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Global existence of smooth solutions to two-dimensional compressible isentropic Euler equations for Chaplygin gases

Dedicated to Professor Yang Lo on the Occasion of his 70th Birthday

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Abstract In this paper we investigate the two-dimensional compressible isentropic Euler equations for Chaplygin gases. Under the assumption that the initial data is *close* to a constant state and the vorticity of the initial velocity vanishes, we prove the global existence of the smooth solution to the Cauchy problem for two-dimensional flow of Chaplygin gases.

Keywords Chaplygin gases, compressible Euler equations, irrotational flow, null condition, Cauchy problem, smooth solution

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1 Introduction

This paper concerns the following two-dimensional compressible isentropic Euler equations

$$\begin{cases} \partial_t \rho + u \cdot \nabla \rho + \rho \nabla \cdot u = 0, \\ \partial_t u + u \cdot \nabla u + \dfrac{1}{\rho} \nabla P = 0, \end{cases} \tag{1.1}$$

where $\rho(x,t) > 0$ is the density, $u(x,t) = (u^1(x,t), u^2(x,t))^T$ stands for the velocity, P denotes the pressure and the state equation reads

$$P = P_0 - A\rho^{-1},$$

in which P_0, A are two positive constants. We are interested in the global existence on the smooth solution to the Cauchy problem for the system (1.1) with the initial data

$$t = 0: \ \rho = \bar{\rho} + \varepsilon \rho_0(x), \ \ u = \varepsilon u_0(x), \tag{1.2}$$

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where ε is a positive small parameter, $\bar{\rho}$ stands for a positive constant and satisfies $P_0 - A\bar{\rho}^{-1} > 0$, where $\rho_0(x)$ is a given smooth function with compact support, and $u_0(x) = (u_0^1(x), u_0^2(x))^T$ is a given vector-valued smooth function with compact support. Without loss of generality, we may assume that there exists a positive number R such that

$$\text{supp}\{\rho_0\}, \text{ supp}\{u_0^1\}, \text{ supp}\{u_0^2\} \subset \{x \in \mathbb{R}^2 \,|\, |x| \leqslant R\}.$$

In what follows, we use the notations: $x = (x_1, x_2) \in \mathbb{R}^2$, $\nabla = (\partial_{x_1}, \partial_{x_2})$ and $\partial_i = \frac{\partial}{\partial x_i}$, $i = 1, 2$, $\partial_t = \frac{\partial}{\partial t} = \partial_0$.

The compressible Euler equations comprise a nonlinear symmetric hyperbolic system of PDEs that model the ideal fluid flow. The characteristic wave speeds of this system are given by the fluid velocity and the local sound speed. The two wave families associated to these speeds may be considered separately. Smooth Eulerian flows of ideal polytropic gases with initial data close to a constant state do not, in gereral, exist for all positive times (see [6,18,19]). Godin [6] obtained the precise information on the asymptotic behavior of the life-span of the smooth solution to three-dimensional spherically symmetric Eulerian flows of ideal polytropic gases with variable entropy, provided that the initial data is a smooth small perturbation with compact support to a constant state. Sideris [18] showed that the classical solution to the three-dimensional Euler equations for a polytropic, ideal fluid must blow up in finite time under some assumption on the initial data. In Sideris [19], the estimates on life-span of smooth solutions in two space dimensions were obtained under suitable assumptions. In particular, we would like to mention that Grassin [8] proved that the suitable initial data, which force particles to spread out, may lead to the global existence in time for ideal polytropic gases.

When the adiabatic constant γ of the gas is equal to -1 (in this case, the equation of state is suitably modified to keep the pressure and its derivative with respect to the density positive), the gas is named Chaplygin gas, or called Karman-Tsien gas (see [21, 22], and in the isentropic case, see [4]). Godin [7] obtained the global existence of a class of smooth three-dimensional spherically symmetric flows of Chaplygin gases with the variable entropy, provided that the initial data is a smooth small perturbation with compact support to a constant state.

By the light-cone gauge, Bordemann and Hoppe [2] simplified the description of relativistic membrane moving in the Minkowski space. By performing variables transformations, they established the relationship between the dynamics of relativistic membrane and two-dimensional fluid dynamics. The equations for the two-dimensional fluid dynamics derived in [2] are nothing but the system (1.1). This implies that the system (1.1) can be used to describe the motion of relativistic membrane moving in the Minkowski space $\mathbb{R}^{1+3}$. On this research topic, we refer to Huang and Kong [13].

In the present paper, we investigate the global existence of the smooth solution of the two-dimensional flow of Chaplygin gases. More precisely, under the assumption that the initial data is *close* to a constant state and the vorticity of the initial velocity vanishes, we show the global existence of the smooth solution of the Cauchy problem (1.1)–(1.2). The main result in this paper is the following theorem.

Theorem 1.1.   *Suppose that the initial velocity is irrotational, i.e., u_0 satisfies*

$$\nabla^\perp \cdot u_0 \triangleq \partial_1 u_0^2 - \partial_2 u_0^1 = 0.$$

Then there exists $\varepsilon_0 > 0$ such that the Cauchy problem (1.1)–(1.2) has a unique global smooth solution for any $\varepsilon \in [0, \varepsilon_0]$.

The key idea to prove Theorem 1.1 is as follows: (i) write the velocity as the gradient of a potential function; (ii) reduce the system (1.1) to a quasilinear wave equation for the potential function; (iii) verify that this quasilinear wave equation satisfies the null condition in two space dimensions; (iv) by null condition, L^2-L^∞ estimates and energy estimates, obtain the global existence on the smooth solution under consideration.

Remark 1.1.   Theorem 1.1 still holds if the initial data $u(x, 0) = \varepsilon u_0(x)$ is replaced by $u(x, 0) = \bar{u} + \varepsilon u_0(x)$, where $\bar{u} \in \mathbb{R}^2$ is a constant vector. In fact, as is well known, replacing x by $x - t\bar{u}$ and u by $u - \bar{u}$ leads to a new solution of (1.1)–(1.2) (with the same pressure law), and so we reduce to the assumptions of Theorem 1.1 immediately.

The paper is organized as follows. In Section 2 we reduce the Cauchy problem (1.1)–(1.2) to a Cauchy problem for a quasilinear wave equation (see (2.16)–(2.17) below). In Section 3, we introduce some notations and prove some lemmas, these lemmas play an important role in the proof of Theorem 1.1. Sections 4 and 5 are devoted to establishing the L^2-L^∞ estimates and energy estimates of the smooth solution, respectively. Theorem 1.1 is proved in Section 6 by the bootstrap argument.

2 Two-dimensional irrotational flow

Let $\nabla^\perp = (-\partial_2, \partial_1)$, $\omega = (\omega^1, \omega^2)$, then $\nabla^\perp \cdot \omega = \partial_1 \omega^2 - \partial_2 \omega^1$. As mentioned before, the key idea of the proof of Theorem 1.1 is to write the velocity u as the gradient of a potential function. To do so, we first prove the following lemma which plays an important roles in the present paper.

Lemma 2.1. *Let T be a given positive real number and assume that (ρ, u) is a smooth solution to the Cauchy problem (1.1)–(1.2) in the domain $\mathbb{R}^2 \times [0, T)$. If the initial velocity $u_0(x)$ satisfies*

$$\nabla^\perp \cdot u_0 = 0, \tag{2.1}$$

then

$$\nabla^\perp \cdot u = 0, \quad \forall (x, t) \in \mathbb{R}^2 \times [0, T). \tag{2.2}$$

Proof. Noting $u = (u^1, u^2)$, by a direct calculation we have

$$\begin{aligned}\nabla^\perp \cdot (u \cdot \nabla u) = {} & \partial_1 u^1 \partial_1 u^2 + u^1 \partial_1^2 u^2 + \partial_1 u^2 \partial_2 u^2 + u^2 \partial_1 \partial_2 u^2 \\ & - \partial_2 u^1 \partial_1 u^1 - u^1 \partial_2 \partial_1 u^1 - \partial_2 u^2 \partial_2 u^1 - u^2 \partial_2^2 u^1,\end{aligned} \tag{2.3}$$

$$(u \cdot \nabla)(\nabla^\perp \cdot u) = u^1 \partial_1^2 u^2 - u^1 \partial_1 \partial_2 u^1 + u^2 \partial_1 \partial_2 u^2 - u^2 \partial_2^2 u^1, \tag{2.4}$$

$$(\nabla \cdot u)(\nabla^\perp \cdot u) = \partial_1 u^1 \partial_1 u^2 - \partial_1 u^1 \partial_2 u^1 + \partial_2 u^2 \partial_1 u^2 - \partial_2 u^1 \partial_2 u^2. \tag{2.5}$$

(2.3)–(2.5) gives

$$\nabla^\perp \cdot (u \cdot \nabla u) = (u \cdot \nabla)(\nabla^\perp \cdot u) + (\nabla \cdot u)(\nabla^\perp \cdot u). \tag{2.6}$$

Noting (2.6) and taking the curl of the second equation in (1.1) gives

$$\partial_t (\nabla^\perp \cdot u) + u \cdot \nabla (\nabla^\perp \cdot u) + (\nabla \cdot u)(\nabla^\perp \cdot u) = 0. \tag{2.7}$$

Combining (2.1) and (2.7) yields (2.2) immediately. Thus, the proof of Lemma 2.1 is completed. □

Remark 2.1. Lemma 2.1 shows that, for two-dimensional compressible isentropic Euler equations for Chaplygin gases, if the flow is irrotational at the initial time, then so is it at any time $t \in [0, T)$, where T stands for the maximal existence time of the smooth solution.

Under the assumptions of Theorem 1.1, for system (1.1) there exists a scalar function $v = v(x, t)$ such that

$$u = \nabla v \tag{2.8}$$

where the scalar function $v(x, t)$ is a smooth function and has a compact support in x (since u has compact support by the finite propagation speed). Substituting $u = \nabla v$ into the second equation in (1.1) and noting Lemma 2.1 lead to

$$\partial_t \nabla v + \frac{1}{2} \nabla |\nabla v|^2 + \nabla f(\rho) = 0, \tag{2.9}$$

where the function f is defined by

$$f'(\rho) = \frac{1}{\rho} \frac{dP}{d\rho} \tag{2.10}$$

and

$$f(\bar{\rho}) = 0. \tag{2.11}$$

It follows from (2.10) and (2.11) that

$$f(\rho) = \int_{\bar{\rho}}^{\rho} \frac{1}{\lambda} \frac{dP}{d\lambda} d\lambda = \frac{\bar{c}^2}{2} - \frac{A}{2\rho^2},$$

where

$$\bar{c} = \left\{ \frac{dP}{d\rho}(\bar{\rho}) \right\}^{\frac{1}{2}}. \tag{2.12}$$

In what follows, we take $\bar{\rho}$ shch that $\bar{c} = 1$. Noting (2.9), we have

$$\partial_t v + \frac{1}{2}|u|^2 + f(\rho) = 0. \tag{2.13}$$

Thus, by (2.8) and (2.13), we obtain

$$\begin{cases} \nabla \cdot u = \triangle v, \\ \partial_t^2 v = -u \cdot \partial_t u - f'(\rho)\partial_t \rho, \\ \partial_t \nabla v = -u \cdot \nabla u - f'(\rho)\nabla \rho. \end{cases} \tag{2.14}$$

It follows from (2.14) that

$$\begin{cases} \nabla \cdot u = \triangle v, \\ \partial_t \rho = -\dfrac{1}{f'(\rho)}(\partial_t^2 v + \nabla v \cdot \partial_t \nabla v), \\ \nabla \rho = -\dfrac{1}{f'(\rho)}(\nabla \partial_t v + \nabla v \cdot \nabla \nabla v). \end{cases} \tag{2.15}$$

Substituting (2.15) into the first equation in (1.1) yields

$$\partial_t^2 v - \triangle v + 2\sum_{i=1}^{2} \partial_i v \partial_t \partial_i v - 2\partial_t v \triangle v + \sum_{i,\,j=1}^{2} \partial_i v \partial_j v \partial_i \partial_j v - |\nabla v|^2 \triangle v = 0. \tag{2.16}$$

On the other hand, noting (1.2), (2.8) and (2.13), we get

$$t = 0: \quad v = \varepsilon \int_R^{x_1} u_0^1(y, x_2) dy, \quad v_t = -\frac{1}{2}\varepsilon^2 |u_0(x)|^2 - f(\bar{\rho} + \varepsilon \rho_0(x)). \tag{2.17}$$

Remark 2.2.   For quasilinear wave equations in two space dimensions satisfying the null condition (resp. both null conditions), Alinhac [1] proved the quasi-global existence (resp. global existence) for their Cauchy problems. His proof relies on the construction of an approximate solution, combined with an energy integral method which displays the null condition(s). The method used in the present paper is different from one in Alinhac [1].

By (2.17), it is easy to see that $v(x, 0)$, $\partial_t v(x, 0)$ are smooth functions of $x \in \mathbb{R}^2$ and has compact support, i.e.,

$$\text{supp}\{v(x, 0)\},\ \text{supp}\{\partial_t v(x, 0)\} \subset \{x \in \mathbb{R}^2 \,|\, |x| \leqslant R\}.$$

Hence by the finite propagation speed, we obtain

$$v(x, t) = 0, \quad \forall\, x \in \{x \in \mathbb{R}^2 \,|\, |x| \geqslant R + t\}. \tag{2.18}$$

Obviously, (2.16) can be rewritten as

$$\partial_t^2 v - \triangle v + \sum_{i,j,k=0}^{2} g_{ij}^k \partial_k v \partial_i \partial_j v + \sum_{i,j,k,l=0}^{2} g_{ij}^{kl} \partial_k v \partial_l v \partial_i \partial_j v = 0, \tag{2.19}$$

where

$$\sum_{i,j,k=0}^{2} g_{ij}^k \partial_k v \partial_i \partial_j v = 2\partial_1 v \partial_0 \partial_1 v + 2\partial_2 v \partial_0 \partial_2 v - 2\partial_0 v \partial_1^2 v - 2\partial_0 v \partial_2^2 v, \tag{2.20}$$

$$\sum_{i,j,k,l=0}^{2} g_{ij}^{kl} \partial_k v \partial_l v \partial_i \partial_j v = 2\partial_1 v \partial_2 v \partial_1 \partial_2 v - (\partial_2 v)^2 \partial_1^2 v - (\partial_1 v)^2 \partial_2^2 v. \tag{2.21}$$

For any given vector $(X_0, X_1, X_2) \in \mathbb{R}^3$ with $X_0^2 = X_1^2 + X_2^2$, we have

$$\sum_{i,j,k=0}^{2} g_{ij}^k X_i X_j X_k = 2X_0 X_1^2 + 2X_0 X_2^2 - 2X_0 X_1^2 - 2X_0 X_2^2 = 0, \tag{2.22}$$

$$\sum_{i,j,k,l=0}^{2} g_{ij}^{kl} X_i X_j X_k X_l = 2X_1 X_2 X_1 X_2 - X_1^2 X_2^2 - X_1^2 X_2^2 = 0. \tag{2.23}$$

This shows that the wave equation (2.16) satisfies the null condition in two space dimensions (see [1]). The concept of null condition was introduced by Christodoulou and Klainerman for nonlinear wave equations in three space dimensions (see [3, 15]). Therefore, we can rewrite the terms

$$\sum_{i,j,k=0}^{2} g_{ij}^k \partial_k v \partial_i \partial_j v \quad \text{and} \quad \sum_{i,j,k,l=0}^{2} g_{ij}^{kl} \partial_k v \partial_l v \partial_i \partial_j v$$

as

$$\sum_{i,j,k=0}^{2} g_{ij}^k \partial_k v \partial_i \partial_j v = 2Q_{01}(\partial_1 v, v) + 2Q_{02}(\partial_2 v, v) \tag{2.24}$$

and

$$\sum_{i,j,k,l=0}^{2} g_{ij}^{kl} \partial_k v \partial_l v \partial_i \partial_j v = -\partial_2 v Q_{12}(\partial_1 v, v) + \partial_1 v Q_{12}(\partial_2 v, v), \tag{2.25}$$

where $Q_{ij}(\varphi, \psi)$ are defined by $Q_{ij}(\varphi, \psi) = \partial_i \varphi \partial_j \psi - \partial_i \psi \partial_j \varphi \ (0 \leqslant i < j \leqslant 2)$, which are null forms.

3 Preliminaries

In order to prove Theorem 1.1, in this section we give some preliminaries.

Following Klainerman [14], we introduce the following vector fields

$$\Gamma = (\Gamma_0, \ldots, \Gamma_6) = (\partial_0, \ \partial_1, \ \partial_2, \ \Omega_{01}, \ \Omega_{02}, \ \Omega_{12}, \ S), \tag{3.1}$$

where

$$\Omega_{01} = t\partial_1 + x_1 \partial_t, \quad \Omega_{02} = t\partial_2 + x_2 \partial_t, \quad \Omega_{12} = x_1 \partial_2 - x_2 \partial_1, \quad S = t\partial_t + r\partial_r. \tag{3.2}$$

It is easy to show that

$$\begin{cases} [\Gamma_i, \Box] = -2\delta_{6i}\Box \quad (i = 0, \ldots, 6), \\ [\partial_i, \partial_j] = 0 \quad (i, j = 0, 1, 2), \quad [S, \partial_i] = -\partial_i \quad (i = 0, 1, 2), \\ [\Omega_{01}, \partial_1] = -\partial_t, \quad [\Omega_{01}, \partial_2] = 0, \quad [\Omega_{01}, \partial_t] = -\partial_1, \quad [\Omega_{01}, \Omega_{02}] = 0, \\ [\Omega_{01}, \Omega_{12}] = 0, \quad [\Omega_{01}, S] = 0, \quad [\Omega_{02}, \partial_1] = 0, \quad [\Omega_{02}, \partial_2] = -\partial_t, \\ [\Omega_{02}, \partial_t] = -\partial_2, \quad [\Omega_{02}, S] = 0, \quad [\Omega_{12}, \partial_1] = -\partial_2, \quad [\Omega_{12}, \partial_2] = \partial_1, \\ [\Omega_{12}, \partial_0] = \partial_0, \quad [S, \Omega_{12}] = 0, \end{cases} \tag{3.3}$$

where $[\cdot, \cdot]$ denotes the usual commutator of linear operators, $\Box = \partial_t^2 - \triangle$ and δ_{ij} is the Kronecker symbol.

In what follows, we shall use the following notations

$$|v(x,t)|_\kappa = \sum_{|a| \leqslant \kappa} |\Gamma^a v(x,t)|, \quad |v(t)|_\kappa = \sup_{x \in \mathbb{R}^2} |v(x,t)|_\kappa,$$

$$[v(x,t)]_\kappa = \sum_{|a| \leqslant \kappa} (1 + |x|)^{\frac{1}{2}} (1 + ||x| - t|) |\Gamma^a v(x,t)|, \quad [v(t)]_\kappa = \sup_{x \in \mathbb{R}^2} [v(x,t)]_\kappa$$

$$\langle v(x,t) \rangle = \sum_{|a| \leqslant \kappa} (1 + |x| + t)^{\frac{1}{2}} |\Gamma^a v(x,t)|,$$

$$\langle v(t)\rangle = \sup_{x\in\mathbb{R}^2} \langle v(x,t)\rangle, \quad \|v(t)\|_\kappa = \sum_{|a|\leqslant \kappa} \|\Gamma^a v(\cdot,t)\|_{L^2},$$

where κ is a nonnegative integer, $a = (a_0, \ldots, a_6)$ is a multi-index and $|a| = |a_0| + \cdots + |a_6|$. Moreover, we also use the following notations

$$|v|_{\kappa,t} = \sup_{0\leqslant s<t} |v(s)|_\kappa, \quad [v]_{\kappa,t} = \sup_{0\leqslant s<t} [v(s)]_\kappa, \quad \langle v\rangle_{\kappa,t} = \sup_{0\leqslant s<t} \langle v(s)\rangle_\kappa, \quad \|v\|_{\kappa,t} = \sup_{0\leqslant s<t} \|v(s)\|_\kappa.$$

Define

$$\Lambda(T) = \left\{ (x,t) \in \mathbb{R}^2 \times [0,T) \mid |\,|x| - t| \leqslant \frac{1}{2}t \right\}. \tag{3.4}$$

Thanks to the null condition, we will be able to show that the null form possesses a good decay in the domain $\Lambda(T)$. In the our argument, the following operators

$$Z = (Z_1, Z_2) = \frac{t - |x|}{t}(\partial_1, \partial_2) + \frac{(x_1,x_2)S + (-x_2, x_1)\Omega_{12}}{|x|t} \tag{3.5}$$

will play an important role. It follows from (3.5) that

$$Z_i = \partial_i + \frac{x_i}{|x|}\partial_t \quad (i=1,2) \tag{3.6}$$

and

$$|Z\varphi(x,t)| \leqslant C\left(\frac{|\,|x| - t|}{t}|\partial\varphi(x,t)| + \frac{1}{t}|\varphi(x,t)|_1 \right), \tag{3.7}$$

here and hereafter C stands for a common absolute constant. We have

Lemma 3.1.   *Assume that h_{ij}^k are constants and it holds that*

$$\sum_{i,j,k=0}^{2} h_{ij}^k X_i X_j X_k = 0 \tag{3.8}$$

for any vector (X_0, X_1, X_2) satisfying $X_0^2 = X_1^2 + X_2^2$. If $\varphi, \psi \in C^\infty(\mathbb{R}^2 \times [0,T))$, then

$$\left| \sum_{i,j,k=0}^{2} h_{ij}^k \partial_k\psi \partial_i\partial_j\varphi(x,t) \right| \leqslant C[|Z\psi(x,t)||\partial^2\varphi(x,t)| + |\partial\psi(x,t)||Z\partial\varphi(x,t)|]. \tag{3.9}$$

Lemma 3.2.   *Assume that h_{ij}^{kl} are constants and it holds that*

$$\sum_{i,j,k,l=0}^{2} h_{ij}^{kl} X_i X_j X_k X_l = 0 \tag{3.10}$$

for any vector (X_0, X_1, X_2) satisfying $X_0^2 = X_1^2 + X_2^2$. If $\varphi, \psi, \phi \in C^\infty(\mathbb{R}^2 \times [0,T))$, then

$$\left| \sum_{i,j,k,l=0}^{2} h_{ij}^{kl} \partial_k\psi \partial_l\phi \partial_i\partial_j\varphi(x,t) \right| \leqslant C[|\partial\psi(x,t)||\partial\phi(x,t)||Z\partial\varphi(x,t)|$$

$$+ |\partial\psi(x,t)||Z\phi(x,t)||\partial^2\varphi(x,t)| + |Z\psi(x,t)||\partial\phi(x,t)||\partial^2\varphi(x,t)|]. \tag{3.11}$$

In what follows, we only prove Lemma 3.2, because the proof of Lemma 3.1 is of the same type and even somewhat simpler.

Proof.   In fact, it follows from (3.10) that

$$\sum_{i,j,k,l=0}^{2} h_{ij}^{kl} \partial_k\psi \partial_l\phi \partial_i\partial_j\varphi$$

$$= \sum_{i,j,k,l=0}^{2} h_{ij}^{kl} (\partial_k + \omega_k\partial_t)\psi \partial_l\phi \partial_i\partial_j\varphi - \sum_{i,j,k.l=0}^{2} h_{ij}^{kl} \omega_k\partial_k\psi(\partial_l + \omega_l\partial_t)\phi \partial_i\partial_j\varphi$$

$$+ \sum_{i,j,k,l=0}^{2} h_{ij}^{kl} \omega_k \omega_l \partial_k \psi \partial_l \phi (\partial_i + \omega_i \partial_t) \partial_j \varphi$$

$$- \sum_{i,j,k,l=0}^{2} h_{ij}^{kl} \omega_k \omega_l \omega_i \partial_k \psi \partial_l \phi (\partial_j + \omega_j \partial_t) \partial_t \varphi + \sum_{i,j,k,l=0}^{2} h_{ij}^{kl} \omega_k \omega_l \omega_i \omega_j \partial_t \psi \partial_t \phi \partial_t^2 \varphi$$

$$= \sum_{i,j,k,l=0}^{2} h_{ij}^{kl} (\partial_k + \omega_k \partial_t) \psi \partial_l \phi \partial_i \partial_j \varphi - \sum_{i,j,k,l=0}^{2} h_{ij}^{kl} \omega_k \partial_k \psi (\partial_l + \omega_l \partial_t) \phi \partial_i \partial_j \varphi$$

$$+ \sum_{i,j,k,l=0}^{2} h_{ij}^{kl} \omega_k \omega_l \partial_k \psi \partial_l \phi (\partial_i + \omega_i \partial_t) \partial_j \varphi - \sum_{i,j,k,l=0}^{2} h_{ij}^{kl} \omega_k \omega_l \omega_i \partial_k \psi \partial_l \phi (\partial_j + \omega_j \partial_t) \partial_t \varphi, \tag{3.12}$$

where $(\omega_0, \omega_1, \omega_2) = (-1, \frac{x_1}{|x|}, \frac{x_2}{|x|})$. Noting (3.6) and (3.12), we obtain (3.11) immediately. This completes the proof of Lemma 3.2.   □

Lemma 3.3. *There exist constants* $\tilde{g}_{ij}^k$ *such that* (i) $\sum_{i,j,k=0}^{2} \tilde{g}_{ij}^k \partial_k v \partial_i \partial_j v$ *satisfies the null condition, that is, for any vector* (X_0, X_1, X_2) *satisfying* $X_0^2 = X_1^2 + X_2^2$, *it holds that*

$$\sum_{i,j,k,l=0}^{2} \tilde{g}_{ij}^{kl} X_i X_j X_k X_l = 0; \tag{3.13}$$

(ii) *the following equality holds*

$$\Gamma \left(\sum_{i,j,k=0}^{2} g_{ij}^k \partial_k v \partial_i \partial_j v \right) = \sum_{i,j,k=0}^{2} g_{ij}^k \partial_k \Gamma v \partial_i \partial_j v + \sum_{i,j,k=0}^{2} g_{ij}^k \partial_k v \partial_i \partial_j \Gamma v + \sum_{i,j,k=0}^{2} \tilde{g}_{ij}^k \partial_k v \partial_i \partial_j v. \tag{3.14}$$

Lemma 3.4. *There exist constants* $\tilde{g}_{ij}^{kl}$ *such that* (I) $\sum_{i,j,k,l=0}^{2} \tilde{g}_{ij}^{kl} \partial_k v \partial_l v \partial_i \partial_j v$ *satisfies the null condition, that is, for any vector* (X_0, X_1, X_2) *satisfying* $X_0^2 = X_1^2 + X_2^2$, *it holds that*

$$\sum_{i,j,k,l=0}^{2} \tilde{g}_{ij}^{kl} X_i X_j X_k X_j = 0; \tag{3.15}$$

(II) *the following equality holds*

$$\Gamma \left(\sum_{i,j,k,l=0}^{2} g_{ij}^{kl} \partial_k v \partial_l v \partial_i \partial_j v \right) = \sum_{i,j,k,l=0}^{2} g_{ij}^{kl} \partial_k \Gamma v \partial_l v \partial_i \partial_j v + \sum_{i,j,k,l=0}^{2} g_{ij}^{kl} \partial_k v \partial_l \Gamma v \partial_i \partial_j v$$

$$+ \sum_{i,j,k,l=0}^{2} g_{ij}^{kl} \partial_k v \partial_l v \partial_i \partial_j \Gamma v + \sum_{i,j,k,l=0}^{2} \tilde{g}_{ij}^{kl} \partial_k v \partial_l v \partial_i \partial_j v. \tag{3.16}$$

As before, in what follows we only prove Lemma 3.4, because the proof of Lemma 3.3 is of the same type and even somewhat simpler.

Proof.   It follows from (3.3) that

$$\partial_m \left(\sum_{i,j,k,l=0}^{2} g_{ij}^{kl} \partial_k v \partial_l v \partial_i \partial_j v \right) = \sum_{i,j,k,l=0}^{2} g_{ij}^{kl} \partial_k \partial_m v \partial_l v \partial_i \partial_j v + \sum_{i,j,k,l=0}^{2} g_{ij}^{kl} \partial_k v \partial_l \partial_m v \partial_i \partial_j v$$

$$+ \sum_{i,j,k,l=0}^{2} g_{ij}^{kl} \partial_k v \partial_l v \partial_i \partial_j \partial_m v, \quad m = 0, 1, 2, \tag{3.17}$$

$$\Omega_{01} \left(\sum_{i,j,k,l=0}^{2} g_{ij}^{kl} \partial_k v \partial_l v \partial_i \partial_j v \right) = 2 \partial_1 \Omega_{01} v \partial_2 v \partial_1 \partial_2 v + 2 \partial_1 v \partial_2 \Omega_{01} v \partial_1 \partial_2 v$$

$$+ 2 \partial_1 v \partial_2 v \partial_1 \partial_2 \Omega_{01} v - 2 \partial_2 \Omega_{01} v \partial_2 v \partial_1^2 v - (\partial_2 v)^2 \partial_1 \Omega_{01} v$$

$$- 2 \partial_1 \Omega_{01} v \partial_1 v \partial_2^2 v - (\partial_1 v)^2 \partial_2^2 \Omega_{01} v - 2 \partial_t v \partial_2 v \partial_1 \partial_2 v$$

$$- 2 \partial_1 v \partial_2 v \partial_t \partial_2 v + 2 (\partial_2 v)^2 \partial_1 \partial_t v + 2 \partial_1 v \partial_t v \partial_2^2 v, \tag{3.18}$$

$$\Omega_{02}\bigg(\sum_{i,j,k,l=0}^{2} g_{ij}^{kl}\partial_k v\partial_l v\partial_i\partial_j v\bigg) = 2\partial_1\Omega_{02}v\partial_2 v\partial_1\partial_2 v + 2\partial_1 v\partial_2\Omega_{02}v\partial_1\partial_2 v$$
$$+ 2\partial_1 v\partial_2 v\partial_1\partial_2\Omega_{02}v - 2\partial_2\Omega_{02}v\partial_2 v\partial_1^2 v - (\partial_2 v)^2\partial_1\Omega_{02}v$$
$$- 2\partial_1\Omega_{02}v\partial_1 v\partial_2^2 v - (\partial_1 v)^2\partial_2^2\Omega_{02}v - 2\partial_1 v\partial_t v\partial_1\partial_2 v$$
$$- 2\partial_1 v\partial_2 v\partial_1\partial_t v + 2\partial_t v\partial_2 v\partial_1^2 v + 2(\partial_1 v)^2\partial_t v\partial_2^2 v, \tag{3.19}$$

$$\Omega_{12}\bigg(\sum_{i,j,k,l=0}^{2} g_{ij}^{kl}\partial_k v\partial_l v\partial_i\partial_j v\bigg) = 2\partial_1\Omega_{12}v\partial_2 v\partial_1\partial_2 v + 2\partial_1 v\partial_2\Omega_{12}v\partial_1\partial_2 v + 2\partial_1 v\partial_2 v\partial_1\partial_2\Omega_{12}v$$
$$- 2\partial_1\Omega_{12}v\partial_1 v\partial_2^2 v - (\partial_1 v)^2\partial_2^2\Omega_{12}v - 2\partial_2 v\partial_2\Omega_{12}v\partial_1^2 v - (\partial_2 v)^2\partial_1^2\Omega_{12}v$$
$$= \sum_{i,j,k,l=0}^{2} g_{ij}^{kl}\partial_k\Omega_{12}v\partial_l v\partial_i\partial_j v + \sum_{i,j,k,l=0}^{2} g_{ij}^{kl}\partial_k v\partial_l\Omega_{12}v\partial_i\partial_j v$$
$$+ \sum_{i,j,k,l=0}^{2} g_{ij}^{kl}\partial_k v\partial_l v\partial_i\partial_j\Omega_{12}v, \tag{3.20}$$

$$S\bigg(\sum_{i,j,k,l=0}^{2} g_{ij}^{kl}\partial_k v\partial_l v\partial_i\partial_j v\bigg) = \sum_{i,j,k,l=0}^{2} g_{ij}^{kl}\partial_k Sv\partial_l v\partial_i\partial_j v + \sum_{i,j,k,l=0}^{2} g_{ij}^{kl}\partial_k v\partial_l Sv\partial_i\partial_j v$$
$$+ \sum_{i,j,k,l=0}^{2} g_{ij}^{kl}\partial_k v\partial_l v\partial_i\partial_j Sv - 4\sum_{i,j,k,l=0}^{2} g_{ij}^{kl}\partial_k v\partial_l v\partial_i\partial_j v. \tag{3.21}$$

The desired (3.15)–(3.16) follows from (3.17)–(3.21) immediately. This proves Lemma 3.4.   □

By Lemmas 3.1–3.4, we can obtain the following lemma.

Lemma 3.5.   *Let $v = v(x,t)$ be a smooth solution of the Cauchy problem (2.16)–(2.17) on the domain $\mathbb{R}^2 \times [0,T)$ and κ be a positive integer, where T is a given positive real number. Then there exists a positive constant C_κ depending on κ but independent of T such that the following inequalities hold for every $(x,t) \in \Lambda(T) \cap \{(y,s) \,|\, s \geqslant 1\}$,*

$$\bigg|\sum_{i,j,k=0}^{2} g_{ij}^{k}\partial_k v\partial_i\partial_j v(x,t)\bigg|_\kappa \leqslant C_\kappa \sum_{|a+b|\leqslant\kappa+1} |Z\Gamma^a v(x,t)||\Gamma^b\partial v(x,t)|, \tag{3.22}$$

$$\sum_{|a|\leqslant\kappa}\bigg|\Gamma^a\bigg(\sum_{i,j,k=0}^{2} g_{ij}^{k}\partial_k v\partial_i\partial_j v\bigg)(x,t) - \sum_{i,j,k=0}^{2} g_{ij}^{k}\partial_k v\partial_i\partial_j\Gamma^a v(x,t)\bigg|$$
$$\leqslant C_\kappa \sum_{|b+c|\leqslant\kappa+1,|b|,|c|\leqslant\kappa} |Z\Gamma^b v(x,t)||\Gamma^c\partial v(x,t)| \tag{3.23}$$

and

$$\bigg|\sum_{i,j,k,l=0}^{2} g_{ij}^{kl}\partial_k v\partial_l v\partial_i\partial_j v(x,t)\bigg|_\kappa \leqslant C_\kappa \sum_{|a+b+c|\leqslant\kappa+1} |Z\Gamma^a v(x,t)||\Gamma^b\partial v(x,t)||\Gamma^c\partial v(x,t)|, \tag{3.24}$$

where the terms

$$\sum_{i,j,k=0}^{2} g_{ij}^{k}\partial_k v\partial_i\partial_j v \quad and \quad \sum_{i,j,k,l=0}^{2} g_{ij}^{kl}\partial_k v\partial_l v\partial_i\partial_j v$$

are defined by (2.20) and (2.21), respectively. Moreover, it holds that

$$\bigg|\sum_{i,j,k=0}^{2} g_{ij}^{k}\partial_k v\partial_i\partial_j v(x,t)\bigg|_\kappa \leqslant C_\kappa \frac{||x|-t|}{1+|x|+t}|\partial v(x,t)|_{[\frac{\kappa+1}{2}]}|\partial v(x,t)|_{\kappa+1}$$
$$+ C_\kappa \frac{1}{1+|x|+t}(|v(x,t)|_{[\frac{\kappa+1}{2}]+1}|\partial v(x,t)|_{\kappa+1} + |v(x,t)|_{[\frac{\kappa+1}{2}]}|v(x,t)|_{\kappa+2}), \tag{3.25}$$

$$\left| \sum_{i,j,k,l=0}^{2} g_{ij}^{kl} \partial_k v \partial_l v \partial_i \partial_j v(x,t) \right|_{\kappa} \leqslant C_\kappa \frac{||x| - t|}{1 + |x| + t} |\partial v(x,t)|^2_{[\frac{\kappa+1}{2}]} |\partial v(x,t)|_{\kappa+1}$$

$$+ C_\kappa \frac{1}{1 + |x| + t} |\partial v(x,t)|_{[\frac{\kappa+1}{2}]} (|v(x,t)|_{[\frac{\kappa+1}{2}]+1} |\partial v(x,t)|_{\kappa+1} + |v(x,t)|_{[\frac{\kappa+1}{2}]} |v(x,t)|_{\kappa+2}) \qquad (3.26)$$

and

$$\sum_{|a| \leqslant \kappa} \left| \Gamma^a \left(\sum_{i,j,k=0}^{2} g_{ij}^{k} \partial_k v \partial_i \partial_j v \right)(x,t) - \sum_{i,j,k=0}^{2} g_{ij}^{k} \partial_k v \partial_i \partial_j \Gamma^a v(x,t) \right|$$

$$\leqslant C_\kappa \frac{||x| - t|}{1 + |x| + t} |\partial v(x,t)|_{[\frac{\kappa+1}{2}]} |\partial v(x,t)|_{\kappa}$$

$$+ C_\kappa \frac{1}{1 + |x| + t} (|v(x,t)|_{[\frac{\kappa+1}{2}]+1} |\partial v(x,t)|_{\kappa} + |v(x,t)|_{[\frac{\kappa+1}{2}]} |v(x,t)|_{\kappa+1}). \qquad (3.27)$$

Proof. We only prove (3.22), (3.23), (3.25) and (3.27), because the proofs of (3.24) and (3.26) are similar.

By Lemma 3.3, we obtain

$$\Gamma^a \left(\sum_{i,j,k=0}^{2} g_{ij}^{k} \partial_k v \partial_i \partial_j v \right) = \sum_{b+c \leqslant a} \sum_{i,j,k=0}^{2} g_{ijbc}^{ka} \partial_k \Gamma^b v \partial_i \partial_j \Gamma^c v, \qquad (3.28)$$

where $\sum_{i,j,k=0}^{2} g_{ijbc}^{ka} \partial_k \Gamma^b v \partial_i \partial_j \Gamma^c v$ satisfies the null condition, that is, for any vector (X_0, X_1, X_2) satisfying $X_0^2 = X_1^2 + X_2^2$, it holds that

$$\sum_{i,j,k=0}^{2} g_{ijbc}^{ka} X_i X_j X_k = 0.$$

Noting (3.3), (3.9) and (3.28), we get (3.22) immediately.

By Lemma 3.3 again, we have

$$\Gamma^a \left(\sum_{i,j,k=0}^{2} g_{ij}^{k} \partial_k v \partial_i \partial_j v \right) - \sum_{i,j,k=0}^{2} g_{ij}^{k} \partial_k v \partial_i \partial_j \Gamma^a v = \sum_{b+c \leqslant a, c < a} \sum_{i,j,k=0}^{2} g_{ijbc}^{ka} \partial_k \Gamma^b v \partial_i \partial_j \Gamma^c v, \qquad (3.29)$$

where $\sum_{i,j,k=0}^{2} g_{ijbc}^{ka} \partial_k \Gamma^b v \partial_i \partial_j \Gamma^c v$ satisfies the null condition, that is, for any vector (X_0, X_1, X_2) satisfying $X_0^2 = X_1^2 + X_2^2$, it holds that

$$\sum_{i,j,k=0}^{2} g_{ijbc}^{ka} X_i X_j X_k = 0.$$

Thus, using (3.3), (3.9) and (3.29), we obtain (3.23).

Finally, noting (3.7), (3.22) and using (3.23), we get (3.25) and (3.27) immediately. Thus, the proof of Lemma 3.5 is completed. □

4 L^2-L^∞ estimates

The L^2-L^∞ estimates play an important role in the proof of Theorem 1.1. This section is devoted to establishing these estimates.

Theorem 4.1. *Let $v = v(x,t)$ be a smooth solution of the Cauchy problem (2.16)–(2.17) on the domain $\mathbb{R}^2 \times [0,T)$, where T is a given positive real number, and let κ be a positive integer, and δ be a positive small constant. Suppose that*

$$[\partial v]_{[\frac{\kappa+6}{2}],T} + \langle v \rangle_{[\frac{\kappa+6}{2}],T} \leqslant \delta, \qquad (4.1)$$

then there exist positive constants C_0, C_1 and ν such that it holds that

$$[\partial v(t)]_\kappa + \langle v(t) \rangle_{\kappa+1} \leqslant C_0 \varepsilon + C_1([\partial v]_{[\frac{\kappa+6}{2}],t} + \langle v \rangle_{[\frac{\kappa+6}{2}],t}) \sup_{0 \leqslant \tau < t} \{(1 + \tau)^{-\nu} \|\partial v(\tau)\|_{\kappa+8}\} \qquad (4.2)$$

for every $t \in [0, T]$.

Proof.   For $t \in [0, 1]$, we have

$$|\partial_t v(x,t)|_\kappa \leqslant C|v(x,t)|_{\kappa+1} \leqslant \frac{C}{1+t}|v(x,t)|_{\kappa+1}. \tag{4.3}$$

On the other hand, noting the fact that $\partial_t = \frac{S}{t} - \frac{x \cdot \nabla}{t}$ and using (2.18), we obtain

$$|\partial_t v(x,t)|_\kappa \leqslant C|\nabla v(x,t)|_\kappa + \frac{C}{1+t}|v(x,t)|_{\kappa+1} \quad \text{for } t \geqslant 1. \tag{4.4}$$

Thus, it follows from (4.3) and (4.4) that, for every $t \in [0, T]$

$$|\partial_t v(x,t)|_\kappa \leqslant C|\nabla v(x,t)|_\kappa + \frac{C}{1+t}|v(x,t)|_{\kappa+1}. \tag{4.5}$$

Thanks to (4.5), in what follows, we only need to prove (4.2) with ∂ replaced by ∂_i $(i = 1, 2)$ in the left-hand side. To do so, we can rewrite the solution $v = v(x,t)$ of the Cauchy problem (2.16)–(2.17) as $v = v_1 + \Phi(F)$, where v_1 is the solution of the following Cauchy problem

$$\begin{cases} \partial_t^2 v_1 - \triangle v_1 = 0, \\ t = 0 : v_1 = \varepsilon \int_R^{x_1} u_0^1(y, x_2)dy, \quad \partial_t v_1 = -\frac{1}{2}\varepsilon^2|u_0(x)|^2 - f(\bar{\rho} + \varepsilon\rho_0(x)), \end{cases} \tag{4.6}$$

while $\Phi(F)$ is the solution of the following Cauchy problem

$$\begin{cases} \partial_t^2 \Phi(F) - \triangle \Phi(F) = F, \\ t = 0 : \Phi(F) = 0, \quad \partial_t \Phi(F) = 0, \end{cases} \tag{4.7}$$

in which

$$F = -\sum_{i,j,k=0}^{2} g_{ij}^k \partial_k v \partial_i \partial_j v - \sum_{i,j,k,l=0}^{2} g_{ij}^{kl} \partial_k v \partial_l v \partial_i \partial_j v. \tag{4.8}$$

On the other hand, it follows from (4.6) that (see [5, 17])

$$[\partial v_1(t)]_\kappa + \langle v_1(t) \rangle_{\kappa+1} \leqslant C\varepsilon \quad \text{for } t \geqslant 0. \tag{4.9}$$

In order to prove (4.2), it suffices to show

$$[\nabla\Phi(F)(t)]_\kappa + \langle \Phi(F)(t) \rangle_{\kappa+1} \leqslant C\varepsilon + C_1 \big([\partial v]_{[\frac{\kappa+6}{2}], t} + \langle v \rangle_{[\frac{\kappa+6}{2}], t}\big) \sup_{0 \leqslant \tau < t} \{(1+\tau)^{-\nu}\|\partial v(\tau)\|_{\kappa+8}\} \tag{4.10}$$

for $t \in [0, T]$. Here and hereafter, we interpret $[\nabla\Phi(F)(t)]_\kappa$ and $\langle \Phi(F)(t) \rangle_{\kappa+1}$ as

$$[\nabla\Phi(F)(t)]_\kappa = \sup_{x \in \mathbb{R}^2} \sum_{|a| \leqslant \kappa} (1 + |x|)^{\frac{1}{2}}(1 + ||x| - t|)|\Gamma^a \nabla\Phi(F)(x,t)|,$$

$$\langle \Phi(F)(t) \rangle_\kappa = \sup_{x \in \mathbb{R}^2} \sum_{|a| \leqslant \kappa} (1 + |x| + t)^{\frac{1}{2}}|\Gamma^a \Phi(F)(x,t)|.$$

To prove (4.10), we need the following lemmas.

Lemma 4.1.   *Suppose that* $F \in C^\infty(\mathbb{R}^2 \times [0,T])$. *Then for any given positive real number* $\mu > 0$ *there exists a positive constant* $C = C(\mu)$ *such that the following inequalities hold*

$$\langle \Phi(F)(x,t) \rangle_k \leqslant CL_{\mu,\,\kappa}(F)(x,t) \tag{4.11}$$

and

$$[\nabla\Phi(F)(x,t)]_k \leqslant CL_{\mu,\,\kappa+1}(F)(x,t), \tag{4.12}$$

where

$$L_{\mu,\,\kappa}(F)(x,t) = \sup_{(y,\tau)\in D(x,t)\cap\Lambda(t)} \{|y|^{\frac{1}{2}}(1+|y|+\tau)^{1+\mu}(1+||y|-\tau|)|F(y,\tau)|_\kappa\}$$

$$+ \sup_{(y,\tau)\in D(x,t)\cap\Lambda(t)^c} \{|y|^{\frac{1}{2}}(1+|y|+\tau)^{1+\mu}(1+|y|)|F(y,\tau)|_\kappa\}, \tag{4.13}$$

in which

$$D(x,t) = \{(y,\tau)\in\mathbb{R}^2\times[0,t] \mid |y-x|\leqslant(t-\tau)\} \tag{4.14}$$

and $\Lambda(t)^c$ denotes the complementary set of $\Lambda(t)$.

Lemma 4.1 can be proved by Propositions 4.1, 4.2 in [10] and (3.3) (see also [11, 12]). The following lemma comes from [16].

Lemma 4.2. *Suppose that $h\in C^2(\mathbb{R}^2\times[0,T))$ and satisfies $\|h(t)\|_2 < \infty$. Then there exits a positive constant C_* independent of T such that the following estimate holds:*

$$(1+|x|+t)^{\frac{1}{2}}(1+||x|-t|)^{\frac{1}{2}}|h(x,t)| \leqslant C_*\|h(t)\|_2, \quad \forall\, t\in[0,T). \tag{4.15}$$

In what follows, we prove a more general lemma (see Lemma 4.3 below), whose special case is nothing but (4.10) (more precisely, the case $\theta=0$ in Lemma 4.3 gives (4.10)). After the proof of Theorem 4.1, we shall prove Lemma 4.3.

Lemma 4.3. *Let $v=v(x,t)$ be the smooth solution of the Cauchy problem (2.16)–(2.17) on the domain $\mathbb{R}^2\times[0,T))$. Suppose that the following estimate holds*

$$[\partial v]_{[\frac{\kappa+6}{2}]-\theta,\,T} + \langle v\rangle_{[\frac{\kappa+6}{2}]-\theta,\,T} \leqslant \delta, \tag{4.16}$$

where the positive constants κ and δ are defined in Theorem 4.1. Then there exists positive constant ν such that for any $t\in[0,T)$, it holds that

$$(1+|x|+t)^{(-\frac{1}{2}+\mu)\theta}([\nabla\Phi(F)(x,t)]_\kappa + \langle\Phi(F)(x,t)\rangle_{\kappa+1})$$

$$\leqslant C([\partial v]_{[\frac{\kappa+6}{2}]-\theta,\,t} + (1-\delta_{2\theta})\langle v\rangle_{[\frac{\kappa+6}{2}]-\theta,\,t})\Big(\varepsilon + \sup_{0\leqslant\tau<t}\{(1+\tau)^{-\nu}\|\partial v(\tau)\|_{\kappa+8-2\theta}\}\Big), \tag{4.17}$$

where μ is a positive constant satisfying $\mu\in(0,\frac{1}{6})$, θ takes its value in $\{0,1,2\}$ and $\delta_{2\theta}$ is the Kronecker symbol.

We will prove Lemma 4.3 after the proof of Theorem 4.1.

Proof of Theorem 4.1. Obviously, taking $\theta=0$ in (4.17) gives the desired (4.10) immediately. This proves Theorem 4.1. $\qquad\square$

We now prove Lemma 4.3.

Proof of Lemma 4.3. We distinguish three cases.

Case 1. $\theta=2$

In this case, it follows from (4.15) and (4.16) that, for every $(y,\tau)\in D(x,t)\cap\Lambda(t)$ it holds that

$$|y|^{\frac{1}{2}}(1+|y|+\tau)^{3\mu}(1+||y|-\tau|)|F(y,\tau)|_{\kappa+1}$$

$$\leqslant C|y|^{\frac{1}{2}}(1+|y|+\tau)^{3\mu}(1+||y|-\tau|)|\partial v(y,\tau)|_{[\frac{\kappa+2}{2}]}|\partial v(y,\tau)|_{\kappa+2}$$

$$\leqslant C|y|^{\frac{1}{2}}(1+|y|+\tau)^{3\mu}(1+|y|)^{-\frac{1}{2}}(1+|y|)^{\frac{1}{2}}(1+||y|-\tau|)|\partial v(y,\tau)|_{[\frac{\kappa+2}{2}]}|\partial v(y,\tau)|_{\kappa+2}$$

$$\leqslant C[\partial v]_{[\frac{\kappa+2}{2}],\,t}\sup_{0\leqslant\tau<t}\{(1+\tau)^{-\frac{1}{2}+3\mu}\|\partial v(\tau)\|_{\kappa+4}\}. \tag{4.18}$$

Similarly, we obtain from (2.18), (4.15) and (4.16) that

$$|y|^{\frac{1}{2}}(1+|y|+\tau)^{3\mu}(1+|y|)|F(y,\tau)|_{\kappa+1} \leqslant C|y|^{\frac{1}{2}}(1+|y|+\tau)^{3\mu}(1+|y|)|\partial v(y,\tau)|_{[\frac{\kappa+2}{2}]}|\partial v(y,\tau)|_{\kappa+2}$$

$$\leqslant C[\partial v]_{[\frac{\kappa+2}{2}]\, t} \sup_{0\leqslant \tau < t} \{(1+\tau)^{-\frac{1}{2}+3\mu}\|\partial v(\tau)\|_{\kappa+4}\} \tag{4.19}$$

for $(y,\tau) \in D(x,t) \cap \Lambda(t)^c$. On the other hand, by (4.11) and (4.12) and the fact that

$$1 + |y| + \tau \leqslant C(1 + |x| + t), \quad \forall\, (y,\tau) \in D(x,t), \tag{4.20}$$

we have

$$(1+|x|+t)^{-1+2\mu}\{[\nabla\Phi(F)(x,t)]_\kappa + \langle\Phi(F)(x,t)\rangle_{\kappa+1}\}$$
$$\leqslant C\Big\{ \sup_{(y,\tau)\in D(x,t)\cap\Lambda(t)} \{|y|^{\frac{1}{2}}(1+|y|+\tau)^{3\mu}(1+||y|-\tau|)|F(y,\tau)|_{\kappa+1}\}$$
$$+ \sup_{(y,\tau)\in D(x,t)\cap\Lambda(t)^c} \{|y|^{\frac{1}{2}}(1+|y|+\tau)^{3\mu}(1+|y|)|F(y,\tau)|_{\kappa+1}\}\Big\}. \tag{4.21}$$

Combining (4.18), (4.19), (4.21) and choosing $\nu \in (0, \frac{1}{2} - 3\mu)$ gives (4.17) in the case $\theta = 2$.

Case 2. $\theta = 1$

In the present situation, it follows from (4.8), (4.11), (4.12) and (4.20) that

$$(1+|x|+t)^{-\frac{1}{2}+\mu}([\nabla\Phi(F)(x,t)]_\kappa + \langle\Phi(F)(x,t)\rangle_{\kappa+1})$$
$$\leqslant C\Big\{ \sup_{(y,\tau)\in D(x,t)\cap\Lambda(t)} A(y,\tau) + \sup_{(y,\tau)\in D(x,t)\cap\Lambda(t)^c} B(y,\tau)\Big\}, \tag{4.22}$$

where

$$A(y,\tau) = |y|^{\frac{1}{2}}(1+|y|+\tau)^{\frac{1}{2}+2\mu}(1+||y|-\tau|)$$
$$\times \Big(\Big|\sum_{i,j,k=0}^{2} g_{ij}^k \partial_k v \partial_i \partial_j v(y,\tau)\Big|_{\kappa+1} + \Big|\sum_{i,j,k,l=0}^{2} g_{ij}^{kl} \partial_k v \partial_l v \partial_i \partial_j v(y,\tau)\Big|_{\kappa+1}\Big)$$

and

$$B(y,\tau) = |y|^{\frac{1}{2}}(1+|y|+\tau)^{\frac{1}{2}+2\mu}(1+|y|)\Big(\Big|\sum_{i,j,k=0}^{2} g_{ij}^k \partial_k v \partial_i \partial_j v(y,\tau)\Big|_{\kappa+1} + \Big|\sum_{i,j,k,l=0}^{2} g_{ij}^{kl} \partial_k v \partial_l v \partial_i \partial_j v(y,\tau)\Big|_{\kappa+1}\Big).$$

When $\tau \in [0,1]$, it follows from (4.15) and (4.16) that, for every $(y,\tau) \in D(x,t) \cap \Lambda(t)$

$$(1+|y|+\tau)^{\frac{1}{2}+2\mu}(1+||y|-\tau|)|y|^{\frac{1}{2}}\Big(\Big|\sum_{i,j,k=0}^{2} g_{ij}^k \partial_k v \partial_i \partial_j v(y,\tau)\Big|_{\kappa+1} + \Big|\sum_{i,j,k,l=0}^{2} g_{ij}^{kl} \partial_k v \partial_l v \partial_i \partial_j v(y,\tau)\Big|_{\kappa+1}\Big)$$
$$\leqslant C[\partial v]_{[\frac{\kappa+2}{2}],t}\|\partial v(\tau)\|_{\kappa+4} \leqslant C[\partial v]_{[\frac{\kappa+2}{2}],t}(1+\tau)^{-\nu}\|\partial v(\tau)\|_{\kappa+4}, \quad \forall\, \nu \in \mathbb{R}. \tag{4.23}$$

Similarly, when $\tau \in [0,1]$, it follows from (2.18), (4.15) and (4.16) that, for every $(y,\tau) \in D(x,t) \cap \Lambda(t)^c$

$$(1+|y|)(1+|y|+\tau)^{\frac{1}{2}+2\mu}|y|^{\frac{1}{2}}\Big(\Big|\sum_{i,j,k=0}^{2} g_{ij}^k \partial_k v \partial_i \partial_j v(y,\tau)\Big|_{\kappa+1} + \Big|\sum_{i,j,k,l=0}^{2} g_{ij}^{kl} \partial_k v \partial_l v \partial_i \partial_j v(y,\tau)\Big|_{\kappa+1}\Big)$$
$$\leqslant C[\partial v]_{[\frac{\kappa+2}{2}],t}(1+\tau)^{-\nu}\|\partial v(\tau)\|_{\kappa+4}, \quad \forall\, \nu \in \mathbb{R}. \tag{4.24}$$

Thus, in what follows, it suffices to investigate the case $\tau \geqslant 1$.

By (3.25), (4.5), (4.9) and (4.17) with $\theta = 2$, for $(y,\tau) \in D(x,t) \cap \Lambda(t) \cap \{(z,s)|s \geqslant 1\}$ we have

$$(1+|y|+\tau)^{\frac{1}{2}+2\mu}|y|^{\frac{1}{2}}(1+||y|-\tau|)\Big|\sum_{i,j,k=0}^{2} g_{ij}^k \partial_k v \partial_i \partial_j v(y,\tau)\Big|_{\kappa+1}$$
$$\leqslant C|y|^{\frac{1}{2}}(1+|y|+\tau)^{-\frac{1}{2}+2\mu}\{(1+||y|-\tau|)^2|\partial v(y,\tau)|_{[\frac{\kappa+2}{2}]}|\partial v(y,\tau)|_{\kappa+2}$$
$$+ (1+||y|-\tau|)(|\partial v(y,\tau)|_{[\frac{\kappa+2}{2}]}|v(y,\tau)|_{\kappa+3} + |v(y,\tau)|_{[\frac{\kappa+4}{2}]}|\partial v(y,\tau)|_{\kappa+2})\}$$

$$\leqslant C([\partial v]_{[\frac{\kappa+2}{2}],\,t}+\langle v\rangle_{[\frac{\kappa+4}{2}],\,t})|y|^{\frac{1}{2}}(1+|y|+\tau)^{-\frac{1}{2}+2\mu}\{(1+|y|)^{-\frac{1}{2}}|v(y,\tau)|_{\kappa+3}$$

$$+(1+|y|)^{-\frac{1}{2}}(1+||y|-\tau|)|\partial v(y,\tau)|_{\kappa+2}+(1+||y|-\tau|)(1+|y|+\tau)^{-\frac{1}{2}}|\partial v(y,\tau)|_{\kappa+2}\}$$

$$\leqslant C([\partial v]_{[\frac{\kappa+2}{2}],\,t}+\langle v\rangle_{[\frac{\kappa+4}{2}],\,t})|y|^{\frac{1}{2}}(1+|y|+\tau)^{-\frac{1}{2}+2\mu}\{(1+|y|)^{-\frac{1}{2}}(1+||y|-\tau|)$$

$$\times(|\nabla v(y,\tau)|_{\kappa+2}+(1+\tau)^{-1}|v(y,\tau)|_{\kappa+3})+(1+|y|)^{-\frac{1}{2}}|v(y,\tau)|_{\kappa+3}$$

$$+(1+||y|-\tau|)(1+|y|+\tau)^{-\frac{1}{2}}(|\nabla v(y,\tau)|_{\kappa+2}+(1+\tau)^{-1}|v(y,\tau)|_{\kappa+3})\}$$

$$\leqslant C([\partial v]_{[\frac{\kappa+2}{2}],\,t}+\langle v\rangle_{[\frac{\kappa+4}{2}],\,t})\{\varepsilon+|y|^{\frac{1}{2}}(1+|y|+\tau)^{-\frac{1}{2}+2\mu}(1+|y|)^{-\frac{1}{2}}(1+||y|-\tau|)$$

$$\times|\nabla\Phi(F)(y,\tau)|_{\kappa+2}+|y|^{\frac{1}{2}}(1+|y|+\tau)^{-\frac{1}{2}+2\mu}(1+|y|)^{-\frac{1}{2}}|\Phi(F)(y,\tau)|_{\kappa+3}\}$$

$$\leqslant C([\partial v]_{[\frac{\kappa+2}{2}],\,t}+\langle v\rangle_{[\frac{\kappa+4}{2}],\,t})\{\varepsilon+(1+|y|+\tau)^{-1+2\mu}([\nabla\Phi(F)(y,\tau)]_{\kappa+2}+\langle\Phi(F)(y,\tau)\rangle_{\kappa+3})\}$$

$$\leqslant C([\partial v]_{[\frac{\kappa+2}{2}],\,t}+\langle v\rangle_{[\frac{\kappa+4}{2}],\,t})(\varepsilon+[\partial v]_{[\frac{\kappa+4}{2}],\,t}\sup_{0\leqslant\tau<t}\{(1+\tau)^{-\nu}\|\partial v(\tau)\|_{\kappa+6}\}). \tag{4.25}$$

Making use of (2.18), (4.5), (4.9) and (4.17) with $\theta=2$ gives

$$(1+|y|+\tau)^{\frac{1}{2}+2\mu}|y|^{\frac{1}{2}}(1+|y|)\left|\sum_{i,j,k=0}^{2}g_{ij}^{k}\partial_k v\partial_i\partial_j v(y,\tau)\right|_{\kappa+1}$$

$$\leqslant C|y|^{\frac{1}{2}}(1+|y|+\tau)^{\frac{1}{2}+2\mu}(1+|y|)|\partial v(y,\tau)|_{[\frac{\kappa+2}{2}]}|\partial v(y,\tau)|_{\kappa+2}$$

$$\leqslant C[\partial v]_{[\frac{\kappa+2}{2}],\,t}|y|^{\frac{1}{2}}(1+|y|+\tau)^{\frac{1}{2}+2\mu}(1+|y|)^{\frac{1}{2}}(1+||y|-\tau|)^{-1}|\partial v(y,\tau)|_{\kappa+2}$$

$$\leqslant C[\partial v]_{[\frac{\kappa+2}{2}],\,t}|y|^{\frac{1}{2}}(1+|y|+\tau)^{\frac{1}{2}+2\mu}(1+|y|)^{\frac{1}{2}}(1+||y|-\tau|)^{-1}$$

$$\times(|\nabla v(y,\tau)|_{\kappa+2}+(1+\tau)^{-1}|v(y,\tau)|_{\kappa+3})$$

$$\leqslant C[\partial v]_{[\frac{\kappa+2}{2}],\,t}\{\varepsilon+|y|^{\frac{1}{2}}(1+|y|+\tau)^{\frac{1}{2}+2\mu}(1+||y|-\tau|)^{-2}[\nabla\Phi(F)(y,\tau)]_{\kappa+2}$$

$$+|y|^{\frac{1}{2}}(1+|y|+\tau)^{2\mu}(1+|y|)^{-\frac{1}{2}}(1+||y|-\tau|)^{-1}\langle\Phi(F)(y,\tau)\rangle_{\kappa+3}\}$$

$$\leqslant C[\partial v]_{[\frac{\kappa+2}{2}],\,t}\{\varepsilon+(1+|y|+\tau)^{-1+2\mu}([\nabla\Phi(F)(y,\tau)]_{\kappa+2}+\langle\Phi(F)(y,\tau)\rangle_{\kappa+3})\}$$

$$\leqslant C[\partial v]_{[\frac{\kappa+2}{2}],\,t}(\varepsilon+[\partial v]_{[\frac{\kappa+4}{2}],\,t}\sup_{0\leqslant\tau<t}\{(1+\tau)^{-\nu}\|\partial v(\tau)\|_{\kappa+6}\}) \tag{4.26}$$

for $(y,\tau)\in D(x,t)\cap\Lambda(t)^c\cap\{(z,s)|s\geqslant1\}$. On the other hand, by (4.15), for $(y,\tau)\in D(x,t)\cap\Lambda(t)\cap\{(z,s)|s\geqslant1\}$ we have

$$(1+|y|+\tau)^{\frac{1}{2}+2\mu}|y|^{\frac{1}{2}}(1+||y|-\tau|)\left|\sum_{i,j,k,l=0}^{2}g_{ij}^{kl}\partial_k v\partial_l v\partial_i\partial_j v(y,\tau)\right|_{\kappa+1}$$

$$\leqslant C[\partial v]_{[\frac{\kappa+2}{2}],t}^2|y|^{\frac{1}{2}}(1+|y|+\tau)^{\frac{1}{2}+2\mu}(1+||y|-\tau|)^{-1}(1+|y|)^{-1}|\partial v(y,\tau)|_{\kappa+2}$$

$$\leqslant C[\partial v]_{[\frac{\kappa+2}{2}],t}^2\sup_{0\leqslant\tau<t}\{(1+\tau)^{-\nu}\|\partial v(\tau)\|_{\kappa+4}\}. \tag{4.27}$$

It follows from (2.18) and (4.15) that, for $(y,\tau)\in D(x,t)\cap\Lambda(t)^c\cap\{(z,s)|s\geqslant1\}$

$$(1+|y|+\tau)^{\frac{1}{2}+2\mu}|y|^{\frac{1}{2}}(1+|y|)\left|\sum_{i,j,k,l=0}^{2}g_{ij}^{kl}\partial_k v\partial_l v\partial_i\partial_j v(y,\tau)\right|_{\kappa+1}$$

$$\leqslant C[\partial v]_{[\frac{\kappa+2}{2}],\,t}^2|y|^{\frac{1}{2}}(1+|y|+\tau)^{\frac{1}{2}+2\mu}(1+||y|-\tau|)^{-2}|\partial v(y,\tau)|_{\kappa+2}$$

$$\leqslant C[\partial v]_{[\frac{\kappa+2}{2}],\,t}^2\sup_{0\leqslant\tau<t}\{(1+\tau)^{-\nu}\|\partial v(\tau)\|_{\kappa+4}\}. \tag{4.28}$$

(4.22)–(4.28) yields (4.17) in the case $\theta=1$.

Case 3. $\theta=0$

For $(y,\tau)\in D(x,t)\cap\Lambda(t)\cap\{(z,s)|s\leqslant1\}$, we obtain from (4.15) and (4.16) that

$$(1+|y|+\tau)^{1+\mu}|y|^{\frac{1}{2}}(1+||y|-\tau|)\left(\left|\sum_{i,j,k=0}^{2}g_{ij}^{k}\partial_k v\partial_i\partial_j v(y,\tau)\right|_{\kappa+1}+\left|\sum_{i,j,k,l=0}^{2}g_{ij}^{kl}\partial_k v\partial_l v\partial_i\partial_j v(y,\tau)\right|_{\kappa+1}\right)$$

$$\leqslant C[\partial v]_{[\frac{\kappa+2}{2}],\,t}|y|^{\frac{1}{2}}(1+|y|+\tau)^{1+\mu}(1+|y|)^{-\frac{1}{2}}|\partial v(y,\tau)|_{\kappa+2}$$

$$\leqslant C[\partial v]_{[\frac{\kappa+2}{2}],\,t}\sup_{0\leqslant\tau<t}\{(1+\tau)^{-\nu}\|\partial v(\tau)\|_{\kappa+4}\}. \tag{4.29}$$

Similarly, it follows from (2.18), (4.15) and (4.16) that, for $(y,\tau)\in D(x,t)\cap\Lambda(t)^c\cap\{(z,s)|s\leqslant 1\}$

$$(1+|y|+\tau)^{1+\mu}|y|^{\frac{1}{2}}(1+|y|)\left(\left|\sum_{i,j,k=0}^2 g_{ij}^k\partial_k v\partial_i\partial_j v(y,\tau)\right|_{\kappa+1}+\left|\sum_{i,j,k,l=0}^2 g_{ij}^{kl}\partial_k v\partial_l v\partial_i\partial_j v(y,\tau)\right|_{\kappa+1}\right)$$

$$\leqslant C[\partial v]_{[\frac{\kappa+2}{2}],\,t}|y|^{\frac{1}{2}}(1+|y|+\tau)^{1+\mu}(1+|y|)^{\frac{1}{2}}(1+||y|-\tau|)^{-1}|\partial v(y,\tau)|_{\kappa+2}$$

$$\leqslant C[\partial v]_{[\frac{\kappa+2}{2}],\,t}\sup_{0\leqslant\tau<t}\{(1+\tau)^{-\nu}\|\partial v(\tau)\|_{\kappa+4}\}. \tag{4.30}$$

We next consider the case $\tau\geqslant 1$.

Noting (3.25), (3.26), (4.5), (4.9), (4.16) and using (4.17) in the case $\theta=1$, for $(y,\tau)\in D(x,t)\cap\Lambda(t)\cap\{(z,s)|s\geqslant 1\}$ we have

$$(1+|y|+\tau)^{1+\mu}(1+||y|-\tau|)|y|^{\frac{1}{2}}\left(\left|\sum_{i,j,k=0}^2 g_{ij}^k\partial_k v\partial_i\partial_j v(y,\tau)\right|_{\kappa+1}+\left|\sum_{i,j,k,l=0}^2 g_{ij}^{kl}\partial_k v\partial_l v\partial_i\partial_j v(y,\tau)\right|_{\kappa+1}\right)$$

$$\leqslant C|y|^{\frac{1}{2}}(1+|y|+\tau)^{\mu}\{(1+||y|-\tau|)^2|\partial v(y,\tau)|_{[\frac{\kappa+2}{2}]}|\partial v(y,\tau)|_{\kappa+2}$$

$$\qquad+(1+||y|-\tau|)(|\partial v(y,\tau)|_{[\frac{\kappa+2}{2}]}|v(y,\tau)|_{\kappa+3}+|v(y,\tau)|_{[\frac{\kappa+4}{2}]}|\partial v(y,\tau)|_{\kappa+2})\}$$

$$\leqslant C([\partial v]_{[\frac{\kappa+2}{2}],\,t}+\langle v\rangle_{[\frac{\kappa+4}{2}],\,t})|y|^{\frac{1}{2}}(1+|y|+\tau)^{\mu}\{(1+|y|)^{-\frac{1}{2}}(1+||y|-\tau|)|\partial v(y,\tau)|_{\kappa+2}$$

$$\qquad+(1+|y|)^{-\frac{1}{2}}|v(y,\tau)|_{\kappa+3}+(1+||y|-\tau|)(1+|y|+\tau)^{-\frac{1}{2}}|\partial v(y,\tau)|_{\kappa+2}\}$$

$$\leqslant C([\partial v]_{[\frac{\kappa+2}{2}],\,t}+\langle v\rangle_{[\frac{\kappa+4}{2}],\,t})|y|^{\frac{1}{2}}(1+|y|+\tau)^{\mu}\{(1+|y|)^{-\frac{1}{2}}(1+||y|-\tau|)$$

$$\qquad\times(|\nabla v(y,\tau)|_{\kappa+2}+(1+\tau)^{-1}|v(y,\tau)|_{\kappa+3})+(1+|y|)^{-\frac{1}{2}}|v(y,\tau)|_{\kappa+3}$$

$$\qquad+(1+||y|-\tau|)(1+|y|+\tau)^{-\frac{1}{2}}(|\nabla v(y,\tau)|_{\kappa+2}+(1+\tau)^{-1}|v(y,\tau)|_{\kappa+3})\}$$

$$\leqslant C([\partial v]_{[\frac{\kappa+2}{2}],\,t}+\langle v\rangle_{[\frac{\kappa+4}{2}],\,t})\{\varepsilon+|y|^{\frac{1}{2}}(1+|y|+\tau)^{\mu}(1+|y|)^{-\frac{1}{2}}(1+||y|-\tau|)$$

$$\qquad\times|\nabla\Phi(F)(y,\tau)|_{\kappa+2}+|y|^{\frac{1}{2}}(1+|y|+\tau)^{\mu}(1+|y|)^{-\frac{1}{2}}|\Phi(F)(y,\tau)|_{\kappa+3}\}$$

$$\leqslant C([\partial v]_{[\frac{\kappa+2}{2}],\,t}+\langle v\rangle_{[\frac{\kappa+4}{2}],\,t})\{\varepsilon+(1+|y|+\tau)^{-\frac{1}{2}+\mu}([\nabla\Phi(F)(y,\tau)]_{\kappa+2}+\langle\Phi(F)(y,\tau)\rangle_{\kappa+3})\}$$

$$\leqslant C([\partial v]_{[\frac{\kappa+2}{2}],\,t}+\langle v\rangle_{[\frac{\kappa+4}{2}],\,t})\Big\{\varepsilon+([\partial v]_{[\frac{\kappa+6}{2}],\,t}+\langle v\rangle_{[\frac{\kappa+6}{2}],\,t})\sup_{0\leqslant\tau<t}\{(1+\tau)^{-\nu}\|\partial v(\tau)\|_{\kappa+8}\}\Big\}. \tag{4.31}$$

Using (2.18), (4.5), (4.9), (4.16) and noting (4.17) with $\theta=1$, for $(y,\tau)\in D(x,t)\cap\Lambda(t)^c\cap\{(z,s)|s\geqslant 1\}$, we obtain

$$(1+|y|+\tau)^{1+\mu}(1+|y|)|y|^{\frac{1}{2}}\left(\left|\sum_{i,j,k=0}^2 g_{ij}^k\partial_k v\partial_i\partial_j v(y,\tau)\right|_{\kappa+1}+\left|\sum_{i,j,k,l=0}^2 g_{ij}^{kl}\partial_k v\partial_l v\partial_i\partial_j v(y,\tau)\right|_{\kappa+1}\right)$$

$$\leqslant C[\partial v]_{[\frac{\kappa+2}{2}],\,t}|y|^{\frac{1}{2}}(1+|y|+\tau)^{1+\mu}(1+|y|)^{\frac{1}{2}}(1+||y|-\tau|)^{-1}|\partial v(y,\tau)|_{\kappa+2}$$

$$\leqslant C[\partial v]_{[\frac{\kappa+2}{2}],\,t}|y|^{\frac{1}{2}}(1+|y|+\tau)^{1+\mu}(1+|y|)^{\frac{1}{2}}(1+||y|-\tau|)^{-1}(|\nabla v(y,\tau)|_{\kappa+2}+(1+\tau)^{-1}|v(y,\tau)|_{\kappa+3})$$

$$\leqslant C[\partial v]_{[\frac{\kappa+2}{2}],\,t}\{\varepsilon+|y|^{\frac{1}{2}}(1+|y|+\tau)^{1+\mu}(1+||y|-\tau|)^{-2}[\nabla\Phi(F)(y,\tau)]_{\kappa+2}$$

$$\qquad+|y|^{\frac{1}{2}}(1+|y|+\tau)^{\frac{1}{2}+\mu}(1+|y|)^{-\frac{1}{2}}(1+||y|-\tau|)^{-1}\langle\Phi(F)(y,\tau)\rangle_{\kappa+3}\}$$

$$\leqslant C[\partial v]_{[\frac{\kappa+2}{2}],\,t}\{\varepsilon+(1+|y|+\tau)^{-\frac{1}{2}+\mu}([\nabla\Phi(F)(y,\tau)]_{\kappa+2}+\langle\Phi(F)(y,\tau)\rangle_{\kappa+3})\}$$

$$\leqslant C[\partial v]_{[\frac{\kappa+2}{2}],\,t}\{\varepsilon+([\partial v]_{[\frac{\kappa+6}{2}],\,t}+\langle v\rangle_{[\frac{\kappa+6}{2}],\,t})(\varepsilon+\sup_{0\leqslant\tau<t}\{(1+\tau)^{-\nu}\|\partial v(\tau)\|_{\kappa+8}\})\}$$

$$\leqslant C([\partial v]_{[\frac{\kappa+6}{2}],\,t}+\langle v\rangle_{[\frac{\kappa+6}{2}],\,t})\Big(\varepsilon+\sup_{0\leqslant\tau<t}\{(1+\tau)^{-\nu}\|\partial v(\tau)\|_{\kappa+8}\}\Big). \tag{4.32}$$

Therefore, combining (4.11)–(4.12) and (4.29)–(4.32) gives (4.17) in the case $\theta=0$ immediately. Thus, the proof of Lemma 4.3 is completed. □

5 Energy estimate

This section is devoted to establishing the energy estimates, which play an important role in the proof of Theorem 1.1.

Theorem 5.1. *Let $v = v(x,t)$ be a smooth solution of the Cauchy problem* (2.16)–(2.17) *on the domain* $\mathbb{R}^2 \times [0,T)$. *If there exists a positive small constant δ such that*

$$[\partial v]_{[\frac{\kappa+1}{2}],T} + \langle v \rangle_{[\frac{\kappa+1}{2}]+1,T} \leqslant \delta, \tag{5.1}$$

then, for every $t \in [0,T)$, it holds that

$$\|\partial v(t)\|_\kappa \leqslant C_2 \varepsilon (1+t)^{C_3([\partial v]_{[\frac{\kappa+1}{2}],T} + \langle v \rangle_{[\frac{\kappa+1}{2}]+1,\,T})}, \tag{5.2}$$

where C_2 and C_3 are two positive constants independent of T.

Proof. Define

$$\psi(\lambda) = \int_{-\infty}^{\lambda} \frac{1}{(1+|\varrho|)^2} d\varrho \tag{5.3}$$

and

$$\varphi(x,t) = \psi(||x|-t|), \quad \dot{\varphi}(x,t) = \psi'(||x|-t|).$$

Obviously,

$$\psi \in C^1(\mathbb{R}), \quad \varphi \in C^1(\mathbb{R}^2 \times \mathbb{R}^+) \quad \text{and} \quad \dot{\varphi} \in C^0(\mathbb{R}^2 \times \mathbb{R}^+).$$

Moreover,

$$0 \leqslant \varphi(x,t) \leqslant 2 \tag{5.4}$$

and

$$\dot{\varphi}(x,t) = \frac{1}{(1+||x|-t|)^2} \tag{5.5}$$

for any $(x,t) \in \mathbb{R}^2 \times (0,+\infty)$.

On the other hand, we introduce the following energy functions

$$E_0(v(t)) = \left\{ \int_{\mathbb{R}^2} (|\partial_t v(x,t)|^2 + |\nabla v(x,t)|^2) dx \right\}^{\frac{1}{2}}, \tag{5.6}$$

$$E_\kappa(v(t)) = \sum_{|a| \leqslant \kappa} E_0(\Gamma^a v(t)), \tag{5.7}$$

$$\tilde{E}_0(v(t)) = \left\{ \int_{\mathbb{R}^2} e^{\varphi}(x,t)(|\partial_t v(x,t)|^2 + |\nabla v(x,t)|^2) dx \right\}^{\frac{1}{2}}, \tag{5.8}$$

$$\tilde{E}_\kappa(v(t)) = \sum_{|a| \leqslant \kappa} \tilde{E}_0(\Gamma^a v(t)). \tag{5.9}$$

Thus, it follows from (5.4), (5.6)–(5.9) that

$$\frac{1}{\hat{C}} \|\partial v(t)\|_\kappa \leqslant \frac{1}{\tilde{C}} E_\kappa(v(t)) \leqslant \tilde{E}_\kappa(v(t)) \leqslant \tilde{C} E_\kappa(v(t)) \leqslant \hat{C} \|\partial v(t)\|_\kappa \tag{5.10}$$

for some positive constant $\hat{C}$ and $\tilde{C}$ which are independent of t. Therefore, in order to prove (5.2), it suffices to show

$$\tilde{E}_\kappa(v(t)) \leqslant C \varepsilon (1+t)^{C([\partial v]_{[\frac{\kappa+1}{2}],T} + \langle v \rangle_{[\frac{\kappa+1}{2}]+1,T})} \tag{5.11}$$

for $t \in [0,T)$.

To do so, noting (2.16), we have

$$\partial_t^2 \Gamma^a v - \triangle \Gamma^a v + 2 \sum_{i=1}^{2} \partial_i v \partial_t \partial_i \Gamma^a v - 2\partial_t v \triangle \Gamma^a v + \sum_{i,j=1}^{2} \partial_i v \partial_j v \partial_i \partial_j \Gamma^a v - |\nabla v|^2 \triangle \Gamma^a v$$

$$= \sum_{b \leqslant a} C_b \Gamma^b \left(-2 \sum_{i=1}^{2} \partial_i v \partial_t \partial_i v + 2\partial_t v \triangle - \sum_{i,\,j=1}^{2} \partial_i v \partial_j v \partial_i \partial_j v + |\nabla v|^2 \triangle \right)$$

$$+2\sum_{i=1}^{2}\partial_i v\partial_t\partial_i\Gamma^a v - 2\partial_t v\triangle\Gamma^a v + \sum_{i,\,j=1}^{2}\partial_i v\partial_j v\partial_i\partial_j\Gamma^a v - |\nabla v|^2\triangle\Gamma^a v \qquad (5.12)$$

for any given multi-index a, where C_b is a constant which is determined by (3.3). Notice that $C_a = 1$. Multiplying $e^\varphi\partial_t\Gamma^a v$ to (5.12) and then integrating with respect to x on $\mathbb{R}^2$ leads to

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^2}\{e^\varphi(|\partial_t\Gamma^a v(x,t)|^2 + |\nabla\Gamma^a v(x,t)|^2)\}dx$$

$$+\int_{\mathbb{R}^2}e^\varphi\Big(\frac{1}{2}\dot\varphi|Z\Gamma^a v|^2 + 2\sum_{i=1}^{2}\partial_i v\partial_t\partial_i\Gamma^a v\partial_t\Gamma^a v - 2\partial_t v\triangle\Gamma^a v\partial_t\Gamma^a v$$

$$+\sum_{i,\,j=1}^{2}\partial_i v\partial_j v\partial_i\partial_j\Gamma^a v\partial_t\Gamma^a v - |\nabla v|^2\triangle\Gamma^a v\partial_t\Gamma^a v\Big)dx = I_1(t), \qquad (5.13)$$

where

$$I_1(t) = \int_{\mathbb{R}^2}e^\varphi\partial_t\Gamma^a v\Big\{\sum_{b\leqslant a}C_b\Gamma^b\Big(-2\sum_{i=1}^{2}\partial_i v\partial_t\partial_i v + 2\partial_t v\triangle v - \sum_{i,\,j=1}^{2}\partial_i v\partial_j v\partial_i\partial_j v + |\nabla v|^2\triangle v\Big)$$

$$+2\sum_{i=1}^{2}\partial_i v\partial_t\partial_i\Gamma^a v - 2\partial_t v\triangle\Gamma^a v + \sum_{i,\,j=1}^{2}\partial_i v\partial_j v\partial_i\partial_j\Gamma^a v - |\nabla v|^2\triangle\Gamma^a v\Big\}dx. \qquad (5.14)$$

Notice that

$$e^\varphi\Big(2\sum_{i=1}^{2}\partial_i v\partial_t\partial_i\Gamma^a v\partial_t\Gamma^a v - 2\partial_t v\triangle\Gamma^a v\partial_t\Gamma^a v + \sum_{i,\,j=1}^{2}\partial_i v\partial_j v\partial_i\partial_j\Gamma^a v\partial_t\Gamma^a v - |\nabla v|^2\triangle\Gamma^a v\partial_t\Gamma^a v\Big)$$

$$=\sum_{i=1}^{2}\partial_i(e^\varphi\partial_i v\partial_t\Gamma^a v\partial_t\Gamma^a v) - 2\sum_{i=1}^{2}\partial_i(e^\varphi\partial_t v\partial_i\Gamma^a v\partial_t\Gamma^a v) + \sum_{i=1}^{2}\partial_t(e^\varphi\partial_i v\partial_i\Gamma^a v\partial_i\Gamma^a v)$$

$$+\sum_{i=1}^{2}\sum_{j=1}^{2}\partial_i(e^\varphi\partial_i v\partial_j v\partial_j\Gamma^a v\partial_t\Gamma^a v) - \frac{1}{2}\sum_{i=1}^{2}\sum_{j=1}^{2}\partial_t(e^\varphi\partial_i v\partial_j v\partial_j\Gamma^a v\partial_i\Gamma^a v)$$

$$-\sum_{i=1}^{2}\partial_i(e^\varphi|\nabla v|^2\partial_i\Gamma^a v\partial_t\Gamma^a v) + \frac{1}{2}\sum_{i=1}^{2}\partial_t(e^\varphi|\nabla v|^2\partial_i\Gamma^a v\partial_i\Gamma^a v)$$

$$-\frac{\dot\varphi}{2}e^\varphi\Big(2\sum_{i=1}^{2}\omega_i\partial_i v\partial_t\Gamma^a v\partial_t\Gamma^a v - 4\sum_{i=1}^{2}\omega_i\partial_t v\partial_i\Gamma^a v\partial_t\Gamma^a v - 2\sum_{i=1}^{2}\partial_t v\partial_i\Gamma^a v\partial_i\Gamma^a v$$

$$+2\sum_{i=1}^{2}\sum_{j=1}^{2}\omega_i\partial_i v\partial_j v\partial_j\Gamma^a v\partial_t\Gamma^a v + \sum_{i=1}^{2}\sum_{j=1}^{2}\partial_i v\partial_j v\partial_j\Gamma^a v\partial_i\Gamma^a v$$

$$-2\sum_{i=1}^{2}\omega_i|\nabla v|^2\partial_i\Gamma^a v\partial_t\Gamma^a v - \sum_{i=1}^{2}|\nabla v|^2\partial_i\Gamma^a v\partial_i\Gamma^a v\Big)$$

$$-\frac{1}{2}e^\varphi\Big(2\sum_{i=1}^{2}\partial_i^2 v\partial_t\Gamma^a v\partial_t\Gamma^a v - 4\sum_{i=1}^{2}\partial_i\partial_t v\partial_i\Gamma^a v\partial_t\Gamma^a v + 2\sum_{i=1}^{2}\partial_t^2 v\partial_i\Gamma^a v\partial_i\Gamma^a v$$

$$+2\sum_{i=1}^{2}\sum_{j=1}^{2}\partial_i(\partial_i v\partial_j v)\partial_j\Gamma^a v\partial_t\Gamma^a v - 2\sum_{i=1}^{2}\sum_{j=1}^{2}\partial_t(\partial_i v\partial_j v)\partial_j\Gamma^a v\partial_i\Gamma^a v$$

$$-2\sum_{i=1}^{2}\partial_i(|\nabla v|^2)\partial_i\Gamma^a v\partial_t\Gamma^a v + \sum_{i=1}^{2}\partial_t(|\nabla v|^2)\partial_i\Gamma^a v\partial_i\Gamma^a v\Big), \qquad (5.15)$$

where $\omega_i = \frac{x_i}{|x|}$ $(i=1,2)$. It follows from (5.13) and (5.15) that

$$I_1(t) = \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^2}\{e^\varphi(|\partial_t\Gamma^a v(x,t)|^2 + |\nabla\Gamma^a v(x,t)|^2)\}dx + \frac{1}{2}\int_{\mathbb{R}^2}e^\varphi\dot\varphi|Z\Gamma^a v|^2 dx$$

$$+\frac{d}{dt}\int_{\mathbb{R}^2}\left(e^\varphi\partial_i v\partial_i\Gamma^a v\partial_i\Gamma^a v-\frac{1}{2}\sum_{i=1}^2\sum_{j=1}^2 e^\varphi\partial_i v\partial_j v\partial_j\Gamma^a v\partial_i\Gamma^a v\right.$$

$$\left.+\frac{1}{2}\sum_{i=1}e^\varphi|\nabla v|^2\partial_i\Gamma^a v\partial_i\Gamma^a v\right)dx+I_2(t)+I_3(t), \tag{5.16}$$

where

$$I_2(t)=-\int_{\mathbb{R}^2}\frac{\dot\varphi}{2}e^\varphi\left(2\sum_{i=1}^2\omega_i\partial_i v\partial_t\Gamma^a v\partial_t\Gamma^a v-4\sum_{i=1}^2\omega_i\partial_t v\partial_i\Gamma^a v\partial_t\Gamma^a v-2\sum_{i=1}^2\partial_t v\partial_i\Gamma^a v\partial_i\Gamma^a v\right.$$

$$+2\sum_{i=1}^2\sum_{j=1}^2\omega_i\partial_i v\partial_j v\partial_j\Gamma^a v\partial_t\Gamma^a v+\sum_{i=1}^2\sum_{j=1}^2\partial_i v\partial_j v\partial_j\Gamma^a v\partial_i\Gamma^a v$$

$$\left.-2\sum_{i=1}^2\omega_i|\nabla v|^2\partial_i\Gamma^a v\partial_t\Gamma^a v-\sum_{i=1}^2|\nabla v|^2\partial_i\Gamma^a v\partial_i\Gamma^a v\right)dx \tag{5.17}$$

and

$$I_3(t)=-\int_{\mathbb{R}^2}\frac{1}{2}e^\varphi\left(2\sum_{i=1}^2\partial_i^2 v\partial_t\Gamma^a v\partial_t\Gamma^a v-4\sum_{i=1}^2\partial_i\partial_t v\partial_i\Gamma^a v\partial_t\Gamma^a v+2\sum_{i=1}^2\partial_t^2 v\partial_i\Gamma^a v\partial_i\Gamma^a v\right.$$

$$+2\sum_{i=1}^2\sum_{j=1}^2\partial_i(\partial_i v\partial_j v)\partial_j\Gamma^a v\partial_t\Gamma^a v-2\sum_{i=1}^2\sum_{j=1}^2\partial_t(\partial_i v\partial_j v)\partial_j\Gamma^a v\partial_i\Gamma^a v$$

$$\left.-2\sum_{i=1}^2\partial_i(|\nabla v|^2)\partial_i\Gamma^a v\partial_t\Gamma^a v+\sum_{i=1}^2\partial_t(|\nabla v|^2)\partial_i\Gamma^a v\partial_i\Gamma^a v\right)dx. \tag{5.18}$$

Combining (5.1) and (5.16) yields

$$\tilde E_\kappa(v(t))^2+\sum_{|a|\leqslant\kappa}\int_0^t\int_{\mathbb{R}^2}e^\varphi\dot\varphi|Z\Gamma^a v(x,\tau)|^2 dxd\tau\leqslant C\tilde E_\kappa(v(0))^2+C\int_0^t|I(\tau)|d\tau \tag{5.19}$$

for $t\in[0,T)$, where

$$I(t)=\sum_{|a|\leqslant\kappa}(|I_1(t)|+|I_2(t)|+|I_3(t)|). \tag{5.20}$$

Now we claim that

$$C|I(t)|\leqslant\frac{1}{2}\sum_{|a|\leqslant\kappa}\int_{\mathbb{R}^2}\dot\varphi e^\varphi|Z\Gamma^a v(x,t)|^2 dx+\frac{C}{1+t}([\partial v]_{[\frac{\kappa+1}{2}],T}+\langle v\rangle_{[\frac{\kappa+1}{2}]+1,T})\int_{\mathbb{R}^2}e^\varphi|\partial v(x,t)|_\kappa^2 dx \tag{5.21}$$

for $t\in[0,T)$.

For the time being, we assume that (5.21) is true. We will prove it later. Thus, it follows from (5.19) and (5.21) that

$$\tilde E_\kappa(v(t))^2+\frac{1}{2}\sum_{|a|\leqslant\kappa}\int_0^t\int_{\mathbb{R}^2}e^\varphi\dot\varphi|Z\Gamma^a v(x,\tau)|^2 dxd\tau$$

$$\leqslant C\tilde E_\kappa(v(0))^2+C\int_0^t\frac{1}{1+\tau}([\partial v]_{[\frac{\kappa+1}{2}],T}+\langle v\rangle_{[\frac{\kappa+1}{2}]+1,T})\tilde E_\kappa(v(\tau))^2 d\tau. \tag{5.22}$$

Noting (2.17), (5.22) and using Gronwall inequality gives (5.11) immediately.

It suffices to prove (5.21) in the rest of this section. To prove (5.21), we distinguish two cases.

Case I. $t\in[0,1]$

In this case, (5.21) comes from (5.14), (5.17) and (5.18) immediately.

Case II. $t\in[1,T)$

We first estimate $|I_1(t)|$.

Noting (2.18), (2.20), (2.21), (3.7), (3.22), (3.23), (5.1), (5.14) and using Cauchy inequality, we have

$$
\begin{aligned}
|I_1(t)| \leqslant & \int_{(x,t)\in\Lambda(T)} \left| \Gamma^a \left(\sum_{i,j,k=0}^{2} g_{ij}^{k} \partial_k v \partial_i \partial_j v \right) - \sum_{i,j,k=0}^{2} g_{ij}^{k} \partial_k v \partial_i \partial_j \Gamma^a v \right| |\partial_t \Gamma^a v| e^\varphi dx \\
& + \sum_{b<a} \int_{(x,t)\in\Lambda(T)} \left| C_b \Gamma^b \left(\sum_{i,j,k=0}^{2} g_{ij}^{k} \partial_k v \partial_i \partial_j v \right) \right| |\partial_t \Gamma^a v| e^\varphi dx \\
& + \sum_{b\leqslant a} \int_{(x,t)\in\Lambda(T)^c} \left| C_b \Gamma^b \left(\sum_{i,j,k=0}^{2} g_{ij}^{k} \partial_k v \partial_i \partial_j v \right) - \sum_{i,j,k=0}^{2} g_{ij}^{k} \partial_k v \partial_i \partial_j \Gamma^a v \right| |\partial_t \Gamma^a v| e^\varphi dx \\
& + \sum_{b\leqslant a} \int_{\mathbb{R}^2} \left| C_b \Gamma^b \left(\sum_{i,j,k,l=0}^{2} g_{ij}^{kl} \partial_k v \partial_l v \partial_i \partial_j v \right) - \sum_{i,j,k,l=0}^{2} g_{ij}^{kl} \partial_k v \partial_l v \partial_i \partial_j \Gamma^a v \right| |\partial_t \Gamma^a v| e^\varphi dx \\
\leqslant & C \int_{(x,t)\in\Lambda(T)} \left(\frac{||x|-t|}{1+|x|+t} |\partial v|_{[\frac{\kappa+1}{2}]} + \frac{1}{1+|x|+t} |v|_{[\frac{\kappa+1}{2}]+1} \right) e^\varphi |\partial v|_\kappa^2 dx \\
& + C \sum_{|b|\leqslant\kappa} \int_{(x,t)\in\Lambda(T)} e^\varphi |\partial v|_{[\frac{\kappa+1}{2}]} |Z\Gamma^b v| |\partial v|_\kappa dx + C \int_{(x,t)\in\Lambda(T)^c} e^\varphi |\partial v|_{[\frac{\kappa+1}{2}]} |\partial v|_\kappa^2 dx \\
& + C \int_{\mathbb{R}^2} e^\varphi |\partial v|_{[\frac{\kappa+1}{2}]}^2 |\partial v|_\kappa^2 dx \\
\leqslant & C \int_{(x,t)\in\Lambda(T)} \left(\frac{||x|-t|}{1+|x|+t} |\partial v|_{[\frac{\kappa+1}{2}]} + \frac{1}{1+|x|+t} |v|_{[\frac{\kappa+1}{2}]+1} \right) e^\varphi |\partial v|_\kappa^2 dx \\
& + C \sum_{|b|\leqslant\kappa} \int_{(x,t)\in\Lambda(T)} e^\varphi |\partial v|_{[\frac{\kappa+1}{2}]} |Z\Gamma^b v| |\partial v|_\kappa dx \\
& + \int_{\mathbb{R}^2} e^\varphi \left(\frac{1}{1+t} [\partial v]_{[\frac{\kappa+1}{2}],T} + |\partial v|_{[\frac{\kappa+1}{2}]}^2 \right) |\partial v|_\kappa^2 dx \\
\leqslant & \epsilon \sum_{|b|\leqslant\kappa} \int_{\mathbb{R}^2} \frac{e^\varphi}{(1+||x|-t|)^2} |Z\Gamma^b v|^2 dx + \frac{C_\epsilon}{1+t} ([\partial v]_{[\frac{\kappa+1}{2}],T} + \langle v \rangle_{[\frac{\kappa+1}{2}]+1,T}) \int_{\mathbb{R}^2} e^\varphi |\partial v(x,t)|_\kappa^2 dx \\
\leqslant & \epsilon \sum_{|b|\leqslant\kappa} \int_{\mathbb{R}^2} \dot\varphi e^\varphi |Z\Gamma^b v|^2 dx + \frac{C_\epsilon}{1+t} ([\partial v]_{[\frac{\kappa+1}{2}],T} + \langle v \rangle_{[\frac{\kappa+1}{2}]+1,T}) \int_{\mathbb{R}^2} e^\varphi |\partial v(x,t)|_\kappa^2 dx, \qquad (5.23)
\end{aligned}
$$

where ϵ is a positive small constant.

We now estimate $|I_2(t)|$.

Noting (2.18), (3.7), (5.1), (5.17) and using the null condition and Cauchy inequality, we obtain

$$
\begin{aligned}
|I_2(t)| \leqslant & \int_{(x,t)\in\Lambda(T)} \dot\varphi e^\varphi \left| \sum_{i=1}^{2} \omega_i \partial_i v \partial_t \Gamma^a v \partial_t \Gamma^a v - 2 \sum_{i=1}^{2} \omega_i \partial_t v \partial_i \Gamma^a v \partial_t \Gamma^a v \right. \\
& \left. - \sum_{i=1}^{2} \partial_t v \partial_i \Gamma^a v \partial_i \Gamma^a v \right| dx + \int_{(x,t)\in\Lambda(T)^c} \dot\varphi e^\varphi \left| \sum_{i=1}^{2} \omega_i \partial_i v \partial_t \Gamma^a v \partial_t \Gamma^a v \right. \\
& \left. -2 \sum_{i=1}^{2} \omega_i \partial_t v \partial_i \Gamma^a v \partial_t \Gamma^a v - \sum_{i=1}^{2} \partial_t v \partial_i \Gamma^a v \partial_i \Gamma^a v \right| dx \\
& + \int_{\mathbb{R}^2} \frac{\dot\varphi}{2} e^\varphi \left| 2 \sum_{i=1}^{2} \sum_{j=1}^{2} \omega_i \partial_i v \partial_j v \partial_j \Gamma^a v \partial_t \Gamma^a v + \sum_{i=1}^{2} \sum_{j=1}^{2} \partial_i v \partial_j v \partial_j \Gamma^a v \partial_i \Gamma^a v \right. \\
& \left. -2 \sum_{i=1}^{2} \omega_i |\nabla v|^2 \partial_i \Gamma^a v \partial_t \Gamma^a v - \sum_{i=1}^{2} |\nabla v|^2 \partial_i \Gamma^a v \partial_i \Gamma^a v \right| dx \\
\leqslant & \int_{(x,t)\in\Lambda(T)} \dot\varphi e^\varphi \left| \sum_{i=1}^{2} \omega_i \partial_i v \partial_t \Gamma^a v \partial_t \Gamma^a v - 2 \sum_{i=1}^{2} \omega_i \partial_t v \partial_i \Gamma^a v \partial_t \Gamma^a v \right.
\end{aligned}
$$

$$-\sum_{i=1}^{2}\partial_t v\partial_i\Gamma^a v\partial_i\Gamma^a v\Big|dx+C\Big(\int_{(x,t)\in\Lambda(T)^c}|\partial v||\partial v|_\kappa^2 dx+\int_{\mathbb{R}^2}|\partial v|^2|\partial v|_\kappa^2 dx\Big)$$

$$\leqslant\int_{(x,t)\in\Lambda(T)}\dot\varphi e^\varphi|\omega_1(\partial_1+\omega_1\partial_t)v(\partial_t\Gamma^a v)^2+\omega_2(\partial_2+\omega_2\partial_t)v(\partial_t\Gamma^a v)^2$$

$$-\partial_t v(\partial_1+\omega_1\partial_t)\Gamma^a v(\partial_1+\omega_1\partial_t)\Gamma^a v-\partial_t v(\partial_2+\omega_2\partial_t)\Gamma^a v(\partial_2+\omega_2\partial_t)\Gamma^a v$$

$$-\omega_1\partial_t v(\partial_1+\omega_1\partial_t)\Gamma^a v\partial_t\Gamma^a v-\omega_2\partial_t v(\partial_2+\omega_2\partial_t)\Gamma^a v\partial_t\Gamma^a v|dx+\frac{C}{1+t}[\partial v]_{0,T}\int_{\mathbb{R}^2}e^\varphi|\partial v|_\kappa^2 dx$$

$$\leqslant C\int_{(x,t)\in\Lambda(T)}\dot\varphi e^\varphi(|Zv||\partial\Gamma^a v|^2+|\partial v||Z\Gamma^a v||\partial\Gamma^a v|)dx+\frac{C}{1+t}[\partial v]_{0,T}\int_{\mathbb{R}^2}e^\varphi|\partial v(x,t)|_\kappa^2 dx$$

$$\leqslant\epsilon\sum_{|a|\leqslant\kappa}\int_{\mathbb{R}^2}\dot\varphi e^\varphi|Z\Gamma^a v(x,t)|^2 dx+\frac{C_\epsilon}{1+t}([\partial v]_{0,T}+\langle v\rangle_{1,T})\int_{\mathbb{R}^2}e^\varphi|\partial v(x,t)|_\kappa^2 dx,\tag{5.24}$$

where $\omega_i=\frac{x_i}{|x|}$ $(i=1,2)$ and ϵ is a positive small constant.

Similarly, using (2.18), (3.7), (5.1), (5.18), the null condition and the Cauchy inequality, we have

$$|I_3(t)|\leqslant\epsilon\sum_{|a|\leqslant\kappa}\int_{\mathbb{R}^2}\dot\varphi e^\varphi|Z\Gamma^a v(x,t)|^2 dx+\frac{C_\epsilon}{1+t}[\partial v]_{1,T}\int_{\mathbb{R}^2}e^\varphi|\partial v(x,t)|_\kappa^2 dx.\tag{5.25}$$

Choosing ϵ suitably small, by (5.23)–(5.25) we see that (5.21) holds for $t\in[1,T)$. Thus the proof of Theorem 5.1 is completed. □

6 Proof of Theorem 1.1

This section is devoted to proving Theorem 1.1. In order to prove Theorem 1.1, it suffices to show the following theorem.

Theorem 6.1. *Under the assumptions of Theorem* 1.1, *there exists a positive constant ε_0 such that, for every $\varepsilon\in[0,\varepsilon_0]$, the smooth solution $v=v(x,t)$ of the Cauchy problem* (2.16)–(2.17) *exists globally in time.*

Proof. The local existence of classical solutions has been proved by the method of Picard iteration (see Sogge [20] and Hörmander [9]). It suffices to show that if $v=v(x,t)$ is a smooth solution of the Cauchy problem (2.16)–(2.17) on the domain $\mathbb{R}^2\times[0,T))$, then $\sum_{|\alpha|\leqslant 4}|\partial^\alpha v(x,t)|\in L^\infty(\mathbb{R}^2\times[0,T))$.

In what follows, we prove Theorem 6.1 by the method of continuous induction, or say, by the bootstrap argument.

Taking an integer $\kappa\geqslant 8$, we have $[\frac{\kappa+9}{2}]\leqslant\kappa$. According to the bootstrap argument, we first assume that

$$[\partial v]_{\kappa,T}+\langle v\rangle_{\kappa+1,T}\leqslant M\varepsilon,\tag{6.1}$$

and then we prove that, by choosing M sufficiently large and ε suitably small we have

$$[\partial v]_{\kappa,T}+\langle v\rangle_{\kappa+1,T}\leqslant\frac{1}{2}M\varepsilon.\tag{6.2}$$

To do so, let δ be a positive and small constant. Choosing $\varepsilon_1\in[0,\frac{\delta}{M}]$, we have

$$[\partial v]_{[\frac{\kappa+6}{2}],T}+\langle v\rangle_{[\frac{\kappa+6}{2}],T}\leqslant[\partial v]_{[\frac{\kappa+9}{2}],T}+\langle v\rangle_{[\frac{\kappa+9}{2}]+1,T}\leqslant[\partial v]_{\kappa,T}+\langle v\rangle_{\kappa+1,T}\leqslant M\varepsilon\leqslant\delta\tag{6.3}$$

for $\varepsilon\in[0,\frac{\delta}{M}]$. Thus, it follows from Theorems 4.1 and 5.1 that

$$[\partial v]_{\kappa,T}+\langle v\rangle_{\kappa+1,T}\leqslant C_0\varepsilon+C_1([\partial v]_{[\frac{\kappa+6}{2}],T}+\langle v\rangle_{[\frac{\kappa+6}{2}],T})\sup_{0\leqslant\tau<T}\{(1+\tau)^{-\nu}\|\partial v(\tau)\|_{\kappa+8}\}$$

$$\leqslant C_0\varepsilon+C_1 C_2 M\varepsilon^2\sup_{0\leqslant\tau<T}\{(1+\tau)^{C_3 M\varepsilon-\nu}\}\leqslant 2C_0\varepsilon\tag{6.4}$$

for $\varepsilon \in [0, \varepsilon_2]$, provided that

$$\varepsilon_2 \in \left[0, \min\left\{\frac{C_0}{C_1 C_2 M}, \ \frac{\nu}{C_3 M}\right\}\right],$$

where ν is the constant appearing in Theorem 4.1. Therefore, taking $\varepsilon_0 = \min\{\varepsilon_1, \varepsilon_2\}$ and $M \geqslant 4C_0$, we obtain (6.2) from (6.4) immediately. This implies that

$$\sum_{|\alpha| \leqslant 4} |\partial^\alpha v(x, t)| \in L^\infty(\mathbb{R}^2 \times [0, T]).$$

Thus, the proof of Theorem 6.1 is completed. □

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