Life-span of classical solutions to hyperbolic geometric flow in two space variables with slow decay initial data

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Abstract

In this paper we investigate the life-span of classical solutions to the hyperbolic geometric flow in two space variables with slow decay initial data. By establishing some new estimates on the solutions of linear wave equations in two space variables, we give a lower bound of the life-span of classical solutions to the hyperbolic geometric flow with asymptotic flat initial Riemann surfaces.

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1 Introduction

Let $\mathscr{M}$ be an n-dimensional complete Riemannian manifold with Riemannian metric g_{ij}. The following evolutionary equation for the metric g_{ij}

$$\frac{\partial^2 g_{ij}}{\partial t^2} = -2R_{ij} \tag{1.1}$$

has been recently introduced by Kong and Liu [8] and named as *hyperbolic geometric flow*, where R_{ij} stands for the Ricci curvature tensor of g_{ij}. For the study on the hyperbolic geometric flow, we refer to the recent papers [1], [2], [7], [8] and [9].

We are interested in the evolution of a Riemannian metric g_{ij} on a Riemann surface $\mathscr{S}$ under the flow (1.1). On a surface, the hyperbolic geometric flow equation (1.1) simplifies, because all of the information about curvature is contained in the scalar curvature function R. In our notation, $R = 2K$ where K is the Gauss curvature. The Ricci curvature is given by

$$R_{ij} = \frac{1}{2} R g_{ij}, \tag{1.2}$$

and the hyperbolic geometric flow equation (1.1) simplifies the following equation for the special metric

$$\frac{\partial^2 g_{ij}}{\partial t^2} = -R g_{ij}. \tag{1.3}$$

The metric for a surface can always be written (at least locally) in the following form

$$g_{ij} = v(t,x,y)\delta_{ij}, \tag{1.4}$$

where $v(t,x,y) > 0$. Therefore, we have

$$R = -\frac{\triangle \ln v}{v}. \tag{1.5}$$

Thus the equation (1.3) becomes

$$\frac{\partial^2 v}{\partial t^2} = \frac{\triangle \ln v}{v} \cdot v,$$

namely,

$$v_{tt} - \triangle \ln v = 0. \tag{1.6}$$

Denote

$$u = \ln v, \tag{1.7}$$

then the wave equation (1.6) reduces to

$$u_{tt} - e^{-u} \Delta u = -u_t^2. \tag{1.8}$$

(1.8) is a quasilinear hyperbolic wave equation. The global existence and the life-span of classical solutions to the Cauchy problem for hyperbolic equations with the initial data with compact support have been studied by many authors (e.g., [6], [15], [3], etc.). However, only a few results have been known for the case of the initial data with non-compact support, which plays an important role in both mathematics and physics.

Recently, Kong, Liu and Xu [9] studies the evolution of a Riemannian metric g_{ij} on a cylinder $\mathscr{C}$ under the hyperbolic geometric flow (1.1). They prove that, for any given initial metric on $\mathbb{R}^2$ in a class of cylinder metrics, one can always choose suitable initial velocity symmetric tensor such that the solution exists for all time, and the scalar curvature corresponding to the solution metric g_{ij} keeps uniformly bounded for all time; moreover, if the initial velocity tensor is suitably "large", then the solution metric g_{ij} converges to the flat metric at an algebraic rate. If the initial velocity tensor does not satisfy the condition, then the solution blows up at a finite time, and the scalar curvature $R(t, x)$ goes to positive infinity as (t, x) tends to the blowup points, and a flow with surgery has to be considered. This result shows that, by comparing to Ricci flow, the hyperbolic geometric flow has the following advantage: the surgery technique may be replaced by choosing suitable initial velocity tensor. Some geometric properties of hyperbolic geometric flow on general open and closed Riemann surfaces are also discussed (see Kong et al [9]).

In this paper, we consider the Cauchy problem for (1.8) with the following initial data

$$t = 0: \quad u = \varepsilon u_0(x), \quad u_t = \varepsilon u_1(x), \tag{1.9}$$

where $\varepsilon > 0$ is a suitably small parameter, $u_0(x)$ and $u_1(x)$ are two smooth functions of $x \in \mathbb{R}^2$ and satisfy that there exist two positive constants $A \in \mathbb{R}^+$ and $k > 1, k \in \mathbb{R}^+$ such that

$$|u_0(x)| \leq \frac{A}{(1+|x|)^k}, \quad |u_1(x)| \leq \frac{A}{(1+|x|)^{k+1}}. \tag{1.10}$$

(1.10) implies that the initial data satisfies the slow decay property, that is, the initial Riemann surface are asymptotic flat. We shall prove the following theorem.

Theorem 1.1 *Suppose that $u_0(x)$, $u_1(x) \in C^\infty(\mathbb{R}^2)$ and satisfy the decay condition (1.10). Then there exist two positive constants δ and ε_0 such that for any fixed $\varepsilon \in [0, \varepsilon_0]$, the Cauchy problem (1.8)-(1.9) has a unique C^∞ solution on the interval $[0, T_\varepsilon]$, where T_ε is given by*

$$T_\varepsilon = \frac{\delta}{\varepsilon^{\frac{4}{3}}}. \tag{1.11}$$

As we know, the flow equation (1.1) is a system of fully nonlinear partial differential equations of second order, it is very difficult to study the global existence or blow-up of the classical solutions of (1.1). An interesting and important question is to investigate the evolution of asymptotic flat initial Riemann surfaces under the flow (1.1). In this case, although the equation (1.1) can simply reduce to (1.8), (1.8) is still a fully nonlinear wave equation, only a few results have been known even for its Cauchy problem. Our main result, Theorem 1.1, gives a lower bound on the life-span of the classical solution of the Cauchy problem (1.8)-(1.9). This theorem shows that the smooth evolution of asymptotic flat initial Riemann surfaces under the flow (1.1) exists at least on the interval $[0, T_\varepsilon]$.

The paper is organized as follows. In Section 2 we establish some new estimates on the solutions of linear wave equations in two space variables, these estimates play an important role in the proof of Theorem 1.1. Based on this, we prove Theorem 1.1 in Section 3, which gives a lower bound of the life-span of classical solutions to the hyperbolic geometric flow with asymptotic flat initial Riemann surfaces.

2 Some useful lemmas

Following Klainerman [11], we introduce a set of partial differential operators

$$Z = \{\partial_i \ \ (i = 0, 1, \cdots, n); \quad L_0; \quad \Omega_{ij} \ \ (1 \le i < j \le n); \quad \Omega_{0i} \ \ (i = 1, \cdots, n)\}, \tag{2.1}$$

where

$$\partial_0 = \frac{\partial}{\partial t}, \quad \partial_i = \frac{\partial}{\partial x_i} \quad (i = 1, \cdots, n), \tag{2.2}$$

$$L_0 = t\partial_0 + \sum_{i=1}^{n} x_i \partial_i, \tag{2.3}$$

$$\Omega_{ij} = x_i \partial_j - x_j \partial_i \quad (1 \le i < j \le n) \tag{2.4}$$

and

$$\Omega_{0i} = t\partial_i + x_i \partial_0 \quad (i = 1, \cdots, n). \tag{2.5}$$

Let Z^I denote a product of $|I|$ of the vector fields (2.2)-(2.5), where $I = (I_1, \cdots, I_\sigma)$ is a multi-index, $|I| = I_1 + \cdots + I_\sigma$, σ is the number of partial differential operators in $Z:\ Z = (Z_1, \cdots, Z_\sigma)$ and

$$Z^I = Z_1^{I_1} \cdots Z_\sigma^{I_\sigma}. \tag{2.6}$$

Throughout this paper, we use the following notations: $L^p(\mathbb{R}^n)$ $(1 \le p \le \infty)$ stands for the usual space of all $L^p(\mathbb{R}^n)$ functions on $\mathbb{R}^n$ with the norm $\|f\|_{L^p}$, H^s denotes s-order Sobolev space on $\mathbb{R}^n$ with the norm

$$\|f\|_{H^s} = \|(1+|\xi|)^{\frac{s}{2}}\hat{f}\|_{L^2},$$

where s is a given real number.

The following lemma has been proved in Li and Zhou [15].

Lemma 2.1 *For any given multi-index $I = (I_1, \cdots, I_\sigma)$, we have*

$$[\Box, Z^I] = \sum_{|J| \le |I|-1} A_{IJ} Z^J \Box \tag{2.7}$$

and

$$[\partial_i, Z^I] = \sum_{|J| \le |I|-1} B_{IJ} Z^J \partial = \sum_{|J| \le |I|-1} \tilde{B}_{IJ} \partial Z^J \quad (i = 0, 1, \cdots, n), \tag{2.8}$$

where $[\cdot,\cdot]$ stands for the Poisson bracket, $J = (J_1, \cdots, J_\sigma)$ a multi-index, $\Box$ denotes the wave operator, $\partial = \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}\right)$ and A_{IJ}, B_{IJ}, $\tilde{B}_{IJ}$ stand for constants.

Lemma 2.2 *Assume that $n \ge 1$. Let u be a solution of the following Cauchy problem*

$$\begin{cases} \phi_{tt} - \triangle \phi = f, \\ t = 0: \quad u = \phi_0(x), \quad u_t = \phi_1(x). \end{cases} \tag{2.9}$$

Then

$$\|\partial \phi(t,\cdot)\|_{H^s} \le C(\|\partial_x \phi_0\|_{H^s} + \|\phi_1\|_{H^s} + \int_0^t \|f(\tau,\cdot)\|_{H^s}), \tag{2.10}$$

provided that all norms appearing in the right-hand side of (2.10) are bounded.

Proof. Taking the Fourier transformation on the variable x in (2.9) leads to

$$\begin{cases} \hat{\phi}_{tt} + |\xi|^2 \hat{\phi} = \hat{f}(t,\xi), \\ t = 0: \hat{\phi} = \hat{\phi}_0(\xi), \hat{\phi}_t = \hat{\phi}_1(\xi). \end{cases} \tag{2.11}$$

Solving the initial value problem (2.11) gives

$$\hat{\phi}(t,\xi) = \cos(t|\xi|)\hat{\phi}_0(\xi) + \frac{\sin(t|\xi|)}{|\xi|}\hat{\phi}_1(\xi) + \int_0^t \frac{\sin((t-\tau)|\xi|)}{|\xi|}\hat{f}(\tau,\xi)d\tau. \tag{2.12}$$

Thanks to (2.12), we obtain

$$\partial_t \hat{\phi}(t,\xi) = -|\xi|\sin(t|\xi|)\hat{\phi}_0(\xi) + \cos(t|\xi|)\hat{\phi}_1(\xi) + \int_0^t \cos((t-\tau)|\xi|)\hat{f}(\tau,\xi)d\tau \tag{2.13}$$

and

$$|\xi|\hat{\phi}(t,\xi) = |\xi|\cos(t|\xi|)\hat{\phi}_0(\xi) + \sin(t|\xi|)\hat{\phi}_1(\xi) + \int_0^t \sin((t-\tau)|\xi|)\hat{f}(\tau,\xi)d\tau. \tag{2.14}$$

It follows from (2.13) and Minkowski inequality that

$$\begin{aligned} \|\partial_t\phi(t,\cdot)\|_{H^s} &\leq \|(1+|\xi|)^{\frac{s}{2}}|\xi|\sin(t|\xi|)\hat{\phi}_0(\xi)\|_{L^2} + \|(1+|\xi|)^{\frac{s}{2}}\cos(t|\xi|)\hat{\phi}_1(\xi)\|_{L^2} \\ &\quad + \int_0^t \|(1+|\xi|)^{\frac{s}{2}}\cos((t-\tau)|\xi|)\hat{f}(\tau,\xi)\|_{L^2}d\tau \\ &\leq C\left(\|\partial_x\phi_0\|_{H^s} + \|\phi_1\|_{H^s} + \int_0^t \|f(\tau,\cdot)\|_{H^s}\right). \end{aligned} \tag{2.15}$$

Similarly, we have

$$\|\partial_x\phi(t,\cdot)\|_{H^s} \leq C\left(\|\partial_x\phi_0\|_{H^s} + \|\phi_1\|_{H^s} + \int_0^t \|f(\tau,\cdot)\|_{H^s}\right). \tag{2.16}$$

Thus, (2.10) comes from (2.15) and (2.16) immediately. This proves Lemma 2.2. ■

Lemma 2.3 *Let ϕ be a solution of the Cauchy problem*

$$\begin{cases} \phi_{tt} - \triangle\phi = \sum_{j=0}^n a_j\partial_j f_j, \\ t=0: \quad \phi = 0, \quad \phi_t = 0 \end{cases} \tag{2.17}$$

Then

$$\|\phi(t,\cdot)\|_{L^2} \leq C\left(\sum_{j=0}^n \int_0^t \|f_j(\tau,\cdot)\|_{L^2}d\tau + \|f_0(0,\cdot)\|_{L^2}\right). \tag{2.18}$$

In particular, for $n \geq 2$ it holds that

$$\begin{aligned} |\phi(t,x)| &\leq C(1+t)^{-\frac{n-1}{2}}\left\{\int_0^t (1+\tau)^{\frac{n-1}{2}}\sum_{j=0}^n \|f_j(\tau,\cdot)\|_{L^\infty}d\tau \right. \\ &\quad \left. + \int_0^t (1+\tau)^{-\frac{n+1}{2}}\sum_{j=0}^n \sum_{|I|\leq n+1} \|Z^I f_j(\tau,\cdot)\|_{L^1}d\tau\right\}. \end{aligned} \tag{2.19}$$

Proof. Taking the Fourier transformation on the variable x in (2.17) yields

$$\begin{cases} \hat{\phi}_{tt} + |\xi|^2\hat{\phi} = \sum_{j=1}^n \sqrt{-1}a_j\xi_j\hat{f}_j + a_0\partial_t\hat{f}_0, \\ t=0: \quad \hat{\phi} = 0, \quad \hat{\phi}_t = 0. \end{cases} \tag{2.20}$$

Solving the initial value problem (2.20) gives

$$\hat{\phi}(t,\xi) = \sum_{j=1}^n a_j \int_0^t \frac{\sin((t-\tau)|\xi|)}{|\xi|}\sqrt{-1}\xi_j\hat{f}_j d\tau + a_0\int_0^t \frac{\sin((t-\tau)|\xi|)}{|\xi|}\partial_t\hat{f}_0 d\tau. \tag{2.21}$$

By Minkowski inequality, we have

$$\left\|\sum_{j=1}^{n} a_j \int_0^t \frac{\sin((t-\tau)|\xi|)}{|\xi|}\sqrt{-1}\xi_j \hat{f}_j d\tau\right\|_{L^2} \leq C\sum_{j=1}^{n} \|f_j(\tau,\cdot)\|_{L^2} d\tau. \tag{2.22}$$

Using the integration by parts, we obtain

$$\begin{aligned} a_0 \int_0^t \frac{\sin((t-\tau)|\xi|)}{|\xi|}\partial_t \hat{f}_0 d\tau &= a_0 \int_0^t \frac{\sin((t-\tau)|\xi|)}{|\xi|} d\hat{f}_0 \\ &= -a_0 \frac{\sin(t|\xi|)}{|\xi|}\hat{f}_0(0,\xi) + a_0 \int_0^t \cos((t-\tau)|\xi|)\hat{f}_0(\tau,\xi)d\tau. \end{aligned}$$

It follows from the Minkowski inequality that

$$\begin{aligned} &\left\|a_0 \int_0^t \frac{\sin((t-\tau)|\xi|)}{|\xi|}\partial_t \hat{f}_0 d\tau\right\|_{L^2} \\ &\leq |a_0| \left\|\frac{\sin(t|\xi|)}{|\xi|}\hat{f}_0(0,\xi)\right\|_{L^2} + |a_0| \left\|\int_0^t \cos((t-\tau)|\xi|)\hat{f}_0(\tau,\xi)d\tau\right\|_{L^2} \\ &\leq C\|f(0,\cdot)\|_{\dot{H}^{-1}} + C\int_0^t \|f_0(\tau,\cdot)\|_{L^2} d\tau. \end{aligned} \tag{2.23}$$

Noting the definition of $\dot{H}^{-1}$ and using Hölder inequality, we have

$$\|f(0,\cdot)\|_{\dot{H}^{-1}} = \sup_{v\in H^1, v\neq 0} \frac{\int_{\mathbb{R}^n} f(0,\xi)v(\xi)d\xi}{\|v\|_{H^1}} \leq \|f(0,\cdot)\|_{L^2}. \tag{2.24}$$

Combining (2.23) and (2.24) yields

$$\left\|a_0 \int_0^t \frac{\sin((t-\tau)|\xi|)}{|\xi|}\partial_t \hat{f}_0 d\tau\right\|_{L^2} \leq C\|f(0,\cdot)\|_{L^2} + C\int_0^t \|f_0(\tau,\cdot)\|_{L^2} d\tau. \tag{2.25}$$

Thus, we obtain (2.18) immediately from (2.21), (2.22), (2.25) and Minkowski inequality.

The proof of (2.19) can be found in Li and Zhou [16], here we omit it. Thus the proof of Lemma 2.3 is completed. ■

The following lemma comes from Klainerman [10].

Lemma 2.4 *Suppose that* ϕ *is* C^2 *smooth and satisfies*

$$\Box\phi + \sum_{j,k=0}^{n} \gamma^{jk}(t,x)\partial_j\partial_k\phi = F \quad (0 \leq t \leq T),$$

and suppose furthermore that

$$\phi \longrightarrow 0 \quad \text{as } |x| \to \infty.$$

If

$$|\gamma| = \sum_{j,k=0}^{n} |\gamma^{jk}| \leq \frac{1}{2} \quad (0 \leq t \leq T),$$

then, for any given $t \in [0,T]$, it holds that

$$\|\partial\phi(t,\cdot)\|_{L^2} \leq 2\exp\left\{\int_0^t 2|\dot{\gamma(\tau)}|d\tau\right\}\|\partial\phi(0,\cdot)\|_{L^2}+2\int_0^t \exp\left\{\int_s^t 2|\dot{\gamma(\tau)}|d\tau\right\}\|F(s,\cdot)\|_{L^2}ds, \tag{2.26}$$

where

$$|\dot{\gamma}(t)| = \sup|\partial_i\gamma^{jk}(t,\cdot)|.$$

Lemma 2.5 *Suppose that $G = G(w)$ is a sufficiently smooth function of $w = (w_1, \cdots, w_m)$ with*

$$G(0) = 0. \tag{2.27}$$

For any given integer $N \geq 0$, if a vector function $w = w(t,x)$ satisfies

$$\sum_{|I|\leq[\frac{N}{2}]} \|Z^I w(t,\cdot)\|_{L^\infty} \leq \nu_0, \quad \forall\, t \in [0,T], \tag{2.28}$$

where $[\cdot]$ stands for the integer part of a real number and ν_0 is a positive constant, then it holds that

$$\sum_{|I|\leq N} \|Z^I G(w(t,\cdot))\|_{L^P} \leq C(\nu_0) \sum_{|I|\leq N} \|Z^I w(t,\cdot)\|_{L^p}, \quad \forall\, t \in [0,T], \tag{2.29}$$

provided that all norms appearing on the right-hand side of (2.29) are bounded, where $C(\nu_0)$ is a positive constant depending on ν_0, and p is a real number with $1 \leq p \leq \infty$.

The proof of Lemma 2.5 can be found in Li and Chen [14].

Lemma 2.6 *Assume that $I = (I_1, \cdots, I_\sigma)$ and $J = (J_1, \cdots, J_\sigma)$ is a multi-index. If a vector function $\phi = \phi(t,x)$ satisfies*

$$\sum_{|J|\leq[\frac{|I|}{2}]} \|Z^J \phi(t,\cdot)\|_{L^\infty} \leq \nu_0, \quad \forall\, t \in [0,T], \tag{2.30}$$

then it holds that

$$\begin{aligned}\|Z^I((e^{-\phi}-1)\partial_i\phi)(t,\cdot)\|_{L^2} \quad &\leq \quad C(\nu_0)\sum_{|I_1|\leq|I|}\ \sum_{|I_2|\leq[\frac{|I|-1}{2}]} \|Z^{I_1}\phi(t,\cdot)\|_{L^2}\|Z^{I_2}\partial_i\phi(t,\cdot)\|_{L^\infty}+\\ &\quad\ \ C(\nu_0)\sum_{|I_2|\leq|I|}\ \sum_{|I_1|\leq[\frac{|I|}{2}]} \|Z^{I_1}\phi(t,\cdot)\|_{L^\infty}\|Z^{I_2}\partial_i\phi(t,\cdot)\|_{L^2}.\end{aligned} \tag{2.31}$$

provided that all norms appearing on the right-hand side of (2.31) are bounded.

Proof. When $|I| = 0$, by Lemma 2.5 we have

$$\begin{aligned} \|(e^{-\phi} - 1)\partial_i\phi(t,\cdot)\|_{L^2} &\leq \|(e^{-\phi} - 1)(t,\cdot)\|_{L^\infty}\|\partial_i\phi(t,\cdot)\|_{L^2} \\ &\leq C(\nu_0)\|\phi(t,\cdot)\|_{L^\infty}\|\partial_i\phi(t,\cdot)\|_{L^2}. \end{aligned} \tag{2.32}$$

For $|I| \geq 1$, it follows from Minkowski inequality and Lemma 2.5 that

$$\begin{aligned} \|Z^I((e^{-\phi} - 1)\partial_i\phi)(t,\cdot)\|_{L^2} &\leq C \sum_{|I_1|+|I_2|\leq|I|,|I_1|>|I_2|} \|Z^{I_1}(e^{-\phi} - 1)(t,\cdot)\|_{L^2}\|Z^{I_2}\partial_i\phi(t,\cdot)\|_{L^\infty}+ \\ &\quad C \sum_{|I_1|+|I_2|\leq|I|,|I_1|\leq|I_2|} \|Z^{I_1}(e^{-\phi} - 1)(t,\cdot)\|_{L^\infty}\|Z^{I_2}\partial_i\phi(t,\cdot)\|_{L^2} \\ &\leq C(\nu_0) \sum_{|I_1|\leq|I|} \sum_{|I_2|\leq[\frac{|I|-1}{2}]} \|Z^{I_1}\phi(t,\cdot)\|_{L^2}\|Z^{I_2}\partial_i\phi(t,\cdot)\|_{L^\infty}+ \\ &\quad C(\nu_0) \sum_{|I_2|\leq|I|} \sum_{|I_1|\leq[\frac{|I|}{2}]} \|Z^{I_1}\phi(t,\cdot)\|_{L^\infty}\|Z^{I_2}\partial_i\phi(t,\cdot)\|_{L^2}. \end{aligned} \tag{2.33}$$

(2.31) follows from (2.32) and (2.33) immediately. Thus the proof of Lemma 2.6 is completed. ■

Lemma 2.7 *Suppose that $\phi_0(x), \phi_1(x) \in C^\infty(\mathbb{R}^2)$ and suppose furthermore that there exist two positive constants $A \in \mathbb{R}^+$ and $k \in \mathbb{R}^+$ such that*

$$|\phi_0(x)| \leq \frac{A}{(1+|x|)^k}, \quad |\phi_1(x)| \leq \frac{A}{(1+|x|)^{k+1}} \quad (k > 1). \tag{H}$$

If $\phi = \phi(t,x)$ is a solution of the following Cauchy problem

$$\begin{cases} \phi_{tt} - \triangle\phi = 0, \\ t = 0: \quad \phi = \phi_0(x), \quad \phi_t = \phi_1(x). \end{cases} \tag{2.34}$$

Then it holds that

$$|\phi(t,x)| \leq \begin{cases} \dfrac{CA}{\sqrt{1+t+|x|}(1+|t-|x||)^{k-\frac{1}{2}}} & (|x| \geq t), \\ \dfrac{CA}{\sqrt{1+t+|x|}\sqrt{1+|t-|x||}} & (|x| \leq t). \end{cases} \tag{2.35}$$

Remark 2.1 *Here we would like to mention that, if the condition (H) is replaced by*

$$|\phi_0(x)| \leq \frac{A}{(1+|x|)^{k+1}}, \quad |\phi_1(x)| \leq \frac{A}{(1+|x|)^{k+1}} \quad (k > 1). \tag{H'}$$

Tsuyata [18] has showed that the solution of the Cauchy problem (2.34) satisfies the following decay estimate

$$|\phi(t,x)| \le \frac{CA}{\sqrt{1+t+|x|}\sqrt{1+|t-|x||}}.$$

Obviously, Lemma 2.7 improve the Tsuyata's result given in [18].

Proof of Lemma 2.7. It is easy to see that the solution of (2.34) reads

$$\phi(t,x) = \frac{1}{2\pi t^2}\int_{|x-y|\le t} \frac{t\phi_0(y) + t^2\phi_1(y) + t\nabla\phi_0(y)\cdot(y-x)}{(t^2-|y-x|^2)^{\frac{1}{2}}}dy. \tag{2.36}$$

We first estimate $|\frac{1}{2\pi t}\int_{|x-y|\le t}\frac{\phi_0(y)}{(t^2-|y-x|^2)^{\frac{1}{2}}}dy|$.

Introduce

$$x = (|x|\cos\theta,\ |x|\sin\theta), \quad y = (r\cos(\theta+\psi),\ r\sin(\theta+\psi))$$

and let χ be the characteristic function of positive numbers. Then

$$\begin{aligned}
&\left|\frac{1}{2\pi t}\int_{|x-y|\le t}\frac{\phi_0(y)}{(t^2-|y-x|^2)^{\frac{1}{2}}}dy\right| \\
&\le \frac{A}{2\pi t}\int_{|x-y|\le t}\frac{1}{\sqrt{t^2-|y-x|^2}(1+|y|)^k}dy \\
&\le \frac{A}{2\pi t}\left(\int_{|t-|x||}^{t+|x|}\frac{r}{(1+r)^k}\int_{-\varphi}^{\varphi}\frac{1}{\sqrt{t^2-|x|^2-r^2+2r|x|\cos\psi}}d\psi dr + \right. \\
&\quad \left. \chi(t-|x|)\int_0^{t-|x|}\frac{r}{(1+r)^k}\int_{-\pi}^{\pi}\frac{1}{\sqrt{t^2-|x|^2-r^2+2r|x|\cos\psi}}d\psi dr\right),
\end{aligned} \tag{2.37}$$

where

$$\varphi = \arccos\frac{|x|^2+r^2-t^2}{2|x|r}.$$

Let $h(y)$ be a continuous function on $\mathbb{R}$ and $y = (r\cos(\theta+\psi),\ r\sin(\theta+\psi))$. Define

$$H(t,|x|,r,\theta,h) = \begin{cases} \displaystyle\int_{-\varphi}^{\varphi}\frac{h(r,\theta+\psi)}{\sqrt{t^2-|x|^2-r^2+2|x|r\cos\psi}}d\psi, & \left|\dfrac{|x|^2+r^2-t^2}{2|x|r}\right| \le 1, \\ \displaystyle\int_{-\pi}^{\pi}\frac{h(r,\theta+\psi)}{\sqrt{t^2-|x|^2-r^2+2|x|r\cos\psi}}d\psi, & \left|\dfrac{|x|^2+r^2-t^2}{2|x|r}\right| \ge 1 \end{cases}$$

and

$$H(t,|x|,r) = H(t,|x|,r,\theta,1),$$

where, as before, φ is given by

$$\varphi = \arccos\frac{|x|^2+r^2-t^2}{2|x|r}.$$

The following proposition has been proved in Kovalyov [13].

Proposition 2.1 *(I) If*

$$t \geq |x| + r \quad \text{and} \quad \left| \frac{|x|^2 + r^2 - t^2}{2|x|r} \right| \geq 1,$$

then $H(t, |x|, r)$ satisfies

$$H(t, |x|, r) \leq C \frac{\ln\left\{2 + \frac{r|x|}{t^2 - (r+|x|)^2}\right\}}{\sqrt{t^2 - |x|^2 - r^2}} \leq \frac{C}{t^2 - (r + |x|)^2}, \tag{2.38}$$

here and hereafter C stands for some constants.

(II) If

$$t \leq |x| + r \quad \text{and} \quad \left| \frac{|x|^2 + r^2 - t^2}{2|x|r} \right| \leq 1,$$

then

$$H(t, |x|, r) \leq \frac{C}{\sqrt{r|x|}} \ln\left\{2 + \frac{r|x|\chi(t - |x|)}{(r + |x|)^2 - t^2}\right\}, \tag{2.39}$$

where χ is the characteristic function of positive numbers.

We now continue to estimate (2.37).

To do so, we distinguish the following two cases: $|x| \geq t$ and $|x| \leq t$.

Case I: $|x| \geq t$

It follows from (2.39) that

$$\left| \frac{1}{2\pi t} \int_{|x-y| \leq t} \frac{\phi_0(y)}{(t^2 - |y - x|^2)^{\frac{1}{2}}} dy \right| \leq \frac{CA}{t\sqrt{|x|}} \int_{|x|-t}^{t+|x|} \frac{1}{(1 + r)^{k-\frac{1}{2}}} dr. \tag{2.40}$$

In the present situation, we distinguish the following cases: $t \geq 1$ and $0 < t < 1$.

Case I-A: $t \geq 1$

In this case, according to k, we distinguish the following three cases:

Case I-A-1: $k > \frac{3}{2}$

In the present situation, it holds that

$$\frac{CA}{t\sqrt{|x|}} \int_{|x|-t}^{t+|x|} \frac{1}{(1 + r)^{k-\frac{1}{2}}} dr = \frac{CA}{t\sqrt{|x|}(1 + |x| - t)^{k-\frac{3}{2}}} \left[1 - \left(\frac{1 + |x| - t}{1 + |x| + t}\right)^{k-\frac{3}{2}}\right].$$

Noting

$$1 - s^{k-\frac{3}{2}} \leq C(1 - s), \quad \forall\, s \in [0, 1]$$

and

$$1 - \frac{1 + |x| - t}{1 + |x| + t} = \frac{2t}{1 + |x| + t},$$

we have

$$\left|\frac{1}{2\pi t}\int_{|x-y|\le t}\frac{\phi_0(y)}{(t^2-|y-x|^2)^{\frac{1}{2}}}dy\right| \le \frac{CA}{\sqrt{|x|}(1+|x|-t)^{k-\frac{3}{2}}(1+|x|+t)}, \le \frac{CA}{\sqrt{|x|+t}(1+|x|-t)^{k-\frac{1}{2}}}. \tag{2.41}$$

Case I-A-2: $k=\frac{3}{2}$

It follows from (2.40) that

$$\left|\frac{1}{2\pi t}\int_{|x-y|\le t}\frac{\phi_0(y)}{(t^2-|y-x|^2)^{\frac{1}{2}}}dy\right| \le \frac{CA}{t\sqrt{|x|}}\int_{|x|-t}^{t+|x|}\frac{1}{(1+r)}dr=\frac{CA}{t\sqrt{|x|}}\ln\left\{1+\frac{2t}{1+|x|-t}\right\}$$
$$\le \frac{CA}{\sqrt{|x|}(1+|x|-t)}\le\frac{CA}{\sqrt{|x|+t}(1+|x|-t)}. \tag{2.42}$$

Case I-A-3: $1<k<\frac{3}{2}$

In the present situation, it follows from (2.40) that

$$\left|\frac{1}{2\pi t}\int_{|x-y|\le t}\frac{\phi_0(y)}{(t^2-|y-x|^2)^{\frac{1}{2}}}dy\right| \le \frac{CA}{t\sqrt{|x|}}\left[(1+t+|x|)^{\frac{3}{2}-k}-(1+|x|-t)^{\frac{3}{2}-k}\right]$$
$$= \frac{CA}{t\sqrt{|x|}(1+|x|-t)^{k-\frac{3}{2}}}\left[\left(\frac{1+t+|x|}{1+|x|-t}\right)^{\frac{3}{2}-k}-1\right].$$

Noting the fact that $1<k<\frac{3}{2}$, we have

$$\left(\frac{1+t+|x|}{1+|x|-t}\right)^{\frac{3}{2}-k}-1\le\frac{Ct}{1+|x|-t}.$$

Hence,

$$\left|\frac{1}{2\pi t}\int_{|x-y|\le t}\frac{\phi_0(y)}{(t^2-|y-x|^2)^{\frac{1}{2}}}dy\right| \le \frac{CA}{\sqrt{|x|+t}(1+|x|-t)^{k-\frac{1}{2}}}. \tag{2.43}$$

Summarizing the above argument, for the case that $|x|\ge t$ and $t\ge 1$, we obtain from (2.41)-(2.43) that

$$\left|\frac{1}{2\pi t}\int_{|x-y|\le t}\frac{\phi_0(y)}{(t^2-|y-x|^2)^{\frac{1}{2}}}dy\right| \le \frac{CA}{\sqrt{1+|x|+t}(1+|x|-t)^{k-\frac{1}{2}}} \quad (k>1). \tag{2.44}$$

Case I-B: $|x|\ge t$ **and** $0<t<1$

We next consider the case that $|x|\ge t$ and $0<t<1$. In this case, we distinguish the following two cases.

Case I-B-1: $|t-|x||\le 1$

Introducing the variable $r = |x-y|$, we have

$$\begin{aligned}\left|\frac{1}{2\pi t}\int_{|x-y|\le t}\frac{\phi_0(y)}{(t^2-|y-x|^2)^{\frac{1}{2}}}dy\right| &\le \frac{1}{2\pi t}\int_{|x-y|\le t}\frac{CA}{(t^2-|y-x|^2)^{\frac{1}{2}}(1+|y|)^k}dy\\ &\le \frac{CA}{\pi t}\int_0^t \frac{r}{\sqrt{t^2-r^2}}dr \le \frac{CA}{\sqrt{1+t+|x|}(1+|x|-t)^{k-\frac{1}{2}}}.\end{aligned} \tag{2.45}$$

Case I-B-2: $|t-|x||>1$

Noting the fact that $|x|\ge t$ and $0<t<1$, we observe

$$|x| > t+1.$$

Thus, by the case (I-A), we have

$$\left|\frac{1}{2\pi t}\int_{|x-y|\le t}\frac{\phi_0(y)}{(t^2-|y-x|^2)^{\frac{1}{2}}}dy\right| \le \frac{CA}{\sqrt{1+t+|x|}(1+|x|-t)^{k-\frac{1}{2}}}. \tag{2.46}$$

Therefore, combining (2.44)-(2.46) gives

$$\left|\frac{1}{2\pi t}\int_{|x-y|\le t}\frac{\phi_0(y)}{(t^2-|y-x|^2)^{\frac{1}{2}}}dy\right| \le \frac{CA}{\sqrt{1+t+|x|}(1+|x|-t)^{k-\frac{1}{2}}}, \tag{2.47}$$

provided that $|x|\ge t$.

Case II: $|x|\le t$

We now consider the case that $|x|\le t$.

It follows from (2.37) that

$$\left|\frac{1}{2\pi t}\int_{|x-y|\le t}\frac{\phi_0(y)}{(t^2-|y-x|^2)^{\frac{1}{2}}}dy\right| \le I + II, \tag{2.48}$$

where

$$I = \frac{A}{2\pi t}\int_{t-|x|}^{t+|x|}\frac{H(t,|x|,r)r}{(1+r)^k}, \quad II = \frac{A}{2\pi t}\int_0^{t-|x|}\frac{H(t,|x|,r)r}{(1+r)^k}.$$

We next estimate I and II by distinguishing the follows cases.

Case II-A: $t+|x|\ge 1$

It follows from (2.39) that

$$I \le \frac{CA}{t\sqrt{|x|}}\int_{t-|x|}^{t+|x|}\ln\left\{2+\frac{|x|}{|x|+r-t}\right\}\frac{1}{(1+r)^{k-\frac{1}{2}}}dr. \tag{2.49}$$

Introducing the variable $\xi = |x| + r - t$, we have

$$
\begin{aligned}
I &\leq \frac{CA}{t\sqrt{|x|}} \int_0^{2|x|} \ln\left\{2 + \frac{|x|}{\xi}\right\} \frac{1}{(1+\xi+t-|x|)^{k-\frac{1}{2}}} d\xi \\
&\leq \frac{CA}{t\sqrt{|x|}(1+t-|x|)^{k-\frac{1}{2}}} \int_0^{2|x|} \ln\left\{\frac{3|x|}{\xi}\right\} d\xi \\
&= \frac{CA}{t\sqrt{|x|}(1+t-|x|)^{k-\frac{1}{2}}} [2|x|\ln(3|x|) - 2|x|\ln(2|x|) + 2|x|] \qquad (2.50) \\
&\leq \frac{CA}{\sqrt{t}(1+t-|x|)^{k-\frac{1}{2}}} \leq \frac{CA}{\sqrt{t+|x|}(1+t-|x|)^{k-\frac{1}{2}}} \\
&\leq \frac{CA}{\sqrt{1+t+|x|}(1+t-|x|)^{k-\frac{1}{2}}}.
\end{aligned}
$$

We now estimate II.

By (2.38), we have

$$
\begin{aligned}
II &\leq \frac{CA}{t} \int_0^{t-|x|} \frac{1}{\sqrt{t^2 - (|x|+r)^2}(1+r)^{k-1}} dr \\
&\leq \frac{CA}{t\sqrt{t+|x|}} \int_0^{t-|x|} \frac{1}{\sqrt{t-|x|-r}(1+r)^{k-1}} dr.
\end{aligned}
$$

Let

$$\rho = \sqrt{t-|x|-r}.$$

Then

$$
\begin{aligned}
II &\leq \frac{CA}{t\sqrt{t+|x|}} \int_0^{\sqrt{t-|x|}} \frac{1}{(1+t-|x|-\rho^2)^{k-1}} d\rho \\
&\leq \frac{CA}{t\sqrt{t+|x|}(1+t-|x|)^{\frac{k-1}{2}}} \int_0^{\sqrt{t-|x|}} \frac{1}{(\sqrt{1+t-|x|}-\rho)^{k-1}} d\rho.
\end{aligned} \qquad (2.51)
$$

In order to estimate II, we distinguish the following three cases.

Case II-A-1: $k > 2$

In the present situation, it follows from (2.51) that

$$
\begin{aligned}
II &\leq \frac{CA}{t\sqrt{t+|x|}(1+t-|x|)^{\frac{k-1}{2}}} \left\{\frac{1}{(\sqrt{1+t-|x|} - \sqrt{t-|x|})^{k-2}} - (\sqrt{1+t-|x|})^{k-2}\right\} \\
&\leq \frac{CA}{t\sqrt{t+|x|}(1+t-|x|)^{\frac{k-1}{2}}} (\sqrt{1+t-|x|} + \sqrt{t-|x|})^{k-2} \\
&\leq \frac{CA}{t\sqrt{t+|x|}\sqrt{1+t-|x|}} \leq \frac{CA}{\sqrt{1+t+|x|}\sqrt{1+t-|x|}}.
\end{aligned}
$$

Case II-A-2: $k = 2$

In this case, by (2.51) we have

$$\begin{aligned} II &\le \frac{CA}{t\sqrt{t+|x|}\sqrt{1+t-|x|}} \times \ln\left\{\frac{\sqrt{1+t-|x|}}{\sqrt{1+t-|x|}-\sqrt{t-|x|}}\right\} \\ &\le \frac{CA}{t\sqrt{t+|x|}\sqrt{1+t-|x|}} \times \frac{\sqrt{1+t-|x|}}{\sqrt{1+t-|x|}-\sqrt{t-|x|}} \\ &\le \frac{CA}{\sqrt{1+t+|x|}\sqrt{1+t-|x|}}. \end{aligned}$$

Case II-A-3: $1 < k < 2$

In this situation, we obtain from (2.51) that

$$\begin{aligned} II &\le \frac{CA}{t\sqrt{t+|x|}(1+t-|x|)^{\frac{k-1}{2}}} \left\{(1+t-|x|)^{\frac{-k+2}{2}} - (\sqrt{1+t-|x|}-\sqrt{t-|x|})^{-k+2}\right\} \\ &\le \frac{CA}{\sqrt{1+t+|x|}\sqrt{1+t-|x|}}. \end{aligned}$$

Summarizing the above argument gives

$$II \le \frac{CA}{\sqrt{1+t+|x|}\sqrt{1+t-|x|}}, \tag{2.52}$$

provided that $t + |x| \ge 1$.

Case II-B: $0 < t + |x| < 1$

As before, introducing the variable $r = |x - y|$, we have

$$\begin{aligned} \left|\frac{1}{2\pi t}\int_{|x-y|\le t} \frac{\phi_0(y)}{(t^2-|y-x|^2)^{\frac{1}{2}}}dy\right| &\le \frac{1}{2\pi t}\int_{|x-y|\le t} \frac{CA}{(t^2-|y-x|^2)^{\frac{1}{2}}(1+|y|)^k}dy \\ &\le \frac{CA}{\pi t}\int_0^t \frac{r}{\sqrt{t^2-r^2}}dr \le \frac{CA}{\sqrt{1+t+|x|}\sqrt{1+t-|x|}}. \end{aligned} \tag{2.53}$$

Combining (2.50) and (2.52)-(2.53) leads to

$$\left|\frac{1}{2\pi t}\int_{|x-y|\le t} \frac{\phi_0(y)}{(t^2-|y-x|^2)^{\frac{1}{2}}}dy\right| \le \frac{CA}{\sqrt{1+t+|x|}\sqrt{1+t-|x|}} \quad (|x| \le t). \tag{2.54}$$

(2.47) and (2.54) imply

$$\left|\frac{1}{2\pi t}\int_{|x-y|\le t} \frac{\phi_0(y)}{(t^2-|y-x|^2)^{\frac{1}{2}}}dy\right| \le \begin{cases} \dfrac{CA}{\sqrt{1+t+|x|}(1+|t-|x||)^{k-\frac{1}{2}}} & (|x| \ge t), \\ \dfrac{CA}{\sqrt{1+t+|x|}\sqrt{1+|t-|x||}} & (|x| \le t). \end{cases} \tag{2.55}$$

By Tsutaya [18], we have

$$\left|\frac{1}{2\pi}\int_{|x-y|\le t}\frac{\phi_1(y)}{(t^2-|y-x|^2)^{\frac{1}{2}}}dy\right| \le \begin{cases} \dfrac{CA}{\sqrt{1+t+|x|}(1+|t-|x||)^{k-\frac{1}{2}}} & (|x|\ge t), \\ \dfrac{CA}{\sqrt{1+t+|x|}\sqrt{1+|t-|x||}} & (|x|\le t) \end{cases} \tag{2.56}$$

and

$$\left|\frac{1}{2\pi t}\int_{|x-y|\le t}\frac{\nabla\phi_0(y)\cdot(y-x)}{(t^2-|y-x|^2)^{\frac{1}{2}}}dy\right| \le \begin{cases} \dfrac{CA}{\sqrt{1+t+|x|}(1+|t-|x||)^{k-\frac{1}{2}}} & (|x|\ge t), \\ \dfrac{CA}{\sqrt{1+t+|x|}\sqrt{1+|t-|x|}} & (|x|\le t). \end{cases} \tag{2.57}$$

(2.35) follows from (2.55)-(2.57) and (2.36) immediately. Thus, the proof of Lemma 2.7 is completed. ■

Lemma 2.8 *Suppose that ϕ is a solution to the Cauchy problem*

$$\phi_{tt}-\triangle\phi=g$$

with zero initial data. Then

$$|\phi(t,x)|\le C(1+t+|x|)^{-\frac{n-1}{2}}\sum_{|I|\le n-1}\int_0^t\|(Z^Ig)(\tau,\cdot)/(1+\tau+|\cdot|)^{\frac{n-1}{2}}\|_{L^1}d\tau. \tag{2.58}$$

In particular, for $n=2$ and $p\in(1,2]$ it holds that

$$\|\phi(t,\cdot)\|_{L^p(\mathbb{R}^2)}\le C(1+t)^{\frac{2}{p}-1}\int_0^t\|g(\tau,\cdot)\|_{L^1(\mathbb{R}^2)}d\tau. \tag{2.59}$$

Proof. The inequality (2.58) comes from Hömander [4] or Klainerman [11] directly, while the proof of (2.59) has been proved by Li and Zhou [15]. ■

Lemma 2.9 *Suppose that $n=2$, and suppose furthermore that $\phi=\phi(t,x)$ is a solution of the wave equation*

$$\phi_{tt}-\Delta\phi=|g_1g_2(t,x)| \tag{2.60}$$

with zero initial data. Then it holds that

$$\|\phi(t,\cdot)\|_{L^2(\mathbb{R}^2)}\le C(1+t)^{\frac{1}{4}}\left\{\sum_{|I|\le 1}\int_0^t(1+\tau)^{-\frac{1}{2}}\|\Gamma g_1(\tau,\cdot)\|^2_{L^2(\mathbb{R}^2)}d\tau\right\}^{\frac{1}{2}}\left\{\int_0^t\|g_2(\tau,\cdot)\|^2_{L^2(\mathbb{R}^2)}d\tau\right\}^{\frac{1}{2}} \tag{2.61}$$

and

$$(1+t)^{\frac{1}{2}}\|\phi(t,\cdot)\|_{L^\infty(\mathbb{R}^2)} \le C\left\{\int_0^t \sum_{|I|\le 1}\|\Gamma^I g_1(\tau,\cdot)\|_{L^2}^2 \frac{d\tau}{\sqrt{1+\tau}}\right\}^{\frac{1}{2}} \left\{\int_0^t \sum_{|I|\le 1}\|\Gamma^I g_2(\tau,\cdot)\|_{L^2}^2 \frac{d\tau}{\sqrt{1+\tau}}\right\}^{\frac{1}{2}}. \tag{2.62}$$

Proof. The proof of (2.61) can be found in [15]. In what follows, we prove (2.62).

Let E be the forward fundamental solution of the wave operator. By the positivity of E and the Hölder inequality, we have

$$\phi(t,x) \le E * |g_1 g_2(t,x)| \le \left(E * g_1^2(t,x)\right)^{\frac{1}{2}} \left(E * g_2^2(t,x)\right)^{\frac{1}{2}}. \tag{2.63}$$

It follows from Lemma 2.8 and Hölder inequality that

$$E * g_1^2(t,x) \le C(1+t)^{-\frac{1}{2}} \sum_{|I|\le 1} \int_0^t \|Z^I g_1(\tau,\cdot)\|_{L^2}^2 \frac{d\tau}{\sqrt{1+\tau}}. \tag{2.64}$$

Similarly,

$$E * g_2^2(t,x) \le C(1+t)^{-\frac{1}{2}} \sum_{|I|\le 1} \int_0^t \|Z^I g_2(\tau,\cdot)\|_{L^2}^2 \frac{d\tau}{\sqrt{1+\tau}}. \tag{2.65}$$

(2.62) comes from (2.63)-(2.65) immediately. This proves Lemma 2.9. ■

The following lemma can be found in Klainerman [12].

Lemma 2.10 *Assume that $p \in [1,\infty)$ and N is an integer satisfying $N > \frac{n}{p}$. Then it holds that*

$$|\phi(t,x)| \le C(1+t+|x|)^{-\frac{n-1}{p}}(1+|t-|x||)^{-\frac{1}{p}} \sum_{|I|\le N} \|Z^I\phi(t,\cdot)\|_{L^p}, \tag{2.66}$$

provided that all norms appearing on the right-hand side of (2.66) are bounded.

3 Lower bound of life-span

This section is devoted to the proof of Theorem 1.1. In order to prove Theorem 1.1, it suffices to show the following theorem.

Theorem 3.1 *Suppose that $u_0(x)$, $u_1(x) \in C^\infty(\mathbb{R}^2)$ and satisfy that there exist two positive constants $A \in \mathbb{R}^+$ and $k \in \mathbb{R}^+$ such that*

$$|u_0(x)| \le \frac{A}{(1+|x|)^k}, \quad |u_1(x)| \le \frac{A}{(1+|x|)^{k+1}} \quad (k>1).$$

Then there exist two positive constants δ and ε_0 such that for any fixed $\varepsilon \in (0, \varepsilon_0]$, the Cauchy problem (1.8)-(1.9) has a unique C^∞ solution on the interval $[0, T_\varepsilon]$, where T_ε is given by

$$T_\varepsilon = \frac{\delta}{\varepsilon^{\frac{4}{3}}} - 1. \tag{3.1}$$

Proof. The local existence of classical solutions has been proved by the method of Picard iteration (see Sogge [17] and Hörmander [5]). In what follows, we prove Theorem 3.1 by the method of continuous induction, or say, the bootstrap argument.

Let l_1 and l_2 be two positive integers such that

$$l_1 - 3 \geq l_2 \geq \frac{1}{2}[l_1] + 1.$$

Introduce

$$\begin{cases} M_1(t) = \sum_{|I| \leq l_1} \|\partial Z^I u(t, \cdot)\|_{L^2(\mathbb{R}^2)}, \\ M_2(t) = \sum_{|I| \leq l_1} \|Z^I u(t, \cdot)\|_{L^2(\mathbb{R}^2)}, \\ N_1(t) = \sum_{|I| \leq l_2} \|\partial Z^I u(t, \cdot)\|_{L^\infty(\mathbb{R}^2)}, \\ N_2(t) = \sum_{|I| \leq l_2} \|Z^I u(t, \cdot)\|_{L^\infty(\mathbb{R}^2)}. \end{cases} \tag{3.2}$$

By the bootstrap argument, for the time being it is supposed that there exist some positive constants M_i, N_i $(i = 1, 2)$ and μ such that

$$\begin{cases} M_1(t) \leq M_1 \varepsilon, \\ M_2(t) \leq M_2 \varepsilon (1 + t)^{\frac{1}{4}}, \\ (1 + t)^{\frac{1}{2}} N_1(t) \leq N_1 \varepsilon, \\ (1 + t)^{\frac{1}{2}} N_2(t) \leq N_2 \varepsilon, \end{cases} \tag{3.3}$$

provided that ε, μ are suitably small and satisfy

$$\varepsilon (1 + t)^{\frac{3}{4}} \leq \mu. \tag{3.3a}$$

According to the bootstrap argument, in what follows we show that, by choosing M_i and N_i $(i=1,2)$ sufficiently large and ε suitably small such that

$$\begin{cases} M_1(t) \le \dfrac{1}{2}M_1\varepsilon, \\ M_2(t) \le \dfrac{1}{2}M_2\varepsilon(1+t)^{\frac{1}{4}}, \\ (1+t)^{\frac{1}{2}}N_1(t) \le \dfrac{1}{2}N_1\varepsilon, \\ (1+t)^{\frac{1}{2}}N_2(t) \le \dfrac{1}{2}N_2\varepsilon, \end{cases} \tag{3.3b}$$

provided that $\varepsilon,\ \mu$ are suitably small and (3.3a) holds.

We first estimate $M_1(t)$.

The equation (1.8) can be rewritten as

$$\Box u = (e^{-u}-1)\Delta u - u_t^2. \tag{3.4}$$

It follows from Lemma 2.1 and (3.4) that

$$\begin{aligned} \Box Z^I u \;\; = \;\; & \sum_{|I_1|+|I_2|\le|I|}\ \sum_{0\le i_1,i_2\le 2} A_{II_1I_2} Z^{I_1}(e^{-u}-1)\partial_{i_1i_2}Z^{I_2}u + \\ & \sum_{|I_1|+|I_2|\le|I|}\ \sum_{0\le i_1,i_2\le 2} \tilde{A}_{II_1I_2}\partial_{i_1}Z^{I_1}u\partial_{i_2}Z^{I_2}u. \end{aligned} \tag{3.5}$$

By Minkowski inequality, (3.2) and Lemma 2.5, for I with $|I|\le l_1-1$ we have

$$\begin{aligned} & \left\| \sum_{|I_1|+|I_2|\le|I|}\ \sum_{0\le i_1,i_2\le 2} A_{II_1I_2} Z^{I_1}(e^{-u}-1)\partial_{i_1i_2}Z^{I_2}u(t,\cdot) \right\|_{L^2} \\ & \le C \sum_{|I_1|+|I_2|\le|I|,|I_1|>|I_2|}\ \sum_{0\le i_1,i_2\le 2} \|Z^{I_1}(e^{-u}-1)\partial_{i_1i_2}Z^{I_2}u(t,\cdot)\|_{L^2} + \\ & \quad C \sum_{|I_1|+|I_2|\le|I|,|I_1|\le|I_2|}\ \sum_{0\le i_1,i_2\le 2} \|Z^{I_1}(e^{-u}-1)\partial_{i_1i_2}Z^{I_2}u(t,\cdot)\|_{L^2} \\ & \le C \sum_{|I_1|+|I_2|\le|I|,|I_1|>|I_2|}\ \sum_{0\le i_1,i_2\le 2} \|Z^{I_1}(e^{-u}-1)\|_{L^2}\|\partial_{i_1i_2}Z^{I_2}u(t,\cdot)\|_{L^\infty} + \\ & \quad C \sum_{|I_1|+|I_2|\le|I|,|I_1|\le|I_2|}\ \sum_{0\le i_1,i_2\le 2} \|Z^{I_1}(e^{-u}-1)\|_{L^\infty}\|\partial_{i_1i_2}Z^{I_2}u(t,\cdot)\|_{L^2} \\ & \le C\{M_2(t)N_1(t)+M_1(t)N_2(t)\}. \end{aligned} \tag{3.6}$$

provided that $\varepsilon_0>0$ is suitably small.

Again by Minkowski inequality and (3.2), for I with $|I| \le l_1 - 1$ we get

$$\begin{aligned} & \left\| \sum_{|I_1|+|I_2|\le|I|} \sum_{0\le i_1,i_2\le 2} \tilde{A}_{II_1I_2} \partial_{i_1} Z^{I_1} u \partial_{i_2} Z^{I_2} u \right\|_{L^2} \\ & \le C \sum_{|I_1|+|I_2|\le|I|, |I_1|>|I_2|} \sum_{0\le i_1,i_2\le 2} \|\partial_{i_1} Z^{I_1} u(t,\cdot)\|_{L^2} \|\partial_{i_2} Z^{I_1} u(t,\cdot)\|_{L^\infty} + \\ & \quad C \sum_{|I_1|+|I_2|\le|I|, |I_1|\le|I_2|} \sum_{0\le i_1,i_2\le 2} \|\partial_{i_1} Z^{I_1} u(t,\cdot)\|_{L^\infty} \|\partial_{i_2} Z^{I_1} u(t,\cdot)\|_{L^2} \\ & \le CM_1(t)N_1(t). \end{aligned} \tag{3.7}$$

On the other hand, noting Lemma 2.2 and using (3.5)-(3.7), for I with $|I| \le l_1 - 1$ we have

$$\|\partial Z^I u(t,\cdot)\|_{L^2} \le C\|\partial Z^I u(0,\cdot)\|_{L^2} + C\int_0^t [M_2(\tau)N_1(\tau) + M_1(\tau)N_2(\tau) + M_1(\tau)N_1(\tau)]d\tau. \tag{3.8}$$

We now estimate $\|\partial Z^I u(t,\cdot)\|_{L^2}$ ($|I| = l_1$).

By Lemma 2.1 and (3.4),

$$\begin{aligned} \Box Z^I u &= Z^I \Box u + \sum_{|J|\le|I|-1} A_{IJ} Z^J \Box u \\ &= \sum_{i,j=0}^{2} (e^{-u}-1)\partial_i\partial_j Z^I u + \sum_{|I_1|+|I_2|\le|I|, |I_2|\le|I|-1} \sum_{0\le i_1,i_2\le 2} \bar{A}_{II_1I_2} Z^{I_1}(e^{-u}-1)\partial_{i_1i_2} Z^{I_2} u \\ &\quad + \sum_{|I_1|+|I_2|\le|I|,} \sum_{0\le i_1,i_2\le 2} \bar{\bar{A}}_{II_1I_2} \partial_{i_1} Z^{I_1} u \partial_{i_2} Z^{I_2} u. \end{aligned}$$

Hence

$$\begin{aligned} \Box Z^I u + \sum_{i,j=0}^{2} (1-e^{-u})\partial_i\partial_j Z^I u &= \sum_{|I_1|+|I_2|\le|I|, |I_2|\le|I|-1} \sum_{0\le i_1,i_2\le 2} \bar{A}_{II_1I_2} Z^{I_1}(e^{-u}-1)\partial_{i_1i_2} Z^{I_2} u \\ &\quad + \sum_{|I_1|+|I_2|\le|I|,} \sum_{0\le i_1,i_2\le 2} \bar{\bar{A}}_{II_1I_2} \partial_{i_1} Z^{I_1} u \partial_{i_2} Z^{I_2} u. \end{aligned} \tag{3.9}$$

Similar to the proof of (3.6), when $\varepsilon_0 > 0$ is suitably small, for I with $|I| = l_1$ it holds that

$$\Big\| \sum_{|I_1|+|I_2|\le|I|, |I_2|\le|I|-1} \sum_{0\le i_1,i_2\le 2} \bar{A}_{II_1I_2} Z^{I_1}(e^{-u}-1)\partial_{i_1i_2} Z^{I_2} u(t,\cdot)\Big\|_{L^2} \le C[M_2(t)N_1(t)+M_1(t)N_2(t)]. \tag{3.10}$$

Similar to the proof of (3.7), for I with $|I| = l_1$ we have

$$\| \sum_{|I_1|+|I_2|\le|I|,\ 0\le i_1,i_2\le 2} \sum \bar{\bar{A}}_{II_1I_2}\partial_{i_1}Z^{I_1}u\partial_{i_2}Z^{I_2}u(t,\cdot)\|_{L^2} \le CM_1(t)N_1(t). \tag{3.11}$$

On the other hand, because of (3.3), it holds that

$$\sum_{i,j=0}^{2}|\gamma^{ij}| \le CN_2\varepsilon < \frac{1}{2}, \tag{3.12}$$

provided that ε is suitably small, where $\gamma^{ij} = (1 - e^{-u})$. Moreover, for $|\dot{\gamma}(t)|$ defined as in Lemma 2.4, we have

$$2\int_0^t |\dot{\gamma}(\tau)|d\tau \le CN_1\mu \le \ln 2, \tag{3.13}$$

provided that ε, μ are suitably small and (3.3a) holds. Thus, noting Lemma 2.4 and using (3.9)-(3.13), for I with $|I| = l_1$ we have

$$\|\partial Z^I u(t,\cdot)\|_{L^2} \le 4\|\partial Z^I u(0,\cdot)\|_{L^2} + C\int_0^t (M_2(\tau)N_1(\tau) + M_1(\tau)N_2(\tau) + M_1(\tau)N_1(\tau))d\tau. \tag{3.14}$$

Combining (3.8) and (3.14) yields

$$M_1(t) \le K_1\varepsilon + C\int_0^t (M_2(\tau)N_1(\tau) + M_1(\tau)N_2(\tau) + M_1(\tau)N_1(\tau))d\tau. \tag{3.15}$$

We next estimate $M_2(t)$.

In the present situation, the equation (1.8) can be rewritten as

$$\Box u = \sum_{i=1}^{2}\partial_i((e^{-u}-1)\partial_i u) - u_t^2 - \sum_{i=1}^{2}\partial_i(e^{-u}-1)\partial_i u. \tag{3.16}$$

By Lemma 2.1 and (3.16), we obtain

$$\begin{aligned}\Box Z^I u \quad = \quad & \sum_{|I_1|+|I_2|\le|I|}\sum_{0\le i_1,i_2\le 2} B_{II_1I_2}\partial_{i_1}\left(Z_1^I(e^{-u}-1)\partial_{i_2}Z^{I_2}u\right)+ \\ & \sum_{|I_1|+|I_2|\le|I|}\sum_{0\le i_1,i_2\le 2} \bar{B}_{II_1I_2}\partial_{i_1}Z^{I_1}u\partial_{i_2}Z^{I_2}u+ \\ & \sum_{|I_1|+|I_2|\le|I|}\sum_{0\le i_1,i_2\le 2} \tilde{B}_{II_1I_2}\partial_{i_1}\left(Z^{I_1}(e^{-u}-1)\right)\partial_{i_2}Z^{I_2}u.\end{aligned}$$

Let

$$Z^I u = w_0 + w_1 + w_2 + w_3, \tag{3.17}$$

where w_0, w_1, w_2 and w_3 satisfy

$$\Box w_0 = 0, \quad w_0|_{t=0} = Z^I u(0,x), \quad \left.\frac{\partial w_0}{\partial t}\right|_{t=0} = \frac{\partial(Z^I u)}{\partial t}(0,x), \tag{3.18}$$

$$\Box w_1 = \sum_{|I_1|+|I_2|\le|I|}\sum_{0\le i_1,i_2\le 2} B_{II_1I_2}\partial_{i_1}\left(Z_1^I(e^{-u}-1)\partial_{i_2}Z^{I_2}u\right), \quad w_1|_{t=0} = \left.\frac{\partial w_1}{\partial t}\right|_{t=0} = 0, \tag{3.19}$$

$$\Box w_2 = \sum_{|I_1|+|I_2|\le|I|}\sum_{0\le i_1,i_2\le 2} \bar{B}_{II_1I_2}\partial_{i_1}Z^{I_1}u\partial_{i_2}Z^{I_2}u,, \quad w_2|_{t=0} = \left.\frac{\partial w_2}{\partial t}\right|_{t=0} = 0, \tag{3.20}$$

and

$$\Box w_3 = \sum_{|I_1|+|I_2|\le|I|}\sum_{0\le i_1,i_2\le 2} \tilde{B}_{II_1I_2}\partial_{i_1}\left(Z^{I_1}(e^{-u}-1)\right)\partial_{i_2}Z^{I_2}u, \quad w_3|_{t=0} = \left.\frac{\partial w_3}{\partial t}\right|_{t=0} = 0, \tag{3.21}$$

respectively. Thanks to Lemma 2.7 and (3.18), we have

$$\|w_0(t,\cdot)\|_{L^2} \le \begin{cases} C\varepsilon\sqrt{\ln(2+t)} \quad (|x|\le t), \\ C\varepsilon \quad (|x|\ge t). \end{cases} \tag{3.22}$$

When $\varepsilon_0 > 0$ is suitably small, noting Lemmas 2.3, 2.5 and using (3.19) and (3.2), we obtain

$$\begin{aligned} \|w_1(t,\cdot)\|_{L^2} &\le C\Bigg(\int_0^t \sum_{|I_1|+|I_2|\le|I|}\sum_{0\le i\le 2}\|Z^{I_1}(e^{-u}-1)\partial_i Z^{I_2}u(\tau,\cdot)\|_{L^2}d\tau + \\ &\qquad \sum_{|I_1|+|I_2|\le|I|}\sum_{0\le i\le 2}\|Z^{I_1}(e^{-u}-1)\partial_i Z^{I_2}u(0,\cdot)\|_{L^2}\Bigg) \\ &\le C\int_0^t \left[M_2(\tau)N_1(\tau)+M_1(\tau)N_2(\tau)\right]d\tau + C\varepsilon. \end{aligned} \tag{3.23}$$

Noting Lemmas 2.9, 2.1 and using (3.20), (3.2) gives

$$\begin{aligned} \|w_2(t,\cdot)\|_{L^2} &\le C(1+t)^{\frac14}\sum_{|I_1|+|I_2|\le|I|,|I_1|>|I_2|}\sum_{|J|\le 1}\sum_{0\le i_1,i_2\le 2}\left\{\int_0^t (1+\tau)^{-\frac12}\|Z^J\partial_{i_1}Z^{I_2}u(\tau,\cdot)\|_{L^2}^2\right\}^{\frac12} \\ &\qquad \times\left\{\int_0^t \|\partial_{i_2}Z^{I_1}u(\tau,\cdot)\|_{L^2}^2 d\tau\right\}^{\frac12} + \\ &\quad C(1+t)^{\frac14}\sum_{|I_1|+|I_2|\le|I|,|I_1|\le|I_2|}\sum_{|J|\le 1}\sum_{0\le i_1,i_2\le 2}\left\{\int_0^t (1+\tau)^{-\frac12}\|Z^J\partial_{i_1}Z^{I_1}u(\tau,\cdot)\|_{L^2}^2\right\}^{\frac12} \\ &\qquad \times\left\{\int_0^t \|\partial_{i_2}Z^{I_2}u(\tau,\cdot)\|_{L^2}^2 d\tau\right\}^{\frac12} \\ &\le C(1+t)^{\frac14}\left\{\int_0^t (1+\tau)^{-\frac12}M_1^2(\tau)d\tau\right\}^{\frac12}\left\{\int_0^t M_1^2(\tau)d\tau\right\}^{\frac12}. \end{aligned} \tag{3.24}$$

By Lemma 2.9, Minkowski inequality, Lemmas 2.1, 2.6 and (3.2), it follows from (3.21) that

$$
\begin{aligned}
\|w_3(t,\cdot)\|_{L^2} \leq C(1+t)^{\frac{1}{4}} & \sum_{|I_1|+|I_2|\leq|I|,|I_1|>|I_2|} \sum_{|J|\leq 1} \sum_{0\leq i_1,i_2\leq 2} \left\{\int_0^t (1+\tau)^{-\frac{1}{2}} \|Z^J\partial_{i_2} Z^{I_2} u(\tau,\cdot)\|_{L^2}^2 d\tau\right\}^{\frac{1}{2}} \\
& \times \left\{\int_0^t \|\partial_{i_1} Z^{I_1}(e^{-u}-1)(\tau,\cdot)\|_{L^2} d\tau\right\}^{\frac{1}{2}} + \\
C(1+t)^{\frac{1}{4}} & \sum_{|I_1|+|I_2|\leq|I|,|I_1|\leq|I_2|} \sum_{|J|\leq 1} \sum_{0\leq i_1,i_2\leq 2} \left\{\int_0^t (1+\tau)^{-\frac{1}{2}} \|Z^J\partial_{i_1} Z^{I_1}(e^{-u}-1)(\tau,\cdot)\|_{L^2}^2 d\tau\right\}^{\frac{1}{2}} \\
& \times \left\{\int_0^t \|\partial_{i_2} Z^{I_2} u(\tau,\cdot)\|_{L^2} d\tau\right\}^{\frac{1}{2}} \\
\leq C(1+t)^{\frac{1}{4}} & \sum_{|I_1|+|I_2|\leq|I|,|I_1|>|I_2|} \sum_{|J|\leq 1} \sum_{0\leq i_1,i_2\leq 2} \left\{\int_0^t (1+\tau)^{-\frac{1}{2}} \|Z^J\partial_{i_2} Z^{I_2} u(\tau,\cdot)\|_{L^2}^2 d\tau\right\}^{\frac{1}{2}} \\
& \times \left\{\int_0^t (\|Z^{I_1}((e^{-u}-1)\partial_{i_1} u)(\tau,\cdot)\|_{L^2}^2 + \|Z^{I_1}\partial_{i_1} u(\tau,\cdot)\|_{L^2}^2) d\tau\right\}^{\frac{1}{2}} + \\
C(1+t)^{\frac{1}{4}} & \sum_{|I_1|+|I_2|\leq|I|,|I_1|\leq|I_2|} \sum_{|J|\leq 1} \sum_{0\leq i_1,i_2\leq 2} \left\{\int_0^t (1+\tau)^{-\frac{1}{2}} \left[\|Z^J Z^{I_1}((e^{-u}-1)\partial_{i_1} u)(\tau,\cdot)\|_{L^2}^2 + \right.\right. \\
& \left.\left. \|Z^J Z^{I_1}\partial_{i_1} u(\tau,\cdot)\|_{L^2}^2\right] d\tau\right\}^{\frac{1}{2}} \times \left\{\int_0^t \|\partial_{i_2} Z^{I_2} u(\tau,\cdot)\|_{L^2} d\tau\right\}^{\frac{1}{2}} \\
\leq C(1+t)^{\frac{1}{4}} & \left\{\int_0^t (1+\tau)^{-\frac{1}{2}} M_1^2(\tau) d\tau\right\}^{\frac{1}{2}} \left\{\int_0^t \left[M_1^2(\tau)N_2^2(\tau) + M_2^2(\tau)N_1^2(\tau) + M_1^2(\tau)\right] d\tau\right\}^{\frac{1}{2}} + \\
C(1+t)^{\frac{1}{4}} & \left\{\int_0^t (1+\tau)^{-\frac{1}{2}} \left[M_1^2(\tau)N_2^2(\tau) + M_2^2(\tau)N_1^2(\tau) + M_1^2(\tau)\right] d\tau\right\}^{\frac{1}{2}} \times \left\{\int_0^t M_1^2(\tau) d\tau\right\}^{\frac{1}{2}} \\
\leq C(1+t)^{\frac{1}{4}} & \left\{\int_0^t (1+\tau)^{-\frac{1}{2}} \left[M_1^2(\tau)N_2^2(\tau) + M_2^2(\tau)N_1^2(\tau) + M_1^2(\tau)\right] d\tau\right\}^{\frac{1}{2}} \\
& \times \left\{\int_0^t \left[M_1^2(\tau)N_2^2(\tau) + M_2^2(\tau)N_1^2(\tau) + M_1^2(\tau)\right] d\tau\right\}^{\frac{1}{2}}.
\end{aligned} \tag{3.25}
$$

provided that $\varepsilon_0 > 0$ is suitably small. Thus, combining (3.17) and (3.22)-(3.25) yields

$$\begin{aligned} M_2(t) \le K_2\varepsilon\sqrt{\ln(2+t)} + C\int_0^t (M_2(\tau)N_1(\tau) + M_1(\tau)N_2(\tau))d\tau + \\ C(1+t)^{\frac{1}{4}}\left\{\int_0^t (1+\tau)^{-\frac{1}{2}}\left[M_1^2(\tau)N_2^2(\tau) + M_2^2(\tau)N_1^2(\tau) + M_1^2(\tau)\right]d\tau\right\}^{\frac{1}{2}} \\ \times\left\{\int_0^t \left[M_1^2(\tau)N_2^2(\tau) + M_2^2(\tau)N_1^2(\tau) + M_1^2(\tau)\right]d\tau\right\}^{\frac{1}{2}}. \end{aligned} \tag{3.26}$$

We now estimate $N_2(t)$.

Using Lemma 2.7, we obtain from (3.18) that

$$(1+t)^{\frac{1}{2}}\|w_0(t,\cdot)\|_{L^\infty} \le C\varepsilon. \tag{3.27}$$

Noting Lemmas 2.3, 2.5 and using (3.19), (3.2), when $\varepsilon_0 > 0$ is suitably small, we have

$$\begin{aligned} (1+t)^{\frac{1}{2}}\|w_1(t,\cdot)\|_{L^\infty} \le C\int_0^t (1+\tau)^{\frac{1}{2}} \sum_{|I_1|+|I_2|\le|I|}\sum_{0\le i\le 2}\|Z^{I_1}(e^{-u}-1)\partial_i Z^{I_2}u(\tau,\cdot)\|_{L^\infty}d\tau + \\ C\int_0^t (1+\tau)^{-\frac{3}{2}}\sum_{|I_1|+|I_2|\le|I|}\sum_{|J|\le 3}\sum_{0\le i\le 2}\|Z^J\left(Z^{I_1}(e^{-u}-1)\partial_i Z^{I_2}u\right)(\tau,\cdot)\|_{L^1}d\tau \\ \le C\left\{\int_0^t (1+\tau)^{\frac{1}{2}}N_1(\tau)N_2(\tau)d\tau + \int_0^t (1+\tau)^{-\frac{3}{2}}M_1(\tau)M_2(t)d\tau\right\}. \end{aligned} \tag{3.28}$$

Noting Lemma 2.9 and using (3.20) and (3.2), we obtain

$$\begin{aligned} (1+t)^{\frac{1}{2}}\|w_2(t,\cdot)\|_{L^\infty} \;\le\; & C\sum_{|I_1|+|I_2|\le|I|}\sum_{|J|=1}\sum_{0\le i_1,i_2\le 2}\left\{\int_0^t \|Z^J(\partial_{i_1}Z^{I_1}u)(\tau,\cdot)\|_{L^2}^2\frac{d\tau}{\sqrt{1+\tau}}\right\}^{\frac{1}{2}} \\ & \times\left\{\int_0^t \|Z^J(\partial_{i_2}Z^{I_2}u)(\tau,\cdot)\|_{L^2}^2\frac{d\tau}{\sqrt{1+\tau}}\right\}^{\frac{1}{2}} \\ \le\; & C\int_0^t M_1^2(\tau)\frac{d\tau}{\sqrt{1+\tau}}. \end{aligned} \tag{3.29}$$

By Lemmas 2.9, 2.1, 2.6, Minkowski inequality and (3.2), when $\varepsilon_0 > 0$ is suitably small, we obtain from (3.21) that

$$
\begin{aligned}
(1+t)^{\frac{1}{2}}\|w_3(t,\cdot)\|_{L^\infty} &\le C \sum_{|I_1|+|I_2|\le|I|}\sum_{|J|\le 1}\sum_{0\le i_1,i_2\le 2}\left\{\int_0^t \|Z^J\left(\partial_{i_1}Z^{I_1}(e^{-u}-1)\right)(\tau,\cdot)\|_{L^2}^2\frac{d\tau}{\sqrt{1+\tau}}\right\}^{\frac{1}{2}} \\
&\qquad\times\left\{\int_0^t \|Z^J(\partial_{i_2}Z^{I_2}u)(\tau,\cdot)\|_{L^2}^2\frac{d\tau}{\sqrt{1+\tau}}\right\}^{\frac{1}{2}} \\
&\le C\left\{\int_0^t (1+\tau)^{-\frac{1}{2}}\left[M_1^2(\tau)N_2^2(\tau)+M_2^2(\tau)N_1^2(\tau)+M_1^2(\tau)\right]d\tau\right\}^{\frac{1}{2}}\left\{\int_0^t(1+\tau)^{-\frac{1}{2}}M_1^2(\tau)d\tau\right\}^{\frac{1}{2}}.
\end{aligned}
\tag{3.30}
$$

Collecting (3.17) and (3.27)-(3.30) gives

$$
\begin{aligned}
(1+t)^{\frac{1}{2}}N_2(t) &\le K_3\varepsilon + C\int_0^t (1+\tau)^{\frac{1}{2}}N_1(\tau)N_2(\tau)d\tau + C\int_0^t(1+\tau)^{-\frac{3}{2}}M_1(\tau)M_2(t)d\tau \\
&\quad + C\left\{\int_0^t(1+\tau)^{-\frac{1}{2}}(M_1^2(\tau)N_2^2(\tau)+M_2^2(\tau)N_1^2(\tau)+M_1^2(\tau))d\tau\right\}^{\frac{1}{2}}\left\{\int_0^t(1+\tau)^{-\frac{1}{2}}M_1^2(\tau)d\tau\right\}^{\frac{1}{2}}.
\end{aligned}
\tag{3.31}
$$

Since, for the time being it supposed that (3.3) holds, (3.12) and (3.13) are true, provided that ε_0 is suitably small, and then, by (3.15), (3.26) and (3.31) it holds that

$$M_1(t) \le K_1\varepsilon + C(M_2N_1 + M_1N_2 + M_1N_1)\varepsilon^2(1+t)^{\frac{3}{4}}, \tag{3.32}$$

$$M_2(t) \le K_2\varepsilon\sqrt{\ln(2+t)} + C(M_2N_1 + M_1N_2 + M_1N_1)\varepsilon^2(1+t) \tag{3.33}$$

and

$$(1+t)^{\frac{1}{2}}N_2(t) \le K_3\varepsilon + C(N_1N_2 + M_1M_2 + M_2N_1 + M_1N_2 + M_1^2)\varepsilon^2\sqrt{1+t}. \tag{3.34}$$

On the other hand, by Lemma 2.10, we have

$$(1+t)^{\frac{1}{2}}N_1(t) \le CM_1(t). \tag{3.35}$$

Thus, choosing

$$M_1 \ge 4K_1, \quad M_2 \ge 4K_2, \quad N_2 \ge 4K_3, N_1 \ge 2CM_1,$$

we obtain from (3.32)-(3.35) that

$$
\begin{cases}
M_1(t) \le \frac{1}{2}M_1\varepsilon, \\
M_2(t) \le \frac{1}{2}M_2\varepsilon(1+t)^{\frac{1}{4}}, \\
(1+t)^{\frac{1}{2}}N_1(t) \le \frac{1}{2}N_1\varepsilon, \\
(1+t)^{\frac{1}{2}}N_2(t) \le \frac{1}{2}N_2\varepsilon,
\end{cases}
\tag{3.36}
$$

provided that ε_0 and μ are suitably small and satisfy

$$\varepsilon_0(1+t)^{\frac{3}{4}} \le \mu.$$

Take $\delta = \mu^{\frac{4}{3}}$. Thus, the proof of Theorem 3.1 is completed. ■

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References

[1] W.-R. Dai, D.-X. Kong and K.-F. Liu, *Hyperbolic geometric flow (I): short- time existence and nonlinear stability*, to appear in Pure and Applied Mathematics Quarterly: Special Issue in honor of Michael Atiyah and Isadore Singer **6** (2010).

[2] W.-R. Dai, D.-X. Kong and K.-F. Liu, *Dissipative hyperbolic geometric flow*, Asian J. Math. **12** (2008), 345-364.

[3] H. Lindblad, *On the lifespan of solutions of nonlinear wave equations with small initial data*, Comm. Pure Appl. Math. **43** (1990), 445-472.

[4] L. Hörmander, L^1-L^∞ estimates for the wave operators, in Analyse Mathématique et Applications, Gauthier-Villars, Pairs, 1988.

[5] L. Hörmander, Lectures on Nonlinear Hyperbolic Differential Equations, Mathématique and Applications, **26**, Spring-Verlag, 1997.

[6] D.-X. Kong, *Life-span of classical solutions to quasilinear reducible hyperbolic systems and its applications*, Chin. Ann. of Math. **13A** (1992), 188-195.

[7] D.-X. Kong, *Hyperbolic geometric flow*, the Proceedings of ICCM 2007, Vol. II, Higher Educationial Press, Beijing, 2007, 95-110.

[8] D.-X. Kong and K.-F. Liu, *Wave character of metrics and hyperbolic geometric flow*, J. Math. Phys. **48** (2007), 103508-1-103508-14.

[9] D.-X. Kong, K.-F. Liu and D.-L. Xu, *The hyperbolic geometric flow on Riemann surfaces* , Comm. in Partial Differential Equation **34** (2009), 553-580.

[10] S. Klainerman, *Global existence for nonlinear wave equation*, Comm. Pure Appl. Math. **33** (1980), 43-101.

[11] S. Klainerman, *Uniform decay estimates and the lorentz invariance of the classical wave equations*, Comm. Pure Appl. Math. **38** (1985), 321-332.

[12] S. Klainerman, *Remarks on the global Sobolev inequalities in the Minkowski space* $\mathbb{R}^{n+1}$, Comm. Pure Appl. Math. **40** (1987), 111-116.

[13] M. Kovalyov, *Long-time behaviour of solutions of systems of nonlinear wave equations*, Comm. in Partial Differential Equation **12** (1987), 471-501.

[14] T.-T. Li and Y.-M. Chen, *Initial value problems for nonlinear wave equations* , Comm. in Partial Differential Equation **13** (1988), 383-422.

[15] T.-T. Li and Y. Zhou, *Life-span of classical solutions to nonlinear wave equations in two space dimensions* , J. Math. Pures Appl. **2** (1994), 223-249.

[16] T.-T. Li and Y. Zhou, *Life-span of classical solutions to fully nonlinear wave equations II* , Nonlinear Analysis, Theory, Methods and Applications **19** (1992), 833-853.

[17] C. D. Sogge, Lectures on Nonlinear Wave Equations, Monographs in Analysis, **II**, International Press, 1995.

[18] K. Tsutaya, *Global existence theorem for semilinear wave equations with non-compact data in two space dimensions*, J. Differential Equations **104** (1993), 332-360.