\mathcal{S}}_{i,j'}/\widehat{\mathcal{S}}_{i,j'-1}\right)\otimes\left(\gamma^{(1)}\oplus\cdots\oplus\gamma^{(a_{i,j}-a_{i,j'})}\right)\right]\\
&\quad+\sum_{i=1}^{I}\sum_{1\le j<I''_A(i,j)\le j'\le K_i}\left[\left(\widehat{\mathcal{S}}_{i,j}/\widehat{\mathcal{S}}_{i,j-1}\right)^\vee\otimes\left(\widehat{\mathcal{S}}_{i,j'}/\widehat{\mathcal{S}}_{i,j'-1}\right)\otimes\left(\gamma^{(a_{i,j}-a_{i,j'}+1)}\oplus\cdots\oplus\gamma^{(-1)}\right)\right].
\end{aligned}$$

Let $\alpha = c_1(\mathcal{O}_{\mathbb{CP}^\infty}(1))$. Putting all these together, applying the rule for Chern roots under tensor products and Lemma 3.3.2 in [L-L-L-Y], one concludes that

$$
\begin{aligned}
& e_{S^1}(\nu(E_{(A;0)}/HQuot_P(\mathcal{E}^n))) \\
&= \left( \prod_{i=1}^{I} \prod_{1\le j''\le j\le K_i} \prod_{I_A(i,j''-1)\le j'\le I'_A(i,j'')} \prod_{k=1}^{m_{i,j}} \prod_{k'=1}^{m_{i+1,j'}} \prod_{l=1}^{a_{i,j}-a_{i+1,j'}} (-y_{i,j;k} + y_{i+1,j';k'} - l\alpha) \right) \\
&\quad \cdot \left( \prod_{i=1}^{I} \prod_{1\le j'<j\le K_i} \prod_{k=1}^{m_{i,j}} \prod_{k'=1}^{m_{i,j'}} \prod_{l=1}^{a_{i,j}-a_{i,j'}} (-y_{i,j;k} + y_{i,j';k'} - l\alpha) \right)^{-1} \\
&\quad \cdot \left( \prod_{i=1}^{I} \prod_{1\le j<I''_A(i,j)\le j'\le K_i} \prod_{k=1}^{m_{i,j}} \prod_{k'=1}^{m_{i,j'}} \prod_{l=a_{i,j}-a_{i,j'}+1}^{-1} (-y_{i,j;k} + y_{i,j';k'} - l\alpha) \right) \\
&= \left( \prod_{i=1}^{I} \prod_{1\le j''\le j\le K_i} \prod_{I_A(i,j''-1)\le j'\le I'_A(i,j'')} \prod_{k=1}^{m_{i,j}} \prod_{k'=1}^{m_{i+1,j'}} \prod_{l=1}^{a_{i,j}-a_{i+1,j'}} (-y_{i,j;k} + y_{i+1,j';k'} - l\alpha) \right) \\
&\quad \cdot \left( \prod_{i=1}^{I} \prod_{1\le j<j'\le K_i} \prod_{k=1}^{m_{i,j}} \prod_{k'=1}^{m_{i,j'}} (-1)^{m_{i,j}m_{i,j'}(a_{i,j'}-a_{i,j}-1)} \Big( -y_{i,j';k'} + y_{i,j;k} - (a_{i,j'}-a_{i,j})\,\alpha \Big) \right)^{-1} .
\end{aligned}
$$

*Remark 3.5.6* [$e_{S^1}(\nu(E_{(A;0)}/HQuot_P(\mathcal{E}^n)))$ *invertible*]. Observe that in the K-group decomposition of $\nu(E_{(A;0)}/HQuot_P(\mathcal{E}^n))$ all the direct summands with null $S^1$-weight are cancelled. Consequencely, $e_{S^1}(\nu(E_{(A;0)}/HQuot_P(\mathcal{E}^n)))$ is an invertible element in $A^*(E_{(A;0)})(\alpha)$, as it should be and is manifest from the final exact expression above.

*Remark 3.5.7* [*Grassmannian manifold*]. For $X = Gr_r(\mathbb{C}^n)$, let $(\alpha_\bullet;0) = (A;0)$, $K = K_1$, $m_j = m_{1,j}$, $a_j = a_{1,j}$, and $y_{j;k} = y_{i,j;k}$. Then the above expression simplifies to

$$
\begin{aligned}
& e_{S^1}(\nu(E_{(\alpha_\bullet;0)}/Quot_P(\mathcal{E}^n))) \\
&= \frac{\prod_{1\le j\le K} \prod_{k=1}^{m_j} \prod_{l=1}^{a_j} (-y_{j;k} - l\alpha)^n}{\prod_{1\le j<j'\le K} \prod_{k=1}^{m_j} \prod_{k'=1}^{m_{j'}} (-1)^{m_j m_{j'}(a_{j'}-a_j-1)} (-y_{i,j';k'} + y_{i,j;k} - (a_{j'}-a_j)\,\alpha)}
\end{aligned}
$$

in [B-CF-K].

## 3.6 An exact computation of $\int_X \tau^* e^{H\cdot t} \cap \mathbf{1}_d$.

Recall the tower of fibrations of $E_{(A;0)}$ obtained by forgetting one by one the subsheaves in an inclusion sequence of subsheaves. It fits into the following commutative diagram:

$$\begin{array}{ccccccccccc}
E_{(A;0)} = E^{(1)}_{(A;0)} & \stackrel{f_1}{\longrightarrow} & \cdots & \stackrel{f_{i-1}}{\longrightarrow} & E^{(i)}_{(A;0)} & \stackrel{f_i}{\longrightarrow} & \cdots & \stackrel{f_{I-1}}{\longrightarrow} & E^{(I)}_{(A;0)} = Fl_{m_{I,\bullet}}(\mathbb{C}^n) & \stackrel{f_I}{\longrightarrow} & pt \\
\downarrow p & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \| \\
Fl_{r_1,\,\ldots,\,r_I}(\mathbb{C}^n) & \longrightarrow & \cdots & \longrightarrow & Fl_{r_i,\,\ldots,\,r_I}(\mathbb{C}^n) & \longrightarrow & \cdots & \longrightarrow & Gr_{r_I}(\mathbb{C}^n) & \longrightarrow & pt
\end{array} .$$

Each $f_i$ is a bundle map with fiber the restrictive flag manifold

$$Fl_{r_{i,1},\,\ldots,\,r_{i,K_i}}(\mathbb{C}^n, \Pi_{i+1,\bullet}) \;=\; Fl_{r_{i,1},\,\ldots r_{i,K_i}}(\Pi_{i+1,K_i}, \Pi_{i+1,\bullet})\,.$$

To integrate a cohomology class over $E_{(A;0)}$ is the same as to push forward that class from $E_{(A;0)}$ to a class on a point. In this section, we shall give an exact expression of this integral via a sequence of push-forwards following the above tower of fibrations.

### The associated roof of the tower of fibrations of $E_{(A;0)}$.

Let $\mathcal{S}_i$ be the tautological subbundle on $Fl_{r_i,\,\ldots,\,r_I}(\mathbb{C}^n)$. Its pull-back to $E^{(i)}_{(A;0)}$ will be denoted the same. Since $\mathcal{P}_{i+1,K_i} = \mathcal{S}_{i+1}$, the restrictive flag manifold bundle $f_i : E^{(i)}_{(A;0)} \rightarrow E^{(i+1)}_{(A;0)}$ over $E^{(i+1)}_{(A;0)}$ is the one associated to the data: (1) inclusion sequence of subbundles of $\mathcal{S}_{i+1}$: $\mathcal{P}_{i+1,\bullet}\,:\,\mathcal{P}_{i+1,1} \hookrightarrow \cdots, \hookrightarrow \mathcal{P}_{i+1,K_i} \hookrightarrow \mathcal{S}_{i+1}$, and (2) sequence of integers: $0 < r_{i,1} < \cdots < r_{i,K_i}$. In the notation of Sec. 3.2, $f_i : E^{(i+1)}_{(A;0)} \rightarrow E^{i+1}_{(A;0)}$ is simply the bundle map $Fl_{r_{i,1},\,\ldots,\,r_{i,K_i}}(\mathcal{S}_{i+1}, \mathcal{P}_{i+1,\bullet}) \rightarrow E^{(i+1)}_{(A;0)}$. Let $E'^{(i)}_{(A;0)} := Fl_{r_{i,1},\,\ldots,\,r_{i,K_i}}(\mathcal{S}_{i+1})$. Then from Sec. 3.2 one has the following commutative diagram:

$$\begin{array}{ccccccccccccc}
 & & & & E'^{(i-1)}_{(A;0)} & & & & E'^{(i)}_{(A;0)} & & & & \\
 & & & \iota_{i-1}\nearrow & & \searrow f'_{i-1} & & \iota_i \nearrow & & \searrow f'_i & & & \\
\cdots & \stackrel{f_{i-2}}{\longrightarrow} & E^{i-1}_{(A;0)} & & \stackrel{f_{i-1}}{\longrightarrow} & & E^{(i)}_{(A;0)} & & \stackrel{f_i}{\longrightarrow} & & E^{(i+1)}_{(A;0)} & \stackrel{f_{i+1}}{\longrightarrow} & \cdots
\end{array} ,$$

where $\iota_i : E^{(i)}_{(A;0)} \rightarrow E'^{(i)}_{(A;0)}$ is the canonical embedding and $f'_i : E'^{(i)}_{(A;0)} \rightarrow E^{(i+1)}_{(A;0)}$ is the natural flag manifold bundle map. We shall call the above diagram the *associated roof* of the tower of fibrations of $E_{(A;0)}$.

### The push-forward/integration formula for $f_i$.

We now discuss an explicit form for each $f_{i*} : A^*(E^{(i)}_{(A;0)}) \rightarrow A^*(E^{(i+1)}_{(A;0)})$ that follows from Lemma 3.2.2 and [Br: Proposition 2.1].

**Fact 3.6.1 [push-forward formula for $f'_i$].** ([Br: Proposition 2.1]; see also [B-CF-K] and [H-B-J: Chapter 4].) *Recall the Chern roots* $\{y_{i,j;k}\}_{k=1}^{m_{i,j}}$ *of* $\mathcal{S}_{i,j}/\mathcal{S}_{i,j-1}$, $1 \leq j \leq K_i$. *Let*

*$\{ y_{i+1;k} \}_{k=1}^{r_{i+1}-r_i}$ be the Chern roots of $\mathcal{S}_{i+1}/\mathcal{S}_i$. For notation uniformality, let $y_{i,K_i+1;k} := y_{i+1;k}$ and $m_{i,K_i+1} := r_{i+1} - r_i$. Let*

$$P \ \in \ A^*(E^{(i+1)}_{(A;0)})\,[\,y_{i,j;k}\,|\,1 \le j \le K_i + 1\,; \ \ \textit{for each } j\,,\ 1 \le k \le m_{i,j}\,]$$

*represent a class in $A^*({E'}^{(i)}_{(A;0)})$. Then*

$$f'_{i\,*}\,P \ = \sum_{\overline{\sigma} \in Sym_{(i+1)}/Sym_{(i,1)}\times\cdots\times Sym_{(i,K_i+1)}} \overline{\sigma}\cdot\left(\frac{P}{\prod_{1\le j<j'\le K_i+1}\prod_{k=1}^{m_{i,j}}\prod_{k'=1}^{m_{i,j'}}(y_{i,j';k'} - y_{i,j;k})}\right)$$

*in $A^*(E^{(i+1)}_{(A;0)})$, where $Sym_{(i+1)}$ is the permutation group of $r_{i+1}$-many letters, acting on the set $\{ y_{i,j;k} \}_{j,k}$, and $Sym_{(i,j)}$ is the permutation group for the set $\{ y_{i,j;k} \}_k$.*

Note that both the numerator and the denominator of the above fraction are invariant under the $Sym_{(i,1)}\times\cdots\times Sym_{(i,K_i+1)}$-action; thus the $Sym_{(i+1)}/Sym_{(i,1)}\times\cdots\times Sym_{(i,K_i+1)}$-action on the fraction is well-defined.

**Corollary 3.6.2 [push-forward formula for $f_i$].** *Let $P \in \iota_i^* A^*({E'}^{(i)}_{(A;0)}) \subset A^*(E^{(i)}_{(A;0)})$ be expressed in terms of the Chern roots as in Fact 3.6.1. Then*

$$f_{i\,*}\,P \ = \sum_{\overline{\sigma} \in Sym_{(i+1)}/Sym_{(i,1)}\times\cdots\times Sym_{(i,K_i+1)}} \overline{\sigma}\cdot\left(\frac{P\ \cdot\ \Omega(\mathcal{P}_{i+1,\bullet})}{\prod_{1\le j<j'\le K_i+1}\prod_{j''=1}^{m_{i,j}}\prod_{j'''=1}^{m_{i,j'}}(y_{i,j';j'''} - y_{i,j;j''})}\right)$$

*in $A^*(E^{(i+1)}_{(A;0)})$, where*

$$\Omega(\mathcal{P}_{i+1,\bullet}) \ = \ \prod_{j=1}^{K_i}\ \prod_{k'=1}^{r_{i+1}-l_{i+1,j}}\ \prod_{k''=1}^{m_{i,j}} (q_{i+1,j;k'} - y_{i,j;k''})\,,$$

*with $\{ q_{i+1,j;k'} \}_{k'}$ being the set of Chern roots of $\mathcal{S}_{i+1}/\mathcal{P}_{i+1,j}$, is the Poincaré dual of $[E^{(i)}_{(A;0)}]$ in $A^*({E'}^{(i)}_{(A;0)})$, described in Lemma 3.2.2.*

**An exact expression of the integral.**

Recall $X = Fl_{r_1,\ldots,r_I}(\mathbb{C}^n)$, the $I$-tuples of integers $d = (d_1,\,\ldots,\,d_I)$, the $I$-tuple of hyperplane classes $H = (H_1,\,\ldots,\,H_I)$ from the embedding $X \hookrightarrow \mathbb{C}\mathrm{P}^{\binom{n}{r_1}-1} \times \cdots \times \mathbb{C}\mathrm{P}^{\binom{n}{r_I}-1}$ that generate $H^2(X,\mathbb{Z})$, and the associated $I$-tuple of Kähler parameters $t = (t_1,\,\ldots,\,t_I)$. Denote $Sym_{(i+1)}/Sym_{(i,1)}\times\cdots\times Sym_{(i,K_i+1)}$ by $\overline{Sym}_{(i+1)}$. Applying Corollary 3.6.2 to the sequence of fibrations $f_i$ as a subfibration of $f'_i$, one concludes that

$$
\begin{aligned}
\int_X \tau^* e^{H\cdot t} \cap \mathbf{1}_d \;=\; & \sum_A \int_{E_{(A;0)}} \frac{g^*\,\psi^*\,e^{\kappa\cdot\zeta}}{e_{S^1}(E_{(A;0)}/HQuot_P(\mathcal{E}^n))} \\
=\; & \sum_A f_{I*}\circ\cdots\circ f_{1*}\left(\frac{e^{-\sum_{i=1}^I \zeta\cdot c_1(\mathcal{S}_{i,K_i})}}{e_{S^1}(E_{(A;0)}/HQuot_P(\mathcal{E}^n))}\right) \\
=\; & \sum_A \sum_{\overline{\sigma}_{I+1}\in\overline{Sym}_{(I+1)}} \overline{\sigma}_{I+1}\cdot\;\cdots \sum_{\overline{\sigma}_2\in\overline{Sym}_{(2)}} \overline{\sigma}_2\,\cdot \\
& \left(\frac{e^{-\sum_{i=1}^I \zeta\cdot c_1(\mathcal{S}_{i,K_i})}\cdot\prod_{i=1}^I \Omega(\mathcal{P}_{i+1,\bullet})}{e_{S^1}(E_{(A;0)})/HQuot_P(\mathcal{E}^n)\cdot\prod_{i=1}^I\prod_{1\le j<j'\le K_i+1}\prod_{k=1}^{m_{i,j}}\prod_{k'=1}^{m_{i,j'}}(y_{i,j';k'}-y_{i,j;k})}\right) \\
=\; & \sum_A \sum_{\overline{\sigma}_{I+1}\in\overline{Sym}_{(I+1)}} \overline{\sigma}_{I+1}\cdot\;\cdots \sum_{\overline{\sigma}_2\in\overline{Sym}_{(2)}} \overline{\sigma}_2\,\cdot \\
& \Bigg[\, e^{-\sum_{i=1}^I (y_{i,1;1}+\cdots+y_{i,1;m_{i,1}}+\cdots+y_{i,K_i;1}+\cdots+y_{i,K_i;m_{i,K_i}})\zeta_i} \\
& \cdot\left(\prod_{i=1}^I \prod_{1\le j''\le j\le K_i}\;\prod_{I_A(i,j''-1)\le j'\le I'_A(i,j'')}\;\prod_{k=1}^{m_{i,j}}\prod_{k'=1}^{m_{i+1,j'}}\prod_{l=1}^{a_{i,j}-a_{i+1,j'}}(-y_{i,j;k}+y_{i+1,j';k'}-l\alpha)\right)^{-1} \\
& \cdot\left(\prod_{i=1}^I\prod_{1\le j<j'\le K_i}\prod_{k=1}^{m_{i,j}}\prod_{k'=1}^{m_{i,j'}}(-1)^{m_{i,j}m_{i,j'}(a_{i,j'}-a_{i,j}-1)}\Big(-y_{i,j';k'}+y_{i,j;k}-(a_{i,j'}-a_{i,j})\,\alpha\Big)\right) \\
& \cdot\left(\prod_{i=1}^I\prod_{j=1}^{K_i}\prod_{k'=1}^{r_{i+1}-l_{i+1,j}}\prod_{k''=1}^{m_{i,j}}(q_{i+1,j;k'}-y_{i,j;k''})\right) \\
& \cdot\left(\prod_{i=1}^I\prod_{1\le j<j'\le K_i+1}\prod_{k=1}^{m_{i,j}}\prod_{k'=1}^{m_{i,j'}}(y_{i,j';k'}-y_{i,j;k})\right)^{-1}\Bigg].
\end{aligned}
$$

*Remark 3.6.3* [*hypergeometric series*]. Note that a fixed $\mathbb{T}^n$-action on $\mathbb{C}^n$ induces a $\mathbb{T}^n$-action on $E_{(A;0)}$ and a compatible $\mathbb{T}^n$-action on the total space all the bundles on $E_{(A;0)}$ whose Chern roots are involved above. Thus, once a $\mathbb{T}^n$-action on $\mathbb{C}^n$ is fixed, all our discussion has a $\mathbb{T}^n$-equivariant extension. In particular, the class $A_d$ is the non-equivariant limit of a $\mathbb{T}^n$-equivariant class. Recall from [L-L-Y1, II: Lemma 2.5] the fact that *the zero class $\omega=0$ is the only class in $H^*_{\mathbb{T}^n}(X)$ such that $\int_X e^{H\cdot\zeta}\cap\omega=0$ for all generic $\zeta\in\mathbb{C}$.* This implies that the integral $\int_X \tau^* e^{H\cdot t}\cap\mathbf{1}_d$ determines the class $\mathbf{1}_d$ in $H^*_{S^1}(X)$ uniquely and, hence, the fundamental hypergeometric series $HG[\mathbf{1}]^X(t)$.

## 4 Remarks on the Hori-Vafa conjecture.

We conclude this paper with some remarks on the Hori-Vafa conjecture.

There are three aspects of stringy dualities that have led to various miraculous conjectural relations among mathematical objects and quantities: the string world-sheet field theory aspect, the string target space-time field theory aspect, and the lower-dimensional effective field theory aspect after compactifications. One important example is the phenomenon of mirror symmetry of Calabi-Yau 3-folds: world-sheet aspects from nonlinear sigma models that give rise to equivalent $d = 2$, $N = (2, 2)$ conformal field theories via a $U(1)$-charge redefinition and effective field theory aspects from compactification of $d = 10$ superstring theories to equivalent $d = 4$, $N = 2$ supersymmetric field theories via a field redefinition. (Cf. Key word search: "*duality*", "*mirror*" from `www.arXiv.org/hep-th`)

In [H-V], Hori and Vafa generalize the world-sheet aspects of mirror symmetry to being the equivalence of $d = 2$, $N = (2, 2)$ supersymmetric field theories (i.e. without imposing the conformal invariance on the theory). This leads them to a much broader encompassing picture of mirror symmetry. (See [HKKPTVVZ] for full explanations.) Putting this in the frame work of abelian gauged linear sigma models (GLSM) ([Wi1]), studying the effective field theories expanded around points in various phases on the theory space of a GLSM, and taking the generalized mirror of these theories enable them to link many $d = 2$ field theories together. Generalization of this setting to nonabelian GLSM ([Wi1: Sec. 5.3]) leads them to the following conjecture, when the physical path integrals are interpreted appropriately mathematically:

**Conjecture 4.1 [Hori-Vafa].** [H-V: Appendix A]. *The hypergeometric series for a given homogeneous space* (*e.g. a Grassmannian manifold*) *can be reproduced from the hypergeometric series of simpler homogeneous spaces* (*e.g. product of projective spaces*). *Similarly for the twisted hypergeometric series that are related to the submanifolds in homogeneous spaces.*

(Cf. [H-V: Appendix A]; see also [B-CF-K].) In other words, different homogeneous spaces (or some simple quotients of them) can give rise to generalized mirror pairs.

### The Hori-Vafa formula for Grassmannian manifolds.

For $X = Gr_r(\mathbb{C}^n)$, $E_{(A;0)} = E_{(\alpha_\bullet;0)} \subset Quot_{(d)}$ is naturally isomorphic to a flag manifold $Fl_{m_1, m_1+m_2, \ldots, m_1+m_2+\cdots+m_k}(\mathbb{C}^n)$, where $\alpha_\bullet$ is a partition of $d$ into nonnegative integers of length $r$, $m_i$ counts the multiplicity of the identical summands in the partition, $m_1 + \cdots + m_k = r$, (cf. [L-L-L-Y]). The tower of fibrations of $E_{(A;0)}$ in the beginning of Sec. 3.6 is shortened to

$$\begin{array}{ccc} E_{(A;0)} = E_{(\alpha_\bullet;0)} & \stackrel{f}{\longrightarrow} & pt \\ \downarrow p & & \| \\ Gr_r(\mathbb{C}^n) & \longrightarrow & pt \end{array} ,$$

in which both $p$ and $f$ are flag manifold bundle maps. Applying push-formula Fact 3.6.1 to $p$ instead, plugging the exact expression of $e_{S^1}(\nu(E_{(\alpha_\bullet)}/Quot_{P(\mathcal{E}^n)}))$ in Remark 3.5.7, and simplifying, one concludes that ([B-CF-K: Theorem 1.5])

$$\begin{aligned} HG[\mathbf{1}]^X(t) \;&=\; e^{-\frac{H\cdot t}{\alpha}} \sum_{d\geq 0} e^{d\cdot t} \sum_{(\alpha_\bullet)} p_* \left( \frac{1}{e_{S^1}(\nu(E_{(\alpha_\bullet;0)})/Quot_P(\mathcal{E}^n))} \right) \\ &=\; e^{-\frac{H\cdot t}{\alpha}} \sum_{d\geq 0} e^{d\cdot t} \sum_{(\alpha_\bullet)} \frac{(-1)^{(r-1)d} \prod_{1\leq j\leq j'\leq r}(-y_j + y_{j'} - (\alpha_j - \alpha_{j'})\,\alpha)}{\prod_{1\leq j\leq j'\leq r}(-y_j+y_{j'}) \prod_{j''=1}^{r} \prod_{l=1}^{\alpha_{j''}}(-y_{j''} - l\alpha)}\,, \end{aligned}$$

where $\{\, y_j \,\}_j$ are the Chern roots of the tautological subbundle on $Gr_r(\mathbb{C}^n)$. Apply this to the simplest Grassmannian manifolds, i.e. projective spaces, one concludes that

**Corollary/Fact 4.2 [Hori-Vafa formula].** *The Hori-Vafa conjecture holds for Grassmannian manifolds. Explicitly, let* $X = Gr_r(\mathbb{C}^n)$ *(with hyperplane class* $H$*) and* $\mathbb{P} = \prod_{i=1}^{r} \mathbb{C}\mathrm{P}^{n-1}$ *(with hyperplane class* $H_i$ *from the* $i$*-th factor). Define*

$$\Delta = \prod_{j<j'}(y_{j'} - y_j) \quad \textit{and} \quad \mathcal{D}_\Delta \;:=\; \prod_{j<j'} \Big(\, \alpha \frac{\partial}{\partial t_{j'}} - \alpha \frac{\partial}{\partial t_j}\,\Big)\,.$$

*Then*

$$HG[\mathbf{1}]^X(t) \;=\; e^{H(r-1)\pi\sqrt{-1}/\alpha}\, \frac{1}{\Delta}\, \Big(\mathcal{D}_\Delta\, HG[\mathbf{1}]^{\mathbb{P}}(t_1,\,\ldots,\,t_r)\Big)\Big|_{t_i = t+(r-1)\pi\sqrt{-1}}\,.$$

The above formula was derived in [B-CF-K] by using the method and key results in [L-L-L-Y].

**General Hori-Vafa formula.**

For general flag manifold $X = Fl_{r_1,\ldots,r_I}(\mathbb{C}^n)$, one way to obtain a push-forward formula for the the natural bundle map $p : E_{(A;0)} \rightarrow X$ is to modify the tower of fibrations of $E_{(A;0)}$ so far used into another tower of fibrations that is compatible with the morphisms on $X$:

$$E_{(A;0)} = \widehat{E}^{(1)}_{(A;0)} \;\xrightarrow{g_1}\; \cdots \;\xrightarrow{g_{I-1}}\; \widehat{E}^{(I)}_{(A;0)} \;\xrightarrow{g_I}\; Fl_{r_1,\ldots,r_I}(\mathbb{C}^n)\,.$$

Indeed there is a god-given one obtained by forgetting all but the last element in a flag that characterizes $S^1$-invariant subsheaves when forgetting the $S^1$-invariant subsheaves of an inlcusion sequence one by one. Unfortunately the $g_i$ thus obtained is no longer a bundle map: though all the fibers of $g_i$ are restrictive flag manifolds, the topology of these restrictive manifolds can change. Thus techniques need to develop to take care of this if one follows this line.

A second line is to work out a clean inversion formula from [L-L-Y1,II: Lemma 2.5] for the case of flag manifolds. $HG[\mathbf{1}]^X(t)$ can then be reconstructed from the integrals

$\int_X \tau^* e^{H\cdot t} \cap \mathbf{1}_d$. Further simplification via Young tableaux combinatorics may hope to lead to a clean exact form for $HG[\mathbf{1}]^X(t)$.

On the other hand, since $HG[\mathbf{1}]^X(t)$ and $\{\int_X \tau^* e^{H\cdot t} \cap \mathbf{1}_d\}_{d\succcurlyeq 0}$ carry informations differ only by the integral over $X$ and are obtainable from each other, one may re-write our expression of $\int_X \tau^* e^{H\cdot t} \cap \mathbf{1}_d$ more suggestively (but only formally) as

$$\begin{aligned}
\int_X \tau^* e^{H\cdot t} \cap \mathbf{1}_d \;&=\; \sum_A \int_{E_{(A;0)}} \frac{g^*\,\psi^*\,e^{\kappa\cdot\zeta}}{e_{S^1}(E_{(A;0)}/HQuot_P(\mathcal{E}^n))} \\
&= \widetilde{\prod}_{i=1}^{I} \left( \sum_{\overline{\sigma}_{i+1} \in \overline{Sym}_{(i+1)}} \overline{\sigma}_{i+1} \cdot \right. \\
&\qquad \left[ e^{-(y_{i,1;1}+\cdots+y_{i,1;m_{i,1}}+\cdots+y_{i,K_i;1}+\cdots+y_{i,K_i;m_{i,K_i}})\zeta_i} \right. \\
&\qquad \cdot \left( \prod_{1\le j''\le j\le K_i} \; \prod_{I_A(i,j''-1)\le j'\le I'_A(i,j'')} \; \prod_{k=1}^{m_{i,j}} \prod_{k'=1}^{m_{i+1,j'}} \prod_{l=1}^{a_{i,j}-a_{i+1,j'}} (-y_{i,j;k} + y_{i+1,j';k'} - l\alpha) \right)^{-1} \\
&\qquad \cdot \left( \prod_{1\le j<j'\le K_i} \prod_{k=1}^{m_{i,j}} \prod_{k'=1}^{m_{i,j'}} (-1)^{m_{i,j}m_{i,j'}(a_{i,j'}-a_{i,j}-1)} \left( -y_{i,j';k'} + y_{i,j;k} - (a_{i,j'}-a_{i,j})\,\alpha \right) \right) \\
&\qquad \cdot \left( \prod_{j=1}^{K_i} \prod_{k'=1}^{r_{i+1}-l_{i+1,j}} \prod_{k''=1}^{m_{i,j}} (q_{i+1,j;k'} - y_{i,j;k''}) \right) \\
&\qquad \left. \left. \cdot \left( \prod_{1\le j<j'\le K_i+1} \prod_{k=1}^{m_{i,j}} \prod_{k'=1}^{m_{i,j'}} (y_{i,j';k'} - y_{i,j;k}) \right)^{-1} \right] \right).
\end{aligned}$$

where $\widetilde{\prod}_{i=1}^{I}$ is a constrained product (i.e. not all summands in each factor can be picked out for multiplication when expanding the product; rather they have to satisfty specific admissible conditions determined by $A$). The level structure indexed by $i$ suggests a version of "*family Hori-Vafa formula*" generalizing the case of Grassmannian manifolds while the appearance of Thom classes in the expression suggests a version of "*quantum submanifold formula*" generalizing the case of complete intersection submanifolds.

Finally, even if all these technicalities are settled and Hori-Vafa conjecture is checked, there is still a final question: *Why do they go this way?* For that one has to turn to the most fundamental understanding of quantum field theories and path integrals.

With all these outlooks - and amazement and puzzles as well -, we conclude this paper temporarily here.

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