SCALAR FLAT METRICS OF EGUCHI-HANSON TYPE

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ABSTRACT. We use a new method to construct a class of asymptotically locally flat, scalar flat metrics. These metrics were constructed via algebraic geometry method by LeBrun before and provide counterexamples to the generalized positive action conjecture of Hawking and Pope.
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1. INTRODUCTION

The scalar flat metrics play an important role in general relativity. In Ref. [1], LeBrun constructed scalar flat metrics on the total spaces of complex line-bundles over CP_1 for which the first Chern class satisfies $c_1 < -2$. Those metrics are asymptotically locally Euclidean and have negative total mass. Therefore they provide counter-examples to the Generalized Positive Action Conjecture of Hawking and Pope [2]. (This conjecture asserts that the total mass is nonnegative for all locally asymptotically flat Riemannian 4-manifolds with zero scalar curvature, and that the total mass vanishes iff the manifolds is Ricci flat with self-dual Weyl curvature.) The metrics constructed by LeBrun are indeed Eguchi-Hanson type. Those with vanishing total mass were obtained by Eguchi and Hanson before and serve as a class of gravitational instantons in Euclidean gravity [3, 4].

In this note we will use a different method to construct those metrics in [1]. We define a class of Eguchi-Hanson type metrics which are parameterized by a real function f. By solving the zero scalar curvature equation directly, we determine the f and obtain the metrics. These metrics are parameterized by real numbers A, B (see Section III for details), and A is proportional to the total mass. We determine all possible A, B which the metrics can be regularized. We find an interesting positive mass phenomena that when $B < 0$, the metrics must have strictly positive mass if they can be regularized. Finally, we compute certain topological invariants of these metrics. Unlike the case of Eguchi-Hanson metrics with vanishing A, the first Pontrjagin class and the signature in the current case relate to the total mass which may not be integer.

2. Eguchi-Hanson metric

Let (x_0, x_1, x_2, x_3) be the four Euclidean coordinates so that the flat metric of $\mathbb{R}^4$ is given by

$$ds_0^2 = dx_0^2 + dx_1^2 + dx_2^2 + dx_3^2.$$

Let θ, ϕ, ψ be the Euler angles on the 3-sphere S^3 with ranges

$$0 \leq \theta \leq \pi, \;\; 0 \leq \phi \leq 2\pi, \;\; 0 \leq \psi \leq 4\pi,$$

and are related to the Cartesian coordinates by

$$\begin{aligned} x_1 &= r\cos\frac{\theta}{2}\cos\frac{\psi+\phi}{2}, \\ x_2 &= r\cos\frac{\theta}{2}\sin\frac{\psi+\phi}{2}, \\ x_3 &= r\sin\frac{\theta}{2}\cos\frac{\psi-\phi}{2}, \\ x_0 &= r\sin\frac{\theta}{2}\sin\frac{\psi-\phi}{2}. \end{aligned}$$

Let σ_1, σ_2 and σ_3 are the Cartan-Maurer forms for $SU(2) \approx S^3$ which are defined by

$$\begin{aligned} \sigma_1 &= \frac{1}{r^2}\big(x_1 dx_0 - x_0 dx_1 + x_2 dx_3 - x_3 dx_2\big) \\ &= \frac{1}{2}\big(\sin\psi d\theta - \sin\theta\cos\psi d\phi\big), \\ \sigma_2 &= \frac{1}{r^2}\big(x_2 dx_0 - x_0 dx_2 + x_3 dx_1 - x_1 dx_3\big) \\ &= \frac{1}{2}\big(-\cos\psi d\theta - \sin\theta\sin\psi d\phi\big), \\ \sigma_3 &= \frac{1}{r^2}\big(x_3 dx_0 - x_0 dx_3 + x_1 dx_2 - x_2 dx_1\big) \\ &= \frac{1}{2}\big(d\psi + \cos\theta d\phi\big). \end{aligned}$$

It is obvious that

$$d\sigma_1 = 2\sigma_2 \wedge \sigma_3, cyclic.$$

The flat metric can be written in polar coordinates as

$$ds_0^2 = dr^2 + r^2\big(\sigma_1^2 + \sigma_2^2 + \sigma_3^2\big). \tag{2.1}$$

In Refs. [3, 4], Eguchi and Hanson found the following self-dual solutions to the Euclidean gravity

$$ds^2 = \Big(1 - \frac{B}{r^4}\Big)^{-1} dr^2 + r^2\Big(\sigma_1^2 + \sigma_2^2 + \big(1 - \frac{B}{r^4}\big)\sigma_3^2\Big) \tag{2.2}$$

with $B \geq 0$. The metric is geodesically complete when $r \geq \sqrt[4]{B}$, $0 \leq \psi \leq 2\pi$. This range of ψ causes the constant-r hypersurfaces as $r \to \infty$ are not 3-sphere, but group manifold of $SO(3) = P_3(\mathbb{R})$. This is an explicit example of a metric whose topology is asymptotically locally Euclidean, but not globally Euclidean. The curvature components of this metric are

$$\begin{aligned}
R^1_{\ 0} = R^2_{\ 3} &= -\frac{2B}{r^6}\big(e^1 \wedge e^0 + e^2 \wedge e^3\big),\\
R^2_{\ 0} = R^3_{\ 1} &= -\frac{2B}{r^6}\big(e^2 \wedge e^0 + e^3 \wedge e^1\big),\\
R^3_{\ 0} = R^1_{\ 2} &= \frac{4B}{r^6}\big(e^3 \wedge e^0 + e^1 \wedge e^2\big).
\end{aligned}$$

where

$$e^0 = f^{-\frac{1}{2}} dr, \;\; e^1 = r\sigma_1, \;\; e^2 = r\sigma_2, \;\; e^3 = r f^{\frac{1}{2}} \sigma_3$$

and

$$f = \sqrt{1 - \frac{B}{r^4}}.$$

The metric is Ricci flat and has zero action, and serves as a class of gravitational instantons.

3. Zero scalar curvature

We shall find a function f such that the following Eguchi-Hanson type metric has zero scalar curvature

$$ds^2 = f^{-2} dr^2 + r^2\Big(\sigma_1^2 + \sigma_2^2 + f^2\sigma_3^2\Big) \tag{3.3}$$

where f is a function of r. Let the coframe $\{e^i\}$ be

$$e^0 = f^{-1} dr, \;\; e^1 = r\sigma_1, \;\; e^2 = r\sigma_2, \;\; e^3 = r f \sigma_3.$$

Define the connection 1-form $\{\omega^i_{\ j}\}$ by $de^i = -\omega^i_{\ j} \wedge e^j$. We find that

$$\begin{aligned}
&\omega^1_{\ 0} = \frac{f}{r} e^1, \;\; \omega^2_{\ 0} = \frac{f}{r} e^2, \;\; \omega^3_{\ 0} = \big(\frac{f}{r} + f'\big) e^3,\\
&\omega^2_{\ 3} = \frac{f}{r} e^1, \;\; \omega^3_{\ 1} = \frac{f}{r} e^2, \;\; \omega^1_{\ 2} = \big(\frac{2}{rf} - \frac{f}{r}\big) e^3.
\end{aligned}$$

The curvature tensor is defined as $R^i_{\ j} = d\omega^i_{\ j} + \omega^i_{\ k} \wedge \omega^k_{\ j}$,

$$
\begin{aligned}
R^1_{\ 0} &= R^2_{\ 3} = -\frac{ff'}{r}\big(e^1 \wedge e^0 + e^2 \wedge e^3\big),\\
R^2_{\ 0} &= R^3_{\ 1} = -\frac{ff'}{r}\big(e^2 \wedge e^0 + e^3 \wedge e^1\big),\\
R^3_{\ 0} &= -\big(ff'' + (f')^2 + \frac{3}{r}ff'\big)e^3 \wedge e^0 + \frac{2ff'}{r}e^1 \wedge e^2,\\
R^1_{\ 2} &= \frac{2ff'}{r}e^3 \wedge e^0 - \frac{4(f^2-1)}{r}e^1 \wedge e^2.
\end{aligned}
$$

So the scalar curvature

$$R = -2\Big(ff'' + \big(f'\big)^2 + \frac{7}{r}ff' + \frac{4}{r^2}\big(f^2 - 1\big)\Big).$$

The zero scalar curvature is given by the following equation

$$ff'' + \big(f'\big)^2 + \frac{7}{r}ff' + \frac{4}{r^2}\big(f^2 - 1\big) = 0.$$

Let $f = \sqrt{1-h}$, the above equation deduces to

$$h'' + \frac{7}{r}h' + \frac{8}{r^2}h = 0. \tag{3.4}$$

It is easy to find the general solutions $h = \frac{2A}{r^2} + \frac{B}{r^4}$ of equation (3.4). Therefore we can take

$$f = \sqrt{1 - \frac{2A}{r^2} - \frac{B}{r^4}} \tag{3.5}$$

in (3.3) to obtain the scalar flat metric.

4. Regularization of metric

The metric (3.3) may have singularity at $r = 0$ and at the zero set of f. The curvature tensors of (3.3) with f given by (3.5) are

$$
\begin{aligned}
R^1_{\ 0} &= R^2_{\ 3} = -2\Big(\frac{A}{r^4} + \frac{B}{r^6}\Big)\big(e^1 \wedge e^0 + e^2 \wedge e^3\big),\\
R^2_{\ 0} &= R^3_{\ 1} = -2\Big(\frac{A}{r^4} + \frac{B}{r^6}\Big)\big(e^2 \wedge e^0 + e^3 \wedge e^1\big),\\
R^3_{\ 0} &= \frac{4B}{r^6}e^3 \wedge e^0 + 4\Big(\frac{A}{r^4} + \frac{B}{r^6}\Big)e^1 \wedge e^2,\\
R^1_{\ 2} &= 4\Big(\frac{A}{r^4} + \frac{B}{r^6}\Big)e^3 \wedge e^0 + 4\Big(\frac{2A}{r^4} + \frac{B}{r^6}\Big)e^1 \wedge e^2.
\end{aligned}
$$

The case $A = B = 0$ gives rise to the flat metric (2.1). So we assume at least one of A, B is nonzero from now on, and, in this case it is easy to see that $r = 0$ is the essential (curvature) singularity which can not be removed by changing coordinates.

Now the metric (3.3) with f given by (3.5) is regular for $r > r_0$, where

(i) $B > 0$, $-\infty < A < \infty$; or $B < 0$, $A > \sqrt{-B}$, we choose

$$r_0 = \sqrt{A + \sqrt{A^2 + B}}. \tag{4.6}$$

(ii) $B = 0$, $A > 0$, we choose $r_0 = \sqrt{2A}$.
(iii) $B < 0$, $A = \sqrt{-B}$, we choose $r_0 = \sqrt{A}$.
(iv) Otherwise, we choose $r_0 = 0$.

A metric in case (ii) has coordinate singularity at r_0 which can be removed by changing coordinates. However, the manifold has an inner boundary. Note that in case (iii), $\frac{A}{r^4} + \frac{B}{r^6}$ is singular at $r = r_0$. This implies that the curvature tensor is singular at $r = r_0$ and the metric can not be regularized. A metric in case (iv) can never be regularized except that $A, B = 0$. Now we study the case (i). Let

$$u^2 = r^2\Big(1 - \frac{2A}{r^2} - \frac{B}{r^4}\Big).$$

Clearly $u \to 0$ as $r \to r_0$. By changing variable, the metric (3.3) becomes

$$ds^2 = \Big(1 + \frac{B}{r^4}\Big)^{-2} du^2 + u^2\sigma_3^2 + r^2\Big(\sigma_1^2 + \sigma_2^2\Big).$$

For fixed θ, ϕ, it approximates to

$$\Big(1 + \frac{B}{r_0^4}\Big)^{-2} du^2 + \frac{u^2}{4} d\psi^2$$

as $u \to 0$. Therefore if we re-choose the range of ψ such that

$$0 \le \psi \le 4\pi\Big(1 + \frac{B}{r_0^4}\Big)^{-1}, \tag{4.7}$$

we can remove the singularity at $r = r_0$. Note that B/r_0^4 must be a rational number in order that M does not collapse along ψ factor. When $A = 0$, $B/r_0^4 = 1$. So we assume $A \neq 0$, and

$$\sqrt{1 + \frac{B}{A^2}} = \frac{k}{l}$$

for certain positive integers k, l. Then (4.7) is equivalent to

$$0 \le \psi \le 2\pi\Big(1 + \frac{l}{k}\Big). \tag{4.8}$$

Summarizing, if $B > 0$, $-\infty < A < \infty$; or $B < 0$, $A > \sqrt{-B}$, we can obtain a geodesically complete metric (3.3) with f given by (3.5), $r \geq r_0$ where r_0 is given by (4.6), and ψ satisfies (4.7) or (4.8).

5. Action and mass

For asymptotically flat geometries, the physical action of a metric g is defined as [5, 6]

$$I(g) = \frac{1}{16\pi}\int_M R + \frac{1}{8\pi}\int_{\partial M}\big(H - H_0\big) \tag{5.9}$$

where R is the scalar curvature of g, H is the trace of the extrinsic curvature of the boundary, and H_0 is the trace of the extrinsic curvature of the boundary embedded in flat space.

Now we compute the action of the metric (3.3). Let $\omega^0{}_i = h_{ij}\omega^j$ at the constant-r hypersurface. We find

$$h_{11} = -\frac{f}{r},\ h_{22} = -\frac{f}{r},\ h_{33} = -\frac{f}{r} - f',\quad h_{ij} = 0\ (i \neq j).$$

Therefore

$$H = -\frac{3f}{r} - f' = -\frac{3}{r} + \frac{A}{r^3} + O\Big(\frac{1}{r^5}\Big),$$

and $H_0 = -\frac{3}{r}$ which is obtained by taking $A, B = 0$. The value of the following integral is used in this note frequently. Let $\mathcal{D} = \big\{0 \leq \theta \leq \pi,\ \ 0 \leq \phi \leq 2\pi,\ \ 0 \leq \psi \leq 4\pi\Big(1 + \frac{B}{r_0^4}\Big)^{-1}\big\}$,

$$V_0 = \int_{\mathcal{D}} \sigma_1\sigma_2\sigma_3 = 2\pi^2\Big(1 + \frac{B}{r_0^4}\Big)^{-1}.$$

Since (3.3) has zero scalar curvature, we obtain

$$I(g) = \frac{1}{8\pi}\lim_{r\to\infty}\int_{\partial M_r}\frac{A}{r^3}r^3\sigma_x\sigma_y\sigma_z = \frac{\pi A}{4}\Big(1 + \frac{B}{r_0^4}\Big)^{-1}.$$

The action is zero if $A = 0$.

The total mass of an asymptotically (locally) flat metric is proportional to

$$\lim_{r\to\infty}\Big(\partial_j g_{ij} - \partial_i g_{jj}\Big)\frac{x^i}{r}\cdot r^3$$

in Cartesian coordinates. In Ref. [1], LeBrun computed the components of (3.3) in Cartesian coordinates and obtained its total mass. By choosing $2A = -(k-1)a^2$, $B = ka^4$, he found a counter-example to the Generalized Positive Action Conjecture of Hawking and Pope [2]. However, in most spacetime geometry, the polar coordinates play much more important role than the Cartesian coordinates. So we use the definition of the total mass via arbitrary orthonormal basis of the background metric [7]. Let $\{\breve{e}^i\}$ be the coframe of (2.1), i.e.,

$$\breve{e}^0 = dr, \quad \breve{e}^1 = r\sigma_1, \quad \breve{e}^2 = r\sigma_2, \quad \breve{e}^3 = r\sigma_3.$$

Let $\{\breve{e}_i\}$ be their dual frame. Denote $\breve{\nabla}$ the Levi-Civita connection of (2.1). Then the total mass of an asymptotically (locally) flat metric g can be defined as follows:

$$E = \frac{1}{4vol(S^3)} \lim_{r\to\infty} \int_{\partial M_r} \Big(\breve{\nabla}^j g_{ij} - \breve{\nabla}_i tr_{g_0}(g)\Big) \star \breve{e}^i \tag{5.10}$$

where $vol(S^3)$ is the volume of unit 3-sphere, $\star$ is the Hodge star operator of the metric (2.1), $\breve{\nabla}_j = \breve{\nabla}_{\breve{e}_j}$ and $g_{ij} = g\big(\breve{e}_i, \breve{e}_j\big)$.

Now we use (5.10) to compute the total mass of metric (3.3). The connection 1-form of (2.1) is

$$\begin{aligned} \breve{\omega}^1{}_0 = \frac{1}{r}\breve{e}^1, \quad \breve{\omega}^2{}_0 = \frac{1}{r}\breve{e}^2, \quad \breve{\omega}^3{}_0 = \frac{1}{r}\breve{e}^3, \\ \breve{\omega}^2{}_3 = \frac{1}{r}\breve{e}^1, \quad \breve{\omega}^3{}_1 = \frac{1}{r}\breve{e}^2, \quad \breve{\omega}^1{}_2 = \frac{1}{r}\breve{e}^3. \end{aligned}$$

The components of the metric (3.3) are

$$g_{00} = f^{-2}, \quad g_{11} = g_{22} = 1, \quad g_{33} = f^2.$$

Thus

$$\begin{aligned} \breve{\nabla}_i g_{i0} &= \breve{e}_i\big(g_{i0}\big) - g_{0l}\breve{\omega}^l{}_i\big(\breve{e}_i\big) - g_{il}\breve{\omega}^l{}_0\big(\breve{e}_i\big) \\ &= \frac{\partial}{\partial r} g_{00} + g_{00}\frac{3}{r} - \big(g_{11} + g_{22} + g_{33}\big)\frac{1}{r} \\ &= \frac{4A}{r^3} + O\Big(\frac{1}{r^5}\Big), \\ \breve{\nabla}_0 tr_{g_0}(g) &= O\Big(\frac{1}{r^6}\Big). \end{aligned}$$

We obtain

$$\begin{aligned} E &= \frac{1}{4vol(S^3)} \lim_{r\to\infty} \int_{\partial M_r} \Big(\breve{\nabla}^j g_{0j} - \breve{\nabla}_0 tr_{g_0}(g)\Big) \star \breve{e}^0 \\ &= \frac{A}{vol(S^3)} V_0 = A\Big(1 + \frac{B}{r_0^4}\Big)^{-1}. \end{aligned}$$

We obtain an interesting positive mass phenomena that any geodesically complete, zero scalar curved Eguchi-Hanson type metric (3.3) with f given by (3.5) and $B < 0$ must have strictly positive total mass.

6. Topological invariants

In Ref. [4], the authors computed the Euler number $\chi(M)$, the first Pontrjagin number $P_1[M]$ and the signature $\tau[M]$ of (2.2) and found $\chi(M)=2$, $P_1[M]=-3$ and $\tau[M]=-1$. Here we will see whether the nonzero A changes these topological invariants.

First we compute the Euler number of (3.3). By Chern's formula [8],

$$\begin{aligned}
\chi(M) &= \frac{1}{32\pi^2}\Big\{\int_M \epsilon_{abcd}R^a{}_b\wedge R^c{}_d - \\
&\quad \int_{\partial M_\infty}\epsilon_{abcd}\Big(2\omega^a{}_b\wedge R^c{}_d-\frac{4}{3}\omega^a{}_b\wedge\omega^c{}_s\wedge\omega^s{}_d\Big)\Big\}\\
&= \frac{8V_0}{\pi^2}\int_{r_0}^{\infty}\Big\{\Big(\frac{A}{r^4}+\frac{B}{r^6}\Big)^2+\frac{B}{r^6}\Big(\frac{A}{r^4}+\frac{B}{r^6}\Big)\Big\}r^3dr\\
&\quad -\frac{2}{\pi^2}V_0\\
&= 4\Big(1-\frac{A}{r_0^2}\Big)\Big(1+\frac{B}{r_0^4}\Big)^{-1}=2.
\end{aligned}$$

Next we compute the first Pontrjagin number of (3.3). Denote $\mathcal{R}$ the curvature tensor. It is straightforward that

$$\begin{aligned}
P_1[M] &= -\frac{1}{8\pi^2}\int_M Tr\Big(\mathcal{R}\wedge\mathcal{R}\Big)\\
&= -\frac{4}{\pi^2}\int_M\Big\{6\Big(\frac{A}{r^4}+\frac{B}{r^6}\Big)^2-\frac{A^2}{r^8}\Big\}r^3dr\sigma_1\sigma_2\sigma_3\\
&= -\frac{4}{\pi^2}2\pi^2\Big(1+\frac{B}{r_0^4}\Big)^{-1}\Big(\frac{5A^2}{4r_0^4}+\frac{2AB}{r_0^6}+\frac{3B^2}{4r_0^8}\Big)\\
&= -3+\frac{A}{r_0^2}.
\end{aligned}$$

In above computations $1-\frac{2A}{r_0^2}-\frac{B}{r_0^4}=0$ is used.

The signature of M can be computed via the Chern-Simons boundary correction [9] (which is zero in the current case) and the signature η-invariant for the boundary (using Proposition 2.12 [10]). It relates to the total mass in an implicit way. We leave it to readers as an exercise.

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